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Article

Symmetric Quantum Inequalities on Finite Rectangular Plane

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Abstract: Finding the range of coordinated convex functions is yet another application for the symmetric Hermite–Hadamard inequality. For any two-dimensional interval $[a_0, a_1] \times [c_0, c_1] \subseteq \Re^2$, we introduce the notion of partial q_θ-, q_ϕ-, and $q_\theta q_\phi$-symmetric derivatives and a $q_\theta q_\phi$-symmetric integral. Moreover, we will construct the $q_\theta q_\phi$-symmetric Hölder's inequality, the symmetric quantum Hermite–Hadamard inequality for the function of two variables in a rectangular plane, and address some of its related applications.

Keywords: coordinate convex functions; symmetric quantum calculus; symmetric quantum Hölder's inequality; symmetric quantum Hermite–Hadamard inequality

MSC: 26D10; 26A51; 05A30; 26B25

1. Introduction

The study of convex functions is an important field in mathematics which occurs almost everywhere in pure and applied branches of mathematics, but mainly it plays a key role in predicting the approximate solutions for linear and non-linear programming. A well known inequality called the Hermite–Hadamard inequality, introduced first by Hadamard [1], states:

Let $\mathfrak{f} : [a_0, a_1] \subseteq \Re \longrightarrow \Re$ be a convex function on $[a_0, a_1]$, then

$$\mathfrak{f}\left(\frac{a_0 + a_1}{2}\right) \leqslant \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \mathfrak{f}(x)\, dx \leqslant \frac{\mathfrak{f}(a_0) + \mathfrak{f}(a_1)}{2} \tag{1}$$

inequality holds.

Over the last three decades, a number of researchers concentrated their research on various types of Hadamard's inequality and its related applications. But we are able to acknowledge with confidence that the actual work on it was diversified by Dragomir in 1992 (see [2]). Many interested researchers have generalized (1) and constructed multiple formulations in different forms as a result of this progressive work. Dragomir established a number of Hermite–Hadamard-type inequalities for many kinds of functions using various assumptions, like convex functions defined on a disc in the plane and on a ball in the space. See Ref. [2] for further details. The Hermite–Hadamard-type inequalities for convex functions on n-dimensional convex bodies were investigated in 2006 by de la Cal and Cárcamo [3]. They derived two main results; the first one is related to mappings on polytopes in $\Re^n$, and the second one is related (1) via symmetric random vectors on arbitrary norms on $\Re^n$. Another extension of (1) for a function defined on a convex subset of $\Re^3$ was introduced by Yang in 2012 [4]. Furthermore, matrix and operator inequalities of the Hermite–Hadamard type were introduced by Moslehian recently in [5]. The idea of a coordinated convex function was given by Dragomir [6]; it is defined as follows:

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For any two-dimensional interval $[a_0, a_1] \times [c_0, c_1] \subseteq \Re^2$, with $a_0 < a_1$ and $c_0 < c_1$. A function $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \longrightarrow \Re$ is a coordinated convex if the partial mappings $\mathfrak{f}_y : [a_0, a_1] \longrightarrow \Re$, $\mathfrak{f}_y(u) = \mathfrak{f}(u, y)$ and $\mathfrak{f}_x : [c_0, c_1] \longrightarrow \Re$, $\mathfrak{f}_x(u) = \mathfrak{f}(x, u)$ are convex for all $x \in [a_0, a_1]$ and $y \in [c_0, c_1]$.

Using this definition, Dragomir derived the Hermite–Hadamard inequality on a coordinated convex function, which is stated as follows:

Theorem 1 ([6])**.** *Let* $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ *be a convex coordinated function, then*

$$\begin{aligned}
\mathfrak{f}\left(\frac{a_0+a_1}{2}, \frac{c_0+c_1}{2}\right) &\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right)dx + \frac{1}{c_1-c_0}\int_{c_0}^{c_1} \mathfrak{f}\left(\frac{a_0+a_1}{2}, y\right)dy\right] \\
&\leqslant \frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} \mathfrak{f}(x,y)dydx \\
&\leqslant \frac{1}{4(a_1-a_0)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_0)dx + \frac{1}{4(a_1-a_0)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_1)dx \\
&+ \frac{1}{4(c_1-c_0)}\int_{c_0}^{c_1} \mathfrak{f}(a_0,y)dy + \frac{1}{4(c_1-c_0)}\int_{c_0}^{c_1} \mathfrak{f}(a_1,y)dy \\
&\leqslant \frac{\mathfrak{f}(a_0,c_0)+\mathfrak{f}(a_1,c_0)+\mathfrak{f}(a_0,c_1)+\mathfrak{f}(a_1,c_1)}{4}
\end{aligned} \tag{2}$$

inequalities hold.

In the current era, the study of quantum calculus is also a center of attention for researchers. Euler (1707–1783) was the first person to propose the concept of quantum calculus after the 17th century. Its role provides a bridge between physics and mathematics. F. H. Jackson and others focused more on quantum calculus in the early 20th century. Quantum calculus (q-calculus), a subfield of time-scale calculus, deals with the study of difference equations and provides solutions to a wide range of dynamical problems. Furthermore, we can state that q-calculus generalizes the derivative and integration of classical calculus and that we retrieve the classical conclusions as $q \to 1$. The Euler notion was first studied by Jackson [7] in 1910. He used it to define q-integration and q-derivatives for continuous functions over an interval $(0, \infty)$, commonly called calculus without limits. Al-Salam [8] introduced the concepts of q-fractional and q-Riemann–Liouville fractional inequalities in 1966. The fundamental principles of q-calculus were first discussed by Kac and Cheung in their book [9]. We recommend reading [10,11] for some details on quantum calculus.

The q-integral and q-derivative of continuous functions over finite intervals were specifically presented in 2013 by Tariboon and Ntouyas.

Definition 1 ([12])**.** *For a continuous function* $\mathfrak{f} : [a_0, a_1] \to \Re$*, suppose that for* $0 < q < 1$

$$\lim_{x \to a_0} \frac{\mathfrak{f}(x) - \mathfrak{f}(qx + (1-q)a_0)}{(1-q)(x-a_0)} = b.$$

Then, ${}_{a_0}D_q\mathfrak{f}(x)$ *is defined as follows:*

$${}_{a_0}D_q\mathfrak{f}(x) = \begin{cases} \dfrac{\mathfrak{f}(x) - \mathfrak{f}(qx + (1-q)a_0)}{(1-q)(x-a_0)}, & \text{if } x \neq a_0, \\ b, & \text{if } x = a_0. \end{cases} \tag{3}$$

If we put $a_0 = 0$, (3) *becomes*

$$D_q\mathfrak{f}(x) = \frac{\mathfrak{f}(x) - \mathfrak{f}(qx)}{(1-q)(x)}.$$

Definition 2 ([12]). *For a continuous function* $\mathfrak{f} : [a_0, a_1] \to \Re$, *the* q*-integral on* $[a_0, a_1]$ *can be obtained by*

$$\int_{a_0}^{x} \mathfrak{f}(t) \ {}_{a_0}d_q t = (1-q)(x-a_0) \sum_{n=0}^{\infty} q^n \mathfrak{f}(q^n x + (1-q^n)a_0), \tag{4}$$

If we put $a_0 = 0$, (4) *becomes*

$$\int_{0}^{x} \mathfrak{f}(t) \ d_q t = (1-q)x \sum_{n=0}^{\infty} q^n \mathfrak{f}(q^n x).$$

Furthermore, Jackson [7] derived its subsequent form:

$$\int_{a_0}^{a_1} \mathfrak{f}(t) \ d_q t = \int_{0}^{a_1} \mathfrak{f}(t) \ d_q t - \int_{0}^{a_0} \mathfrak{f}(t) \ d_q t.$$

Utilizing the above essentials of quantum calculus, Tariboon and Ntouyas developed various well-known inequalities on finite intervals in [13] and the Hermite–Hadamard inequality is one of them. But there is an error in the first form of the quantum Hermite–Hadamard inequality, which was pointed out by Alp et al. in [14], where they gave a counter-example to the previously established form of this inequality. Thus, they formulated the corrected form of Hermite–Hadamard inequality in quantum calculus, which can be presented as

Theorem 2 ([14]). *For* $\mathfrak{f} : [a_0, a_1] \to \Re$ *being a convex function on* $[a_0, a_1]$, *then*

$$\mathfrak{f}\left(\frac{qa_0 + a_1}{1+q}\right) \le \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \mathfrak{f}(t) \ {}_{a_0}d_q t \ \le \frac{q\mathfrak{f}(a_0) + \mathfrak{f}(a_1)}{1+q},$$

inequality holds for $q \in (0, 1)$.

Dragomir et al., in [15], published some work on quantum calculus. They provided some definitions of the partial q_θ-derivative, q_ϕ-derivative, $q_\theta q_\phi$-derivative, and $q_\theta q_\phi$-integral and shifted the notion of quantum calculus on finite intervals with work on bi-intervals (i.e., coordinates in plane).

Definition 3 ([15]). *Let* $\mathfrak{f}$ *be a continuous function of two variables from* $[a_0, a_1] \times [c_0, c_1] \subseteq \Re^2$ *to* $\Re$ *and* $0 < q_\theta < 1$, $0 < q_\phi < 1$, *then*

$$\frac{{}_{a_0}\partial_{q_\theta} \mathfrak{f}(x,y)}{{}_{a_0}\partial_{q_\theta} x} = \frac{\mathfrak{f}(q_\theta x + (1-q_\theta)a_0, y) - \mathfrak{f}(x,y)}{(1-q_\theta)(x-a_0)}, \quad x \neq a_0,$$

$$\frac{{}_{c_0}\partial_{q_\phi} \mathfrak{f}(x,y)}{{}_{c_0}\partial_{q_\phi} y} = \frac{\mathfrak{f}(x, q_\phi y + (1-q_\phi)c_0) - \mathfrak{f}(x,y)}{(1-q_\phi)(y-c_0)}, \quad y \neq c_0,$$

and

$$\frac{{}_{a_0,c_0}\partial^2_{q_\theta,q_\phi} \mathfrak{f}(x,y)}{{}_{a_0}\partial_{q_\theta} x \ {}_{c_0}\partial_{q_\phi} y} = \frac{1}{(1-q_\theta)(1-q_\phi)(x-a_0)(y-c_0)} \times \big[\mathfrak{f}(q_\theta x + (1-q_\theta)a_0, q_\phi y + (1-q_\phi)c_0)$$
$$- \mathfrak{f}(q_\theta x + (1-q_\theta)a_0, y) - \mathfrak{f}(x, q_\phi y + (1-q_\phi)c_0) + \mathfrak{f}(x,y)\big], \quad x \neq a_0, y \neq c_0,$$

are called the partial q_θ*-derivative,* q_ϕ*-derivative, and* $q_\theta q_\phi$*-derivative at* $(x, y) \in [a_0, a_1] \times [c_0, c_1]$.

Definition 4 ([15])**.** *Let $\mathfrak{f}$ be a continuous function from $[a_0, a_1] \times [c_0, c_1] \subseteq \Re^2$ to $\Re$, then*

$$\int_{c_0}^{y}\int_{a_0}^{x} \mathfrak{f}(x,y)\ {}_{a_0}d_{q_\theta}x\ {}_{c_0}d_{q_\phi}y = (x-a_0)(y-c_0)(1-q_\theta)(1-q_\phi) \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_\theta^n q_\phi^m \mathfrak{f}\Big(q_\theta^n x + (1-q_\theta^n)a_0, q_\phi^m y + \Big(1-q_\phi^m\Big)c_0\Big)$$

is called the $q_\theta q_\phi$-integral on $[a_0, a_1] \times [c_0, c_1]$.

With the help of these definitions, they derived the Hermite–Hadamard inequality for the convex function of two variables.

Theorem 3. *Let $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ be a convex coordinated function on $[a_0, a_1] \times [c_0, c_1]$, then for $0 < q_\theta < 1$ and $0 < q_\phi < 1$, the following inequalities hold:*

$$\begin{aligned}
&\mathfrak{f}\left(\frac{a_0+a_1}{2}, \frac{c_0+c_1}{2}\right) \\
&\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1} \mathfrak{f}(x, \frac{c_0+c_1}{2})\ {}_{a_0}d_{q_\theta}x + \frac{1}{c_1-c_0}\int_{c_0}^{c_1} \mathfrak{f}(\frac{a_0+a_1}{2}, y)\ {}_{c_0}d_{q_\phi}y\right] \\
&\leqslant \frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} \mathfrak{f}(x,y)\ {}_{c_0}d_{q_\phi}y\ {}_{a_0}d_{q_\theta}x \\
&\leqslant \frac{q_\phi}{2(a_1-a_0)(1+q_\phi)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_0)\ {}_{a_0}d_{q_\theta}x + \frac{1}{2(a_1-a_0)(1+q_\phi)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_1)\ {}_{a_0}d_{q_\theta}x \\
&+ \frac{q_\theta}{2(c_1-c_0)(1+q_\theta)}\int_{c_0}^{c_1} \mathfrak{f}(a_0,y)\ {}_{c_0}d_{q_\phi}y + \frac{1}{2(c_1-c_0)(1+q_\theta)}\int_{c_0}^{c_1} \mathfrak{f}(a_1,y)\ {}_{c_0}d_{q_\phi}y \\
&\leqslant \frac{q_\theta q_\phi \mathfrak{f}(a_0,c_0) + q_\phi \mathfrak{f}(a_1,c_0) + q_\theta \mathfrak{f}(a_0,c_1) + \mathfrak{f}(a_1,c_1)}{(1+q_\theta)(1+q_\phi)}.
\end{aligned} \tag{5}$$

In July 2020, N. Alp and M. Z. Sarikaya [16] published an article in which they gave a counter-example that disproves the (5) and derived the correct form of the Hermite–Hadamard inequality.

Theorem 4 ([16])**.** *Let $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ be a convex coordinated function on $[a_0, a_1] \times [c_0, c_1]$, then for $0 < q_\theta < 1$ and $0 < q_\phi < 1$, the following inequalities hold:*

$$\begin{aligned}
&\mathfrak{f}\left(\frac{a_0 q_\theta + a_1}{1+q_\theta}, \frac{c_0 q_\phi + c_1}{1+q_\phi}\right) \\
&\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1} \mathfrak{f}(x, \frac{c_0 q_\phi+c_1}{1+q_\phi})\ {}_{a_0}d_{q_\theta}x + \frac{1}{c_1-c_0}\int_{c_0}^{c_1} \mathfrak{f}(\frac{a_0 q_\theta+a_1}{1+q_\theta}, y)\ {}_{c_0}d_{q_\phi}y\right] \\
&\leqslant \frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} \mathfrak{f}(x,y)\ {}_{c_0}d_{q_\phi}y\ {}_{a_0}d_{q_\theta}x \\
&\leqslant \frac{q_\phi}{2(a_1-a_0)(1+q_\phi)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_0)\ {}_{a_0}d_{q_\theta}x + \frac{1}{2(a_1-a_0)(1+q_\phi)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_1)\ {}_{a_0}d_{q_\theta}x \\
&+ \frac{q_\theta}{2(c_1-c_0)(1+q_\theta)}\int_{c_0}^{c_1} \mathfrak{f}(a_0,y)\ {}_{c_0}d_{q_\phi}y + \frac{1}{2(c_1-c_0)(1+q_\theta)}\int_{c_0}^{c_1} \mathfrak{f}(a_1,y)\ {}_{c_0}d_{q_\phi}y \\
&\leqslant \frac{q_\theta q_\phi \mathfrak{f}(a_0,c_0) + q_\phi \mathfrak{f}(a_1,c_0) + q_\theta \mathfrak{f}(a_0,c_1) + \mathfrak{f}(a_1,c_1)}{(1+q_\theta)(1+q_\phi)}.
\end{aligned} \tag{6}$$

In the recent past, a number of publications have been presented regarding the improvement and development of different variants of the quantum Hermite–Hadamard and related inequalities (see [17–20], and references therein). However, in the present article we are interested in exploring such findings under the new and latest perspective of symmetric quantum calculus.

2. Preliminaries and New Results in Symmetric Quantum Calculus

The symmetric partial derivatives of the function are defined below.

Definition 5 ([21]). *Let $\mathfrak{f}(x, y)$ be a function which is defined on $\Re^2$, then*

$$\frac{\partial^* \mathfrak{f}}{\partial x} = \lim_{h \to 0} \frac{\mathfrak{f}(x+h, y) - \mathfrak{f}(x-h, y)}{2h} \tag{7}$$

$$\frac{\partial^* \mathfrak{f}}{\partial y} = \lim_{h \to 0} \frac{\mathfrak{f}(x, y+h) - \mathfrak{f}(x, y-h)}{2h} \tag{8}$$

are said to be symmetric partial derivatives of $\mathfrak{f}$ with respect to x and y, respectively.

Da Cruz et al. introduced a new concept called symmetric quantum calculus in [22]. It plays an important role in developing hypergeometric and a class of harmonic functions in complex analysis [23]. The symmetric quantum analogue of any number m can be defined as [9]

$$\widetilde{[m]} = \frac{q^m - q^{-m}}{q - q^{-1}} \qquad \textit{where } m \in N.$$

In addition to this, the q-differential and h-differential for $q \neq 1$ and $h \neq 0$ can be defined as

$$\tilde{d}_q \mathfrak{f}_1(x) = \mathfrak{f}_1(qx) - \mathfrak{f}_1(q^{-1}x), \quad x \neq a_0,$$
$$\tilde{d}_h \mathfrak{f}_2(x) = \mathfrak{f}_2(x+h) - \mathfrak{f}_2(x-h), \quad x \neq a_0.$$

The derivative and integral in symmetric quantum calculus can be derived as

Definition 6 ([24]). *For a continuous function $\mathfrak{f} : [a_0, a_1] \to \Re$, then the $\tilde{q}$-derivative or ${}_{a_0}q$-symmetric derivative on $[a_0, a_1]$ is defined as*

$${}_{a_0}\tilde{D}_q \mathfrak{f}(x) = \frac{\tilde{d}_q \mathfrak{f}(x)}{\tilde{d}_q x} = \frac{\mathfrak{f}(qx + (1-q)a_0) - \mathfrak{f}(q^{-1}x + (1-q^{-1})a_0)}{(q - q^{-1})(x - a_0)}, \quad x \neq a_0,$$

if $a_0 = 0$, this becomes

$$\tilde{D}_q \mathfrak{f}(x) = \frac{\tilde{d}_q \mathfrak{f}(x)}{\tilde{d}_q x} = \frac{\mathfrak{f}(qx) - \mathfrak{f}(q^{-1}x)}{(q - q^{-1})x}, \quad x \neq 0.$$

Definition 7 ([24]). *For a continuous function $\mathfrak{f} : [a_0, a_1] \to \Re$, then the $\tilde{q}$-integral or ${}_{a_0}q$-symmetric integral on $[a_0, a_1]$ is defined as*

$$\int_{a_0}^{x} \mathfrak{f}(t) \, {}_{a_0}\tilde{d}_q t = (q^{-1} - q)(x - a_0) \sum_{n=0}^{\infty} q^{2n+1} \mathfrak{f}\left(q^{2n+1}x + \left(1 - q^{2n+1}\right)a_0\right);$$

here, $x \in [a_0, a_1]$ or

$$\int_{a_0}^{x} \mathfrak{f}(t) \, {}_{a_0}\tilde{d}_q t = (1 - q^2)(x - a_0) \sum_{n=0}^{\infty} q^{2n} \mathfrak{f}\left(q^{2n+1}x + \left(1 - q^{2n+1}\right)a_0\right).$$

In this section, with the help of all of the notions above, we will define some new definitions in symmetric quantum calculus that assist in constructing the $q_\theta q_\phi$-symmetric Hölder's inequality and Hermite–Hadamard inequalities.

We introduce and provide definitions below.

Definition 8. *Let $\mathfrak{f}$ be a continuous function of two variables from* $[a_0, a_1] \times [c_0, c_1] \subseteq \Re^2$ *to* $\Re$ *and* $0 < q_\theta < 1, 0 < q_\phi < 1$*, then*

$$\frac{{}_{a_0}\tilde{\partial}_{q_\theta}\mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta}x} = \frac{\mathfrak{f}(q_\theta x + (1-q_\theta)a_0, y) - \mathfrak{f}(q_\theta^{-1}x + (1-q_\theta^{-1})a_0, y)}{(q_\theta - q_\theta^{-1})(x - a_0)}, \quad x \neq a_0,$$

$$\frac{{}_{c_0}\tilde{\partial}_{q_\phi}\mathfrak{f}(x,y)}{{}_{c_0}\tilde{\partial}_{q_\phi}y} = \frac{\mathfrak{f}(x, q_\phi y + (1-q_\phi)c_0) - \mathfrak{f}(x, q_\phi^{-1}y + (1-q_\phi^{-1})c_0)}{(q_\phi - q_\phi^{-1})(y - c_0)}, \quad y \neq c_0,$$

and

$$\begin{aligned}\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}\mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta}x\,{}_{c_0}\tilde{\partial}_{q_\phi}y} &= \frac{1}{(q_\theta - q_\theta^{-1})(q_\phi - q_\phi^{-1})(x-a_0)(y-c_0)} \times \Big[\mathfrak{f}(q_\theta x + (1-q_\theta)a_0, q_\phi y \\ &\quad + (1-q_\phi)c_0) - \mathfrak{f}(q_\theta x + (1-q_\theta)a_0, q_\phi^{-1}y + (1-q_\phi^{-1})c_0) \\ &\quad - \mathfrak{f}(q_\theta^{-1}x + (1-q_\theta^{-1})a_0, q_\phi y + (1-q_\phi)c_0) \\ &\quad + \mathfrak{f}(q_\theta^{-1}x + (1-q_\theta^{-1})a_0, q_\phi^{-1}y + (1-q_\phi^{-1})c_0)\Big], \quad x \neq a_0, y \neq c_0, \end{aligned} \tag{9}$$

are called the partial q_θ-, q_ϕ-, *and* $q_\theta q_\phi$-*symmetric derivatives at* $(x, y) \in [a_0, a_1] \times [c_0, c_1]$.

If $\frac{{}_{a_0}\tilde{\partial}_{q_\theta}\mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta}x}$, $\frac{{}_{c_0}\tilde{\partial}_{q_\phi}\mathfrak{f}(x,y)}{{}_{c_0}\tilde{\partial}_{q_\phi}y}$, and $\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}\mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta}x\,{}_{c_0}\tilde{\partial}_{q_\phi}y}$ exist for all $(x, y) \in [a_0, a_1] \times [c_0, c_1]$, then $\mathfrak{f}$ is called partially q_θ-, q_ϕ-, and $q_\theta q_\phi$-symmetric differentiable on $[a_0, a_1] \times [c_0, c_1]$.

Example 1. *Let* $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ *be a function.*
Case 1. *If $\mathfrak{f}$ is a non-convex function. Suppose that* $\mathfrak{f}(x, y) = xy$*, then the partial* q_θ-*symmetric derivative of* $\mathfrak{f}(x, y)$ *is*

$$\begin{aligned}\frac{{}_{a_0}\tilde{\partial}_{q_\theta}\mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta}x} &= \frac{(q_\theta x + (1-q_\theta)a_0)y - (q_\theta^{-1}x + (1-q_\theta^{-1})a_0)y}{(q_\theta - q_\theta^{-1})(x-a_0)} \\ &= \frac{y[q_\theta x + a_0 - q_\theta a_0 - q_\theta^{-1}x - a_0 + q_\theta^{-1}a_0]}{(q_\theta - q_\theta^{-1})(x-a_0)} = y.\end{aligned}$$

The partial q_ϕ-*symmetric derivative of* $\mathfrak{f}(x,y)$ *is*

$$\begin{aligned}\frac{{}_{c_0}\tilde{\partial}_{q_\phi}\mathfrak{f}(x,y)}{{}_{c_0}\tilde{\partial}_{q_\phi}y} &= \frac{x(q_\phi y + (1-q_\phi)c_0) - x(q_\phi^{-1}y + (1-q_\phi^{-1})c_0)}{(q_\phi - q_\phi^{-1})(y-c_0)} \\ &= \frac{x[q_\phi y + c_0 - q_\phi c_0 - q_\phi^{-1}y - c_0 + q_\phi^{-1}c_0)]}{(q_\phi - q_\phi^{-1})(y-c_0)} = x.\end{aligned}$$

And also, the partial $q_\theta q_\phi$-*symmetric derivative of* $\mathfrak{f}(x,y)$ *is*

$$\begin{aligned}\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}\mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta}x\,{}_{c_0}\tilde{\partial}_{q_\phi}y} &= \frac{1}{(q_\theta - q_\theta^{-1})(q_\phi - q_\phi^{-1})(x-a_0)(y-c_0)}\Big[\{q_\theta x + (1-q_\theta)a_0\}\{q_\phi y \\ &\quad + (1-q_\phi)c_0\} - \{q_\theta x + (1-q_\theta)a_0\}\{q_\phi^{-1}y + (1-q_\phi^{-1})c_0\} \\ &\quad - \{q_\theta^{-1}x + (1-q_\theta^{-1})a_0\}\{q_\phi y + (1-q_\phi)c_0\} \\ &\quad + \{q_\theta^{-1}x + (1-q_\theta^{-1})a_0\}\{q_\phi^{-1}y + (1-q_\phi^{-1})c_0\}\Big]\end{aligned}$$

$$= \frac{1}{(q_\theta - q_\theta^{-1})(q_\phi - q_\phi^{-1})(x-a_0)(y-c_0)} \left[\{q_\theta x + (1-q_\theta)a_0\}\{q_\phi y + (1-q_\phi)c_0 - q_\phi^{-1}y - (1-q_\phi^{-1})c_0\} - \{q_\theta^{-1}x + (1-q_\theta^{-1})a_0\} \times \{q_\phi y + (1-q_\phi)c_0 - q_\phi^{-1}y - (1-q_\phi^{-1})c_0\} \right]$$

$$= \frac{1}{(q_\theta - q_\theta^{-1})(q_\phi - q_\phi^{-1})(x-a_0)(y-c_0)} \left[\{q_\theta x + (1-q_\theta)a_0 - q_\theta^{-1}x - (1-q_\theta^{-1})a_0\}\{q_\phi y + (1-q_\phi)c_0 - q_\phi^{-1}y - (1-q_\phi^{-1})c_0\} \right]$$

$$= \frac{1}{(q_\theta - q_\theta^{-1})(q_\phi - q_\phi^{-1})(x-a_0)(y-c_0)} \left[\{(q_\theta - q_\theta^{-1})x + (1 - q_\theta - 1 + q_\theta^{-1})a_0\} \times \{(q_\phi - q_\phi^{-1})y + (1 - q_\phi - 1 + q_\phi^{-1})c_0\} \right]$$

$$= \frac{1}{(q_\theta - q_\theta^{-1})(q_\phi - q_\phi^{-1})(x-a_0)(y-c_0)} \left[\{(q_\theta - q_\theta^{-1})x - (q_\theta - q_\theta^{-1})a_0\} \times \{(q_\phi - q_\phi^{-1})y - (q_\phi - q_\phi^{-1})c_0\} \right],$$

and finally,

$$\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi} \mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta} x \; {}_{c_0}\tilde{\partial}_{q_\phi} y} = 1.$$

Case 2. *If $\mathfrak{f}$ is a convex function. Suppose that $\mathfrak{f}(x, y) = x^2 + y^2$, then the partial q_θ-symmetric derivative of $\mathfrak{f}(x, y)$ is*

$$\begin{aligned}
\frac{{}_{a_0}\tilde{\partial}_{q_\theta} \mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta} x} &= \frac{(q_\theta x + (1-q_\theta)a_0)^2 + y^2 - (q_\theta^{-1}x + (1-q_\theta^{-1})a_0)^2 - y^2}{(q_\theta - q_\theta^{-1})(x-a_0)} \\
&= \frac{1}{(q_\theta - q_\theta^{-1})(x-a_0)}[(q_\theta)^2x^2 + a_0^2 + (q_\theta)^2a_0^2 - 2q_\theta a_0^2 + 2q_\theta x a_0 - 2(q_\theta)^2 x a_0 \\
&- (q_\theta^{-1})^2x^2 - a_0^2 - (q_\theta^{-1})^2a_0^2 + 2q_\theta^{-1}a_0^2 - 2q_\theta^{-1}xa_0 + 2(q_\theta^{-1})^2xa_0] \\
&= \frac{1}{(q_\theta - q_\theta^{-1})(x-a_0)}[\{(q_\theta)^2 - (q_\theta^{-1})^2\}x^2 + \{(q_\theta)^2 - (q_\theta^{-1})^2\}a_0^2 - 2a_0^2(q_\theta - q_\theta^{-1}) \\
&+ 2(q_\theta - q_\theta^{-1})xa_0 - 2\{(q_\theta)^2 - (q_\theta^{-1})^2\}xa_0] \\
&= \frac{(q_\theta + q_\theta^{-1})x^2 + (q_\theta + q_\theta^{-1})a_0^2 - 2a_0^2 + 2xa_0 - 2(q_\theta + q_\theta^{-1})xa_0}{x - a_0}.
\end{aligned}$$

Similarly, the partial q_ϕ-symmetric derivative of $\mathfrak{f}(x,y)$ is

$$\frac{{}_{c_0}\tilde{\partial}_{q_\phi} \mathfrak{f}(x,y)}{{}_{c_0}\tilde{\partial}_{q_\phi} y} = \frac{(q_\phi + q_\phi^{-1})y^2 + (q_\phi + q_\phi^{-1})c_0^2 - 2c_0^2 + 2yc_0 - 2(q_\phi + q_\phi^{-1})yc_0}{y - c_0}.$$

And also, the partial $q_\theta q_\phi$*-symmetric derivative of* $\mathfrak{f}(x,y)$ *is*

$$\begin{aligned}\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}\mathfrak{f}(x,y)}{{}_{a_0}\tilde{\partial}_{q_\theta}x\; {}_{c_0}\tilde{\partial}_{q_\phi}y} &= \frac{1}{(q_\theta-q_\theta^{-1})(q_\phi-q_\phi^{-1})(x-a_0)(y-c_0)}\Big[\{q_\theta x+(1-q_\theta)a_0\}^2+\{q_\phi y\\ &+(1-q_\phi)c_0\}^2-\{q_\theta x+(1-q_\theta)a_0\}^2-\{q_\phi^{-1}y+(1-q_\phi^{-1})c_0\}^2\\ &-\{q_\theta^{-1}x+(1-q_\theta^{-1})a_0\}^2-\{q_\phi y+(1-q_\phi)c_0\}^2\\ &+\{q_\theta^{-1}x+(1-q_\theta^{-1})a_0\}^2+\{q_\phi^{-1}y+(1-q_\phi^{-1})c_0\}^2\Big]=0.\end{aligned}$$

Remark 1. *If* $q_\theta, q_\phi \longrightarrow 1$ *in both cases of Example 1, then the results will coincide with the symmetric partial derivatives of such functions coming from (7) and (8), respectively.*

Definition 9. *Let* $\mathfrak{f}$ *be a continuous function from* $[a_0, a_1]\times[c_0, c_1] \subseteq \Re^2$ *to* $\Re$*, then for* $(x, y)\in[a_0, a_1]\times[c_0, c_1]$*,*

$$\begin{aligned}\int_{c_0}^{y}\int_{a_0}^{x}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y &= (x-a_0)(y-c_0)(q_\theta^{-1}-q_\theta)(q_\phi^{-1}-q_\phi)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n+1}\\ &\times q_\phi^{2m+1}\mathfrak{f}\Big(q_\theta^{2n+1}x+\Big(1-q_\theta^{2n+1}\Big)a_0, q_\phi^{2m+1}y+\Big(1-q_\phi^{2m+1}\Big)c_0\Big)\end{aligned}$$

or

$$\begin{aligned}\int_{c_0}^{y}\int_{a_0}^{x}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y &= (x-a_0)(y-c_0)(1-q_\theta^2)(1-q_\phi^2)\\ &\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n+1}x+\Big(1-q_\theta^{2n+1}\Big)a_0, q_\phi^{2m+1}y+\Big(1-q_\phi^{2m+1}\Big)c_0\Big)\end{aligned}\tag{10}$$

is called $q_\theta q_\phi$*-symmetric integral on* $[a_0, a_1]\times[c_0, c_1]$.

From (10), we can say that

$$\int_{c_0}^{y}\int_{a_0}^{x}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y = \int_{a_0}^{x}\int_{c_0}^{y}\mathfrak{f}(x,y)\; {}_{c_0}\tilde{d}_{q_\phi}y\; {}_{a_0}\tilde{d}_{q_\theta}x.$$

Moreover, for any point $(x_0, y_0)\in(a_0, x)\times(c_0, y)$, we can write

$$\begin{aligned}\int_{y_0}^{y}\int_{x_0}^{x}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y &= \int_{y_0}^{y}\int_{a_0}^{x}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y - \int_{y_0}^{y}\int_{a_0}^{x_0}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y\\ &= \int_{c_0}^{y}\int_{a_0}^{x}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y - \int_{c_0}^{y_0}\int_{a_0}^{x}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y\\ &-\int_{c_0}^{y}\int_{a_0}^{x_0}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y + \int_{c_0}^{y_0}\int_{a_0}^{x_0}\mathfrak{f}(x,y)\; {}_{a_0}\tilde{d}_{q_\theta}x\; {}_{c_0}\tilde{d}_{q_\phi}y.\end{aligned}\tag{11}$$

Some results that are given below hold for Definitions 8 and 9.

Theorem 5. *Let* $\mathfrak{f}$ *be a continuous function from* $[a_0, a_1]\times[c_0, c_1]\subseteq\Re^2$ *to* $\Re$*, then*

(1) $\dfrac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}}{{}_{a_0}\tilde{\partial}_{q_\theta}x\; {}_{c_0}\tilde{\partial}_{q_\phi}y}\displaystyle\int_{c_0}^{y}\int_{a_0}^{x}\mathfrak{f}(u,v)\; {}_{a_0}\tilde{d}_{q_\theta}u\; {}_{c_0}\tilde{d}_{q_\phi}v = \mathfrak{f}(x,y)$

(2) $\displaystyle\int_{c_0}^{y}\int_{a_0}^{x}\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}\mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta}u\; {}_{c_0}\tilde{\partial}_{q_\phi}v}\; {}_{a_0}\tilde{d}_{q_\theta}u\; {}_{c_0}\tilde{d}_{q_\phi}v = \mathfrak{f}(x,y)$

(3) $\displaystyle\int_{y_0}^{y}\int_{x_0}^{x}\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}\mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta}u\; {}_{c_0}\tilde{\partial}_{q_\phi}v}\; {}_{a_0}\tilde{d}_{q_\theta}u\; {}_{c_0}\tilde{d}_{q_\phi}v = \mathfrak{f}(x,y)-\mathfrak{f}(x,y_0)-\mathfrak{f}(x_0,y)+\mathfrak{f}(x_0,y_0)$

for $(x_0, y_0) \in (a_0, a_1) \times (c_0, c_1)$.

Proof. (1) Using (10), and then, (9),

$$\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}}{{}_{a_0}\tilde{\partial}_{q_\theta}x\ {}_{c_0}\tilde{\partial}_{q_\phi}y}\int_{c_0}^{y}\int_{a_0}^{x}\mathfrak{f}(u,v)\ {}_{a_0}\tilde{d}_{q_\theta}u\ {}_{c_0}\tilde{d}_{q_\phi}v = \frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}}{{}_{a_0}\tilde{\partial}_{q_\theta}x\ {}_{c_0}\tilde{\partial}_{q_\phi}y}\Big[(x-a_0)(y-c_0)(1-q_\theta^2)(1-q_\phi^2)$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n+1}x+\Big(1-q_\theta^{2n+1}\Big)a_0, q_\phi^{2m+1}y+\Big(1-q_\phi^{2m+1}\Big)c_0\Big)\Big]$$
$$=\frac{1}{(q_\theta-q_\theta^{-1})(q_\phi-q_\phi^{-1})(x-a_0)(y-c_0)}\Big[q_\theta q_\phi(x-a_0)(y-c_0)(1-q_\theta^2)(1-q_\phi^2)$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n+2}x+\Big(1-q_\theta^{2n+2}\Big)a_0, q_\phi^{2m+2}y+\Big(1-q_\phi^{2m+2}\Big)c_0\Big)$$
$$-q_\theta q_\phi^{-1}(x-a_0)(y-c_0)(1-q_\theta^2)(1-q_\phi^2)$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n+2}x+\Big(1-q_\theta^{2n+2}\Big)a_0, q_\phi^{2m}y+\Big(1-q_\phi^{2m}\Big)c_0\Big)$$
$$-q_\theta^{-1}q_\phi(x-a_0)(y-c_0)(1-q_\theta^2)(1-q_\phi^2)$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n}x+\Big(1-q_\theta^{2n}\Big)a_0, q_\phi^{2m+2}y+\Big(1-q_\phi^{2m+2}\Big)c_0\Big)$$
$$+q_\theta^{-1}q_\phi^{-1}(x-a_0)(y-c_0)(1-q_\theta^2)(1-q_\phi^2)$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n}x+\Big(1-q_\theta^{2n}\Big)a_0, q_\phi^{2m}y+\Big(1-q_\phi^{2m}\Big)c_0\Big)\Big]$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n+2}q_\phi^{2m+2}\mathfrak{f}\Big(q_\theta^{2n+2}x+\Big(1-q_\theta^{2n+2}\Big)a_0, q_\phi^{2m+2}y+\Big(1-q_\phi^{2m+2}\Big)c_0\Big)$$
$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n+2}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n+2}x+\Big(1-q_\theta^{2n+2}\Big)a_0, q_\phi^{2m}y+\Big(1-q_\phi^{2m}\Big)c_0\Big)$$
$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m+2}\mathfrak{f}\Big(q_\theta^{2n}x+\Big(1-q_\theta^{2n}\Big)a_0, q_\phi^{2m+2}y+\Big(1-q_\phi^{2m+2}\Big)c_0\Big)$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\theta^{2n}q_\phi^{2m}\mathfrak{f}\Big(q_\theta^{2n}x+\Big(1-q_\theta^{2n}\Big)a_0, q_\phi^{2m}y+\Big(1-q_\phi^{2m}\Big)c_0\Big)$$
$$=\mathfrak{f}(x,y).$$

(2) Using (9), and then, (10),

$$\int_{c_0}^{y}\int_{a_0}^{x}\frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi}\mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta}x\ {}_{c_0}\tilde{\partial}_{q_\phi}y}\ {}_{a_0}\tilde{d}_{q_\theta}u\ {}_{c_0}\tilde{d}_{q_\phi}v$$
$$=\int_{c_0}^{y}\int_{a_0}^{x}\frac{1}{(q_\theta-q_\theta^{-1})(q_\phi-q_\phi^{-1})(x-a_0)(y-c_0)}\big[\mathfrak{f}(q_\theta u+(1-q_\theta)a_0, q_\phi v+(1-q_\phi)c_0)$$
$$-\mathfrak{f}\Big(q_\theta^{-1}u+\Big(1-q_\theta^{-1}\Big)a_0, q_\phi v+(1-q_\phi)c_0\Big)-\mathfrak{f}\Big(q_\theta u+(1-q_\theta)a_0, q_\phi^{-1}v+\Big(1-q_\phi^{-1}\Big)c_0\Big)$$
$$+\mathfrak{f}\Big(q_\theta^{-1}u+\Big(1-q_\theta^{-1}\Big)a_0, q_\phi^{-1}v+\Big(1-q_\phi^{-1}\Big)c_0\Big)\Big]$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big[\mathfrak{f}\Big(q_\theta^{2n+2} x + \Big(1 - q_\theta^{2n+2}\Big) a_0, q_\phi^{2m+2} y + \Big(1 - q_\phi^{2m+2}\Big) c_0 \Big) \\
&- \mathfrak{f}\Big(q_\theta^{2n} x + \Big(1 - q_\theta^{2n}\Big) a_0, q_\phi^{2m+2} y + \Big(1 - q_\phi^{2m+2}\Big) c_0 \Big) \\
&- \mathfrak{f}\Big(q_\theta^{2n+2} x + \Big(1 - q_\theta^{2n+2}\Big) a_0, q_\phi^{2m} y + \Big(1 - q_\phi^{2m}\Big) c_0 \Big) \\
&- \mathfrak{f}\Big(q_\theta^{2n} x + \Big(1 - q_\theta^{2n}\Big) a_0, q_\phi^{2m} y + \Big(1 - q_\phi^{2m}\Big) c_0 \Big) \Big] \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathfrak{f}\Big(q_\theta^{2n} x + \Big(1 - q_\theta^{2n}\Big) a_0, q_\phi^{2m} y + \Big(1 - q_\phi^{2m}\Big) c_0 \Big) \\
&- \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \mathfrak{f}\Big(q_\theta^{2n} x + \Big(1 - q_\theta^{2n}\Big) a_0, q_\phi^{2m} y + \Big(1 - q_\phi^{2m}\Big) c_0 \Big) \\
&- \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \mathfrak{f}\Big(q_\theta^{2n} x + \Big(1 - q_\theta^{2n}\Big) a_0, q_\phi^{2m} y + \Big(1 - q_\phi^{2m}\Big) c_0 \Big) \\
&+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathfrak{f}\Big(q_\theta^{2n} x + \Big(1 - q_\theta^{2n}\Big) a_0, q_\phi^{2m} y + \Big(1 - q_\phi^{2m}\Big) c_0 \Big) \\
&= \mathfrak{f}(x, y).
\end{aligned}
$$

(3) Using (11), and then, (2) results in

$$
\begin{aligned}
&\int_{y_0}^{y} \int_{x_0}^{x} \frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi} \mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta} x \; {}_{c_0}\tilde{\partial}_{q_\phi} y} \; {}_{a_0}\tilde{d}_{q_\theta} u \; {}_{c_0}\tilde{d}_{q_\phi} v \\
&= \int_{c_0}^{y} \int_{a_0}^{x} \frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi} \mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta} x \; {}_{c_0}\tilde{\partial}_{q_\phi} y} \; {}_{a_0}\tilde{d}_{q_\theta} u \; {}_{c_0}\tilde{d}_{q_\phi} v - \int_{c_0}^{y_0} \int_{a_0}^{x} \frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi} \mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta} x \; {}_{c_0}\tilde{\partial}_{q_\phi} y} \; {}_{a_0}\tilde{d}_{q_\theta} u \; {}_{c_0}\tilde{d}_{q_\phi} v \\
&- \int_{c_0}^{y} \int_{a_0}^{x_0} \frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi} \mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta} x \; {}_{c_0}\tilde{\partial}_{q_\phi} y} \; {}_{a_0}\tilde{d}_{q_\theta} u \; {}_{c_0}\tilde{d}_{q_\phi} v + \int_{c_0}^{y_0} \int_{a_0}^{x_0} \frac{{}_{a_0,c_0}\tilde{\partial}^2_{q_\theta,q_\phi} \mathfrak{f}(u,v)}{{}_{a_0}\tilde{\partial}_{q_\theta} x \; {}_{c_0}\tilde{\partial}_{q_\phi} y} \; {}_{a_0}\tilde{d}_{q_\theta} u \; {}_{c_0}\tilde{d}_{q_\phi} v \\
&= \mathfrak{f}(x, y) - \mathfrak{f}(x, y_0) - \mathfrak{f}(x_0, y) + \mathfrak{f}(x_0, y_0).
\end{aligned}
$$

□

Hölder's inequality for the double sum can be stated as

Theorem 6 ([15])**.** *For any two real or complex sequences* $(a_{st})_{s,t\in N}, (c_{st})_{s,t\in N}$ *and* $\frac{1}{r_\theta} + \frac{1}{r_\phi} = 1$, *with* $r_\theta > 1, r_\phi > 1$

$$
\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} |a_{st} c_{st}| \leqslant \left(\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} |a_{st}|^{r_\theta} \right)^{\frac{1}{r_\theta}} \left(\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} |c_{st}|^{r_\phi} \right)^{\frac{1}{r_\phi}} \tag{12}
$$

inequality holds for finite sums.

Now, with the help of Theorem 5, we construct Hölder's inequality for the function of two variables in symmetric quantum calculus.

Theorem 7. *($q_\theta q_\phi$-Symmetric Hölder's Inequality). Let* $\mathfrak{f}_1$ *and* $\mathfrak{f}_2$ *be continuous functions from* $[a_0, a_1] \times [c_0, c_1] \subseteq \Re^2$ *to* $\Re$, *then for* $0 < q_\theta < 1, 0 < q_\phi < 1$, *and* $\frac{1}{r_\theta} + \frac{1}{r_\phi}$*=1, with* $r_\theta > 1, r_\phi > 1$, *then*

$$
\begin{aligned}
&\int_{a_0}^{a_1} \int_{c_0}^{c_1} |\mathfrak{f}_1(x, y) \mathfrak{f}_2(x, y)| \; {}_{c_0}\tilde{d}_{q_\phi} y \; {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \left(\int_{a_0}^{a_1} \int_{c_0}^{c_1} |\mathfrak{f}_1(x, y)|^{r_\theta} \; {}_{c_0}\tilde{d}_{q_\phi} y \; {}_{a_0}\tilde{d}_{q_\theta} x \right)^{\frac{1}{r_\theta}} \left(\int_{a_0}^{a_1} \int_{c_0}^{c_1} |\mathfrak{f}_2(x, y)|^{r_\phi} \; {}_{c_0}\tilde{d}_{q_\phi} y \; {}_{a_0}\tilde{d}_{q_\theta} x \right)^{\frac{1}{r_\phi}}
\end{aligned} \tag{13}
$$

inequality holds.

Proof. Using the definition of a $q_\theta q_\phi$-symmetric integral and (12), we have

$$\begin{aligned}
&\int_{a_0}^{a_1}\int_{c_0}^{c_1} |f_1(x,y)f_2(x,y)|\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x \\
&= (a_1-a_0)(c_1-c_0)(1-q_\theta^2)(1-q_\phi^2) \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_\theta^{2n}q_\phi^{2m}\left|f_1\left(q_\theta^{2n+1}a_1+\left(1-q_\theta^{2n+1}\right)a_0, q_\phi^{2m+1}c_1+\left(1-q_\phi^{2m+1}\right)c_0\right)\right| \\
&\times \left|f_2\left(q_\theta^{2n+1}a_1+\left(1-q_\theta^{2n+1}\right)a_0, q_\phi^{2m+1}c_1+\left(1-q_\phi^{2m+1}\right)c_0\right)\right| \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{(a_1-a_0)(c_1-c_0)(1-q_\theta^2)(1-q_\phi^2)\right\}^{\frac{1}{r_\theta}}\left\{q_\theta^{2n}q_\phi^{2m}\right\}^{\frac{1}{r_\theta}} \\
&\times \left|f_1\left(q_\theta^{2n+1}a_1+\left(1-q_\theta^{2n+1}\right)a_0, q_\phi^{2m+1}c_1+\left(1-q_\phi^{2m+1}\right)c_0\right)\right| \\
&\times \left\{(a_1-a_0)(c_1-c_0)(1-q_\theta^2)(1-q_\phi^2)\right\}^{\frac{1}{r_\phi}}\left\{q_\theta^{2n}q_\phi^{2m}\right\}^{\frac{1}{r_\phi}} \\
&\times \left|f_2\left(q_\theta^{2n+1}a_1+\left(1-q_\theta^{2n+1}\right)a_0, q_\phi^{2m+1}c_1+\left(1-q_\phi^{2m+1}\right)c_0\right)\right| \\
&\leqslant \Bigg[(a_1-a_0)(c_1-c_0)(1-q_\theta^2)(1-q_\phi^2) \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_\theta^{2n}q_\phi^{2m}\left|f_1\left(q_\theta^{2n+1}a_1+\left(1-q_\theta^{2n+1}\right)a_0, q_\phi^{2m+1}c_1+\left(1-q_\phi^{2m+1}\right)c_0\right)\right|_\theta^{r}\Bigg]^{\frac{1}{r_\theta}} \\
&\times \Bigg[(a_1-a_0)(c_1-c_0)(1-q_\theta^2)(1-q_\phi^2) \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_\theta^{2n}q_\phi^{2m}\left|f_2\left(q_\theta^{2n+1}a_1+\left(1-q_\theta^{2n+1}\right)a_0, q_\phi^{2m+1}c_1+\left(1-q_\phi^{2m+1}\right)c_0\right)\right|_\phi^{r}\Bigg]^{\frac{1}{r_\phi}} \\
&= \left(\int_{a_0}^{a_1}\int_{c_0}^{c_1} |f_1(x,y)|^{r_\theta}\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x\right)^{\frac{1}{r_\theta}}\left(\int_{a_0}^{a_1}\int_{c_0}^{c_1} |f_2(x,y)|^{r_\phi}\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x\right)^{\frac{1}{r_\phi}}.
\end{aligned}$$

□

In our previous work [25], we derived some results in symmetric quantum calculus which are given below.

For any function $f : [a_0, a_1] \to \Re$ which is convex differentiable on (a_0, a_1) and $0 < q < 1$, then

$$f\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}\right) \leqslant \frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\ {}_{a_0}\tilde{d}_q x \leqslant \frac{(1-q+q^2)f(a_0)+qf(a_1)}{1+q^2} \quad (14)$$

$$\begin{aligned}
&f\left(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\right) - \frac{(1-q)^2(a_1-a_0)}{1+q^2}f'\left(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\right) \\
&\leqslant \frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\ {}_{a_0}\tilde{d}_q x \leqslant \frac{(1-q+q^2)f(b_0)+qf(a_1)}{1+q^2}
\end{aligned} \quad (15)$$

$$\begin{aligned}
&f\left(\frac{a_0+a_1}{2}\right) - \frac{(1-q)^2(a_1-a_0)}{2(1+q^2)}f'\left(\frac{a_0+a_1}{2}\right) \\
&\leqslant \frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\ {}_{a_0}\tilde{d}_q x \leqslant \frac{(1-q+q^2)f(a_0)+qf(a_1)}{1+q^2}.
\end{aligned} \quad (16)$$

3. Hermite–Hadamard Inequalities for Coordinated Convex Function in Symmetric Quantum Calculus

In this section, we will construct the Hermite–Hadamard inequality for the convex coordinated function in symmetric quantum calculus and its types. For this, let Ω be a class of continuous functions from a one- or two-dimensional interval to $\Re$ that satisfies some of the following properties of symmetric quantum integrals:

(i) For $\mathfrak{f}_1, \mathfrak{f}_2 \in \Omega$ with $\mathfrak{f}_1 \leqslant \mathfrak{f}_2$ and $[a_0, a_1] \subseteq \Re$, then $\int_{a_0}^{a_1} \mathfrak{f}_1(x)\ {}_{a_0}\tilde{d}_q x \leqslant \int_{a_0}^{a_1} \mathfrak{f}_2(x)\ {}_{a_0}\tilde{d}_q x$ for all $x \in [a_0, a_1]$.

(ii) For $\mathfrak{f}_1 \in \Omega$, then $\left|\int_{a_0}^{a_1} \mathfrak{f}_1(x)\ {}_{a_0}\tilde{d}_q x\right| \leqslant \int_{a_0}^{a_1} |\mathfrak{f}_1(x)|\ {}_{a_0}\tilde{d}_q x$.

It is important to mention here that throughout the rest of the paper all the functions that we will consider belong to the class Ω.

Theorem 8. *Let* $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ *be a convex coordinated function on* $[a_0, a_1] \times [c_0, c_1]$*, then for* $0 < q_\theta < 1$ *and* $0 < q_\phi < 1$*,*

$$\begin{aligned}
&\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2}, \frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right) \\
&\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1} \mathfrak{f}(x, \frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2})\ {}_{a_0}\tilde{d}_{q_\theta} x \right.\\
&\left. + \frac{1}{c_1-c_0}\int_{c_0}^{c_1} \mathfrak{f}(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2}, y)\ {}_{c_0}\tilde{d}_{q_\phi} y\right] \\
&\leqslant \frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} \mathfrak{f}(x,y)\ {}_{c_0}\tilde{d}_{q_\phi} y\ {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_0)\ {}_{a_0}\tilde{d}_{q_\theta} x + \frac{q_\phi}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1} \mathfrak{f}(x,c_1)\ {}_{a_0}\tilde{d}_{q_\theta} x \\
&+ \frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1} \mathfrak{f}(a_0,y)\ {}_{c_0}\tilde{d}_{q_\phi} y + \frac{q_\theta}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1} \mathfrak{f}(a_1,y)\ {}_{c_0}\tilde{d}_{q_\phi} y \\
&\leqslant \frac{1}{(1+q_\theta^2)(1+q_\phi^2)}\Big[(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0,c_0) + q_\theta(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1,c_0) \\
&+ q_\phi(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_1) + q_\theta q_\phi \mathfrak{f}(a_1,c_1)\Big]
\end{aligned} \tag{17}$$

inequalities hold.

Proof. Let $\mathfrak{h}_x$ be a convex function from $[c_0, c_1]$ to $\Re$ and defined as $\mathfrak{h}_x(y) = \mathfrak{f}(x,y)$. Then, using (14), we have

$$\mathfrak{h}_x\left(\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right) \leqslant \frac{1}{c_1-c_0}\int_{c_0}^{c_1} \mathfrak{h}_x(y)\ {}_{c_0}\tilde{d}_{q_\phi} y \leqslant \frac{(1-q_\phi+q_\phi^2)\mathfrak{h}_x(c_0)+q_\phi\mathfrak{h}_x(c_1)}{1+q_\phi^2},$$

$$\mathfrak{f}\left(x, \frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right) \leqslant \frac{1}{c_1-c_0}\int_{c_0}^{c_1} \mathfrak{f}(x,y)\ {}_{c_0}\tilde{d}_{q_\phi} y \leqslant \frac{(1-q_\phi+q_\phi^2)\mathfrak{f}(x,c_0)+q_\phi\mathfrak{f}(x,c_1)}{1+q_\phi^2} \tag{18}$$

for $0 < q_\phi < 1$ and for all $x \in [a_0, a_1]$. For any $0 < q_\theta < 1$, taking the symmetric q_θ-integral on $[a_0, a_1]$

$$\begin{aligned}
&\frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{(1 - q_\phi + q_\phi^2)c_0 + q_\phi c_1}{1 + q_\phi^2}\right) {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1}{(a_1 - a_0)(c_1 - c_0)} \int_{a_0}^{a_1} \int_{c_0}^{c_1} \mathfrak{f}(x, y) \; {}_{c_0}\tilde{d}_{q_\phi} y \, {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \frac{(1 - q_\phi + q_\phi^2)\mathfrak{f}(x, c_0) + q_\phi \mathfrak{f}(x, c_1)}{1 + q_\phi^2} \; {}_{a_0}\tilde{d}_{q_\theta} x \\
&= \frac{1 - q_\phi + q_\phi^2}{(a_1 - a_0)(1 + q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x, c_0) \; {}_{a_0}\tilde{d}_{q_\theta} x + \frac{q_\phi}{(a_1 - a_0)(1 + q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x, c_1) \; {}_{a_0}\tilde{d}_{q_\theta} x.
\end{aligned} \tag{19}$$

Similarly, let $\mathfrak{h}_y$ be a convex function from $[a_0, a_1]$ to $\Re$ and defined as $\mathfrak{h}_y(x) = \mathfrak{f}(x, y)$. Then, again using (14), we have

$$\mathfrak{h}_y\left(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}\right) \leqslant \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \mathfrak{h}_y(x) \; {}_{a_0}\tilde{d}_{q_\theta} x \leqslant \frac{(1 - q_\theta + q_\theta^2)\mathfrak{h}_y(a_0) + q_\theta \mathfrak{h}_y(a_1)}{1 + q_\theta^2},$$

$$\mathfrak{f}\left(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}, y\right) \leqslant \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \mathfrak{f}(x, y) \; {}_{a_0}\tilde{d}_{q_\theta} x \leqslant \frac{(1 - q_\theta + q_\theta^2)\mathfrak{f}(a_0, y) + q_\theta \mathfrak{f}(a_1, y)}{1 + q_\theta^2} \tag{20}$$

for $0 < q_\theta < 1$ and for all $y \in [c_0, c_1]$, for any $0 < q_\phi < 1$.

Taking the symmetric q_ϕ-integral on $[c_0, c_1]$,

$$\begin{aligned}
&\frac{1}{c_1 - c_0} \int_{c_0}^{c_1} \mathfrak{f}\left(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}, y\right) {}_{c_0}\tilde{d}_{q_\phi} y \\
&\leqslant \frac{1}{(a_1 - a_0)(c_1 - c_0)} \int_{c_0}^{c_1} \int_{a_0}^{a_1} \mathfrak{f}(x, y) \; {}_{a_0}\tilde{d}_{q_\theta} x \; {}_{c_0}\tilde{d}_{q_\phi} y \\
&\leqslant \frac{1}{c_1 - c_0} \int_{c_0}^{c_1} \frac{(1 - q_\theta + q_\theta^2)\mathfrak{f}(a_0, y) + q_\theta \mathfrak{f}(a_1, y)}{1 + q_\theta^2} \; {}_{c_0}\tilde{d}_{q_\phi} y \\
&= \frac{1 - q_\theta + q_\theta^2}{(c_1 - c_0)(1 + q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_0, y) \; {}_{c_0}\tilde{d}_{q_\phi} y + \frac{q_\theta}{(c_1 - c_0)(1 + q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_1, y) \; {}_{c_0}\tilde{d}_{q_\phi} y.
\end{aligned} \tag{21}$$

Adding (19) and (21):

$$\begin{aligned}
&\frac{1}{2}\Bigg[\frac{1}{a_1 - a_0} \int_{a_0}^{a_1} \mathfrak{f}(x, \frac{(1 - q_\phi + q_\phi^2)c_0 + q_\phi c_1}{1 + q_\phi^2}) \; {}_{a_0}\tilde{d}_{q_\theta} x \\
&+ \frac{1}{c_1 - c_0} \int_{c_0}^{c_1} \mathfrak{f}(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}, y) \; {}_{c_0}\tilde{d}_{q_\phi} y\Bigg] \\
&\leqslant \frac{1}{(a_1 - a_0)(c_1 - c_0)} \int_{a_0}^{a_1} \int_{c_0}^{c_1} \mathfrak{f}(x, y) \; {}_{c_0}\tilde{d}_{q_\phi} y \, {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1 - q_\phi + q_\phi^2}{2(a_1 - a_0)(1 + q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x, c_0) \; {}_{a_0}\tilde{d}_{q_\theta} x + \frac{q_\phi}{2(a_1 - a_0)(1 + q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x, c_1) \; {}_{a_0}\tilde{d}_{q_\theta} x \\
&+ \frac{1 - q_\theta + q_\theta^2}{2(c_1 - c_0)(1 + q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_0, y) \; {}_{c_0}\tilde{d}_{q_\phi} y + \frac{q_\theta}{2(c_1 - c_0)(1 + q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_1, y) \; {}_{c_0}\tilde{d}_{q_\phi} y.
\end{aligned} \tag{22}$$

Replacing x by $\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2}$ in (18) and y by $\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}$ in (20), then adding, we have

$$
\begin{aligned}
&\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)\\
&\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1}\mathfrak{f}(x,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2})\ {}_{a_0}\tilde{d}_{q_\theta}x\right.\\
&\left.+\frac{1}{c_1-c_0}\int_{c_0}^{c_1}\mathfrak{f}(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},y)\ {}_{c_0}\tilde{d}_{q_\phi}y\right]\\
&\leqslant \frac{(1-q_\phi+q_\phi^2)\mathfrak{f}(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_0)+q_\phi\mathfrak{f}(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_1)}{1+q_\phi^2}\\
&+\frac{(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2})+q_\theta\mathfrak{f}(a_1,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2})}{2(1+q_\theta^2)}.
\end{aligned}
\tag{23}
$$

Applying (14) on the right-hand side of (22):

$$
\begin{aligned}
&\frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1}\mathfrak{f}(x,c_0)\ {}_{a_0}\tilde{d}_{q_\theta}x+\frac{q_\phi}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1}\mathfrak{f}(x,c_1)\ {}_{a_0}\tilde{d}_{q_\theta}x\\
&+\frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1}\mathfrak{f}(a_0,y)\ {}_{c_0}\tilde{d}_{q_\phi}y+\frac{q_\theta}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1}\mathfrak{f}(a_1,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\\
&\leqslant \frac{1-q_\phi+q_\phi^2}{2(1+q_\phi^2)}\left[\frac{(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_0)+q_\theta\mathfrak{f}(a_1,c_0)}{1+q_\theta^2}\right]\\
&+\frac{q_\phi}{2(1+q_\phi^2)}\left[\frac{(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_1)+q_\theta\mathfrak{f}(a_1,c_1)}{1+q_\theta^2}\right]\\
&+\frac{1-q_\theta+q_\theta^2}{2(1+q_\theta^2)}\left[\frac{(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0,c_0)+q_\phi\mathfrak{f}(a_0,c_1)}{1+q_\phi^2}\right]\\
&+\frac{q_\theta}{2(1+q_\theta^2)}\left[\frac{(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1,c_0)+q_\phi\mathfrak{f}(a_1,c_1)}{1+q_\phi^2}\right]\\
&=\frac{1}{(1+q_\theta^2)(1+q_\phi^2)}\Big[(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0,c_0)+q_\theta(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1,c_0)\\
&+q_\phi(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_1)+q_\theta q_\phi\mathfrak{f}(a_1,c_1)\Big].
\end{aligned}
\tag{24}
$$

Joining (22)–(24), we obtain the desired result. □

Remark 2. *If q_θ, q_ϕ approach 1 in Theorem 8, then it is reconstructed into the classical one* (2).

Now, we derive another new result of symmetric quantum calculus which will assist in proving some related results of Hermite–Hadamard inequalities.

Theorem 9. *Let* $\mathfrak{f} : [a_0, a_1] \longrightarrow \Re$ *be a convex function on* (a_0, a_1) *and differentiable as well, then for* $0 < q < 1$

$$\begin{aligned}
\mathfrak{f}\left(\frac{(1-q+q^2)a_0 + qa_1}{1+q^2}\right) &\leqslant \frac{1}{a_1 - a_0}\int_{a_0}^{a_1} \mathfrak{f}(x)\ {}_{a_0}\tilde{d}_q x \\
&\leqslant \frac{q(1-q+q^2)\mathfrak{f}(a_0) + q^2\mathfrak{f}(a_1)}{(1+q^2)^2} + \left(\frac{1-q+q^2}{1+q^2}\right)\mathfrak{f}\left(\frac{(1-q+q^2)a_0 + qa_1}{1+q^2}\right) \\
&\leqslant \frac{(1-q+q^2)\mathfrak{f}(a_0) + q\mathfrak{f}(a_1)}{1+q^2}
\end{aligned} \tag{25}$$

inequalities hold.

Proof. Let $\mathfrak{g}_1(x)$ and $\mathfrak{g}_2(x)$ be the line segments joining the points $(a_0, \mathfrak{f}(a_0)), (\frac{(1-q+q^2)a_0+qa_1}{1+q^2}, \mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}))$ and $(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}, \mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2})), (a_1, \mathfrak{f}(a_1))$, respectively. Owing to the convexity of $\mathfrak{f}$ on $[a_0, a_1]$, $\mathfrak{f}$ is always below the line segments $\mathfrak{g}_1(x)$ and $\mathfrak{g}_2(x)$. The equations of these line segments are $\mathfrak{g}_1(x) = \frac{\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)}{\frac{(1-q+q^2)a_0+qa_1}{1+q^2} - a_0}(x - a_0) + \mathfrak{f}(a_0)$ and $\mathfrak{g}_2(x) = \frac{\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_1)}{\frac{(1-q+q^2)a_0+qa_1}{1+q^2} - a_1}(x - a_1) + \mathfrak{f}(a_1)$.

From Figure 1, we can write the equations of $\mathfrak{g}_1(x), \mathfrak{g}_2(x)$, and $\mathfrak{g}_3(x)$ using the two-point form formula of the equation of a line as

$$\begin{aligned}
\mathfrak{g}_1(x) &= (1+q^2)\frac{\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)}{q(a_1 - a_0)}(x - a_0) + \mathfrak{f}(a_0), \\
\mathfrak{g}_2(x) &= (1+q^2)\frac{\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_1)}{(1-q+q^2)(a_0 - a_1)}(x - a_1) + \mathfrak{f}(a_1), \\
\mathfrak{g}_3(x) &= \frac{\mathfrak{f}(a_1) - \mathfrak{f}(a_0)}{a_1 - a_0}(x - a_0) + \mathfrak{f}(a_0).
\end{aligned}$$

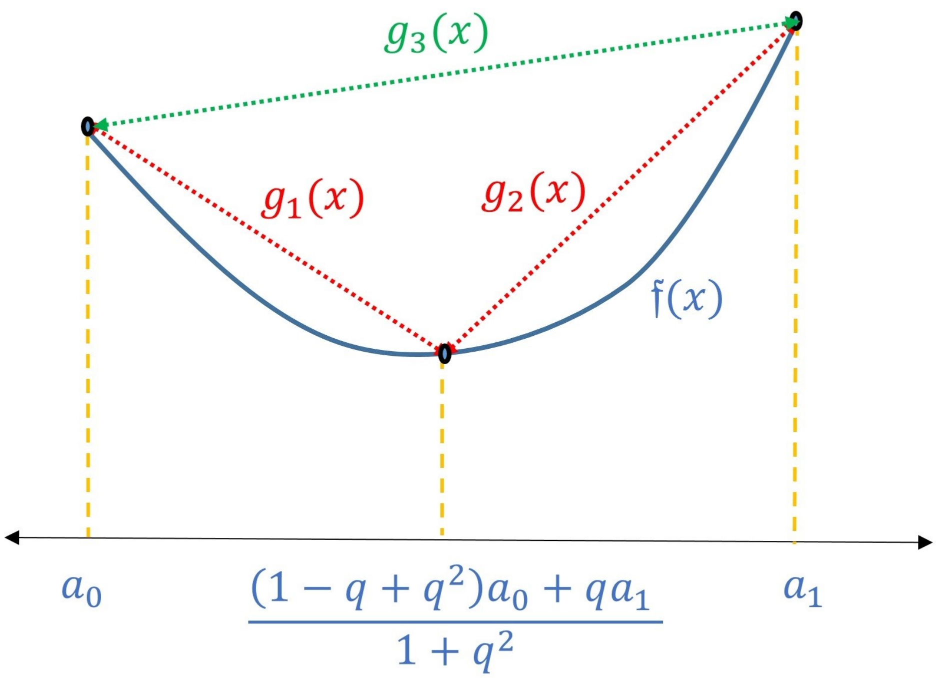

Figure 1. Continuous convex function defined on $[a_0, a_1]$.

Also, from Figure 1, from a_0 to $\frac{(1-q+q^2)a_0+qa_1}{1+q^2}$,

$$\mathfrak{f}(x) \leqslant \mathfrak{g}_1(x) = (1+q^2)\frac{\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)}{q(a_1-a_0)}(x-a_0) + \mathfrak{f}(a_0). \tag{26}$$

Similarly, from $\frac{(1-q+q^2)a_0+qa_1}{1+q^2}$ to a_1,

$$\mathfrak{f}(x) \leqslant \mathfrak{g}_2(x) = (1+q^2)\frac{\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_1)}{(1-q+q^2)(a_0-a_1)}(x-a_1) + \mathfrak{f}(a_1). \tag{27}$$

Taking the symmetric q-integral, the inequalities (26) and (27) from a_0 to $\frac{(1-q+q^2)a_0+qa_1}{1+q^2}$ and $\frac{(1-q+q^2)a_0+qa_1}{1+q^2}$ to a_1, respectively, then from Figure 1,

$$\int_{a_0}^{a_1} \mathfrak{f}(x)\ {}_{a_0}\tilde{d}_q x \leqslant \int_{a_0}^{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}} \mathfrak{g}_1(x)\ {}_{a_0}\tilde{d}_q x + \int_{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}}^{a_1} \mathfrak{g}_2(x)\ {}_{a_0}\tilde{d}_q x, \tag{28}$$

where $x \in [a_0, a_1]$.

$$\begin{aligned}
&\int_{a_0}^{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}} \mathfrak{g}_1(x)\ {}_{a_0}\tilde{d}_q x = \frac{1+q^2}{q(a_1-a_0)}\left[\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)\right] \\
&\times \left[\int_{a_0}^{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}} x\ {}_{a_0}\tilde{d}_q x - a_0\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2} - a_0\right)\right] \\
&+\mathfrak{f}(a_0)\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2} - a_0\right) \\
&= \frac{1+q^2}{q(a_1-a_0)}\left[\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)\right]\left[(1-q^2)\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2} - a_0\right)\sum_{n=0}^{\infty} q^{2n}\right. \\
&\times \left.\left\{(1-q^{2n+1})a_0 + q^{2n+1}\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}\right)\right\} - a_0 q\left(\frac{a_1-a_0}{1+q^2}\right)\right] + q\mathfrak{f}(a_0)\left(\frac{a_1-a_0}{1+q^2}\right) \\
&= \frac{1+q^2}{q(a_1-a_0)}\left[\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)\right]\left[\frac{(1-q^2)q(a_1-a_0)}{1+q^2}\left\{\left(\frac{1}{1-q^2} - \frac{q}{1-q^4}\right)a_0\right.\right. \\
&+ \left.\left.\frac{q}{1-q^4}\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}\right)\right\} - a_0 q\left(\frac{a_1-a_0}{1+q^2}\right)\right] + q\mathfrak{f}(a_0)\left(\frac{a_1-a_0}{1+q^2}\right) \\
&= \left[\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)\right]\left[\frac{(1-q+q^2)a_0}{1+q^2} + \frac{q}{1+q^2}\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}\right) - a_0\right] \\
&+ q\mathfrak{f}(a_0)\left(\frac{a_1-a_0}{1+q^2}\right) \\
&= \left[\mathfrak{f}(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}) - \mathfrak{f}(a_0)\right]\left[\frac{q^2(a_1-a_0)}{(1+q^2)^2}\right] + q\mathfrak{f}(a_0)\left(\frac{a_1-a_0}{1+q^2}\right) \\
&= \frac{q(a_1-a_0)}{(1+q^2)^2}\left[q\mathfrak{f}\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}\right) - q\mathfrak{f}(a_0) + \mathfrak{f}(a_0)(1+q^2)\right] \\
&= \frac{q(a_1-a_0)}{(1+q^2)^2}\left[q\mathfrak{f}\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}\right) + (1-q+q^2)\mathfrak{f}(a_0)\right].
\end{aligned} \tag{29}$$

Similarly,

$$\int_{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}}^{a_1} \mathfrak{g}_2(x)\ {}_{a_0}\tilde{d}_q x = \frac{a_1-a_0}{(1+q^2)^2}\left[q^2\mathfrak{f}(a_1) + (1-q+q^2-q^3+q^4)\mathfrak{f}\left(\frac{(1-q+q^2)a_0+qa_1}{1+q^2}\right)\right]. \tag{30}$$

Adding (29) and (30):

$$\int_{a_0}^{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}} \mathfrak{g}_1(x)\ {}_{a_0}\tilde{d}_q x + \int_{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}}^{a_1} \mathfrak{g}_2(x)\ {}_{a_0}\tilde{d}_q x$$
$$= (a_1 - a_0)\left[\frac{q(1-q+q^2)\mathfrak{f}(a_0) + q^2\mathfrak{f}(a_1)}{(1+q^2)^2} + \left(\frac{1-q+q^2}{1+q^2}\right)\mathfrak{f}\left(\frac{(1-q+q^2)a_0 + qa_1}{1+q^2}\right)\right]. \tag{31}$$

So, inequality (28) becomes

$$\int_{a_0}^{a_1} \mathfrak{f}(x)\ {}_{a_0}\tilde{d}_q x \leqslant (a_1 - a_0)\left[\frac{q(1-q+q^2)\mathfrak{f}(a_0) + q^2\mathfrak{f}(a_1)}{(1+q^2)^2} + \left(\frac{1-q+q^2}{1+q^2}\right)\mathfrak{f}\left(\frac{(1-q+q^2)a_0 + qa_1}{1+q^2}\right)\right]. \tag{32}$$

From Figure 1, we can also conclude that

$$\int_{a_0}^{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}} \mathfrak{g}_1(x)\ {}_{a_0}\tilde{d}_q x + \int_{\frac{(1-q+q^2)a_0+qa_1}{1+q^2}}^{a_1} \mathfrak{g}_2(x)\ {}_{a_0}\tilde{d}_q x \leqslant \int_{a_0}^{a_1} \mathfrak{g}_3(x)\ {}_{a_0}\tilde{d}_q x$$
$$= (a_1 - a_0)\frac{(1-q+q^2)\mathfrak{f}(a_0) + q\mathfrak{f}(a_1)}{1+q^2}. \tag{33}$$

Finally, if we combine the left-hand side of (14) with (32) and (33), we obtain the desired result. □

Remark 3. *If* q *approaches 1 in Theorem 9, then it is reconstructed into corollary 2.1 from* [26].

Theorem 10. *Let* $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ *be a convex coordinated function on* $[a_0, a_1] \times [c_0, c_1]$*, then for* $0 < q_\theta < 1$ *and* $0 < q_\phi < 1$*,*

$$\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0 + q_\theta a_1}{1+q_\theta^2}, \frac{(1-q_\phi+q_\phi^2)c_0 + q_\phi c_1}{1+q_\phi^2}\right)$$
$$\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{(1-q_\phi+q_\phi^2)c_0 + q_\phi c_1}{1+q_\phi^2}\right)\ {}_{a_0}\tilde{d}_{q_\theta} x\right.$$
$$\left. + \frac{1}{c_1-c_0}\int_{c_0}^{c_1} \mathfrak{f}(\frac{(1-q_\theta+q_\theta^2)a_0 + q_\theta a_1}{1+q_\theta^2}, y)\ {}_{c_0}\tilde{d}_{q_\phi} y\right]$$
$$\leqslant \frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} \mathfrak{f}(x,y)\ {}_{c_0}\tilde{d}_{q_\phi} y\ {}_{a_0}\tilde{d}_{q_\theta} x \leqslant \frac{q_\phi(1-q_\phi+q_\phi^2)}{2(a_1-a_0)(1+q_\phi^2)^2}\int_{a_0}^{a_1} \mathfrak{f}(x, c_0)\ {}_{a_0}\tilde{d}_{q_\theta} x$$
$$+ \frac{q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)^2}\int_{a_0}^{a_1} \mathfrak{f}(x, c_1)\ {}_{a_0}\tilde{d}_{q_\theta} x$$
$$+ \frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{(1-q_\phi+q_\phi^2)c_0 + q_\phi c_1}{1+q_\phi^2}\right)\ {}_{a_0}\tilde{d}_{q_\theta} x$$
$$+ \frac{q_\theta(1-q_\theta+q_\theta^2)}{2(c_1-c_0)(1+q_\theta^2)^2}\int_{c_0}^{c_1} \mathfrak{f}(a_0, y)\ {}_{c_0}\tilde{d}_{q_\phi} y + \frac{q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)^2}\int_{c_0}^{c_1} \mathfrak{f}(a_1, y)\ {}_{c_0}\tilde{d}_{q_\phi} y$$
$$+ \frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1} \mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0 + q_\theta a_1}{1+q_\theta^2}, y\right)\ {}_{c_0}\tilde{d}_{q_\phi} y$$

$$\leqslant \frac{1}{(1+q_\theta^2)^2(1+q_\phi^2)^2}\Big[q_\theta(1-q_\theta+q_\theta^2)q_\phi(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0,c_0)+q_\theta(1-q_\theta+q_\theta^2)q_\phi^2\mathfrak{f}(a_0,c_1)$$
$$+q_\theta^2 q_\phi(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1,c_0)+q_\theta^2q_\phi^2\mathfrak{f}(a_1,c_1)\Big]$$
$$+\frac{(1-q_\theta+q_\theta^2)q_\phi(1-q_\phi+q_\phi^2)}{(1+q_\theta^2)(1+q_\phi^2)^2}\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_0\right)$$
$$+\frac{(1-q_\theta+q_\theta^2)q_\phi^2}{(1+q_\theta^2)(1+q_\phi^2)^2}\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_1\right)$$
$$+\frac{q_\theta(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)}{(1+q_\theta^2)^2(1+q_\phi^2)}\mathfrak{f}\left(a_0,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)$$
$$+\frac{q_\theta^2(1-q_\phi+q_\phi^2)}{(1+q_\theta^2)^2(1+q_\phi^2)}\mathfrak{f}\left(a_1,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)$$

$$+\frac{(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)}{(1+q_\theta^2)(1+q_\phi^2)}\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right) \tag{34}$$

inequalities hold.

Proof. Let $\mathfrak{h}_x$ be a convex function from $[c_0,c_1]$ to $\Re$ and defined as $\mathfrak{h}_x(y)=\mathfrak{f}(x,y)$. Then, using (32), we have

$$\frac{1}{c_1-c_0}\int_{c_0}^{c_1}\mathfrak{h}_x(y)\ {}_{c_0}\tilde{d}_{q_\phi}y\leqslant\frac{q_\phi(1-q_\phi+q_\phi^2)\mathfrak{h}_x(c_0)+q_\phi^2\mathfrak{h}_x(c_1)}{(1+q_\phi^2)^2}$$
$$+\left(\frac{1-q_\phi+q_\phi^2}{1+q_\phi^2}\right)\mathfrak{h}_x\left(\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)$$
$$\frac{1}{c_1-c_0}\int_{c_0}^{c_1}\mathfrak{f}(x,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\leqslant\frac{q_\phi(1-q_\phi+q_\phi^2)\mathfrak{f}(x,c_0)+q_\phi^2\mathfrak{f}(x,c_1)}{(1+q_\phi^2)^2}$$
$$+\left(\frac{1-q_\phi+q_\phi^2}{1+q_\phi^2}\right)\mathfrak{f}\left(x,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)$$

holds for all $x\in[a_0,a_1]$ and $0<q_\phi,q_\theta<1$. Taking the symmetric q_θ-integral from a_0 to a_1:

$$\frac{1}{c_1-c_0}\int_{a_0}^{a_1}\int_{c_0}^{c_1}\mathfrak{f}(x,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x$$
$$\leqslant\frac{q_\phi(1-q_\phi+q_\phi^2)}{(1+q_\phi^2)^2}\int_{a_0}^{a_1}\mathfrak{f}(x,c_0)\ {}_{a_0}\tilde{d}_{q_\theta}x+\frac{q_\phi^2}{(1+q_\phi^2)^2}\int_{a_0}^{a_1}\mathfrak{f}(x,c_1)\ {}_{a_0}\tilde{d}_{q_\theta}x$$
$$+\left(\frac{1-q_\phi+q_\phi^2}{1+q_\phi^2}\right)\int_{a_0}^{a_1}\mathfrak{f}\left(x,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x. \tag{35}$$

Similarly, let $\mathfrak{h}_y$ be a convex function from $[a_0, a_1]$ to $\Re$ and defined as $\mathfrak{h}_y(x) = \mathfrak{f}(x, y)$. Then, again using (32), we have

$$\frac{1}{a_1 - a_0}\int_{a_0}^{a_1} \mathfrak{h}_y(x)\ {}_{a_0}\tilde{d}_{q_\theta}x \leqslant \frac{q_\theta(1 - q_\theta + q_\theta^2)\mathfrak{h}_y(a_0) + q_\theta^2\mathfrak{h}_y(a_1)}{(1 + q_\theta^2)^2}$$
$$+ \left(\frac{1 - q_\theta + q_\theta^2}{1 + q_\theta^2}\right)\mathfrak{h}_y\left(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}\right)$$
$$\frac{1}{a_1 - a_0}\int_{a_0}^{a_1} \mathfrak{f}(x, y)\ {}_{a_0}\tilde{d}_{q_\theta}x \leqslant \frac{q_\theta(1 - q_\theta + q_\theta^2)\mathfrak{f}(a_0, y) + q_\theta^2\mathfrak{f}(a_1, y)}{(1 + q_\theta^2)^2}$$
$$+ \left(\frac{1 - q_\theta + q_\theta^2}{1 + q_\theta^2}\right)\mathfrak{f}\left(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}, y\right)$$

holds for all $y \in [c_0, c_1]$ and $0 < q_\phi, q_\theta < 1$. Now, taking the symmetric q_ϕ-integral from c_0 to c_1.

$$\frac{1}{a_1 - a_0}\int_{c_0}^{c_1}\int_{a_0}^{a_1} \mathfrak{f}(x, y)\ {}_{a_0}\tilde{d}_{q_\theta}x\ {}_{c_0}\tilde{d}_{q_\phi}y$$
$$\leqslant \frac{q_\theta(1 - q_\theta + q_\theta^2)}{(1 + q_\theta^2)^2}\int_{c_0}^{c_1} \mathfrak{f}(a_0, y)\ {}_{c_0}\tilde{d}_{q_\phi}y + \frac{q_\theta^2}{(1 + q_\theta^2)^2}\int_{c_0}^{c_1} \mathfrak{f}(a_1, y)\ {}_{c_0}\tilde{d}_{q_\phi}y$$
$$+ \left(\frac{1 - q_\theta + q_\theta^2}{1 + q_\theta^2}\right)\int_{c_0}^{c_1} \mathfrak{f}\left(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}, y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y. \tag{36}$$

Now, $\frac{1}{a_1 - a_0}$ (35) + $\frac{1}{c_1 - c_0}$(36), we have

$$\frac{1}{(a_1 - a_0)(c_1 - c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} \mathfrak{f}(x, y)\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x \leqslant \frac{q_\phi(1 - q_\phi + q_\phi^2)}{2(a_1 - a_0)(1 + q_\phi^2)^2}\int_{a_0}^{a_1} \mathfrak{f}(x, c_0)\ {}_{a_0}\tilde{d}_{q_\theta}x$$
$$+ \frac{q_\phi^2}{2(a_1 - a_0)(1 + q_\phi^2)^2}\int_{a_0}^{a_1} \mathfrak{f}(x, c_1)\ {}_{a_0}\tilde{d}_{q_\theta}x$$
$$+ \frac{1 - q_\phi + q_\phi^2}{2(a_1 - a_0)(1 + q_\phi^2)}$$
$$\times \int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{(1 - q_\phi + q_\phi^2)c_0 + q_\phi c_1}{1 + q_\phi^2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x$$
$$+ \frac{q_\theta(1 - q_\theta + q_\theta^2)}{2(c_1 - c_0)(1 + q_\theta^2)^2}\int_{c_0}^{c_1} \mathfrak{f}(a_0, y)\ {}_{c_0}\tilde{d}_{q_\phi}y$$
$$+ \frac{q_\theta^2}{2(c_1 - c_0)(1 + q_\theta^2)^2}\int_{c_0}^{c_1} \mathfrak{f}(a_1, y)\ {}_{c_0}\tilde{d}_{q_\phi}y$$
$$+ \frac{1 - q_\theta + q_\theta^2}{2(c_1 - c_0)(1 + q_\theta^2)}$$
$$\times \int_{c_0}^{c_1} \mathfrak{f}\left(\frac{(1 - q_\theta + q_\theta^2)a_0 + q_\theta a_1}{1 + q_\theta^2}, y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y. \tag{37}$$

Now, applying (32) on the right-hand side of (37), we have

$$\frac{q_\phi(1-q_\phi+q_\phi^2)}{2(a_1-a_0)(1+q_\phi^2)^2}\int_{a_0}^{a_1}\mathfrak{f}(x,c_0)\,{}_{a_0}\tilde{d}_{q_\theta}x \leqslant \frac{q_\phi(1-q_\phi+q_\phi^2)}{2(1+q_\phi^2)^2}\left[\frac{q_\theta(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_0)+q_\theta^2\mathfrak{f}(a_1,c_0)}{(1+q_\theta^2)^2}\right.$$
$$\left.+\left(\frac{1-q_\theta+q_\theta^2}{1+q_\theta^2}\right)\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_0\right)\right], \quad (38)$$

$$\frac{q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)^2}\int_{a_0}^{a_1}\mathfrak{f}(x,c_1)\,{}_{a_0}\tilde{d}_{q_\theta}x \leqslant \frac{q_\phi^2}{2(1+q_\phi^2)^2}\left[\frac{q_\theta(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_1)+q_\theta^2\mathfrak{f}(a_1,c_1)}{(1+q_\theta^2)^2}\right.$$
$$\left.+\left(\frac{1-q_\theta+q_\theta^2}{1+q_\theta^2}\right)\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_1\right)\right], \quad (39)$$

$$\frac{q_\theta(1-q_\theta+q_\theta^2)}{2(c_1-c_0)(1+q_\theta^2)^2}\int_{c_0}^{c_1}\mathfrak{f}(a_0,y)\,{}_{c_0}\tilde{d}_{q_\phi}y \leqslant \frac{q_\theta(1-q_\theta+q_\theta^2)}{2(1+q_\theta^2)^2}\left[\frac{q_\phi(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0,c_0)+q_\phi^2\mathfrak{f}(a_0,c_1)}{(1+q_\phi^2)^2}\right.$$
$$\left.+\left(\frac{1-q_\phi+q_\phi^2}{1+q_\phi^2}\right)\mathfrak{f}\left(a_0,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)\right], \quad (40)$$

$$\frac{q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)^2}\int_{c_0}^{c_1}\mathfrak{f}(a_1,y)\,{}_{c_0}\tilde{d}_{q_\phi}y \leqslant \frac{q_\theta^2}{2(1+q_\theta^2)^2}\left[\frac{q_\phi(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1,c_0)+q_\phi^2\mathfrak{f}(a_1,c_1)}{(1+q_\phi^2)^2}\right.$$
$$\left.+\left(\frac{1-q_\phi+q_\phi^2}{1+q_\phi^2}\right)\mathfrak{f}\left(a_1,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)\right], \quad (41)$$

$$\frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1}\mathfrak{f}\left(x,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)\,{}_{a_0}\tilde{d}_{q_\theta}x$$
$$\leqslant \frac{1-q_\phi+q_\phi^2}{2(1+q_\phi^2)}\left[\frac{q_\theta(1-q_\theta+q_\theta^2)\mathfrak{f}\left(a_0,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)+q_\theta^2\mathfrak{f}\left(a_1,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)}{(1+q_\theta^2)^2}\right.$$
$$\left.+\left(\frac{1-q_\theta+q_\theta^2}{1+q_\theta^2}\right)\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)\right], \quad (42)$$

$$\frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1}\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},y\right)\,{}_{c_0}\tilde{d}_{q_\phi}y$$
$$\leqslant \frac{1-q_\theta+q_\theta^2}{2(1+q_\theta^2)}\left[\frac{q_\phi(1-q_\phi+q_\phi^2)\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_0\right)+q_\phi^2\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},c_1\right)}{(1+q_\phi^2)^2}\right.$$
$$\left.+\left(\frac{1-q_\phi+q_\phi^2}{1+q_\phi^2}\right)\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)\right]. \quad (43)$$

Adding (38)–(43), and then, combining the obtained result with (37) and the left-hand sides of (22) and (23), we obtain the desired result. □

Remark 4. *If q_θ, q_ϕ approach 1 in Theorem 10, then it is reconstructed into the classical one of theorem 2.2 from* [26].

Theorem 11. *Let $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ be a convex coordinated function on $[a_0, a_1] \times [c_0, c_1]$, then for $0 < q_\theta < 1$ and $0 < q_\phi < 1$,*

$$\begin{aligned}
&\mathfrak{f}\left(\frac{a_0+a_1}{2}, \frac{c_0+c_1}{2}\right) - \frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)} \frac{\partial \mathfrak{f}\left(\frac{a_0+a_1}{2}, \frac{c_0+c_1}{2}\right)}{\partial x} \\
&- \frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)} \frac{\partial \mathfrak{f}\left(\frac{a_0+a_1}{2}, \frac{c_0+c_1}{2}\right)}{\partial y} - \frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)(c_1-c_0)} \int_{c_0}^{c_1} \frac{\partial}{\partial x} \mathfrak{f}\left(\frac{a_0+a_1}{2}, y\right) {}_{c_0}\tilde{d}_{q_\phi} y \\
&- \frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)(a_1-a_0)} \int_{a_0}^{a_1} \frac{\partial}{\partial y} \mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right) {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0} \int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right) {}_{a_0}\tilde{d}_{q_\theta} x + \frac{1}{c_1-c_0} \int_{c_0}^{c_1} \mathfrak{f}\left(\frac{a_0+a_1}{2}, y\right) {}_{c_0}\tilde{d}_{q_\phi} y\right] \\
&- \frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)(c_1-c_0)} \int_{c_0}^{c_1} \frac{\partial}{\partial x} \mathfrak{f}\left(\frac{a_0+a_1}{2}, y\right) {}_{c_0}\tilde{d}_{q_\phi} y \\
&- \frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)(a_1-a_0)} \int_{a_0}^{a_1} \frac{\partial}{\partial y} \mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right) {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1}{(a_1-a_0)(c_1-c_0)} \int_{a_0}^{a_1} \int_{c_0}^{c_1} \mathfrak{f}(x, y) {}_{c_0}\tilde{d}_{q_\phi} y \, {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x, c_0) {}_{a_0}\tilde{d}_{q_\theta} x + \frac{q_\phi}{2(a_1-a_0)(1+q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x, c_1) {}_{a_0}\tilde{d}_{q_\theta} x \\
&+ \frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_0, y) {}_{c_0}\tilde{d}_{q_\phi} y + \frac{q_\theta}{2(c_1-c_0)(1+q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_1, y) {}_{c_0}\tilde{d}_{q_\phi} y \\
&\leqslant \frac{1}{(1+q_\theta^2)(1+q_\phi^2)}\Big[(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0, c_0) + q_\theta(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1, c_0) \\
&+ q_\phi(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0, c_1) + q_\theta q_\phi \mathfrak{f}(a_1, c_1)\Big] \qquad (44)
\end{aligned}$$

inequalities hold.

Proof. Let $\mathfrak{h}_x$ be a convex function from $[c_0, c_1]$ to $\Re$ and defined as $\mathfrak{h}_x(y) = \mathfrak{f}(x, y)$. Then, using (16), we have

$$\mathfrak{h}_x\left(\frac{c_0+c_1}{2}\right) - \frac{(1-q_\phi)^2(c_1-c_0)}{2(1+q_\phi^2)} \frac{\partial \mathfrak{h}_x\left(\frac{c_0+c_1}{2}\right)}{\partial y} \leqslant \frac{1}{c_1-c_0} \int_{c_0}^{c_1} \mathfrak{h}_x(y) {}_{c_0}\tilde{d}_{q_\phi} y,$$

$$\mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right) - \frac{(1-q_\phi)^2(c_1-c_0)}{2(1+q_\phi^2)} \frac{\partial \mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right)}{\partial y} \leqslant \frac{1}{c_1-c_0} \int_{c_0}^{c_1} \mathfrak{f}(x, y) {}_{c_0}\tilde{d}_{q_\phi} y, \qquad (45)$$

for $0 < q_\phi < 1$ and for all $x \in [a_0, a_1]$. For any $0 < q_\theta < 1$.
Taking the symmetric q_θ-integral on $[a_0, a_1]$ and dividing by $(a_1 - a_0)$, we have

$$\begin{aligned}
&\frac{1}{a_1-a_0} \int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right) {}_{a_0}\tilde{d}_{q_\theta} x - \frac{(1-q_\phi)^2(c_1-c_0)}{2(1+q_\phi^2)(a_1-a_0)} \int_{a_0}^{a_1} \frac{\partial}{\partial y} \mathfrak{f}\left(x, \frac{c_0+c_1}{2}\right) {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1}{(c_1-c_0)(a_1-a_0)} \int_{a_0}^{a_1} \int_{c_0}^{c_1} \mathfrak{f}(x, y) {}_{c_0}\tilde{d}_{q_\phi} y \, {}_{a_0}\tilde{d}_{q_\theta} x. \qquad (46)
\end{aligned}$$

Similarly, let $\mathfrak{h}_y$ be a convex function from $[a_0, a_1]$ to $\Re$ and defined as $\mathfrak{h}_y(x) = \mathfrak{f}(x, y)$. Then, again using (16), we have

$$\mathfrak{h}_y\left(\frac{a_0+a_1}{2}\right) - \frac{(1-q_\theta)^2(a_1-a_0)}{2(1+q_\theta^2)} \frac{\partial \mathfrak{h}_y\left(\frac{a_0+a_1}{2}\right)}{\partial x} \leqslant \frac{1}{a_1-a_0} \int_{a_0}^{a_1} \mathfrak{h}_y(x) {}_{a_0}\tilde{d}_{q_\theta} x,$$

$$f\left(\frac{a_0+a_1}{2},y\right)-\frac{(1-q_\theta)^2(a_1-a_0)}{2(1+q_\theta^2)}\frac{\partial f\left(\frac{a_0+a_1}{2},y\right)}{\partial x}\leqslant\frac{1}{a_1-a_0}\int_{a_0}^{a_1}f(x,y)\ {}_{a_0}\tilde{d}_{q_\theta}x, \tag{47}$$

for $0<q_\theta<1$ and for all $y\in[c_0,c_1]$, for any $0<q_\phi<1$.

Taking the symmetric q_ϕ-integral on $[c_0,c_1]$ and dividing by (c_1-c_0), we have

$$\begin{aligned}&\frac{1}{c_1-c_0}\int_{c_0}^{c_1}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y-\frac{(1-q_\theta)^2(a_1-a_0)}{2(1+q_\theta^2)(c_1-c_0)}\int_{c_0}^{c_1}\frac{\partial}{\partial x}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y\\&\leqslant\frac{1}{(c_1-c_0)(a_1-a_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1}f(x,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x.\end{aligned} \tag{48}$$

Adding (46) and (48), we have

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1}f\left(x,\frac{c_0+c_1}{2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x+\frac{1}{c_1-c_0}\int_{c_0}^{c_1}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y\right]\\&-\frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)(c_1-c_0)}\int_{c_0}^{c_1}\frac{\partial}{\partial x}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y\\&-\frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)(a_1-a_0)}\int_{a_0}^{a_1}\frac{\partial}{\partial y}f\left(x,\frac{c_0+c_1}{2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x\\&\leqslant\frac{1}{(c_1-c_0)(a_1-a_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1}f(x,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x.\end{aligned} \tag{49}$$

Replacing x by $\frac{a_0+a_1}{2}$ and y by $\frac{c_0+c_1}{2}$ in (45) and (47), respectively, then adding, we have

$$\begin{aligned}&f\left(\frac{a_0+a_1}{2},\frac{c_0+c_1}{2}\right)-\frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)}\frac{\partial f\left(\frac{a_0+a_1}{2},\frac{c_0+c_1}{2}\right)}{\partial x}-\frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)}\frac{\partial f\left(\frac{a_0+a_1}{2},\frac{c_0+c_1}{2}\right)}{\partial y}\\&\leqslant\frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1}f\left(x,\frac{c_0+c_1}{2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x+\frac{1}{c_1-c_0}\int_{c_0}^{c_1}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y\right]\\&\Rightarrow f\left(\frac{a_0+a_1}{2},\frac{c_0+c_1}{2}\right)-\frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)}\frac{\partial f\left(\frac{a_0+a_1}{2},\frac{c_0+c_1}{2}\right)}{\partial x}\\&-\frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)}\frac{\partial f\left(\frac{a_0+a_1}{2},\frac{c_0+c_1}{2}\right)}{\partial y}-\frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)(c_1-c_0)}\int_{c_0}^{c_1}\frac{\partial}{\partial x}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y\\&-\frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)(a_1-a_0)}\int_{a_0}^{a_1}\frac{\partial}{\partial y}f\left(x,\frac{c_0+c_1}{2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x\\&\leqslant\frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1}f\left(x,\frac{c_0+c_1}{2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x+\frac{1}{c_1-c_0}\int_{c_0}^{c_1}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y\right]\\&-\frac{(1-q_\theta)^2(a_1-a_0)}{4(1+q_\theta^2)(c_1-c_0)}\int_{c_0}^{c_1}\frac{\partial}{\partial x}f\left(\frac{a_0+a_1}{2},y\right)\ {}_{c_0}\tilde{d}_{q_\phi}y\\&-\frac{(1-q_\phi)^2(c_1-c_0)}{4(1+q_\phi^2)(a_1-a_0)}\int_{a_0}^{a_1}\frac{\partial}{\partial y}f\left(x,\frac{c_0+c_1}{2}\right)\ {}_{a_0}\tilde{d}_{q_\theta}x.\end{aligned} \tag{50}$$

Finally, by (22), (24), (49), and (50) we obtain the desired result. □

Theorem 12. *Let* $\mathfrak{f} : [a_0, a_1] \times [c_0, c_1] \subseteq \Re^2 \longrightarrow \Re$ *be a convex coordinated function on* $[a_0, a_1] \times [c_0, c_1]$*, then for* $0 < q_\theta < 1$ *and* $0 < q_\phi < 1$*,*

$$
\begin{aligned}
&\mathfrak{f}\left(\frac{q_\theta a_0 + (1-q_\theta+q_\theta^2)a_1}{1+q_\theta^2}, \frac{q_\phi c_0 + (1-q_\phi+q_\phi^2)c_1}{1+q_\phi^2}\right) \\
&- \frac{(1-q_\theta)^2(a_1-a_0)}{2(1+q_\theta^2)} \frac{\partial \mathfrak{f}\left(\frac{q_\theta a_0 + (1-q_\theta+q_\theta^2)a_1}{1+q_\theta^2}, \frac{q_\phi c_0 + (1-q_\phi+q_\phi^2)c_1}{1+q_\phi^2}\right)}{\partial x} \\
&- \frac{(1-q_\phi)^2(c_1-c_0)}{2(1+q_\phi^2)} \frac{\partial \mathfrak{f}\left(\frac{q_\theta a_0 + (1-q_\theta+q_\theta^2)a_1}{1+q_\theta^2}, \frac{q_\phi c_0 + (1-q_\phi+q_\phi^2)c_1}{1+q_\phi^2}\right)}{\partial y} \\
&- \frac{(1-q_\theta)^2(a_1-a_0)}{2(1+q_\theta^2)(c_1-c_0)} \int_{c_0}^{c_1} \frac{\partial}{\partial x}\mathfrak{f}\left(\frac{q_\theta a_0 + (1-q_\theta+q_\theta^2)a_1}{1+q_\theta^2}, y\right) {}_{c_0}\tilde{d}_{q_\phi} y \\
&- \frac{(1-q_\phi)^2(c_1-c_0)}{2(1+q_\phi^2)(a_1-a_0)} \int_{a_0}^{a_1} \frac{\partial}{\partial y}\mathfrak{f}\left(x, \frac{q_\phi c_0 + (1-q_\phi+q_\phi^2)c_1}{1+q_\phi^2}\right) {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1}{2}\left[\frac{1}{a_1-a_0} \int_{a_0}^{a_1} \mathfrak{f}\left(x, \frac{q_\phi c_0 + (1-q_\phi+q_\phi^2)c_1}{1+q_\phi^2}\right) {}_{a_0}\tilde{d}_{q_\theta} x \right. \\
&\left. + \frac{1}{c_1-c_0} \int_{c_0}^{c_1} \mathfrak{f}\left(\frac{q_\theta a_0 + (1-q_\theta+q_\theta^2)a_1}{1+q_\theta^2}, y\right) {}_{c_0}\tilde{d}_{q_\phi} y \right] \\
&- \frac{(1-q_\theta)^2(a_1-a_0)}{2(1+q_\theta^2)(c_1-c_0)} \int_{c_0}^{c_1} \frac{\partial}{\partial x}\mathfrak{f}\left(\frac{q_\theta a_0 + (1-q_\theta+q_\theta^2)a_1}{1+q_\theta^2}, y\right) {}_{c_0}\tilde{d}_{q_\phi} y \\
&- \frac{(1-q_\phi)^2(c_1-c_0)}{2(1+q_\phi^2)(a_1-a_0)} \int_{a_0}^{a_1} \frac{\partial}{\partial y}\mathfrak{f}\left(x, \frac{q_\phi c_0 + (1-q_\phi+q_\phi^2)c_1}{1+q_\phi^2}\right) {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1}{(a_1-a_0)(c_1-c_0)} \int_{a_0}^{a_1}\int_{c_0}^{c_1} \mathfrak{f}(x,y) \, {}_{c_0}\tilde{d}_{q_\phi} y \, {}_{a_0}\tilde{d}_{q_\theta} x \\
&\leqslant \frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x,c_0) \, {}_{a_0}\tilde{d}_{q_\theta} x + \frac{q_\phi}{2(a_1-a_0)(1+q_\phi^2)} \int_{a_0}^{a_1} \mathfrak{f}(x,c_1) \, {}_{a_0}\tilde{d}_{q_\theta} x \\
&+ \frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_0,y) \, {}_{c_0}\tilde{d}_{q_\phi} y + \frac{q_\theta}{2(c_1-c_0)(1+q_\theta^2)} \int_{c_0}^{c_1} \mathfrak{f}(a_1,y) \, {}_{c_0}\tilde{d}_{q_\phi} y \\
&\leqslant \frac{1}{(1+q_\theta^2)(1+q_\phi^2)}\Big[(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0,c_0) + q_\theta(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1,c_0) \\
&+ q_\phi(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_1) + q_\theta q_\phi \mathfrak{f}(a_1,c_1)\Big]
\end{aligned}
\tag{51}
$$

inequalities hold.

Proof. Using (15) and same methodology of Theorem 5, we will get the desired result. □

Now, we will investigate our main result (17) through examples.

Example 2. *Let* $\mathfrak{f} : [1, 2] \times [3, 4] \subseteq \Re^2 \longrightarrow \Re$ *be a convex coordinated function on* $[1, 2] \times [3, 4]$ *and defined as* $\mathfrak{f}(x, y) = x^2 + y^2$*, applying it on* (17)*. Let* $a_0 = 1$*,* $a_1 = 2$*,* $c_0 = 3$*, and* $c_1 = 4$*.*

Case 1. *If* $q_\theta < q_\phi$*, put* $q_\theta = \frac{1}{3}$ *and* $q_\phi = \frac{1}{2}$*.*

Then, the term on the left-hand side of (17) *becomes*

$$\mathfrak{f}\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2}, \frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)=\mathfrak{f}\left(\frac{13}{10},\frac{17}{5}\right)=\frac{169}{100}+\frac{289}{25}=13.25. \tag{52}$$

The right-hand part of the extreme left term of (17) *becomes*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1}\mathfrak{f}(x,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2})\ {}_{a_0}\tilde{d}_{q_\theta}x\right.\\
&\left.+\frac{1}{c_1-c_0}\int_{c_0}^{c_1}\mathfrak{f}(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},y)\ {}_{c_0}\tilde{d}_{q_\phi}y\right]\\
&=\frac{1}{2}\left[\int_1^2\mathfrak{f}(x,\frac{17}{5})\ {}_1\tilde{d}_{q_\theta}x+\int_3^4\mathfrak{f}(\frac{13}{10},y)\ {}_3\tilde{d}_{q_\phi}y\right]\\
&=\frac{1}{2}\left[\int_1^2 x^2\ {}_1\tilde{d}_{q_\theta}x+\left(\frac{17}{5}\right)^2(2-1)+\left(\frac{13}{10}\right)^2(4-3)+\int_3^4 y^2\ {}_3\tilde{d}_{q_\phi}y\right]\\
&=\frac{1}{2}\left[(2-1)(1-q_\theta^2)\sum_{n=0}^{\infty}q_\theta^{2n}\left\{(1-q_\theta^{2n+1}).1+q_\theta^{2n+1}.2\right\}^2+\frac{289}{25}+\frac{169}{100}\right.\\
&\left.+(4-3)(1-q_\phi^2)\sum_{m=0}^{\infty}q_\phi^{2m}\left\{(1-q_\phi^{2m+1}).3+q_\phi^{2m+1}.4\right\}^2\right]\\
&=\frac{1}{2}\left[(1-q_\theta^2)\left\{\frac{1}{1-q_\theta^2}+\frac{q_\theta^2}{1-q_\theta^6}+\frac{2q_\theta}{1-q_\theta^4}\right\}+\frac{1325}{100}+(1-q_\phi^2)\left\{\frac{9}{1-q_\phi^2}+\frac{q_\phi^2}{1-q_\phi^6}+\frac{6q_\phi}{1-q_\phi^4}\right\}\right]\\
&=\frac{1}{2}\left[1+\frac{q_\theta^2}{1+q_\theta^2+q_\theta^4}+\frac{2q_\theta}{1+q_\theta^2}+\frac{1325}{100}+9+\frac{q_\phi^2}{1+q_\phi^2+q_\phi^4}+\frac{6q_\phi}{1+q_\phi^2}\right]\\
&=\frac{1}{2}\left[10+\frac{9}{91}+\frac{3}{5}+\frac{1325}{100}+\frac{4}{21}+\frac{12}{5}\right]=13.27.
\end{aligned}\tag{53}$$

The middle term of (17) *becomes*

$$\begin{aligned}
&\frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1}\mathfrak{f}(x,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\ {}_{a_0}\tilde{d}_{q_\theta}x\\
&=\frac{1}{(2-1)(4-3)}\int_1^2\int_3^4(x^2+y^2)\ {}_3\tilde{d}_{q_\phi}y\ {}_1\tilde{d}_{q_\theta}x\\
&=(1-q_\theta^2)(1-q_\phi^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\phi^{2m}q_\theta^{2n}\left[\left\{1-q_\theta^{2n+1}+2q_\theta^{2n+1}\right\}^2+\left\{3-3q_\phi^{2m+1}+4q_\phi^{2m+1}\right\}^2\right]\\
&=(1-q_\theta^2)(1-q_\phi^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_\phi^{2m}\left[q_\theta^{2n}+q_\theta^{6n+2}+2q_\theta^{4n+1}+9q_\theta^{2n}+6q_\theta^{2n}q_\phi^{2m+1}+q_\theta^{2n}q_\phi^{4m+2}\right]\\
&=(1-q_\theta^2)(1-q_\phi^2)\sum_{m=0}^{\infty}q_\phi^{2m}\left[\frac{1}{1-q_\theta^2}+\frac{q_\theta^2}{1-q_\theta^6}+\frac{2q_\theta}{1-q_\theta^4}+\frac{9}{1-q_\theta^2}+\frac{6q_\phi^{2m+1}}{1-q_\theta^2}+\frac{q_\phi^{4m+2}}{1-q_\theta^2}\right]\\
&=(1-q_\phi^2)\sum_{m=0}^{\infty}\left[10q_\phi^{2m}+\frac{q_\theta^2q_\phi^{2m}}{1+q_\theta^2+q_\theta^4}+\frac{2q_\theta q_\phi^{2m}}{1+q_\theta^2}+6q_\phi^{4m+1}+q_\phi^{6m+2}\right]\\
&=10+\frac{q_\theta^2}{1+q_\theta^2+q_\theta^4}+\frac{2q_\theta}{1+q_\theta^2}+\frac{6q_\phi}{1+q_\phi^2}+\frac{q_\phi^2}{1+q_\phi^2+q_\phi^4}=10+\frac{9}{91}+\frac{3}{5}+\frac{12}{5}+\frac{4}{21}=13.29.
\end{aligned}\tag{54}$$

The second term from the right-hand side of (17) *becomes*

$$
\begin{aligned}
&\frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1} f(x,c_0)\ {}_{a_0}\tilde{d}_{q_\theta}x+\frac{q_\phi}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1} f(x,c_1)\ {}_{a_0}\tilde{d}_{q_\theta}x\\
&+\frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1} f(a_0,y)\ {}_{c_0}\tilde{d}_{q_\phi}y+\frac{q_\theta}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1} f(a_1,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\\
&=\frac{3}{10}\left[\int_1^2 x^2\ {}_1\tilde{d}_{q_\theta}x+3^2(2-1)\right]+\frac{1}{5}\left[\int_1^2 x^2\ {}_1\tilde{d}_{q_\theta}x+4^2(2-1)\right]+\frac{7}{20}\left[1^2(4-3)+\int_3^4 y^2\ {}_3\tilde{d}_{q_\phi}y\right]\\
&+\frac{3}{20}\left[2^2(4-3)+\int_3^4 y^2\ {}_3\tilde{d}_{q_\phi}y\right]\\
&=\frac{3}{10}\left[1+\frac{q_\theta^2}{1+q_\theta^2+q_\theta^4}+\frac{2q_\theta}{1+q_\theta^2}+9\right]+\frac{1}{5}\left[1+\frac{q_\theta^2}{1+q_\theta^2+q_\theta^4}+\frac{2q_\theta}{1+q_\theta^2}+16\right]\\
&+\frac{7}{20}\left[1+9+\frac{q_\phi^2}{1+q_\phi^2+q_\phi^4}+\frac{6q_\phi}{1+q_\phi^2}\right]+\frac{3}{20}\left[4+9+\frac{q_\phi^2}{1+q_\phi^2+q_\phi^4}+\frac{6q_\phi}{1+q_\phi^2}\right]\\
&=\frac{3}{10}\left(10+\frac{9}{91}+\frac{3}{5}\right)+\frac{1}{5}\left(17+\frac{9}{91}+\frac{3}{5}\right)+\frac{7}{20}\left(10+\frac{4}{21}+\frac{12}{5}\right)+\frac{3}{20}\left(13+\frac{4}{21}+\frac{12}{5}\right)\\
&=13.49.
\end{aligned}
\tag{55}
$$

The extreme right-hand term of (17) *becomes*

$$
\begin{aligned}
&\frac{1}{(1+q_\theta^2)(1+q_\phi^2)}\Big[(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)f(a_0,c_0)+q_\theta(1-q_\phi+q_\phi^2)f(a_1,c_0)\\
&+q_\phi(1-q_\theta+q_\theta^2)f(a_0,c_1)+q_\theta q_\phi f(a_1,c_1)\Big]\\
&=\frac{(\frac{7}{9})(\frac{3}{4})(10)+(\frac{1}{3})(\frac{3}{4})(13)+(\frac{1}{2})(\frac{7}{9})(17)+(\frac{1}{3})(\frac{1}{2})(20)}{(\frac{10}{9})(\frac{5}{4})}=13.7.
\end{aligned}
\tag{56}
$$

From (52)–(56), *we have*

$$13.25<13.27<13.29<13.49<13.7.$$

Which endorses our result.

Case 2. *If* $q_\theta>q_\phi$, *put* $q_\theta=\frac{1}{2}$ *and* $q_\phi=\frac{1}{3}$.
Then, the term on the left-hand side of (17) *becomes*

$$
f\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)=f\left(\frac{7}{5},\frac{33}{10}\right)=12.85.
\tag{57}
$$

The right-hand part of the extreme left term of (17) *becomes*

$$
\begin{aligned}
&\frac{1}{2}\Bigg[\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2})\ {}_{a_0}\tilde{d}_{q_\theta}x\\
&+\frac{1}{c_1-c_0}\int_{c_0}^{c_1} f(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},y)\ {}_{c_0}\tilde{d}_{q_\phi}y\Bigg]\\
&=\frac{1}{2}\left[10+\frac{4}{21}+\frac{9}{91}+\frac{1089}{100}+\frac{49}{25}+\frac{13}{5}\right]=12.87.
\end{aligned}
\tag{58}
$$

The middle term of (17) *becomes*

$$\frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} f(x,y)\,{}_{c_0}\tilde{d}_{q_\phi}y\,{}_{a_0}\tilde{d}_{q_\theta}x = 10+\frac{4}{21}+\frac{4}{5}+\frac{9}{91}+\frac{9}{5}=12.89. \quad (59)$$

The second term from the right-hand side of (17) *becomes*

$$\begin{aligned}
&\frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1} f(x,c_0)\,{}_{a_0}\tilde{d}_{q_\theta}x+\frac{q_\phi}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1} f(x,c_1)\,{}_{a_0}\tilde{d}_{q_\theta}x\\
&+\frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1} f(a_0,y)\,{}_{c_0}\tilde{d}_{q_\phi}y+\frac{q_\theta}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1} f(a_1,y)\,{}_{c_0}\tilde{d}_{q_\phi}y\\
&=\frac{7}{20}\left(10+\frac{4}{21}+\frac{4}{5}\right)+\frac{3}{20}\left(17+\frac{4}{21}+\frac{4}{5}\right)+\frac{3}{10}\left(10+\frac{9}{91}+\frac{9}{5}\right)+\frac{1}{5}\left(13+\frac{9}{91}+\frac{9}{5}\right)\\
&=13.09. \qquad (60)
\end{aligned}$$

The extreme right-hand term of (17) *becomes*

$$\begin{aligned}
&\frac{1}{(1+q_\theta^2)(1+q_\phi^2)}\Big[(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)f(a_0,c_0)+q_\theta(1-q_\phi+q_\phi^2)f(a_1,c_0)\\
&+q_\phi(1-q_\theta+q_\theta^2)f(a_0,c_1)+q_\theta q_\phi f(a_1,c_1)\Big]\\
&=\frac{(\frac{3}{4})(\frac{7}{9})(10)+(\frac{1}{2})(\frac{7}{9})(13)+(\frac{1}{3})(\frac{3}{4})(17)+(\frac{1}{2})(\frac{1}{3})(20)}{(\frac{5}{4})(\frac{10}{9})}=13.3. \qquad (61)
\end{aligned}$$

From (57)–(61), *we have*

$$12.85<12.87<12.89<13.09<13.3.$$

Which also endorses our result.

Case 3. *If* $q_\theta = q_\phi$, *put* $q_\theta = q_\phi = \frac{1}{2}$.
Then, the term on the left-hand side of (17) *becomes*

$$f\left(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2}\right)=f\left(\frac{7}{5},\frac{17}{5}\right)=13.52. \quad (62)$$

The right-hand part of the extreme left term of (17) *becomes*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x,\frac{(1-q_\phi+q_\phi^2)c_0+q_\phi c_1}{1+q_\phi^2})\,{}_{a_0}\tilde{d}_{q_\theta}x\right.\\
&\left.+\frac{1}{c_1-c_0}\int_{c_0}^{c_1} f(\frac{(1-q_\theta+q_\theta^2)a_0+q_\theta a_1}{1+q_\theta^2},y)\,{}_{c_0}\tilde{d}_{q_\phi}y\right]\\
&=\frac{1}{2}\left[10+\frac{4}{21}+\frac{4}{5}+\frac{289}{25}+\frac{49}{25}+\frac{4}{21}+\frac{12}{5}\right]=13.55. \qquad (63)
\end{aligned}$$

The middle term of (17) *becomes*

$$\frac{1}{(a_1-a_0)(c_1-c_0)}\int_{a_0}^{a_1}\int_{c_0}^{c_1} f(x,y)\,{}_{c_0}\tilde{d}_{q_\phi}y\,{}_{a_0}\tilde{d}_{q_\theta}x = 10+\frac{4}{21}+\frac{4}{5}+\frac{4}{21}+\frac{12}{5}=13.58. \quad (64)$$

The second term from the right-hand side of (17) *becomes*

$$\begin{aligned}
&\frac{1-q_\phi+q_\phi^2}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1}\mathfrak{f}(x,c_0)\ {}_{a_0}\tilde{d}_{q_\theta}x+\frac{q_\phi}{2(a_1-a_0)(1+q_\phi^2)}\int_{a_0}^{a_1}\mathfrak{f}(x,c_1)\ {}_{a_0}\tilde{d}_{q_\theta}x\\
&+\frac{1-q_\theta+q_\theta^2}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1}\mathfrak{f}(a_0,y)\ {}_{c_0}\tilde{d}_{q_\phi}y+\frac{q_\theta}{2(c_1-c_0)(1+q_\theta^2)}\int_{c_0}^{c_1}\mathfrak{f}(a_1,y)\ {}_{c_0}\tilde{d}_{q_\phi}y\\
&=\frac{3}{10}\left(10+\frac{4}{21}+\frac{4}{5}\right)+\frac{1}{5}\left(17+\frac{4}{21}+\frac{4}{5}\right)+\frac{3}{10}\left(10+\frac{4}{21}+\frac{12}{5}\right)+\frac{1}{5}\left(13+\frac{4}{21}+\frac{12}{5}\right)\\
&=13.79.
\end{aligned}\tag{65}$$

The extreme right-hand term of (17) *becomes*

$$\begin{aligned}
&\frac{1}{(1+q_\theta^2)(1+q_\phi^2)}\Big[(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)\mathfrak{f}(a_0,c_0)+q_\theta(1-q_\phi+q_\phi^2)\mathfrak{f}(a_1,c_0)\\
&+q_\phi(1-q_\theta+q_\theta^2)\mathfrak{f}(a_0,c_1)+q_\theta q_\phi\mathfrak{f}(a_1,c_1)\Big]\\
&=\frac{(\frac{3}{4})(\frac{3}{4})(10)+(\frac{1}{2})(\frac{3}{4})(13)+(\frac{1}{2})(\frac{3}{4})(17)+(\frac{1}{2})(\frac{1}{2})(20)}{(\frac{5}{4})(\frac{5}{4})}=14.
\end{aligned}\tag{66}$$

From (62)–(66), *we have*

$$13.52<13.55<13.58<13.79<14.$$

Which also endorses our result.

4. Application

Like a quantum Hermite–Hadamard inequality [15], the inequality (17) can also be used to find the range for coordinated convex functions whose symmetric quantum integrals cannot be calculated.

Example 3. *Let* $\mathfrak{f}(x,y)=e^{x^2+y^2}$ *be a coordinated convex function whose quantum and symmetric quantum integrals cannot be calculated on* $\triangle=[0,1]\times[0,1]$. *But the range of this function for* $q_\theta q_\phi$- *and* $q_\theta q_\phi$-*symmetric integrals can be calculated with the help of the Hermite–Hadamard inequality in quantum and symmetric quantum calculus, respectively.*

In Quantum Calculus:

In example 2 *from* [15], *we have*

$$e^{\frac{1}{(1+q_\theta)^2}+\frac{1}{(1+q_\phi)^2}}\leqslant\int_0^1\int_0^1 e^{x^2+y^2}d_{q_\phi}yd_{q_\theta}x\leqslant\frac{q_\theta q_\phi+q_\theta e+q_\phi e+e^2}{(1+q_\theta)(1+q_\phi)}\tag{67}$$

Case 1. *If* $q_\theta=q_\phi=\frac{1}{2}$, *then* (67) *becomes*

$$2.432\leqslant\int_0^1\int_0^1 e^{x^2+y^2}d_{q_\phi}yd_{q_\theta}x\leqslant 4.603.\tag{68}$$

Case 2. *If* $q_\theta\neq q_\phi$ *and* $q_\theta=\frac{1}{3}$, $q_\phi=\frac{1}{2}$, *then* (67) *becomes*

$$2.737\leqslant\int_0^1\int_0^1 e^{x^2+y^2}d_{q_\phi}yd_{q_\theta}x\leqslant 4.911.\tag{69}$$

In Symmetric Quantum Calculus:

We have

$$e^{\frac{q_\theta^2}{(1+q_\theta^2)^2}+\frac{q_\phi^2}{(1+q_\phi^2)^2}} \leqslant \int_0^1\int_0^1 e^{x^2+y^2}\tilde{d}_{q_\phi}y\tilde{d}_{q_\theta}x$$
$$\leqslant \frac{(1-q_\theta+q_\theta^2)(1-q_\phi+q_\phi^2)e+q_\theta(1-q_\phi+q_\phi^2)e+q_\phi(1-q_\theta+q_\theta^2)e+q_\theta q_\phi e^2}{(1+q_\theta^2)(1+q_\phi^2)}. \quad (70)$$

Case 1. *If* $q_\theta = q_\phi = \frac{1}{2}$, *then* (70) *becomes*

$$1.338 \leqslant \int_0^1\int_0^1 e^{x^2+y^2}\tilde{d}_{q_\phi}y\tilde{d}_{q_\theta}x \leqslant 2.847. \quad (71)$$

Case 2. *If* $q_\theta \neq q_\phi$ *and* $q_\theta = \frac{1}{3}$, $q_\phi = \frac{1}{2}$, *then* (70) *becomes*

$$1.28 \leqslant \int_0^1\int_0^1 e^{x^2+y^2}\tilde{d}_{q_\phi}y\tilde{d}_{q_\theta}x \leqslant 2.234. \quad (72)$$

5. Conclusions

In this work, we develop the calculus of symmetric quantum calculus on coordinates in plane that helps us to construct the $q_\theta q_\phi$-symmetric Hölder's and Hermite–Hadamard inequalities for functions of two variables. We provide examples to justify our novel results. Moreover, the $q_\theta q_\phi$-symmetric Hermite–Hadamard inequality can also be used for finding the range of coordinated convex functions. In the second example, we can see that the range set of coordinated convex functions in symmetric quantum calculus is smaller than in quantum calculus.

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References

1. Hadamard, J. Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann . *J. Math. Pures Appl.* **1893**, *58*, 171–215.
2. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*; RGMIA Monographs; Victoria University: Melbourne, VIC, Australia , 2000.
3. De la Cal, J.; Carcamo, J. Multidimensional Hermite-Hadamard inequalities and the convex order. *J. Math. Anal. Appl.* **2006**, *324*, 248–261. [CrossRef]
4. Yang, W.H. A generalization of Hadamard's inequality for convex functions. *Appl. Math. Lett.* **2008**, *21*, 254–257. [CrossRef]
5. Moslehian, M.S. Matrix Hermite-Hadamard type inequalities. *Houst. J. Math.* **2013**, *39*, 177–189.
6. Dragomir, S.S. On the Hadamard's inequality for functions on the co-ordinates in a rectangle from the plane. *Taiwan. J. Math.* **2001**, *5*, 775–788. [CrossRef]
7. Jackson, F.H. On a q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
8. Al-Salam, W. Some fractional q-integrals and q-derivatives. *Proc. Edinb. Math. Soc.* **1966**, *15*, 135–140. [CrossRef]
9. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
10. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Springer: Berlin/Heidelberg, Germany, 2012.
11. Ernst, T. A method for q-calculus. *J. Nonlinear Math. Phys.* **2003**, *10*, 487–525. [CrossRef]
12. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121. [CrossRef]
13. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *2013*, 282. [CrossRef]

14. Alp, N.; Sarikaya, M.Z.; Kunt, M.; Iscan, I. q-Hermite-Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud. Univ. Sci.* **2018**, *30*, 193–203. [CrossRef]
15. Latif, M.A.; Dragomir, S.S.; Momoniat, E. Some q-analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane. *J. King Saud Univ. Sci.* **2017**, *29*, 263–273. [CrossRef]
16. Alp, N.; Sarikaya, M.Z. Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral. *Appl. Math. e-Notes* **2020**, *20*, 341–356.
17. Bermudo, S.; Kórus, P.; Valdés, J.N. On *q*-Hermite–Hadamard Inequalities for General Convex Functions. *Acta Math. Hung.* **2020**, *160*, 364–374. [CrossRef]
18. Rashid, S.; Butt, S.I.; Kanwal, S.; Ahmad, H.; Wang, M.K. Quantum integral inequalities with respect to Raina's function via coordinated generalized-convex functions with applications. *J. Funct. Spaces* **2021**, *2021*, 6631474.
19. Alqudah, M.A.; Kashuri, A.; Mohammed, P.O.; Abdeljawad, T.; Raees, M.; Anwar, M.; Hamed, Y.S. Hermite-Hadamard Integral Inequalities on Co-ordinated Convex Functions in Quantum Calculus. *Adv. Differ. Equ.* **2021**, *2021*, 264. [CrossRef]
20. Butt, S.I.; Budak, H.; Nonlaopon, K. New Quantum Mercer Estimates of Simpson-Newton like Inequalities via Convexity. *Symmetry* **2022**, *14*, 1935. [CrossRef]
21. Haines, S.L. The Symmetric Derivative. Master's Thesis, Bowling Green State University, Bowling Green, OH, USA, 1965. Available online: https://etd.ohiolink.edu/acprod/odb_etd/etd/r/1501/10?clear=10&p10_accession_num=bgsu1670603448495953 (accessed on 7 April 2024)
22. Da Cruz, A.M.B.; Martins, N. The q-symmetric variational calculus. *Comput. Math. Appl.* **2012**, *64*, 2241–2250. [CrossRef]
23. Lavagno, A.; Gervino, G. Quantum mechanics in-deformed calculus. *J. Phys. Conf. Ser.* **2009**, *174*, 012071. [CrossRef]
24. Nosheen, A.; Ijaz, S.; Khan, K.A.; Awan, K.M.; Albahar, M.A.; Thanoon, M. Some q-Symmetric Integral Inequalities Involving s-Convex Functions. *Symmetry* **2023**, *15*, 1169. [CrossRef]
25. Butt, S.I.; Aftab, M.N.; Nabwey, H.A.; Etemad, S. Some Hermite-Hadamard and midpoint type inequalities in symmetric quantum calculus. *AIMS Math.* **2024**, *9*, 5523–5549. [CrossRef]
26. Bakula, M.K. An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates. *Aust. J. Math. Anal. Appl.* **2014**, *11*, 7.