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Birth–Death Processes with Two-Type Catastrophes

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Abstract: This paper concentrates on the general birth–death processes with two different types of catastrophes. The Laplace transform of transition probability function for birth–death processes with two-type catastrophes is successfully expressed with the Laplace transform of transition probability function of the birth–death processes without catastrophe. The first effective catastrophe occurrence time is considered. The Laplace transform of its probability density function, expectation and variance are obtained.

Keywords: birth–death process; catastrophe; Laplace transform; probability density function

MSC: 60J27; 60J35

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1. Introduction

The Markov process is a very important branch of stochastic processes that has a very wide range of applications. Standard references are Anderson [1], Asmussen [2], Chen [3] and others.

The birth–death process, as a very important class of Markov processes, has been widely applied in finance, communications, population science and queueing theory. In the past few decades, there have been many works on generalizing the ordinary birth–death process, making the theory of birth–death processes more and more fruitful. Recently, the stochastic models with catastrophe have aroused much research interest. For example, Chen Zhang and Liu [4], Economou and Fakinos [5] and Pakes [6] considered the instantaneous distribution of continuous-time Markov chains with catastrophes. Chen and Renshaw [7,8] analyzed the effect of catastrophes on the $M/M/1$ queuing model. Zhang and Li [9] extended these results to the $M/M/c$ queuing model with catastrophes. Li and Zhang [10] further considered the effect of catastrophes on the $M^X/M/c$ queuing model. Di Crescenzo et al. [11] discussed the probability distribution and the relevant numerical characteristics of the first occurrence time of an effective disaster for a general birth–death process with catastrophes. Other related works can be found in Artalejo [12], Bayer and Boxma [13], Chen, Pollett, Li and Zhang [14], Dudin and Karolik [15], Gelenbe [16], Gelenbe, Glynn and Sigman [17], Jain and Sigman [18], Zeifman and Korotysheva [19] and many others.

The models considered in the above references involve only one type of catastrophe. However, in real situations, there may be multiple types of catastrophes involved in a stochastic model. For example, earthquakes and fires have a huge influence on a biological population. Wars and epidemics affect population in a country. In general, different catastrophes may have completely different effects. Therefore, a natural and important problem is considering the first occurrence time of an effective catastrophe (including different types of catastrophes) and determiining the type of the first effective catastrophe.

The purpose of this paper is to consider the general birth–death processes with two-type catastrophes. We mainly discuss the property of the first occurrence time of effective catastrophe and the type of the first effective catastrophe.

We start our discussion by presenting the infinitesimal generator, i.e., the so-called q-matrix.

Definition 1. *Let $\{N_t : t \geq 0\}$ be a continuous-time Markov chain on state space $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$, if its q-matrix $Q = (q_{ij} : i, j \in \mathbf{Z}_+)$ is given by*

$$Q = \hat{Q} + Q_d, \tag{1}$$

where $\hat{Q} = (\hat{q}_{ij} : i, j \in \mathbf{Z}_+)$ and $Q_d = (q_{ij}^{(d)} : i, j \in \mathbf{Z}_+)$ are given by

$$\hat{q}_{ij} = \begin{cases} \lambda_i, & i \geq 0,\ j = i+1, \\ \mu_i, & i \geq 1,\ j = i-1, \\ -\lambda_0, & i = j = 0, \\ -\omega_i, & i = j \geq 1, \\ 0, & otherwise. \end{cases} \tag{2}$$

and

$$q_{ij}^{(d)} = \begin{cases} \beta, & i = 0 \text{ or } i \geq 2, j = 1, \\ \alpha, & i \geq 1,\ j = 0, \\ -\beta, & i = j = 0, \\ -\alpha, & i = j = 1, \\ -\gamma, & i = j \geq 2, \\ 0, & otherwise. \end{cases} \tag{3}$$

with $\alpha, \beta \geq 0, \lambda_i > 0\ (i \geq 0),\ \mu_i > 0\ (i \geq 1)$ and $\omega_i = \lambda_i + \mu_i\ (i \geq 1),\ \gamma = \alpha + \beta$, respectively.

Then, $\{N_t : t \geq 0\}$ is called a birth–death processes with two-type catastrophes. Its probability transition function is denoted by $P(t) = (p_{ij}(t) : i, j \in \mathbf{Z}_+)$ and the corresponding resolvent is denoted by $\Pi(\lambda) = (\pi_{j,n}(\lambda) : j, n \in \mathbf{Z}_+)$.

Remark 1. *By Definition 1, α and β describe the rates of catastrophes. We call them α-catastrophe and β-catastrophe, respectively. That is, α-catastrophe kills all the individuals in the system, while β-catastrophe partially kills the individuals in the system with only one individual left. If $\alpha = \beta = 0$, i.e., there is no catastrophe, then $\{N_t : t \geq 0\}$ degenerates into an ordinary birth–death process, which is denoted by $\{\hat{N}(t) : t \geq 0\}$, and its q-matrix is denoted by $\hat{Q}$. The probability transition function of $\{\hat{N}_t : t \geq 0\}$ is denoted by $\hat{P}(t) = (\hat{p}_{ij}(t) : i, j \in \mathbf{Z}_+)$ and the corresponding resolvent is denoted by $\hat{\Pi}(\lambda) = (\hat{\pi}_{j,n}(\lambda) : j, n \in \mathbf{Z}_+)$.*

The rest of this paper is organized as follows. In the following Section 2, we reveal the relationship of the transition probability of $\{N_t : t \geq 0\}$ and the transition probability of $\{\hat{N}_t : t \geq 0\}$ in Laplace transform version (Theorem 1). In Section 3, the first occurrence time of an effective catastrophe is discussed. We first construct a new process, $\{M_t : t \geq 0\}$, which coincides with $\{N_t : t \geq 0\}$ until the occurrence of catastrophe and can distinguish what type of catastrophe occurs, and then reveal the relationship of the transition probability of $\{M_t : t \geq 0\}$ and the transition probability of $\{\hat{N}_t : t \geq 0\}$) in Laplace transform version (Theorems 2 and 3). Finally, we obtain the probability distribution of the first occurrence time of an effective catastrophe in a Laplace transform version and the probabilities of the first effective catastrophe being of the α-type or the β-type.

2. Probability Transition Function

From Definition 1, we see that a catastrophe may reduce the system state to zero or one. However, since natural death rate $\mu_1,\ \mu_2 > 0$, when the system state transfers to zero from one or transfers to one from two, it is difficult to distinguish whether it was a catastrophe or a natural death. Therefore, it is important to discuss such effective catastrophe. For this purpose, we first construct the relationship of $P(t)$ and $\hat{P}(t)$ (or, equivalently, $\Pi(\lambda)$ and $\hat{\Pi}(\lambda)$).

The following lemma presents the basic properties of $P(t)$ (or $\Pi(\lambda)$) and $\hat{P}(t)$ (or $\hat{\Pi}(\lambda)$).

Lemma 1. *(i) $P(t) = (p_{j,n}(t) : j, n \in \mathbf{Z}_+)$ satisfies the following Kolmogorov forward equations: for any $j, n \in \mathbf{Z}_+$ and $t \geq 0$,*

$$\begin{cases} p'_{j,0}(t) = -(\lambda_0 + \gamma)p_{j,0}(t) + \mu_1 p_{j,1}(t) + \alpha, \\ p'_{j,1}(t) = \lambda_0 p_{j,0}(t) - (\omega_1 + \gamma)p_{j,1}(t) + \mu_2 p_{j,2}(t) + \beta, \\ p'_{j,n}(t) = \lambda_{n-1} p_{j,n-1}(t) - (\omega_n + \gamma)p_{j,n}(t) + \mu_{n+1} p_{j,n+1}(t), \quad n \geq 2, \end{cases} \tag{4}$$

or equivalently, in the resolvent version

$$\begin{cases} (\lambda + \lambda_0 + \gamma)\pi_{j,0}(\lambda) - \delta_{j,0} = \mu_1 \pi_{j,1}(\lambda) + \frac{\alpha}{\lambda}, \\ (\lambda + \omega_1 + \gamma)\pi_{j,1}(\lambda) - \delta_{j,1} = \lambda_0 \pi_{j,0}(\lambda) + \mu_2 \pi_{j,2}(\lambda) + \frac{\beta}{\lambda}, \\ (\lambda + \omega_n + \gamma)\pi_{j,n}(\lambda) - \delta_{j,n} = \lambda_{n-1}\pi_{j,n-1}(\lambda) + \mu_{n+1}\pi_{j,n+1}(\lambda), \quad n \geq 2. \end{cases} \tag{5}$$

(ii) $\hat{P}(t) = (\hat{p}_{j,n}(t) : j, n \in \mathbf{Z}_+)$ satisfies the following Kolmogorov forward equations: for any $j, n \in \mathbf{Z}_+$ and $t \geq 0$,

$$\begin{cases} \hat{p}'_{j,0}(t) = -\lambda_0 \hat{p}_{j,0}(t) + \mu_1 \hat{p}_{j,1}(t), \\ \hat{p}'_{j,n}(t) = \lambda_{n-1}\hat{p}_{j,n-1}(t) - (\lambda_n + \mu_n)\hat{p}_{j,n}(t) + \mu_{n+1}\hat{p}_{j,n+1}(t), \quad n \geq 1, \end{cases}$$

or, equivalently, in the resolvent version

$$\begin{cases} (\lambda + \lambda_0)\hat{\pi}_{j,0}(\lambda) - \delta_{j,0} = \mu_1 \hat{\pi}_{j,1}(\lambda), \\ (\lambda + \lambda_n + \mu_n)\hat{\pi}_{j,n}(\lambda) - \delta_{j,n} = \lambda_{n-1}\hat{\pi}_{j,n-1}(\lambda) + \mu_{n+1}\hat{\pi}_{j,n+1}(\lambda), \quad n \geq 1. \end{cases}$$

Proof. (i) By Kolmogorov forward equations and the honesty of $P(t)$, we know that

$$\begin{aligned} p'_{j,0}(t) &= -(\lambda_0 + \beta)p_{j,0}(t) + (\mu_1 + \alpha)p_{j,1}(t) + \sum_{k=2}^{\infty} \alpha p_{j,k}(t) \\ &= -(\lambda_0 + \beta)p_{j,0}(t) + \mu_1 p_{j,1}(t) + \sum_{k=1}^{\infty} \alpha p_{j,k}(t) \\ &= -(\lambda_0 + \beta)p_{j,0}(t) + \mu_1 p_{j,1}(t) + \alpha(1 - p_{j,0}(t)) \\ &= -(\lambda_0 + \gamma)p_{j,0}(t) + \mu_1 p_{j,1}(t) + \alpha \end{aligned}$$

and

$$\begin{aligned} p'_{j,1}(t) &= (\lambda_0 + \beta)p_{j,0}(t) - (\lambda_1 + \mu_1 + \alpha)p_{j,1}(t) + (\mu_2 + \beta)p_{j,2}(t) + \sum_{k=3}^{\infty} \beta p_{j,k}(t) \\ &= \lambda_0 p_{j,0}(t) - (\omega_1 + \alpha)p_{j,1}(t) + \mu_2 p_{j,2}(t) + \beta(1 - p_{j,1}(t)) \\ &= \lambda_0 p_{j,0}(t) - (\omega_1 + \gamma)p_{j,1}(t) + \mu_2 p_{j,2}(t) + \beta. \end{aligned}$$

The other equalities of (i) and (ii) follow directly from Kolmogorov forward equations and the Laplace transform. The proof is complete. □

The following theorem plays an important role in later discussion. It reveals the relationship of $P(t)$ and $\hat{P}(t)$ (or, equivalently, $\Pi(\lambda)$ and $\hat{\Pi}(\lambda)$).

Theorem 1. *For any $j, n \in \mathbf{Z}_+$, we have*

$$p_{j,n}(t) = e^{-\gamma t}\hat{p}_{j,n}(t) + \alpha \int_0^t e^{-\gamma s}\hat{p}_{0,n}(s)ds + \beta \int_0^t e^{-\gamma s}\hat{p}_{1,n}(s)ds \tag{6}$$

or, equivalently, in the resolvent version

$$\pi_{j,n}(\lambda) \quad = \quad \hat{\pi}_{j,n}(\lambda+\gamma) + \frac{1}{\lambda} \cdot [\alpha\hat{\pi}_{0,n}(\lambda+\gamma) + \beta\hat{\pi}_{1,n}(\lambda+\gamma)] \tag{7}$$

Proof. We first assume $\alpha = 0$. The corresponding process is denoted by $\tilde{N}_t$ and its probability transition function is denoted by $\tilde{P}(t) = (\tilde{p}_{j,n}(t) : j, n \in \mathbf{Z}_+)$. Denote $\{A_t : t \geq 0\} = \{\hat{N}_t : t \geq 0\}$. Let $\{K_t : t \geq 0\}$ be a Poisson process with parameter β, which is independent of $\{A_t : t \geq 0\}$. Note that $\{K_t : t \geq 0\}$ can be viewed as a catastrophe flow. We let $l(t)$ be the time until the first catastrophe before time t. Then, $l(t)$ has the truncated exponential law,

$$P(l(t) \leq u) = 1 - e^{-\beta u} I_{[0,t)}(u).$$

We denote $\{A_t^{(0)} : t \geq 0\} := \{A_t : t \geq 0\}$. We let $\{A_t^{(n)} : t \geq 0\}_{n\geq 1}$ be an independent sequence of copies of $\{A_t^{(0)} : t \geq 0\}$ but with $A_0^{(n)} = 1$. We define $\{R_t : t \geq 0\}$ by

$$R_t = A_{l(t)}^{(K_t)}, \quad t \geq 0.$$

Then, $\{R_t : t \geq 0\}$ is a continuous-time Markov chain. It evolves like $A_t^{(0)}$. At the first catastrophe time, it jumps to State 1, and then evolves like $A_t^{(1)}$. At the next catastrophe time, it jumps to State 1 again, and so on. We let $\bar{P}(t) = (\bar{p}_{jn}(t) : j, n \in \mathbf{Z}_+))$ be the probability transition function of $\{R_t : t \geq 0\}$. Then,

$$p_{jn}(t) = P(R_t = n|R_0 = j) = P_j(R_t = n) = E_j[I_{\{n\}}(R_t)] = E_j[E_j[I_{\{n\}}(A_{l(t)}^{(K_t)})|K_t, l(t)]],$$

where $P_j = P(\cdot|R_0 = j)$ and E_j is the mathematical expectation under P_j. We denote $G(K_t, l(t)) := E_j[I_{\{n\}}(A_{l(t)}^{(K_t)})|K_t, l(t)]$ for a moment. Then, the above equality equals to

$$\begin{aligned}
& E_j[G(K_t, l(t))] \\
= \quad & E_j[E_j[G(K_t, l(t))|l(t)]] \\
= \quad & P_j(l(t) = t)E_j[G(K_t, l(t))|l(t) = t] + \beta\xi \int_0^t e^{-\beta s} E_j[G(K_t, l(t))|l(t) = s]ds.
\end{aligned}$$

Since $l(t) = t \Leftrightarrow K_t = 0$ and $R_0 = j \Leftrightarrow A_0 = j$, we have

$$P_j(l(t) = t) = P_j(K_t = 0) = e^{-\beta t}$$

and

$$E_j[G(K_t, l(t))|l(t) = t] = E_j[I_{\{n\}}(A_t^{(0)})] = E_j[I_{\{n\}}(A_t)] = \hat{p}_{jn}(t).$$

If $s < t$, then

$$\begin{aligned}
& E_j[G(K_t, l(t))|l(t) = s] \\
= \quad & \sum_{k=1}^{\infty} P_j(K_t = k|l(t) = s)G(k, s) \\
= \quad & \sum_{k=1}^{\infty} P_j(K_t = k|l(t) = s)E_j[I_{\{n\}}(A_{l(t)}^{(K_t)})|K_t = k, l(t) = s] \\
= \quad & \sum_{k=1}^{\infty} P_j(K_t = k|l(t) = s)E_j[I_{\{n\}}(A_s^{(k)})]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} P_j(K_t = k | l(t) = s) E[I_{\{n\}}(A_s^{(k)}) | A_0 = j, A_0^{(k)} = 1] \\
&= \sum_{k=1}^{\infty} P_j(K_t = k | l(t) = s) P(A_s^{(k)} = n | A_0 = j, A_0^{(k)} = 1] \\
&= \sum_{k=1}^{\infty} P_j(K_t = k | l(t) = s) P(A_s^{(k)} = n | A_0^{(k)} = 1] \\
&= \sum_{k=1}^{\infty} P_j(K_t = k | l(t) = s) P(A_s = n | A_0 = 1] \\
&= \hat{p}_{1,n}(s).
\end{aligned}
$$

Therefore,

$$\bar{p}_{j,n}(t) = e^{-\beta t} \hat{p}_{j,n}(t) + \beta \xi \int_0^t e^{-\beta s} \hat{p}_{1,n}(s) ds.$$

It is easy to check that $\bar{p}'_{j,n}(0) = \tilde{p}'_{j,n}(0)$. This implies that R_t and $\tilde{N}_t$ are same in the sense of distribution. Hence,

$$\tilde{p}_{j,n}(t) = e^{-\beta t} \hat{p}_{j,n}(t) + \beta \int_0^t e^{-\beta s} \hat{p}_{1,n}(s) ds. \tag{8}$$

Now, we consider the general case of $\alpha > 0$. Denote $\{\tilde{A}_t : t \geq 0\} := \{\tilde{N}_t : t \geq 0\}$. Let $\{\tilde{K}_t : t \geq 0\}$ be a Poisson process with parameter $\alpha\xi$, which is independent of $\{\tilde{A}_t : t \geq 0\}$. $\{\tilde{K}_t : t \geq 0\}$ can be viewed as a catastrophe flow with parameter α. Let $\tilde{l}(t)$ be the time until the first catastrophe before time t. Then, $l(t)$ has the truncated exponential law

$$P(\tilde{l}(t) \leq u) = 1 - e^{-\alpha u} I_{[0,t)}(u).$$

We denote $\{\tilde{A}_t^{(0)} : t \geq 0\} := \{\tilde{A}_t : t \geq 0\}$. Let $\{\tilde{A}_t^{(n)} : t \geq 0\}_{n \geq 1}$ be an independent sequence of copies of $\{\tilde{A}_t^{(0)} : t \geq 0\}$ but with $\tilde{A}_0^{(n)} = 0$ $(n \geq 1)$. We define $\{\tilde{R}_t : t \geq 0\}$ by

$$\tilde{R}_t = \tilde{A}_{\tilde{l}(t)}^{(\tilde{K}_t)}, \quad t \geq 0.$$

Let $\check{P}(t) = (\check{p}_{j,n}(t) : j, n \in \mathbf{Z}_+)$ be the probability transition function of $\{\tilde{R}_t : t \geq 0\}$. By a similar argument as above, we know that

$$\check{p}_{j,n}(t) = e^{-\alpha t} \bar{p}_{j,n}(t) + \alpha \int_0^t e^{-\alpha s} \bar{p}_{0,n}(s) ds.$$

By (8),

$$
\begin{aligned}
\check{p}_{j,n}(t) &= e^{-\alpha t} [e^{-\beta t} \hat{p}_{j,n}(t) + \beta \int_0^t e^{-\beta s} \hat{p}_{1,n}(s) ds] \\
&\quad + \alpha \int_0^t e^{-\alpha s} [e^{-\beta s} \hat{p}_{0,n}(s) + \beta \int_0^s e^{-\beta u} \hat{p}_{1,n}(u) du] ds \\
&= e^{-(\alpha+\beta)t} \hat{p}_{j,n}(t) + \alpha \int_0^t e^{-(\alpha+\beta)s} \hat{p}_{0,n}(s) ds + \beta \int_0^t e^{-(\alpha+\beta)s} \hat{p}_{1,n}(s) ds.
\end{aligned}
$$

It is easy to check that $\check{p}'_{j,n}(0) = p'_{j,n}(0)$. This implies that $\tilde{R}_t$ and N_t are same in sense of distribution. Hence,

$$p_{j,n}(t) = e^{-(\alpha+\beta)t}\hat{p}_{j,n}(t) + \alpha\int_0^t e^{-(\alpha+\beta)s}\hat{p}_{0,n}(s)ds + \beta\int_0^t e^{-(\alpha+\beta)s}\hat{p}_{1,n}(s)ds.$$

Equation (6) is proven. Taking Laplace transform on (6) implies (7). The proof is complete. □

3. The First Occurrence Time of Effective Catastrophe

We now consider the first effective catastrophe of $\{N_t : t \geq 0\}$. We let C_j be the first occurrence time of an effective catastrophe for $\{N_t : t \geq 0\}$ starting from state j. The probability density function of C_j is denoted by $d_j(t)$. We let $C_{j,0}$ and $C_{j,1}$ be the first occurrence time of an effective α-catastrophe and an effective β-catastrophe, respectively. It is obvious that $C_j = C_{j,0} \wedge C_{j,1}$.

In particular, if $\alpha = 0$ or $\beta = 0$, then $C_j = C_{j,1}$ or $C_j = C_{j,0}$, respectively, and the current model deduces to the model discussed in Di Crescenzo et al. [11]. In this paper, we mainly consider the property of C_j and probabilities $P(C_j \leq t, C_{j,0} < C_{j,1})$ and $P(C_j \leq t, C_{j,1} < C_{j,0})$. For this purpose, we construct a new process, $\{M_t : t \geq 0\}$, such that $\{M_t : t \geq 0\}$ coincides with $\{N_t : t \geq 0\}$ until the occurrence of catastrophe, but $\{M_t : t \geq 0\}$ enters into an absorbing state -1 if the first effective catastrophe is a β-type and enters into another absorbing state -2 if the first effective catastrophe is an α-type. Therefore, the state space of $\{M_t : t \geq 0\}$ is $\mathbf{S} := \{-2, -1, 0, 1, \cdots\}$ and its q-matrix $\tilde{Q} = (\tilde{q}_{jn} : j, n \in \mathbf{S})$ is given by

$$\tilde{q}_{ij} = \begin{cases} \lambda_i, & i \geq 0,\ j = i+1, \\ \mu_i, & i \geq 1,\ j = i-1, \\ \alpha, & i \geq 1,\ j = -2, \\ \beta, & i = 0,\ j = -1, \\ \beta, & i \geq 2,\ j = -1, \\ -(\lambda_0 + \beta), & i = j = 0, \\ -(\omega_1 + \alpha), & i = j = 1, \\ -(\omega_i + \gamma), & i = j \geq 2, \\ 0, & \text{otherwise}. \end{cases}$$

Different from Q, $\tilde{Q}$ can reveal the different effects of different types of catastrophes. More specifically, an α-catastrophe or a β-catastrophe occur at state $j \geq 0$ if and only if the system state jumps to -2 or -1 from $j \geq 0$, respectively. Since $\{M_t : t \geq 0\}$ coincides with $\{N_t : t \geq 0\}$ until the occurrence of catastrophe and both -2 and -1 are absorbing states for $\{M_t : t \geq 0\}$, we know that C_j and the absorbing time of $\{M_t : t \geq 0\}$ are same in the sense of distribution.

Let $H(t) = (h_{j,n}(t) : j, n \in \mathbf{S})$ and $\Phi(\lambda) = (\phi_{j,n}(\lambda) : j, n \in \mathbf{S})$ be the $\tilde{Q}$-transition function and the $\tilde{Q}$-resolvent.

Lemma 2. *For any $j \geq 0$, we have*

$$\begin{cases} h'_{j,-2}(t) = \alpha(1 - h_{j,-2}(t) - h_{j,-1}(t) - h_{j,0}(t)), \\ h'_{j,-1}(t) = \beta(1 - h_{j,-2}(t) - h_{j,-1}(t) - h_{j,1}(t)), \\ h'_{j,0}(t) = -(\lambda_0 + \beta)h_{j,0}(t) + \mu_1 h_{j,1}(t), \\ h'_{j,1}(t) = \lambda_0 h_{j,0}(t) - (\omega_1 + \alpha)h_{j,1}(t) + \mu_2 h_{j,2}(t), \\ h'_{j,n}(t) = \lambda_{n-1}h_{j,n-1}(t) - (\omega_n + \gamma)h_{j,n}(t) + \mu_{n+1}h_{j,n+1}(t), \quad n \geq 2, \end{cases} \tag{9}$$

or, equivalently, in the resolvent version

$$\begin{cases} \lambda\phi_{j,-2}(\lambda) = \alpha(\frac{1}{\lambda} - \phi_{j,-2}(\lambda) - \phi_{j,-1}(\lambda) - \phi_{j,0}(\lambda)), \\ \lambda\phi_{j,-1}(\lambda) = \beta(\frac{1}{\lambda} - \phi_{j,-2}(\lambda) - \phi_{j,-1}(\lambda) - \phi_{j,1}(\lambda)), \\ (\lambda + \lambda_0 + \beta)\phi_{j,0}(\lambda) - \delta_{j,0} = \mu_1\phi_{j,1}(\lambda), \\ (\lambda + \omega_1 + \alpha)\phi_{j,1}(\lambda) - \delta_{j,1} = \lambda_0\phi_{j,0}(\lambda) + \mu_2\phi_{j,2}(\lambda), \\ (\lambda + \omega_n + \gamma)\phi_{j,n}(\lambda) - \delta_{j,n} = \lambda_{n-1}\phi_{j,n-1}(\lambda) + \mu_{n+1}\phi_{j,n+1}(\lambda), \quad n \geq 2. \end{cases} \tag{10}$$

Proof. By Kolmogorov forward equation,

$$\begin{aligned} h'_{j,-2}(t) &= \sum_{k=1}^{\infty} \alpha h_{j,k}(t) \\ &= \alpha(1 - h_{j,-2}(t) - h_{j,-1}(t) - h_{j,0}(t)). \end{aligned}$$

$$\begin{aligned} h'_{j,-1}(t) &= \beta h_{j,0}(t) + \sum_{k=2}^{\infty} \beta h_{j,k}(t) \\ &= \beta(1 - h_{j,-2}(t) - h_{j,-1}(t) - h_{j,1}(t)). \end{aligned}$$

The other equalities of (9) follow directly from Kolmogorov forward equations and (10) follows from the Laplace transform of (9). The proof is complete. □

We now investigate the relationship of $\Phi(\lambda)$ and $\Pi(\lambda)$. For this purpose, we define

$$A_{ij}(\lambda) := 1 - \lambda\pi_{i,j}(\lambda), \quad i,j \geq 0 \tag{11}$$

and

$$H(\lambda) := \lambda^{-1}\{[\lambda + \alpha A_{00}(\lambda)][\lambda + \beta A_{11}(\lambda)] - \alpha\beta A_{10}(\lambda)A_{01}(\lambda)\}. \tag{12}$$

The following theorem reveals that $\Phi(\lambda)$ can be reexpressed with $\Pi(\lambda)$.

Theorem 2. *Let $\Phi(\lambda) = (\phi_{j,n}(\lambda) : j, n \in \mathbf{S})$ be the $\tilde{Q}$-resolvent and $\Pi(\lambda) = (\pi_{j,n}(\lambda) : j, n \in \mathbf{Z}_+)$ be the Q-resolvent. Then,*

$$\phi_{0,n}(\lambda) = \frac{(\lambda + \beta A_{11}(\lambda))\pi_{0,n}(\lambda) - \beta A_{01}(\lambda)\pi_{1,n}(\lambda)}{H(\lambda)}, \quad n \geq 0, \tag{13}$$

$$\phi_{1,n}(\lambda) = \frac{-\alpha A_{10}(\lambda)\pi_{0,n}(\lambda) + (\lambda + \alpha A_{00}(\lambda))\pi_{1,n}(\lambda)}{H(\lambda)}, \quad n \geq 0 \tag{14}$$

and

$$\phi_{j,n}(\lambda) = \pi_{j,n}(\lambda) + F_j(\lambda)\pi_{0,n}(\lambda) + G_j(\lambda)\pi_{1,n}(\lambda), \quad j \geq 2, \ n \geq 0, \tag{15}$$

where

$$F_j(\lambda) := \frac{\alpha\beta A_{10}(\lambda)A_{j1}(\lambda) - \alpha(\lambda + \beta A_{11}(\lambda))A_{j0}(\lambda)}{\lambda H(\lambda)} \tag{16}$$

and

$$G_j(\lambda) := \frac{\alpha\beta A_{01}(\lambda)A_{j0}(\lambda) - \beta(\lambda + \alpha A_{00}(\lambda))A_{j1}(\lambda)}{\lambda H(\lambda)} \tag{17}$$

with $(\pi_{j,n}(\lambda) : j, n \geq 0)$ being given by (7).

Proof. By (10) with $j = 0, 1$,

$$\begin{cases} (\lambda + \lambda_0 + \beta)\phi_{0,0}(\lambda) - 1 = \mu_1 \phi_{0,1}(\lambda), \\ (\lambda + \omega_1 + \alpha)\phi_{0,1}(\lambda) = \lambda_0 \phi_{0,0}(\lambda) + \mu_2 \phi_{0,2}(\lambda), \\ (\lambda + \omega_n + \gamma)\phi_{0,n}(\lambda) = \lambda_{n-1}\phi_{0,n-1}(\lambda) + \mu_{n+1}\phi_{0,n+1}(\lambda), \quad n \geq 2, \end{cases} \tag{18}$$

$$\begin{cases} (\lambda + \lambda_0 + \beta)\phi_{1,0}(\lambda) = \mu_1 \phi_{1,1}(\lambda), \\ (\lambda + \omega_1 + \alpha)\phi_{1,1}(\lambda) - 1 = \lambda_0 \phi_{1,0}(\lambda) + \mu_2 \phi_{1,2}(\lambda), \\ (\lambda + \omega_n + \gamma)\phi_{1,n}(\lambda) = \lambda_{n-1}\phi_{1,n-1}(\lambda) + \mu_{n+1}\phi_{1,n+1}(\lambda), \quad n \geq 2 \end{cases} \tag{19}$$

and by (5) with $j = 0, 1$,

$$\begin{cases} (\lambda + \lambda_0 + \gamma)\pi_{0,0}(\lambda) - 1 = \mu_1 \pi_{0,1}(\lambda) + \frac{\alpha}{\lambda}, \\ (\lambda + \omega_1 + \gamma)\pi_{0,1}(\lambda) = \lambda_0 \pi_{0,0}(\lambda) + \mu_2 \pi_{0,2}(\lambda) + \frac{\beta}{\lambda}, \\ (\lambda + \omega_n + \gamma)\pi_{0,n}(\lambda) = \lambda_{n-1}\pi_{0,n-1}(\lambda) + \mu_{n+1}\pi_{0,n+1}(\lambda), \ \ n \geq 2. \end{cases} \tag{20}$$

$$\begin{cases} (\lambda + \lambda_0 + \gamma)\pi_{1,0}(\lambda) = \mu_1 \pi_{1,1}(\lambda) + \frac{\alpha}{\lambda}, \\ (\lambda + \omega_1 + \gamma)\pi_{1,1}(\lambda) - 1 = \lambda_0 \pi_{1,0}(\lambda) + \mu_2 \pi_{1,2}(\lambda) + \frac{\beta}{\lambda}, \\ (\lambda + \omega_n + \gamma)\pi_{1,n}(\lambda) = \lambda_{n-1}\pi_{1,n-1}(\lambda) + \mu_{n+1}\pi_{1,n+1}(\lambda), \ \ n \geq 2. \end{cases} \tag{21}$$

We let

$$\phi_{0,n}(\lambda) = A(\lambda)\pi_{0,n}(\lambda) + B(\lambda)\pi_{1,n}(\lambda), \quad n \geq 0. \tag{22}$$

Substituting (22) into (18) and using (20), we have

$$\begin{cases} (\lambda + \alpha A_{00}(\lambda))A(\lambda) + \alpha A_{10}(\lambda)B(\lambda) = \lambda \\ \beta A_{01}(\lambda)A(\lambda) + (\lambda + \beta A_{11}(\lambda))B(\lambda) = 0. \end{cases} \tag{23}$$

Indeed, by the first equality of (18),

$$(\lambda + \lambda_0 + \beta)[A(\lambda)\pi_{0,0}(\lambda) + B(\lambda)\pi_{1,0}(\lambda)] - 1 = \mu_1[A(\lambda)\pi_{0,1}(\lambda) + B(\lambda)\pi_{1,1}(\lambda)],$$

i.e.,

$$A(\lambda)[(\lambda + \lambda_0 + \beta)\pi_{0,0}(\lambda) - \mu_1\pi_{0,1}(\lambda)] + B(\lambda)[(\lambda + \lambda_0 + \beta)\pi_{1,0}(\lambda) - \mu_1\pi_{1,1}(\lambda)] = 1.$$

It follows from the first equality of (20) and the first equality of (21) that

$$(\lambda + \alpha A_{00}(\lambda))A(\lambda) + \alpha A_{10}(\lambda)B(\lambda) = \lambda.$$

By the second equality of (18),

$$\begin{aligned} & (\lambda + \omega_1 + \alpha)[A(\lambda)\pi_{0,1}(\lambda) + B(\lambda)\pi_{1,1}(\lambda)] \\ = \ & \lambda_0[A(\lambda)\pi_{0,0}(\lambda) + B(\lambda)\pi_{1,0}(\lambda)] + \mu_2[A(\lambda)\pi_{0,2}(\lambda) + B(\lambda)\pi_{1,2}(\lambda)], \end{aligned}$$

i.e.,

$$\begin{aligned} & A(\lambda)[(\lambda + \omega_1 + \alpha)\pi_{0,1}(\lambda) - \lambda_0\pi_{0,0}(\lambda) - \mu_2\pi_{0,2}(\lambda)] \\ & + B(\lambda)[(\lambda + \omega_1 + \alpha)\pi_{1,1}(\lambda) - \lambda_0\pi_{1,0}(\lambda) - \mu_2\pi_{1,2}(\lambda)] = 0. \end{aligned}$$

It follows from the second equality of (20) and the second equality of (21) that

$$\beta A_{01}(\lambda)A(\lambda) + (\lambda + \beta A_{11}(\lambda))B(\lambda) = 0.$$

Therefore, (23) holds. It follows from (23) that

$$A(\lambda) = \frac{\lambda + \beta A_{11}(\lambda)}{H(\lambda)} \quad and \quad B(\lambda) = \frac{-\beta A_{01}(\lambda)}{H(\lambda)}.$$

The other equalities of (18) also hold.

We let

$$\phi_{1,n}(\lambda) = C(\lambda)\pi_{0,n}(\lambda) + D(\lambda)\pi_{1,n}(\lambda), \quad n \geq 0. \tag{24}$$

Substituting (24) into (19) and using (21), we have

$$\begin{cases} (\lambda + \alpha - \alpha\lambda\pi_{0,0}(\lambda))C(\lambda) + \alpha(1 - \lambda\pi_{1,0}(\lambda))D(\lambda) = 0 \\ \beta(1 - \lambda\pi_{0,1}(\lambda))C(\lambda) + (\lambda + \beta - \beta\lambda\pi_{1,1}(\lambda))D(\lambda) = \lambda. \end{cases} \tag{25}$$

Indeed, by the second equality of (19),

$$\begin{aligned} & (\lambda + \omega_1 + \alpha)[C(\lambda)\pi_{0,1}(\lambda) + D(\lambda)\pi_{1,1}(\lambda)] - 1 \\ = \quad & \lambda_0[C(\lambda)\pi_{0,0}(\lambda) + D(\lambda)\pi_{1,0}(\lambda)] + \mu_2[C(\lambda)\pi_{0,2}(\lambda) + D(\lambda)\pi_{1,2}(\lambda)], \end{aligned}$$

i.e.,

$$\begin{aligned} & [(\lambda + \omega_1 + \alpha)\pi_{0,1}(\lambda) - \lambda_0\pi_{0,0}(\lambda) - \mu_2\pi_{0,2}(\lambda)]C(\lambda) \\ & +[(\lambda + \omega_1 + \alpha)\pi_{1,1}(\lambda) - \lambda_0\pi_{1,0}(\lambda) - \mu_2\pi_{1,2}(\lambda)]D(\lambda) = 1 \end{aligned}$$

It follows from the second equality of (20) and the second equality of (21) that

$$\beta A_{01}(\lambda)C(\lambda) + (\lambda + \beta A_{11}(\lambda))D(\lambda) = \lambda.$$

By the first equality of (19),

$$(\lambda + \lambda_0 + \beta)[C(\lambda)\pi_{0,0}(\lambda) + D(\lambda)\pi_{1,0}(\lambda)] = \mu_1[C(\lambda)\pi_{0,1}(\lambda) + D(\lambda)\pi_{1,1}(\lambda)],$$

i.e.,

$$[(\lambda + \lambda_0 + \beta)\pi_{0,0}(\lambda) - \mu_1\pi_{0,1}(\lambda)]C(\lambda) + B(\lambda)[(\lambda + \lambda_0 + \beta)\pi_{1,0}(\lambda) - \mu_1\pi_{1,1}(\lambda)] = 0.$$

It follows from the first equality of (20) and the first equality of (21) that

$$(\lambda + \alpha A_{00}(\lambda))C(\lambda) + \alpha A_{10}(\lambda))D(\lambda) = 0.$$

Therefore, (25) holds. It follows from (25) that

$$C(\lambda) = \frac{-\alpha A_{10}(\lambda)}{H(\lambda)} \quad and \quad D(\lambda) = \frac{\lambda + \alpha A_{00}(\lambda)}{H(\lambda)}.$$

The other equalities of (19) also hold.

By (10) with $j \geq 2$,

$$\begin{cases} (\lambda + \lambda_0 + \beta)\phi_{j,0}(\lambda) = \mu_1\phi_{j,1}(\lambda), \\ (\lambda + \omega_1 + \alpha)\phi_{j,1}(\lambda) = \lambda_0\phi_{j,0}(\lambda) + \mu_2\phi_{j,2}(\lambda), \\ (\lambda + \omega_n + \gamma)\phi_{j,n}(\lambda) - \delta_{j,n} = \lambda_{n-1}\phi_{j,n-1}(\lambda) + \mu_{n+1}\phi_{j,n+1}(\lambda), \quad n \geq 2 \end{cases} \tag{26}$$

and by (5) with $j \geq 2$,

$$\begin{cases} (\lambda + \lambda_0 + \gamma)\pi_{j,0}(\lambda) = \mu_1 \pi_{j,1}(\lambda) + \frac{\alpha}{\lambda}, \\ (\lambda + \omega_1 + \gamma)\pi_{j,1}(\lambda) = \lambda_0 \pi_{j,0}(\lambda) + \mu_2 \pi_{j,2}(\lambda) + \frac{\beta}{\lambda}, \\ (\lambda + \omega_n + \gamma)\pi_{j,n}(\lambda) - \delta_{j,n} = \lambda_{n-1}\pi_{j,n-1}(\lambda) + \mu_{n+1}\pi_{j,n+1}(\lambda), \ \ n \geq 2. \end{cases} \tag{27}$$

Let

$$\phi_{j,n}(\lambda) = D_j(\lambda)\pi_{j,n}(\lambda) + F_j(\lambda)\pi_{0,n}(\lambda) + G_j(\lambda)\pi_{1,n}(\lambda). \tag{28}$$

Substituting (28) into the last equality of (26), we have

$$\begin{aligned} &D_j(\lambda)[(\lambda + \omega_n + \gamma)\pi_{j,n}(\lambda) - \lambda_{n-1}\pi_{j,n-1}(\lambda) - \mu_{n+1}\pi_{j,n+1}(\lambda)] - \delta_{j,n} \\ &+F_j(\lambda)[(\lambda + \omega_n + \gamma)\pi_{0,n}(\lambda) - \lambda_{n-1}\pi_{0,n-1}(\lambda) - \mu_{n+1}\pi_{0,n+1}(\lambda)] \\ &+G_j(\lambda)[(\lambda + \omega_n + \gamma)\pi_{1,n}(\lambda) - \lambda_{n-1}\pi_{1,n-1}(\lambda) - \mu_{n+1}\pi_{1,n+1}(\lambda)] = 0, \quad n \geq 2. \end{aligned} \tag{29}$$

By the last equalities of (20), (21) and (27), we have $D_j(\lambda)\delta_{j,n} = \delta_{j,n}$ for $n \geq 2$ and hence $D_j(\lambda) = 1$.

Substituting (28) into the first and second equalities of (26) and using (20) and (21), we have

$$\begin{cases} (\lambda + \alpha A_{00}(\lambda))F_j(\lambda) + \alpha A_{10}(\lambda)G_j(\lambda) = \alpha\lambda\pi_{j,0}(\lambda) - \alpha, \\ \beta A_{01}(\lambda)F_j(\lambda) + (\lambda + \beta A_{11}(\lambda))G_j(\lambda) = \beta\lambda\pi_{j,1}(\lambda) - \beta. \end{cases} \tag{30}$$

Solving (30) yields (16) and (17). The proof is complete. □

By Theorem 1, we know that

$$\lambda\pi_{j,n}(\lambda) = \lambda\hat{\pi}_{j,n}(\lambda + \gamma) + \alpha\hat{\pi}_{0,n}(\lambda + \gamma) + \beta\hat{\pi}_{1,n}(\lambda + \gamma).$$

Denote

$$a_n(\lambda) = 1 - \alpha\hat{\pi}_{0,n}(\lambda + \gamma) - \beta\hat{\pi}_{1,n}(\lambda + \gamma), \ \ n \geq 0. \tag{31}$$

Then, $A_{jn}(\lambda)$ can be represented as

$$A_{jn}(\lambda) = a_n(\lambda) - \lambda\hat{\pi}_{j,n}(\lambda + \gamma), \tag{32}$$

Hence, by some algebra, $H(\lambda)$ can be represented as

$$\begin{aligned} &H(\lambda) \\ = \ &\alpha\beta[a_0(\lambda)\hat{\pi}_{0,1}(\lambda + \gamma) + a_1(\lambda)\hat{\pi}_{1,0}(\lambda + \gamma) - \lambda\hat{\pi}_{1,0}(\lambda + \gamma)\hat{\pi}_{0,1}(\lambda + \gamma)] \\ &+\alpha a_0(\lambda)\beta(\lambda) + \beta a_1(\lambda)\alpha(\lambda) + \lambda\alpha(\lambda)\beta(\lambda), \end{aligned} \tag{33}$$

where $\alpha(\lambda) = 1 - \alpha\hat{\pi}_{0,0}(\lambda + \gamma)$, $\beta(\lambda) = 1 - \beta\hat{\pi}_{1,1}(\lambda + \gamma)$. Indeed,

$$\begin{aligned} \lambda H(\lambda) \ = \ &(\alpha a_0(\lambda) + \lambda\alpha(\lambda))(\beta a_1(\lambda) + \lambda\beta(\lambda)) \\ &-\alpha\beta(a_0(\lambda) - \lambda\hat{\pi}_{1,0}(\lambda + \gamma))(a_1(\lambda) - \lambda\hat{\pi}_{0,1}(\lambda + \gamma)) \\ = \ &\alpha\beta a_0(\lambda)a_1(\lambda) + \alpha\lambda a_0(\lambda)\beta(\lambda) \\ &+\beta\lambda a_1(\lambda)\alpha(\lambda) + \lambda^2\alpha(\lambda)\beta(\lambda) \\ &-\alpha\beta a_0(\lambda)a_1(\lambda) + \alpha\beta a_0(\lambda)\lambda\hat{\pi}_{0,1}(\lambda + \gamma) + \alpha\beta a_1(\lambda)\lambda\hat{\pi}_{1,0}(\lambda + \gamma) \\ &-\alpha\beta\lambda^2\hat{\pi}_{1,0}(\lambda + \gamma)\hat{\pi}_{0,1}(\lambda + \gamma) \\ = \ &\lambda\alpha\beta[a_0(\lambda)\hat{\pi}_{0,1}(\lambda + \gamma) + a_1(\lambda)\hat{\pi}_{1,0}(\lambda + \gamma) - \lambda\hat{\pi}_{1,0}(\lambda + \gamma)\hat{\pi}_{0,1}(\lambda + \gamma)] \\ &+\lambda[\alpha a_0(\lambda)\beta(\lambda) + \beta a_1(\lambda)\alpha(\lambda) + \lambda\alpha(\lambda)\beta(\lambda)] \end{aligned}$$

which implies (33).

The following theorem further reveals that $\Phi(\lambda)$ can be reexpressed with $\hat{\Pi}(\lambda)$.

Theorem 3. *Let $\Phi(\lambda) = (\phi_{j,n}(\lambda) : j,n \in \mathbf{S})$ be the $\tilde{Q}$-resolvent and $\hat{\Pi}(\lambda) = (\hat{\pi}_{j,n}(\lambda) : j,n \in \mathbf{Z}_+)$ be the $\hat{Q}$-resolvent. Then,*

$$\phi_{j,n}(\lambda) = \hat{\pi}_{j,n}(\lambda+\gamma) + \frac{U_j(\lambda)\hat{\pi}_{0,n}(\lambda+\gamma) + V_j(\lambda)\hat{\pi}_{1,n}(\lambda+\gamma)}{H(\lambda)}, \quad j,n \geq 0, \tag{34}$$

where

$$U_j(\lambda) = \alpha(\lambda+\alpha+\beta)\beta(\lambda)\hat{\pi}_{j,0}(\lambda+\gamma) + \alpha\beta(\lambda+\alpha+\beta)\hat{\pi}_{1,0}(\lambda+\gamma)\hat{\pi}_{j,1}(\lambda+\gamma),$$

and

$$V_j(\lambda) = \beta(\lambda+\alpha+\beta)\alpha(\lambda)\hat{\pi}_{j,1}(\lambda+\gamma) + \alpha\beta(\lambda+\alpha+\beta)\hat{\pi}_{0,1}(\lambda+\gamma)\hat{\pi}_{j,0}(\lambda+\gamma).$$

Proof. By (11) and (12) and Theorem 1, we know that for any $j,n \geq 0$,

$$\begin{aligned} A_{jn}(\lambda) &= 1 - \lambda\hat{\pi}_{j,n}(\lambda+\gamma) - \alpha\hat{\pi}_{0,n}(\lambda+\gamma) - \beta\hat{\pi}_{1,n}(\lambda+\gamma) \\ &= \lambda[\hat{\pi}_{0,n}(\lambda+\gamma) - \hat{\pi}_{j,n}(\lambda+\gamma)] + A_{0n}(\lambda) \\ &= \lambda[\hat{\pi}_{1,n}(\lambda+\gamma) - \hat{\pi}_{j,n}(\lambda+\gamma)] + A_{1n}(\lambda). \end{aligned}$$

Note that the right-hand sides of (16) and (17) are well defined. We can define $F_j(\lambda)$ and $G_j(\lambda)$ for $j = 0,1$. Hence, it follows from Theorem 2 that for any $j \geq 0$,

$$\begin{aligned} \lambda H(\lambda)F_j(\lambda) &= \alpha\beta A_{10}(\lambda)A_{01}(\lambda) + \alpha\beta\lambda A_{10}(\lambda)[\hat{\pi}_{0,1}(\lambda+\gamma) - \hat{\pi}_{j,1}(\lambda+\gamma)] \\ &\quad -\alpha(\lambda+\beta A_{11}(\lambda))A_{00}(\lambda) - \alpha\lambda(\lambda+\beta A_{11}(\lambda))[\hat{\pi}_{0,0}(\lambda+\gamma) - \hat{\pi}_{j,0}(\lambda+\gamma)] \end{aligned}$$

and

$$\begin{aligned} \lambda H(\lambda)G_j(\lambda) &= -\beta\lambda A_{01}(\lambda) + \alpha\beta\lambda A_{01}(\lambda)[\hat{\pi}_{0,0}(\lambda+\gamma) - \hat{\pi}_{j,0}(\lambda+\gamma)] \\ &\quad -\beta\lambda(\lambda+\alpha A_{00}(\lambda))[\hat{\pi}_{0,1}(\lambda+\gamma) - \hat{\pi}_{j,1}(\lambda+\gamma)]. \end{aligned}$$

Therefore, by some algebra, we can obtain

$$\begin{aligned} &\lambda H(\lambda)[F_j(\lambda) + \frac{\alpha}{\lambda}(1 + F_j(\lambda) + G_j(\lambda))] \\ =\ &\alpha\lambda(\lambda+\alpha+\beta)(1 - \beta\hat{\pi}_{1,1}(\lambda+\gamma))\hat{\pi}_{j,0}(\lambda+\gamma) + \alpha\beta\lambda(\lambda+\alpha+\beta)\hat{\pi}_{1,0}(\lambda+\gamma)\hat{\pi}_{j,1}(\lambda+\gamma) \\ =:\ &\lambda U_j(\lambda), \quad j \geq 0. \end{aligned} \tag{35}$$

Similarly,

$$\begin{aligned} &\lambda H(\lambda)[G_j(\lambda) + \frac{\beta}{\lambda}(1 + F_j(\lambda) + G_j(\lambda))] \\ =\ &\beta\lambda(\lambda+\alpha+\beta)(1 - \alpha\hat{\pi}_{0,0}(\lambda+\gamma))\hat{\pi}_{j,1}(\lambda+\gamma) + \alpha\beta\lambda(\lambda+\alpha+\beta)\hat{\pi}_{0,1}(\lambda+\gamma)\hat{\pi}_{j,0}(\lambda+\gamma) \\ =:\ &\lambda V_j(\lambda), \quad j \geq 0. \end{aligned} \tag{36}$$

By Theorems 1 and 2, for any $j \geq 2, n \geq 0$,

$$
\begin{aligned}
\phi_{j,n}(\lambda) &= \pi_{j,n}(\lambda) + F_j(\lambda)\pi_{0,n}(\lambda) + G_j(\lambda)\pi_{1,n}(\lambda) \\
&= \hat{\pi}_{j,n}(\lambda+\gamma) + \frac{\alpha\hat{\pi}_{0,n}(\lambda+\gamma) + \beta\hat{\pi}_{1,n}(\lambda+\gamma)}{\lambda} \\
&\quad + F_j(\lambda)[\hat{\pi}_{0,n}(\lambda+\gamma) + \frac{\alpha\hat{\pi}_{0,n}(\lambda+\gamma) + \beta\hat{\pi}_{1,n}(\lambda+\gamma)}{\lambda}] \\
&\quad + G_j(\lambda)[\hat{\pi}_{1,n}(\lambda+\gamma) + \frac{\alpha\hat{\pi}_{0,n}(\lambda+\gamma) + \beta\hat{\pi}_{1,n}(\lambda+\gamma)}{\lambda}] \\
&= \hat{\pi}_{j,n}(\lambda+\gamma) + [F_j(\lambda) + \frac{\alpha}{\lambda}(1 + F_j(\lambda) + G_j(\lambda))] \cdot \hat{\pi}_{0,n}(\lambda+\gamma) \\
&\quad + [G_j(\lambda) + \frac{\beta}{\lambda}(1 + F_j(\lambda) + G_j(\lambda))] \cdot \hat{\pi}_{1,n}(\lambda+\gamma),
\end{aligned}
$$

where $F_j(\lambda)$ and $G_j(\lambda)$ are given in (16) and (17). By (35) and (36), we know (34) holds for $j \geq 2, n \geq 0$.

As for $j = 0$, by (13) and Theorem 1,

$$
\begin{aligned}
&\phi_{0,n}(\lambda) \\
= &\frac{\lambda(\lambda+\beta A_{11}(\lambda))\pi_{0,n}(\lambda) - \beta\lambda A_{01}(\lambda)\pi_{1,n}(\lambda)}{\lambda H(\lambda)} \\
= &\frac{(\lambda+\beta A_{11}(\lambda))[(\lambda+\alpha)\hat{\pi}_{0,n}(\lambda+\gamma) + \beta\hat{\pi}_{1,n}(\lambda+\gamma)] - \beta A_{01}(\lambda)[(\lambda+\beta)\hat{\pi}_{1,n}(\lambda+\gamma) + \alpha\hat{\pi}_{0,n}(\lambda+\gamma)]}{\lambda H(\lambda)} \\
= &\frac{(\lambda+\alpha)(\lambda+\beta A_{11}(\lambda)) - \alpha\beta A_{01}(\lambda)}{\lambda H(\lambda)}\hat{\pi}_{0,n}(\lambda+\gamma) + \frac{\beta[\lambda+\beta A_{11}(\lambda) - (\lambda+\beta)A_{01}(\lambda)]}{\lambda H(\lambda)}\hat{\pi}_{1,n}(\lambda+\gamma) \\
= &\hat{\pi}_{0,n}(\lambda+\gamma) + \frac{(\lambda+\alpha)(\lambda+\beta A_{11}(\lambda)) - \alpha\beta A_{01}(\lambda) - \lambda H(\lambda)}{\lambda H(\lambda)}\hat{\pi}_{0,n}(\lambda+\gamma) \\
&+ \frac{\beta[\lambda+\beta A_{11}(\lambda) - (\lambda+\beta)A_{01}(\lambda)]}{\lambda H(\lambda)}\hat{\pi}_{1,n}(\lambda+\gamma).
\end{aligned}
$$

By the definition of $H(\lambda)$,

$$
\begin{aligned}
&(\lambda+\alpha)(\lambda+\beta A_{11}(\lambda)) - \alpha\beta A_{01}(\lambda) - \lambda H(\lambda) \\
= &(\lambda+\alpha)(\lambda+\beta A_{11}(\lambda)) - \alpha\beta A_{01}(\lambda) - (\lambda+\alpha A_{00}(\lambda))(\lambda+\beta A_{11}(\lambda)) + \alpha\beta A_{10}(\lambda)A_{01}(\lambda) \\
= &\alpha(\lambda+\beta A_{11}(\lambda))(1 - A_{00}(\lambda)) - \alpha\beta A_{01}(\lambda)(1 - A_{10}(\lambda)).
\end{aligned}
$$

On the other hand, by some algebra, we can see that

$$
\begin{aligned}
\lambda U_0(\lambda) &= \lambda H(\lambda)[F_0(\lambda) + \frac{\alpha}{\lambda}(1 + F_0(\lambda) + G_0(\lambda))] \\
&= \alpha\beta A_{10}(\lambda)A_{01}(\lambda)) - \alpha(\lambda+\beta A_{11}(\lambda))A_{00}(\lambda)) + \alpha(\lambda+\beta A_{11}(\lambda)) - \alpha\beta A_{01}(\lambda) \\
&= \alpha(\lambda+\beta A_{11}(\lambda))(1 - A_{00}(\lambda)) - \alpha\beta A_{01}(\lambda)(1 - A_{10}(\lambda)),
\end{aligned}
$$

$$
\begin{aligned}
\lambda V_0(\lambda) &= \lambda H(\lambda)[G_0(\lambda) + \frac{\beta}{\lambda}(1 + F_0(\lambda) + G_0(\lambda))] \\
&= \alpha\beta A_{01}(\lambda)A_{00}(\lambda) - \beta(\lambda+\alpha A_{00}(\lambda))A_{01}(\lambda) + \beta(\lambda+\beta A_{11}(\lambda)) - \beta^2 A_{01}(\lambda) \\
&= \beta[\lambda+\beta A_{11}(\lambda) - (\lambda+\beta)A_{01}(\lambda)].
\end{aligned}
$$

Therefore, (34) holds for $j = 0$. By a similar argument, (34) also holds for $j = 1$. The proof is complete. □

We now consider the probability distribution of C_j and the related probabilities $P(C_j \leq t, C_{j,0} < C_{j,1})$ and $P(C_j \leq t, C_{j,1} < C_{j,0})$. It is easy to see that $P(C_j \leq t, C_{j,k} < C_{j,1-k})$ is differentiable in t for $k = 0, 1$. We let $d_{j,k}(t) = \frac{d}{dt}P(C_j \leq t, C_{j,k} < C_{j,1-k})$ for $k = 0, 1$.

Also, we let $\Delta_{j,k}(\lambda)$ denote the Laplace transform of $d_{j,k}(t)$ for $k = 0, 1$ and $\Delta_j(\lambda)$ denote the Laplace transform of $d_j(t)$.

The following theorem presents the probability distribution of C_j $(j \geq 0)$ in the Laplace transform version and the probability that the first effective catastrophe is an α-type or a β-type.

Theorem 4. *For any $j \geq 0$, we have*

$$\Delta_{j,0}(\lambda) = \frac{\alpha(\lambda+\beta)(1-\lambda\phi_{j0}(\lambda)) - \alpha\beta(1-\lambda\phi_{j,1}(\lambda))}{\lambda^2 + (\alpha+\beta)\lambda},$$

$$\Delta_{j,1}(\lambda) = \frac{\beta(\lambda+\alpha)(1-\lambda\phi_{j1}(\lambda)) - \alpha\beta(1-\lambda\phi_{j,0}(\lambda))}{\lambda^2 + (\alpha+\beta)\lambda}$$

and

$$\Delta_j(\lambda) = \frac{\alpha(1-\lambda\phi_{j0}(\lambda)) + \beta(1-\lambda\phi_{j1}(\lambda))}{\lambda+\alpha+\beta},$$

where $\phi_{j,0}(\lambda)$ and $\phi_{j,1}(\lambda)$ are given in Theorem 3. In particular,

$$P(C_{j,0} < C_{j,1}) = \frac{\alpha[1+\beta(\phi_{j,1}(0) - \phi_{j0}(0))]}{\alpha+\beta},$$

$$P(C_{j,1} < C_{j,0}) = \frac{\beta[1+\alpha(\phi_{j0}(0) - \phi_{j,1}(0))]}{\alpha+\beta},$$

where $\phi_{j,0}(\lambda)$ and $\phi_{j,1}(\lambda)$ are given by (34).

Proof. By the definitions of $\{M_t : t \geq 0\}$ and $\{N_t : t \geq 0\}$, we know that for any $j \geq 0$,

$$P(C_{j,0} \leq t, C_{j,0} < C_{j,1}) = \int_0^t d_{j,0}(\tau)d\tau = h_{j,-2}(t),$$
$$P(C_{j,1} \leq t, C_{j,1} < C_{j,0}) = \int_0^t d_{j,1}(\tau)d\tau = h_{j,-1}(t)$$

and

$$P(C_j \leq t) = \int_0^t d_j(\tau)d\tau = h_{j,-2}(t) + h_{j,-1}(t).$$

Therefore, $d_{j,0}(t) = h'_{j,-2}(t)$, $d_{j,1}(t) = h'_{j,-1}(t)$ and $d_j(t) = h'_{j,-2}(t) + h'_{j,-1}(t)$. Hence,

$$\Delta_{j,0}(\lambda) = \lambda\phi_{j,-2}(\lambda), \quad \Delta_{j,1}(\lambda) = \lambda\phi_{j,-1}(\lambda)$$

and

$$\Delta_j(\lambda) \quad = \quad \lambda\phi_{j,-2}(\lambda) + \lambda\phi_{j,-1}(\lambda).$$

By (10) of Lemma 2, we know that

$$(\lambda+\alpha)\lambda\phi_{j,-2}(\lambda) + \alpha\lambda\phi_{j,-1}(\lambda) = \alpha(1-\lambda\phi_{j,0}(\lambda))$$

and

$$\beta\lambda\phi_{j,-2}(\lambda) + (\lambda+\beta)\lambda\phi_{j,-1}(\lambda) = \beta(1-\lambda\phi_{j,1}(\lambda)).$$

Therefore, by the first two equalities of (10),

$$\Delta_{j,0}(\lambda) = \lambda\phi_{j,-2}(\lambda) = \frac{\alpha[(\lambda+\beta)(1-\lambda\phi_{j0}(\lambda)) - \beta(1-\lambda\phi_{j,1}(\lambda))]}{\lambda^2+(\alpha+\beta)\lambda},$$

$$\Delta_{j,1}(\lambda) = \lambda\phi_{j,-1}(\lambda) = \frac{\beta[(\lambda+\alpha)(1-\lambda\phi_{j1}(\lambda)) - \alpha(1-\lambda\phi_{j,0}(\lambda))]}{\lambda^2+(\alpha+\beta)\lambda}$$

and hence

$$\Delta_j(\lambda) = \frac{\alpha(1-\lambda\phi_{j0}(\lambda)) + \beta(1-\lambda\phi_{j1}(\lambda))}{\lambda+\alpha+\beta}.$$

Note that $P(C_j < \infty) = \Delta_j(0) = 1$; the last two assertions hold. The proof is complete. □

The following theorem gives the mathematical expectation and the second moment of C_j.

Theorem 5. *For any* $j \geq 0$,

$$E[C_j] = \frac{1+\alpha\phi_{j,0}(0)+\beta\phi_{j,1}(0)}{\alpha+\beta}$$

and

$$E[C_j^2] = \frac{2[1+\alpha\phi_{j,0}(0)+\beta\phi_{j,1}(0) - (\alpha+\beta)(\alpha\phi'_{j,0}(0)+\beta\phi'_{j,1}(0))]}{(\alpha+\beta)^2},$$

where $\phi_{j,0}(\lambda)$ *and* $\phi_{j,1}(\lambda)$ *are given by* (34).

Proof. By Theorem 4, we have

$$(\lambda+\alpha+\beta)\Delta_j(\lambda) = \alpha(1-\lambda\phi_{j,0}(\lambda)) + \beta(1-\lambda\phi_{j,1}(\lambda)).$$

Differentiating the above equality yields

$$(\lambda+\alpha+\beta)\Delta'_j(\lambda) + \Delta_j(\lambda) = -\alpha(\lambda\phi_{j,0}(\lambda))' - \beta(\lambda\phi_{j,1}(\lambda))'. \tag{37}$$

Let $\lambda = 0$ and note that $\Delta_j(0) = 1$. We have

$$E[C_j] = -\Delta'_j(0) = \frac{1+\alpha\phi_{j,0}(0)+\beta\phi_{j,1}(0)}{\alpha+\beta}.$$

Differentiating (37) yields

$$\begin{aligned} & (\lambda+\alpha+\beta)\Delta''_j(\lambda) + 2\Delta'_j(\lambda) \\ = & -\alpha(\lambda\phi_{j,0}(\lambda))'' - \beta(\lambda\phi_{j,1}(\lambda))'' \\ = & -\alpha[\lambda\phi''_{j,0}(\lambda) + 2\phi'_{j,0}(\lambda)] - \beta[\lambda\phi''_{j,1}(\lambda) + 2\phi'_{j,1}(\lambda)]. \end{aligned}$$

Let $\lambda = 0$ in the above equality yield

$$(\alpha+\beta)\Delta''_j(0) + 2\Delta'_j(0) = -2\alpha\phi'_{j,0}(0) - 2\beta\phi'_{j,1}(0).$$

Therefore,

$$\begin{aligned}E[C_j^2] &= \Delta_j''(0) \\ &= \frac{2(-\Delta_j'(0) - \alpha\phi_{j,0}'(0) - \beta\phi_{j,1}'(0))}{\alpha+\beta} \\ &= \frac{2[1 + \alpha\phi_{j,0}(0) + \beta\phi_{j,1}(0) - (\alpha+\beta)(\alpha\phi_{j,0}'(0) + \beta\phi_{j,1}'(0))]}{(\alpha+\beta)^2}.\end{aligned}$$

The proof is complete. □

Finally, if $\alpha = 0$ or $\beta = 0$, we obtain the following result which is due to Di Crescenzo et al. [11].

Corollary 1. *(i) If $\beta = 0$, then for any $j \geq 0$,*

$$E[C_j] = \frac{1}{\alpha} + \frac{\hat{\pi}_{j,0}(\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\alpha)}$$

and

$$E[C_j^2] = \frac{2}{\alpha^2}\left(1 + \frac{\alpha\hat{\pi}_{j,0}(\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\alpha)} - \frac{\alpha^2\hat{\pi}_{j,0}'(\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\alpha)} - \frac{\alpha^3\hat{\pi}_{j,0}(\alpha)\hat{\pi}_{0,0}'(\alpha)}{(1 - \alpha\hat{\pi}_{0,0}(\alpha))^2}\right).$$

(ii) If $\alpha = 0$, then for any $j \geq 0$,

$$E[C_j] = \frac{1}{\beta} + \frac{\hat{\pi}_{j,1}(\beta)}{1 - \beta\hat{\pi}_{1,1}(\beta)}$$

and

$$E[C_j^2] = \frac{2}{\beta^2}\left(1 + \frac{\beta\hat{\pi}_{j,1}(\beta)}{1 - \beta\hat{\pi}_{1,1}(\beta)} - \frac{\beta^2\hat{\pi}_{j,1}'(\beta)}{1 - \beta\hat{\pi}_{1,1}(\beta)} - \frac{\beta^3\hat{\pi}_{j,1}(\beta)\hat{\pi}_{1,1}'(\beta)}{(1 - \beta\hat{\pi}_{1,1}(\beta))^2}\right).$$

Proof. If $\beta = 0$, by Theorem 3,

$$\phi_{j,0}(\lambda) = \hat{\pi}_{j,0}(\lambda+\alpha) + \frac{\alpha\hat{\pi}_{j,0}(\lambda+\alpha)\hat{\pi}_{0,0}(\lambda+\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\lambda+\alpha)} = \frac{\hat{\pi}_{j,0}(\lambda+\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\lambda+\alpha)}.$$

Therefore,

$$\phi_{j,0}(0) = \frac{\hat{\pi}_{j,0}(\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\alpha)}$$

and

$$\phi_{j,0}'(0) = \frac{\hat{\pi}_{j,0}'(\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\alpha)} + \frac{\alpha\hat{\pi}_{j,0}(\alpha)\hat{\pi}_{0,0}'(\alpha)}{(1 - \alpha\hat{\pi}_{0,0}(\alpha))^2}.$$

Hence, by Theorem 4,

$$E[C_j] = \frac{1 + \alpha\phi_{j,0}(0)}{\alpha} = \frac{1}{\alpha} + \frac{\hat{\pi}_{j,0}(\alpha)}{1 - \alpha\hat{\pi}_{0,0}(\alpha)}$$

and

$$\begin{aligned} E[C_j^2] &= \frac{2}{\alpha^2}[1+\alpha\phi_{j,0}(0)-\alpha^2\phi'_{j,0}(0)] \\ &= \frac{2}{\alpha^2}\left(1+\frac{\alpha\hat{\pi}_{j,0}(\alpha)}{1-\alpha\hat{\pi}_{0,0}(\alpha)}-\frac{\alpha^2\hat{\pi}'_{j,0}(\alpha)}{1-\alpha\hat{\pi}_{0,0}(\alpha)}-\frac{\alpha^3\hat{\pi}_{j,0}(\alpha)\hat{\pi}'_{0,0}(\alpha)}{(1-\alpha\hat{\pi}_{0,0}(\alpha))^2}\right). \end{aligned}$$

(i) is proven. The proof of (ii) is similar. □

4. Summary

In this paper, we mainly considered the influence of two-type catastrophes in the general birth–death processes. We first revealed the relationship of transition probability of the process with catastrophe and transition probability of the process without catastrophe in the Laplace transform version. Then, we constructed a new process, $\{M_t : t \geq 0\}$, which coincides with $\{N_t : t \geq 0\}$ until the occurrence of catastrophe and can distinguish what type the first effective catastrophe is when it occurs. By discussing the relationship of the transition probability of $\{M_t : t \geq 0\}$ and the transition probability of the process with catastrophe, we established the relationship of the transition probability of $\{M_t : t \geq 0\}$ and the transition probability of the process without catastrophe in the Laplace transform version. Finally, we obtained the probability distribution of the first occurrence time of an effective catastrophe in the Laplace transform version and the probabilities of that the first effective catastrophe is an α-type or a β-type. In particular, if $\alpha = 0$ or $\beta = 0$, we then obtained the results in Di Crescenzo et al. [11].

Relevant to the model considered in this paper, there are some interesting and important problems. For example, we let $C_j(n)$ $(n \geq 1)$ denote the occurrence time of the n'th catastrophe. What is the probability distribution of $C_j(n)$? And also, how do the multi-type catastrophes affect a branching system?

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References

1. Anderson, W. *Continuous-Time Markov Chains: An Applications-Oriented Approach*; Springer: New York, NY, USA, 1991.
2. Asmussen, S. *Applied Probability and Queues*, 2nd ed.; John Wiley: New York, NY, USA, 2003.
3. Chen, M.F. *Eigenvalues, Inequalities, and Ergodic Theory*; Springer: London, UK, 2004.
4. Chen, A.Y.; Zhang, H.J.; Liu, K. Birth-death processes with disasters and instantaneous resurrection. *Adv. Appl. Probab.* **2004**, *36*, 267–292. [CrossRef]
5. Economou, A.; Fakinos, D. A continuous-time Markov chain under the influence of a regulating point process and applications in stochastic models with catastrophes. *Eur. J. Oper. Res.* **2003**, *149*, 625–640. [CrossRef]
6. Pakes, A.G. Killing and resurrection of Markov processes. *Commun. Statist. Stoch. Model.* **1997**, *13*, 255–269. [CrossRef]
7. Chen, A.Y.; Renshaw, E. The $M/M/1$ queue with mass exodus and mass arrives when empty. *J. Appl. Probab.* **1997**, *34*, 192–207. [CrossRef]
8. Chen, A.Y.; Renshaw, E. Markov bulk-arriving queues with state-dependent control at idle time. *Adv. Appl. Probab.* **2004**, *36*, 499–524. [CrossRef]
9. Zhang, L.N.; Li, J.P. The $M/M/c$ queue with mass exodus and mass arrivals when empty. *J. Appl. Probab.* **2015**, *52*, 990–1002. [CrossRef]
10. Li, J.P.; Zhang, L.N. $M^X/M/c$ Queue with catastrophes and state-dependent control at idle time. *Front. Math. China* **2017**, *12*, 1427–1439. [CrossRef]
11. Di Crescenzo, A.; Giorno, V.; Nobile, A.G.; Ricciardi, L.M. A note on birth-death processes with catastrophes. *Stat. Probab. Lett.* **2008**, *78*, 2248–2257. [CrossRef]
12. Artalejo, J. G-networks: A versatile approach for work removal in queueing networks. *Eur. J. Oper. Res.* **2000**, *126*, 233–249. [CrossRef]
13. Bayer, N.; Boxma, O.J. Wiener-Hopf analysis of an $M/G/1$ queue with negative customers and of a related class of random walks. *Queueing Syst.* **1996**, *23*, 301–316. [CrossRef]

14. Chen, A.Y.; Pollett, P.; Li, J.P.; Zhang, H.J. Markovian bulk-arrival and bulk-service queues with state-dependent control. *Queueing Syst.* **2010**, *64*, 267–304. [CrossRef]
15. Dudin, A.; Karolik, A. $BMAP/SM/1$ queue with Markovian input of disasters and non-instantaneous recovery. *Perform. Eval.* **2001**, *45*, 19–32. [CrossRef]
16. Gelenbe, E. Product-form queueing networks with negative and positive customers. *J. Appl. Probab.* **1991**, *28*, 656–663. [CrossRef]
17. Gelenbe, E.; Glynn, P.; Sigman, K. Queues with negative arrivals. *J. Appl. Probab.* **1991**, *28*, 245–250. [CrossRef]
18. Jain, G.; Sigman, K. A Pollaczek-Khintchine formula for $M/G/1$ queues with disasters. *J. Appl. Probab.* **1996**, *33*, 1191–1200. [CrossRef]
19. Zeifman, A.; Korotysheva, A. Perturbation Bounds for $M_t/M_t/N$ Queue with Catastrophes. *Stoch. Model.* **2012**, *28*, 49–62. [CrossRef]