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On Curvature Pinching for Submanifolds with Parallel Normalized Mean Curvature Vector

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Abstract: In this note, we investigate the pinching problem for oriented compact submanifolds of dimension n with parallel normalized mean curvature vector in a space form $F^{n+p}(c)$. We first prove a codimension reduction theorem for submanifolds under lower Ricci curvature bounds. Moreover, if the submanifolds have constant normalized scalar curvature $R \geq c$, we obtain a classification theorem for submanifolds under lower Ricci curvature bounds. It should be emphasized that our Ricci pinching conditions are sharp for even n and $p = 2$.

Keywords: submanifold; rigidity theorems; Ricci curvature; scalar curvature

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1. Introduction

The geometric rigidity of compact submanifolds plays an important role in submanifold geometry. In 1968, Simons [1] first studied the rigidity for minimal submanifolds in spheres. Later, a series of striking rigidity theorems for minimal submanifolds were obtained by some geometers [2–6]. Let M^n be an n-dimensional oriented compact submanifold in the complete and simply connected space form $F^{n+p}(c)$ with constant curvature c. In 1979, Ejiri [7] proved a rigidity result for minimal submanifolds with pinched Ricci curvatures in spheres.

Theorem 1 ([7]). *Let $M^n (n \geq 4)$ be a simply connected, compact oriented minimal submanifold in S^{n+p}. If the Ricci curvature of M satisfies $Ric_M \geq n-2$, then M is either the totally geodesic submanifold S^n, the Clifford torus $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ in S^{n+1} with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3})$ in S^7. Here $\mathbb{C}P^2(\frac{4}{3})$ denotes the two-dimensional complex projective space minimally immersed into S^7 with constant holomorphic sectional curvature $\frac{4}{3}$.*

Later, Shen [8] and Li [9] obtained that if M^3 is a compact oriented minimal submanifold in S^{3+p}, and $Ric_M \geq 1$, then M is totally geodesic. Afterward, Xu and Tian [10] got a refined version of Theorem 1, where the condition "M is simply connected" was removed. In 2013, Xu and Gu [11] generalized the Ejiri rigidity theorem to compact submanifolds with parallel mean curvature vector in space forms.

Theorem 2 ([11]). *Let $M^n (n \geq 3)$ be an oriented compact submanifold with parallel mean curvature vector in $F^{n+p}(c)$ with $c + H^2 > 0$. If*

$$Ric_M \geq (n-2)(c + H^2),$$

then M is either the totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$, the Clifford hypersurface $S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}})$ in the totally umbilic sphere $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3}(c + H^2))$

in $S^7(\frac{1}{\sqrt{c+H^2}})$. *Here* $\mathbb{C}P^2(\frac{4}{3}(c+H^2))$ *denotes the two-dimensional complex projective space minimally immersed in* $S^7(\frac{1}{\sqrt{c+H^2}})$ *with constant holomorphic sectional curvature* $\frac{4}{3}(c+H^2)$.

Further discussions for submanifolds with parallel mean curvature vector have been carried out by many authors (see [12–14], etc.).

On the other hand, it is important to study the rigidity problem for compact submanifolds with constant scalar curvature. In 1977, Cheng and Yau [15] constructed a self-adjoint second-order differential operator to study n-dimensional closed hypersurfaces with constant scalar curvature in the space form $F^{n+1}(c)$, and obtained a classification result.

Theorem 3 ([15])**.** *Let M^n be a compact hypersurface with constant normalized scalar curvature R in the space form $F^{n+1}(c)$ with constant curvature c. If $R-c\geq 0$, and the sectional curvature of M satisfies $K_M\geq 0$, then M is either a totally umbilical hypersurface, or a (Riemannian) product of two totally umbilical constantly curved submanifolds.*

In 1996, Li [16] studied Cheng-Yau's self-adjoint operator, and proved a rigidity theorem for submanifolds with pinched scalar curvature.

Theorem 4 ([16])**.** *Let $M^n(n\geq 3)$ be a compact hypersurface with constant normalized scalar curvature R in the unit sphere S^{n+1}. If $\bar{R}=R-1\geq 0$, and the norm square S of the second fundamental form of M satisfies*

$$n\bar{R}\leq S\leq \frac{n}{(n-2)(n\bar{R}+2)}\Big[n(n-1)\bar{R}^2+4(n-1)\bar{R}+n\Big],$$

then either $S\equiv n\bar{R}$, and M is a totally umbilical hypersurface, or

$$S=\frac{n}{(n-2)(n\bar{R}+2)}\Big[n(n-1)\bar{R}^2+4(n-1)\bar{R}+n\Big],$$

and $M=S^1\left(\sqrt{1-r^2}\right)\times S^{n-1}(r), r=\sqrt{\frac{n-2}{n(R+1)}}$ *and* $\bar{R}=R-1$.

After that, some rigidity theorems for submanifolds with constant scalar curvature were obtained [17,18]. In 2013, Guo and Li [19] generalized Theorem 4 to the case of $n(\geq 4)$-dimensional submanifolds with parallel normalized mean curvature vector in spheres.

The pinching problem of submanifolds with parallel normalized mean curvature vector seems interesting. In this paper, we first study the compact submanifolds with parallel normalized mean curvature vector in space forms. Using Li-Li's inequality [20] and the DDVV inequality proved by Lu, Ge-Tang [21,22], we obtain a codimension reduction theorem.

Theorem 5. *Let $M^n(n\geq 3)$ be an oriented compact submanifold with parallel normalized mean curvature vector in the space form $F^{n+p}(c)(p>1)$. If the Ricci curvature of M satisfies*

$$Ric_M>(n-2+\delta(n,p))(c+H^2)>0,$$

then M lies in the totally geodesic space form $F^{n+1}(c)$. Here

$$\delta(n,p)=\begin{cases}0, & \text{for } p=2,\\ \frac{n-1}{3n-5}, & \text{for } p\geq 3.\end{cases}$$

If $p\geq 3$, $\frac{1}{3}<\delta(n,p)\leq\frac{1}{2}$. Moreover, we investigate the compact submanifolds with constant scalar curvature and parallel normalized mean curvature vector, and prove a rigidity result.

Theorem 6. *Let $M^n (n \geq 3)$ be an oriented compact submanifold with constant normalized scalar curvature R and parallel normalized mean curvature vector in the space form $F^{n+p}(c)$. If $R \geq c$, and*

$$Ric_M > (n - 2 + \delta(n, p))(c + H^2) > 0,$$

then M must be the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{c+H^2}}\right)$. Here $\delta(n, p)$ is defined as in Theorem 5.

2. Notation and Lemmas

Let M^n be an n-dimensional oriented compact submanifold in the $(n+p)$-dimensional complete and simply connected space form $F^{n+p}(c)$ with constant curvature c. We make the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n + p,\ 1 \leq i, j, k, \cdots \leq n,\ n + 1 \leq \alpha, \beta, \gamma \leq n + p.$$

For an arbitrary fixed point $x \in M \subset F^{n+p}(c)$, we choose an orthonormal local frame field $\{e_A\}$ in $F^{n+p}(c)$, where $\{e_i\}$ are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of $F^{n+p}(c)$ respectively. Let h, ξ and Rm be the second fundamental form, mean curvature vector and the Riemannian curvature tensor of M, respectively, and $\bar{R}m$ the Riemannian curvature tensor of $F^{n+p}(c)$. Then

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j,\ h^\alpha_{ij} = h^\alpha_{ji},$$

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,\ \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha,$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right), \tag{1}$$

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \tag{2}$$

The squared norm S of the second fundamental form of M are give by $S := \sum_{\alpha,i,j}(h^\alpha_{ij})^2$. Define $H_\alpha = (h^\alpha_{ij})_{n\times n}$ and choose e_{n+1} such that it is parallel to ξ. Hence, we have

$$trH_{n+1} = nH,\ trH_\alpha = 0, \text{ for } \alpha \neq n+1.$$

Here H is the mean curvature of M. Denote by $Ric(u)$ the Ricci curvature of M in the direction of $u \in UM$, where UM is the unit tangent bundle. From (1) we have

$$Ric(e_i) = (n-1)c + \sum_{\alpha,j}\left[h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2\right], \tag{3}$$

and

$$n(n-1)(R - c) = n^2H^2 - S, \tag{4}$$

where R is the normalized scalar curvature, given by $R = \frac{1}{n(n-1)}\sum_{i,j} R_{ijij}$. Denoting the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl} respectively. Then, by definition

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha},$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$.

Now, we assume that M has parallel normalized mean curvature vector. Then, $\omega_{n+1,\alpha} = 0$ for any α. Therefore, $\sum_i \omega_{n+1,i} \wedge \omega i\alpha = 0$, for any α. Thus,

$$H_{n+1}H_\alpha = H_\alpha H_{n+1},$$

and

$$R_{n+1,\alpha kl} = 0, \text{ for any } \alpha, k, l.$$

Following [5,6], we have

$$\Delta h_{ij}^{n+1} = \sum_k h_{kkij}^{n+1} + \sum_{k,m} (h_{mk}^{n+1} R_{mijk} + h_{im}^{n+1} R_{mkjk}), \tag{5}$$

$$\Delta h_{ij}^{\alpha} = \sum_k h_{kkij}^{\alpha} + \sum_{k,m} h_{km}^{\alpha} R_{mijk} + \sum_{k,m} h_{mi}^{\alpha} R_{mkjk} - \sum_{\beta \neq n+1,k} h_{ki}^{\beta} R_{\alpha\beta jk}, \; for \; \alpha \neq n+1. \tag{6}$$

We define the gradient and Hessian of f by

$$df = \sum_i f_i \omega_i, \sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji},$$

where f is a function f defined on M. We make appeal to the differential operator due to Cheng and Yau [15], acting on any C^2-function f by

$$\Box f = \sum_{i,j} \left(nH\delta_{ij} - h_{ij}^{n+1} \right) f_{ij}. \tag{7}$$

It follows from [15] that the operator $\Box$ is self-adjoint, that is,

$$\int_M \Box f dv = 0, \text{ for any } f \in C^2(M). \tag{8}$$

The following lemma will be used to prove our main theorems, which is essentially due to Cheng-Yau [15], and see also [16,19].

Lemma 1. *Assume the normalized scalar curvature R =constant and $R - c \geq 0$, then*

$$|\nabla h|^2 \geq n^2 |\nabla H|^2. \tag{9}$$

Denote by $N(A)$ the square of the norm of A, where $A = (a_{ij})$ is a matrix. Then, $N(A) = \mathrm{tr}(A^t A) = \sum_{i,j} a_{ij}^2$, and we have the following lemmas.

Lemma 2 ([20])**.** *Let $B_1, \ldots, B_m$ be symmetric $(n \times n)$-matrices. Set $S_{\alpha\beta} = tr\left(B_\alpha^t B_\beta\right)$, $S_\alpha = S_{\alpha\alpha}$, then*

$$\sum_{\alpha,\beta}^{m} N(B_\alpha B_\beta - B_\beta B_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq (1 + \frac{1}{2} sgn(m-1)) \Big(\sum_\alpha S_\alpha \Big)^2,$$

and the equality holds if and only if one of the following conditions holds:

(1) $B_1 = B_2 = \ldots = B_m = 0$

(2) only two of the matrices $B_1, B_2, \ldots, B_m$ are different from zero. Moreover, assuming $B_1 \neq 0, B_1 \neq 0, B_3 = \ldots = B_m = 0$, one has $S_1 = S_2$, and there exists an orthogonal $(n \times n)$-matrix T such that

$$TB_1^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right),$$

$$TB_2^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right).$$

The following DDVV inequality is proved by Lu and Ge-Tang [21,22].

Lemma 3 (DDVV Inequality)**.** *Let $B_1, \ldots, B_m$ be symmetric $(n \times n)$-matrices. Then,*

$$\sum_{\alpha,\beta} N(B_\alpha B_\beta - B_\beta B_\alpha) \leq \Big(\sum_\alpha N(B_\alpha) \Big)^2,$$

where the equality holds if and only if under some rotation all B_r's are zero except two matrices which can be written as

$$\tilde{B}_r = P \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t, \qquad \tilde{B}_s = P \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t,$$

for an orthogonal $(n \times n)$-matrix P.

3. A Codimension Reduction Theorem

In this section, we assume M is a compact submanifold with parallel normalized mean curvature vector in $F^{n+p}(c)$ for $p > 1$. Since $trH_\alpha = 0$, we have $\sum_k h^\alpha_{kkij} = 0$ for $\alpha \neq n+1$. Then, we obtain from (6) that

$$\begin{aligned} \sum_{i,j,\alpha\neq n+1} h^\alpha_{ij} \Delta h^\alpha_{ij} = & \sum_{i,j,k,m,\alpha\neq n+1} (h^\alpha_{ij} h^\alpha_{km} R_{mijk} + h^\alpha_{ij} h^\alpha_{mi} R_{mkjk}) \\ & - \sum_{i,j,k,\alpha,\beta\neq n+1} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk}. \end{aligned} \tag{10}$$

We obtain from (1) and (2) that

$$\begin{aligned} & \sum_{i,j,k,m,\alpha\neq n+1} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{i,j,k,m,\alpha\neq n+1} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} \\ = \; & ncS_I + \sum_{\alpha\neq n+1,\beta} trH_\beta \cdot tr(H^2_\alpha H_\beta) - \sum_{\alpha\neq n+1,\beta} [tr(H_\alpha H_\beta)]^2 \\ & - \sum_{\beta,\alpha\neq n+1} [tr(H^2_\alpha H^2_\beta) - tr(H_\alpha H_\beta)^2], \end{aligned}$$

and

$$\sum_{i,j,k,\alpha,\beta\neq n+1} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk} = \sum_{\alpha,\beta\neq n+1} [tr(H^2_\alpha H^2_\beta) - tr(H_\alpha H_\beta)^2].$$

For submanifolds with parallel normalized mean curvature, $H_{n+1} H_\alpha = H_\alpha H_{n+1}$ for any α. $(tr(H_\alpha H_\beta))$ is a symmetric $(p-1) \times (p-1)$-matrix for $\alpha, \beta \neq n+1$. Then, we choose the normal vector fields $\{e_\alpha\}_{\alpha\neq n+1}$ such that

$$tr(H_\alpha H_\beta) = trH^2_\alpha \cdot \delta_{\alpha\beta}.$$

This implies

$$\sum_{\alpha,\beta\neq n+1} [tr(H_\alpha H_\beta)]^2 = \sum_{\alpha\neq n+1} tr(H^2_\alpha)^2. \tag{11}$$

Then, we have

$$\begin{aligned} \sum_{i,j,\alpha\neq n+1} h^\alpha_{ij} \Delta h^\alpha_{ij} = & \; ncS_I - 2\sum_{\alpha,\beta\neq n+1} [tr(H^2_\alpha H^2_\beta) - tr(H_\alpha H_\beta)^2] - \sum_{\alpha\neq n+1} (trH^2_\alpha)^2 \\ & + \sum_{\alpha\neq n+1} tr(H^2_\alpha H_{n+1}) \cdot trH_{n+1} - \sum_{\alpha\neq n+1} [tr(H_\alpha H_{n+1})]^2. \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned}\frac{1}{2}\triangle S_I &= \sum_{i,j,k,\alpha\neq n+1}(h_{ijk}^{\alpha})^2+\sum_{i,j,\alpha\neq n+1}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \\ &= X_1+Y_1,\end{aligned} \tag{13}$$

where

$$\begin{aligned}X_1 &:= -\sum_{\alpha,\beta\neq n+1}N(H_\alpha H_\beta-H_\beta H_\alpha)-\sum_{\alpha\neq n+1}(trH_\alpha^2)^2 \\ &\quad +\sum_{\alpha\neq n+1}tr(H_\alpha^2H_{n+1})\cdot trH_{n+1}-\sum_{\alpha\neq n+1}[tr(H_\alpha H_{n+1})]^2+ncS_I, \\ Y_1 &:= \sum_{i,j,k,\alpha\neq n+1}(h_{ijk}^{\alpha})^2.\end{aligned}$$

Lemma 4. $X_1\geq nS_I[Ric_{\min}-(n-2)(c+H^2)]-sgn(p-2)\frac{n-1}{2(n-2)}S_I^2.$

Proof. Since $H_{n+1}H_\alpha=H_\alpha H_{n+1}$, H_{n+1} and H_α can be simultaneously diagonalized for every fixed α:

(i) If $p=2$, we choose $\{e_i\}$ such that H_{n+2} is a diagonal matrix, i.e., $h_{ij}^{n+2}=0$ for $i\neq j$. Then, it can be seen from (3) that

$$(n-2)H\left(h_{ii}^{n+1}-H\right)\geq Ric_{\min}-(n-1)\left(c+H^2\right)+\left(h_{ii}^{n+1}-H\right)^2+\left(h_{ii}^{n+2}\right)^2.$$

Hence, we obtain

$$\begin{aligned}& tr\left(H_{n+2}^2H_{n+1}\right)\cdot trH_{n+1}-[tr(H_{n+2}H_{n+1})]^2 \\ =\ & -[tr(H_{n+1}-HI)H_{n+2}]^2+nHtr\left[(H_{n+1}-HI)H_{n+2}^2\right]+nH^2S_I \\ =\ & nH\sum_i\left(h_{ii}^{n+1}-H\right)\left(h_{ii}^{n+2}\right)^2-\left[\sum_i\left(h_{ii}^{n+1}-H\right)\left(h_{ii}^{n+2}\right)\right]^2+nH^2S_I \\ \geq\ & \frac{n}{n-2}[Ric_{\min}-(n-1)(c+H^2)]S_I+\frac{1}{n-2}[\left(\sum_i(h_{ii}^{n+1}-H)h_{ii}^{n+2}\right)^2 \\ & +(trH_{n+2}^2)^2]-[\sum_i(h_{ii}^{n+1}-H)h_{ii}^{n+2}]^2+nH^2S_I \\ \geq\ & \frac{n}{n-2}[Ric_{\min}-(n-1)(c+H^2)]S_I \\ & +\frac{1}{n-2}(trH_{n+2}^2)^2-\frac{n-3}{n-2}(S_H-nH^2)S_I+nH^2S_I,\end{aligned} \tag{14}$$

where I is the unit $(n\times n)$-matrix. Then, it follows from (4) and (14) that

$$\begin{aligned}X_1 &\geq \frac{n}{n-2}[Ric_{\min}-(n-1)(c+H^2)]S_I \\ &\quad -\frac{n-3}{n-2}(trH_{n+2}^2)^2-\frac{n-3}{n-2}(S_H-nH^2)S_I+nH^2S_I+ncS_I \\ &\geq \frac{n}{n-2}[Ric_{\min}-(n-1)(c+H^2)]S_I \\ &\quad -\frac{n-3}{n-2}(S-nH^2)S_I+nH^2S_I+ncS_I \\ &\geq nS_I[Ric_{\min}-(n-2)(c+H^2)].\end{aligned}$$

(ii) If $p \geq 3$, for a fixed α, let $\{e_i\}$ be a frame diagonalizing the matrix H_α such that $h_{ij}^\alpha = 0$ for $i \neq j$. So we observe that these terms can be written as follows:

$$\begin{aligned} & \sum_{\alpha\neq n+1} tr(H_\alpha^2 H_{n+1}) \cdot trH_{n+1} - \sum_{\alpha\neq n+1} [tr(H_\alpha H_{n+1})]^2 \\ = \ & nH \sum_{\alpha\neq n+1} [tr(H_{n+1} - HI)H_\alpha^2] - \sum_{\alpha\neq n+1} [tr(H_{n+1} - HI)H_\alpha]^2 + nH^2 S_I \\ = \ & nH \sum_{\alpha\neq n+1} [\sum_i (h_{ii}^{n+1} - H)(h_{ii}^\alpha)^2] - \sum_{\alpha\neq n+1} [\sum_i (h_{ii}^{n+1} - H)h_{ii}^\alpha]^2 + nH^2 S_I. \end{aligned} \tag{15}$$

We also obtain from (3) that

$$(n-1)(c+H^2) + (n-2)H(h_{ii}^{n+1} - H) - (h_{ii}^{n+1} - H)^2 - (h_{ii}^\alpha)^2 - \sum_{j,\beta\neq\alpha,n+1} (h_{ij}^\beta)^2 \geq Ric_{\min}.$$

This implies that

$$\begin{aligned} nH\sum_i \left(h_{ii}^{n+1} - H\right)(h_{ii}^\alpha)^2 \ \geq \ & \frac{n}{n-2}\sum_i \left(h_{ii}^{n+1} - H\right)^2 (h_{ii}^\alpha)^2 + \frac{n}{n-2}\sum_i (h_{ii}^\alpha)^4 \\ & + \frac{n}{n-2} \sum_{\substack{\beta\neq\alpha,n+1 \\ i,j}} \left(h_{ij}^\beta\right)^2 (h_{ii}^\alpha)^2 \\ & + \frac{n}{n-2}\left[Ric_{\min} - (n-1)\left(c+H^2\right)\right] trH_\alpha^2 \\ \geq \ & \frac{1}{n-2}\left[\sum_i \left(h_{ii}^{n+1} - H\right)(h_{ii}^\alpha)\right]^2 + \frac{1}{n-2}\left(trH_\alpha^2\right)^2 \\ & + \frac{n}{n-2}\left[Ric_{\min} - (n-1)\left(c+H^2\right)\right] trH_\alpha^2. \end{aligned} \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned} & - \sum_{\beta,\alpha\neq n+1} N\left(H_\alpha H_\beta - H_\beta H_\alpha\right) - \sum_{\alpha\neq n+1} \left(trH_\alpha^2\right)^2 \\ & \quad + \sum_{\alpha\neq n+1} tr\left(H_\alpha^2 H_{n+1}\right) \cdot trH_{n+1} - \sum_{\alpha\neq n+1} [tr(H_\alpha H_{n+1})]^2 \\ \geq \ & - \sum_{\beta,\alpha\neq n+1} N\left(H_\alpha H_\beta - H_\beta H_\alpha\right) - \sum_{\alpha\neq n+1} \left(trH_\alpha^2\right)^2 + nH \sum_{\alpha\neq n+1} \sum_i \left(h_{ii}^{n+1} - H\right)(h_{ii}^\alpha)^2 \\ & - \sum_{\alpha\neq n+1} \left[\sum_i \left(h_{ii}^{n+1} - H\right) h_{ii}^\alpha\right]^2 + nH^2 S_I \\ \geq \ & -\left(\frac{1}{n-2} + \frac{n-3}{n-2}\right) \sum_{\beta,\alpha\neq n+1} N\left(H_\alpha H_\beta - H_\beta H_\alpha\right) - \frac{n-3}{n-2} \sum_{\alpha\neq n+1} \left(trH_\alpha^2\right)^2 + nH^2 S_I \\ & - \frac{n-3}{n-2} \sum_{\alpha\neq n+1} \left[\sum_i \left(h_{ii}^{n+1} - H\right)(h_{ii}^\alpha)\right]^2 + \frac{n}{n-2}\left[Ric_{\min} - (n-1)\left(c+H^2\right)\right] S_I. \end{aligned} \tag{17}$$

It follows from Lemmas 2 and 3 that

$$\sum_{\beta,\alpha\neq n+1} N\left(H_\alpha H_\beta - H_\beta H_\alpha\right) - \sum_{\alpha\neq n+1} \left(trH_\alpha^2\right)^2 \leq \frac{3}{2} S_I^2,$$

and

$$\sum_{\beta,\alpha\neq n+1} N\left(H_\alpha H_\beta - H_\beta H_\alpha\right) \leq S_I^2.$$

These together with (4) and (17) imply that

$$\begin{aligned} X_1 &\geq S_I\Big\{n(c+H^2) - \frac{n-3}{n-2}\Big(S_H - nH^2\Big) + \frac{n}{n-2}\Big[Ric_{\min} - (n-1)\Big(c+H^2\Big)\Big]\Big\} \\ &\quad -\Big[\frac{1}{n-2} + \frac{3(n-3)}{2(n-2)}\Big]S_I^2 \\ &\geq nS_I\Big[Ric_{\min} - (n-2)(c+H^2)\Big] + \frac{n-1}{2(n-2)}S_I^2. \end{aligned} \tag{18}$$

This proves the lemma. □

Theorem 7. *If $M^n(n \geq 3)$ is an oriented compact submanifold with parallel normalized mean curvature in the space forms $F^{n+p}(c)(p \geq 2)$, then*

$$\int_M \Big\{nS_I[Ric_{\min} - (n-2)(c+H^2) - sgn(p-2)\frac{n-1}{2(n-2)}S_I^2\Big\}dM \leq 0.$$

Proof. Since M has parallel normalized mean curvature, $H_{n+1}H_\alpha = H_\alpha H_{n+1}$. This implies that $Y_1 = \sum_{i,j,k,\alpha \neq n+1}(h_{ijk}^\alpha)^2 \geq 0$. Then, it follows from Lemma 4 that

$$\frac{1}{2}\triangle S_I = X_1 + Y_1 \geq nS_I[Ric_{\min} - (n-2)(c+H^2)] - sgn(p-2)\frac{n-1}{2(n-2)}S_I^2. \tag{19}$$

Hence,

$$\begin{aligned} 0 &= \frac{1}{2}\int_M \triangle S_I dM \\ &\geq \int_M \Big\{nS_I[Ric_{\min} - (n-2)(c+H^2)] - sgn(p-2)\frac{n-1}{2(n-2)}S_I^2\Big\}dM. \end{aligned}$$

This proves Theorem 7. □

Now, we are in the position to prove Theorem 5.

Proof of Theorem 5. It follows from (4) that

$$S_I \leq S - nH^2 \leq n[(n-1)(c+H^2) - Ric_{\min}]. \tag{20}$$

This together with (19) implies that

$$\begin{aligned} &\frac{1}{2}\triangle S_I \\ &\geq nS_I[Ric_{\min} - (n-2)(c+H^2) + sgn(p-2)\frac{n-1}{2(n-2)}(Ric_{\min} - (n-1)(c+H^2))] \\ &\geq nS_I\Big[1 + sgn(p-2)\frac{n-1}{2(n-2)}\Big] \times \Big[Ric_{\min} - \Big(n-2+sgn(p-2)\frac{n-1}{3n-5}\Big)(c+H^2)\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= \frac{1}{2}\int_M \triangle S_I dM \\ &\geq \int_M \Big\{nS_I\Big[1 + sgn(p-2)\frac{n-1}{2(n-2)}\Big] \\ &\quad \times \Big[Ric_{\min} - \Big(n-2+sgn(p-2)\frac{n-1}{3n-5}\Big)\Big(c+H^2\Big)\Big]\Big\}dM. \end{aligned}$$

If $Ric_{\min} > \left[n-2+sgn(p-2)\frac{n-1}{3n-5}\right](c+H^2)$, then $S_I = 0$. It follows from a theorem due to Erbacher [23] that M lies in the totally geodesic submanifold $F^{n+1}(c)$ of $F^{n+p}(c)$. This proves Theorem 5. □

4. Submanifolds with Constant Scalar Curvature

In this section, we assume M is an oriented compact submanifold with constant normalized scalar curvature R and parallel normalized mean curvature vector in the space form $F^{n+p}(c)$. Since $trH_{n+1} = nH$, we obtain from (5) that

$$\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} = \sum_{i,j} nH_{ij}h_{ij}^{n+1} + \sum_{i,j,k,m}(h_{ij}^{n+1}h_{mk}^{n+1}R_{mijk} + h_{ij}^{n+1}h_{im}^{n+1}R_{mkjk}). \tag{21}$$

Applying (1) and (2), we obtain

$$\begin{aligned}
&\sum_{i,j,k,m} h_{ij}^{n+1}h_{km}^{n+1}R_{mijk} + \sum_{i,j,k,m} h_{ij}^{n+1}h_{mi}^{n+1}R_{mkjk} \\
= \;& nc(S_H - nH^2) + \sum_\alpha trH_\alpha \cdot tr(H_{n+1}^2 H_\alpha) - \sum_\alpha [tr(H_{n+1}H_\alpha)]^2 \\
& - \sum_\alpha [tr(H_{n+1}^2H_\alpha^2) - tr(H_{n+1}H_\alpha)^2].
\end{aligned}$$

Since $H_{n+1}H_\alpha = H_\alpha H_{n+1}$ for any α,

$$\begin{aligned}
\frac{1}{2}\triangle S_H &= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}\triangle h_{ij}^{n+1} \\
&= X_2 + Y_2,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
X_2 &:= nHtrH_{n+1}^3 - (trH_{n+1}^2)^2 - \sum_{\alpha\neq n+1}[tr(H_{n+1}H_\alpha)]^2 + nc\Big(S_H - nH^2\Big), \\
Y_2 &:= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} nH_{ij}h_{ij}^{n+1}.
\end{aligned}$$

Lemma 5. $X_2 \geq n\big(S_H - nH^2\big)\left[Ric_{min} - (n-2)(c+H^2)\right]$.

Proof. We choose the orthonormal frame fields $\{e_i\}$ such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$. Let

$$\begin{aligned}
&f_k = \sum_i(\lambda_i^{n+1})^k,\ \mu_i^{n+1} = H - \lambda_i^{n+1},\ i = 1,2,\ldots,n, \\
&B_k = \sum_i(\mu_i^{n+1})^k,
\end{aligned}$$

and we have

$$B_1 = 0,\ B_2 = S_H - nH^2,\ B_3 = 3HS_H - 2nH^3 - f_3.$$

Then

$$\begin{aligned}
X_2 &= -S_H^2 + nHf_3 - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^\alpha\Big)^2 + nc\Big(S_H - nH^2\Big) \\
&= -S_H^2 + nH(3HS_H - 2nH^3 - B_3) - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^\alpha\Big)^2 \\
&\quad + nc\Big(S_H - nH^2\Big) \\
&= B_2[nc + 2nH^2 - S_H] - nHB_3 - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^\alpha\Big)^2.
\end{aligned} \tag{23}$$

According to Equation (3) in space form $F^{n+p}(c)$, we have

$$(n-1)c + nH\lambda_i^{n+1} - (\lambda_i^{n+1})^2 - \sum_{\alpha\neq n+1,j}(h_{ij}^\alpha)^2 = Ric(e_i) \geq Ric_{\min},$$

$$S - nH^2 \leq n[(n-1)(c+H^2) - Ric_{\min}], \tag{24}$$

and

$$(n-2)H(\lambda_i^{n+1} - H) - (\lambda_i^{n+1} - H)^2 + (n-1)(c+H^2) - \sum_{\alpha\neq n+1,j}(h_{ij}^\alpha)^2 - Ric_{\min} \geq 0,$$

from which it can be deduced that

$$H(\lambda_i^{n+1} - H) \geq \frac{(\lambda_i^{n+1} - H)^2}{n-2} + \frac{\sum_{\alpha\neq n+1,j}(h_{ij}^\alpha)^2}{n-2} + \frac{Ric_{\min}}{n-2} - \frac{n-1}{n-2}(c+H^2).$$

So,

$$\begin{aligned} -nHB_3 \quad \geq \quad & \frac{n}{n-2}\sum_i \left(\mu_i^{n+1}\right)^4 + \frac{n}{n-2}\sum_{\alpha\neq n+1}\sum_{i,j}\left(h_{ij}^\alpha\right)^2\left(\mu_i^{n+1}\right)^2 \\ & + \frac{n}{n-2}\left[Ric_{\min} - (n-1)\left(c+H^2\right)\right]B_2. \end{aligned}$$

This together with (23) and (24) implies that

$$\begin{aligned} X_2 \quad \geq \quad & B_2\left\{nc + 2nH^2 - S_H + \frac{n}{n-2}[Ric_{\min} - (n-1)(c+H^2)]\right\} \\ & + \frac{n}{n-2}\sum_i(\mu_i^{n+1})^4 + \sum_{\alpha\neq n+1}\left[\frac{n}{n-2}\sum_i(h_{ii}^\alpha)^2(\mu_i^{n+1})^2 - \left(\sum_i \mu_i^{n+1}h_{ii}^\alpha\right)^2\right] \\ \geq \quad & B_2\left\{nc + 2nH^2 - S_H + \frac{n}{n-2}[Ric_{\min} - (n-1)(c+H^2)]\right\} \\ & \frac{B_2^2}{n-2} - \frac{n-3}{n-2}\sum_{\alpha\neq n+1}\left(\sum_i \mu_i^{n+1}h_{ii}^\alpha\right)^2 \\ \geq \quad & B_2\left\{nc + nH^2 - \frac{n-3}{n-2}(S - nH^2)\right. \\ & \left. + \frac{n}{n-2}[Ric_{\min} - (n-1)(c+H^2)]\right\} \\ \geq \quad & \frac{n}{n-2}B_2\Big\{(n-2)(c+H^2) \\ & -(n-3)[(n-1)(c+H^2) - Ric_{\min}] + [Ric_{\min} - (n-1)(c+H^2)]\Big\} \\ = \quad & n(S_H - nH^2)[Ric_{\min} - (n-2)(c+H^2)]. \end{aligned} \tag{25}$$

This completes the proof of the lemma. □

Proof of Theorem 6. Since the normalized scalar curvature R is constant, we obtain from (4) that

$$n^2\Delta H^2 = 2n^2 H\Delta H + 2n^2|\nabla H|^2 = \Delta S. \tag{26}$$

On the other hand, we obtain from (13) and (22) that

$$\frac{1}{2}\Delta S = |\nabla h|^2 + \sum_{i,j,} nH_{ij}^{n+1}h_{ij}^{n+1} + X_1 + X_2. \tag{27}$$

This together with (7) implies that

$$\begin{aligned}\Box(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i,j} nh^{n+1}_{ij}H_{ij} \\ &= (|\nabla h|^2 - n^2|\nabla H|^2) + X_1 + X_2.\end{aligned} \tag{28}$$

Therefore, we obtain from Lemmas 1, 4 and 5 that

$$\begin{aligned}\Box(nH) &\geq n\Big(S - nH^2\Big)\Big[Ric_{\min} - (n-2)\Big(c + H^2\Big)\Big] - sgn(p-2)\frac{n-1}{2(n-2)}S_I^2 \\ &\geq n\Big(S - nH^2\Big)\Big[Ric_{\min} - (n-2)\Big(c + H^2\Big) - \frac{sgn(p-2)(n-1)}{2n(n-2)}\Big(S - nH^2\Big)\Big] \\ &\geq n\Big(S - nH^2\Big)\Big\{Ric_{\min} - (n-2)\Big(c + H^2\Big) \\ &\quad + sgn(p-2)\frac{n-1}{2(n-2)}\Big[Ric_{\min} - (n-1)\Big(c + H^2\Big)\Big]\Big\} \\ &= n\Big(S - nH^2\Big)\Big[1 + sgn(p-2)\frac{n-1}{2(n-2)}\Big] \\ &\quad \times \Big[Ric_{\min} - \Big(n - 2 + sgn(p-2)\frac{n-1}{3n-5}\Big)\Big(c + H^2\Big)\Big].\end{aligned} \tag{29}$$

Since the operator $\Box$ is self-adjoint, we conclude

$$\begin{aligned}0 \geq &\int_M n(S - nH^2)\Big[1 + sgn(p-2)\frac{n-1}{2(n-2)}\Big] \\ &\times \Big[Ric_{\min} - \Big(n - 2 + sgn(p-2)\frac{n-1}{3n-5}\Big)\Big(c + H^2\Big)\Big].\end{aligned}$$

Thus, we obtain from the assumption $Ric_M > (n - 2 + sgn(p-2)\frac{n-1}{3n-5})(c + H^2)$ that

$$S - nH^2 = 0.$$

This means M must be the totally umbilcal sphere $S^n(\frac{1}{\sqrt{c+H^2}})$. This proves Theorem 6. $\Box$

5. Discussion

The following example shows the pinching constant is the best possible in even dimensions and $p = 2$.

Example 1. *Let $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ be the totally umbilic sphere in $F^{n+p}(c)$ with $c + H^2 > 0$. Here the mean curvature H is a constant.*

Let $M = S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}})$ be a Clifford hypersurface in $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$. Then, M is a compact submanifold in $F^{n+2}(c)$ with parallel normalized mean curvature vector, the Ricci curvature $Ric_M \equiv (n-2)(c + H^2)$, and the normalized scalar curvature $R = \frac{n-2}{n-1}(c + H^2)$.

More generally, M is also a submanifold in $F^{n+p}(c)$ with parallel normalized mean curvature vector for $p \geq 3$, and the Ricci curvature of M satisfies $Ric_M \equiv (n-2)(c + H^2)$.

For $n = 4$ and $p \geq 4$, we have the following example.

Example 2. *Let $S^7(\frac{1}{\sqrt{c+H^2}})$ be the totally umbilic sphere in $F^{4+p}(c)$ with $c + H^2 > 0$. Here the mean curvature H is a constant. Let $\mathbb{C}P^2(\frac{4}{3}(c + H^2))$ be the two-dimensional complex projective space minimally immersed in $S^7(\frac{1}{\sqrt{c+H^2}})$ with constant holomorphic sectional curvature $\frac{4}{3}(c + H^2)$. Then, M is a compact submanifold in $F^{4+p}(c)$ with parallel normalized mean curvature vector for $p \geq 4$, the Ricci curvature $Ric_M \equiv 2(c + H^2)$, and the normalized scalar curvature $R = \frac{2}{3}(c + H^2)$.*

Motivated by Theorem 6, Examples 1 and 2, we propose the following conjecture.

Conjecture 1. *Let M be an $n(\geq 3)$-dimensional oriented compact submanifold with constant normalized scalar curvature R and parallel normalized mean curvature vector in the space form $F^{n+p}(c)$. If $R \geq c$, and*

$$Ric_M > (n-2)(c + H^2) > 0,$$

then M must be the totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$.

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