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Stability Analysis of Linear Time-Varying Delay Systems via a Novel Augmented Variable Approach

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Abstract: This paper investigates the stability issues of time-varying delay systems. Firstly, a novel augmented Lyapunov functional is constructed for a class of bounded time-varying delays by introducing new double integral terms. Subsequently, a time-varying matrix-dependent zero equation is introduced to relax the constraints of traditional constant matrix-dependent zero equations. Secondly, for a class of periodic time-varying delays, considering the monotonicity of the delay and combining it with an augmented variable approach, Lyapunov functionals are constructed for monotonically increasing and monotonically decreasing delay intervals, respectively. Based on the constructed augmented Lyapunov functionals and the employed time-varying zero equation, less conservative stability criteria are obtained separately for bounded and periodic time-varying delays. Lastly, three examples are used to verify the superiority of the stability conditions obtained in this paper.

Keywords: stability analysis; augmented variable; time-delay systems; time-varying delay

MSC: 93D05

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1. Introduction

The time delay phenomenon is ubiquitous in control systems, stemming from the nature of real-world control environments where the transmission, processing, and execution of information necessitate a specific duration, thereby inducing a temporal lag between inputs and outputs [1]. This delay, commonly called a time delay, is observable in various systems, including networked control systems, electronic and biological systems, and economic models. Within these systems, time delays affect dynamic characteristics such as the stability and response speed and can lead to a spectrum of complex behaviors, including oscillations, instability, and even chaos. Even a small delay may have a great impact on the performance and security of systems that are sensitive to time delays. Therefore, conducting stability analyses for delayed systems is crucial. This analysis helps pre-emptively predict and mitigate potential issues caused by time delays and provides crucial insights for developing effective control strategies and optimizing system designs. Currently, research on time-delay systems has achieved numerous advancements and breakthroughs, laying a solid foundation for addressing more complex time-delay issues and paving new paths for future technological development and innovation [2–10].

One of the most commonly used approaches in the stability analysis of delay systems is the Lyapunov functional method, which is primarily characterized by the construction of specific functionals to analyze a system's stability. These functionals are generally non-negative real-valued functions closely related to the system's state and are primarily employed to quantify the system's energy or level of stability. There is no widely applicable Lyapunov functional construction framework, which that urges researchers to explore and develop new functional construction approaches to reduce the conservatism of stability conditions. Examples of such approaches are the piecewise Lyapunov functional [11,12], the augmented Lyapunov functional [13,14], the delay-product-type Lyapunov functional [15] and the time-varying Lyapunov functional [16] methods. Ding et al. [17] constructed a new delay-partitioning Lyapunov functional to study the stability issues of neutral-type delay systems. In [18], the stability of linear systems with differentiable time-varying delays was studied using an auxiliary equation-based method. Among these functional construction methods, augmented functionals and delay-product-type functionals are widely used because they can capture more effective system information or delay information. However, when augmented functional methods are combined with delay-product-type functional methods, the functional derivative is likely of a quadratic form with delayed higher-order terms. Such nonlinear high-order delay terms cannot be directly solved with the help of linear matrix inequality tools. As a result, researchers have developed some methods to determine the high-order delay inequality, such as second-order delay inequality determination methods [19–22], third-order delay inequality determination methods [23,24], and n-th $(n \geq 2)$ order delay inequality determination methods [25]. Although these determination methods effectively address the problem of determining high-order inequalities, they may introduce an additional manual computational complexity or numerous redundant decision variables. Fortunately, it has been revealed in [26] that generating high-order time-delay terms can be avoided by augmenting additional variables. This eliminates the complicated calculation process of transforming high-order delay inequalities into linear delay inequalities. Since the augmented variable method can avoid high-order delay terms, some new augmented terms can be introduced to improve traditional augmented Lyapunov functionals based on this method, which motivates the research in this article.

Researchers are also focusing on improving the accuracy of the integral term estimates in Lyapunov functional derivatives to further lower the conservatism in the stability determination criteria of delay systems. Many inequalities for integral term estimation have been developed, such as the Wirtinger-based inequality [27], the Jensen inequality [28], the Bessel–Legendre inequality [29], the reciprocally convex inequality [30–33] and the free-matrix-based inequality [34]. These advanced integral inequality methods improve the accuracy of stability analyses and broaden the application of Lyapunov functionals in complex time-delay systems. Some integral inequalities are used to construct complex Lyapunov functionals or to relax the positive definiteness requirement of Lyapunov functionals. Therefore, in the field of stability analyses for delay systems, the development of more effective Lyapunov functionals and the enhancement in the accuracy of integral term estimates have emerged as two crucial and urgent directions for improvement, underscoring the relevance and importance of our research.

Based on the above discussion, this paper aims to improve the stability determination criteria from a functional construction perspective. We will primarily analyze two types of delay systems: those with a class of bounded time-varying delays and those with periodic time-varying delays. The augmented variable and delay-product-type methods are used to construct the Lyapunov functional. The relationship between augmented and traditional variables is established using the time-varying matrix dependence zero equation. Based on this, less conservative stability criteria are derived for these two time-varying delay cases. Finally, three numerical examples verify the advantages of the constructed Lyapunov functional.

Throughout this paper, $\mathbb{R}^n$ represents the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n_+$ are the set of $n \times m$ real matrices and of $n \times n$ symmetric positive definite matrices, respectively; $\mathbb{N}$ represents a set of positive integers; diag$\{\cdots\}$ is a block diagonal matrices; and $\mathrm{Sym}\{H\} = H + H^T$.

2. Problem Statement and Preliminaries

This paper considers linear systems with time-varying state delays, which are described as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-\ell(t)) \\ x(t) = \phi(t), t \in [-d_M, 0] \end{cases} \tag{1}$$

where A and A_d are system matrices and $\phi(t)$, $\ell(t)$ and $x(t)$ are the initial condition, time-varying delay, and system state, respectively.

This article aims to establish a sufficient stability condition for linear systems with delays, aiming to maximize the stability region of the delays. Before unveiling our main findings, let us first introduce a lemma that is fundamental to developing these primary results.

Lemma 1 ([34]). *Define differentiable function χ: $[\lambda_1, \lambda_2] \to \mathbb{R}^n$ and $\bar{\xi} \in \mathbb{R}^m$. For a matrix $\mathbf{E} \in \mathbb{S}_+^n$ and any matrix $M \in \mathbb{R}^{3n\times m}$, inequality (2) holds:*

$$-\int_{\lambda_1}^{\lambda_2} \dot{\chi}^T(v)\mathbf{E}\dot{\chi}(v)\mathrm{d}v \leq\ 2\tilde{\xi}^T\tilde{\Pi}^T M\bar{\xi} + (\lambda_2-\lambda_1)\bar{\xi}^T M^T \tilde{\mathbf{E}}^{-1} M\bar{\xi} \tag{2}$$

where

$$\begin{aligned} \tilde{\xi} =& \left[\chi^T(\lambda_2),\ \chi^T(\lambda_1),\ \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\chi^T(s)\mathrm{d}s,\ \frac{1}{(\lambda_2-\lambda_1)^2}\int_{\lambda_1}^{\lambda_2}\int_{\theta}^{\lambda_2}\chi^T(s)\mathrm{d}s\mathrm{d}\theta\right]^T, \\ \tilde{\Pi} =& [\tilde{g}_1^T - \tilde{g}_2^T\ \ \tilde{g}_1^T + \tilde{g}_2^T - 2\tilde{g}_3^T\ \ \tilde{g}_1^T - \tilde{g}_2^T + 6\tilde{g}_3^T - 12\tilde{g}_4^T]^T, \\ \tilde{\mathbf{E}} =& diag\{\mathbf{E}, 3\mathbf{E}, 5\mathbf{E}\}, \\ \tilde{g}_\kappa =& [0_{n\times(\kappa-1)n}\ I_n\ 0_{n\times(4-\kappa)n}],\ \ \kappa = 1, 2, \ldots, 4. \end{aligned}$$

3. Main Results

3.1. A Class of Bounded Time-Varying Delays

In this subsection, we investigate the stability issues of systems with bounded time-varying delays. It is crucial to emphasize that our research, while unable to determine the specific characteristics of the delay, has rigorously defined upper and lower bounds for the delays and their derivatives considered in this paper. These bounds are based on the following assumptions:

$$0 \leq \ell(t) \leq d_M,\ \ -\mu \leq \dot{\ell}(t) \leq \mu \tag{3}$$

where μ and d_M are real numbers.

Next, some stability conditions will be obtained for system (1) meeting delay condition (3). First, the following simplified symbols are given.

$$\begin{aligned} \rho_0(t) &= \left[x^T(t)\ \ x^T(t-\ell(t))\ \ x^T(t-d_M)\right]^T \\ \rho_1(t) &= \left[\int_{t-\ell(t)}^{t} x^T(\varsigma)d\varsigma\ \ \frac{1}{\ell(t)}\int_{t-\ell(t)}^{t}\int_{s}^{t} x^T(\varsigma)d\varsigma ds\right]^T \\ \rho_2(t) &= \left[\int_{t-d_M}^{t-\ell(t)} x^T(\varsigma)d\varsigma\ \ \frac{1}{d_M-\ell(t)}\int_{t-d_M}^{t-\ell(t)}\int_{s}^{t-\ell(t)} x^T(\varsigma)d\varsigma ds\right]^T \\ \rho_3(t) &= \left[\int_{t-\ell(t)}^{t}\int_{s}^{t} x^T(\varsigma)d\varsigma ds\ \ \int_{t-d_M}^{t-\ell(t)}\int_{s}^{t-\ell(t)} x^T(\varsigma)d\varsigma ds\right]^T \\ \varphi_1(t) &= \left[\rho_0^T(t)\ \ \rho_1^T(t)\ \ \rho_2^T(t)\ \ \rho_3^T(t)\right]^T,\ \ \varphi_2(t) = \left[x^T(t)\ \ \dot{x}^T(t)\right]^T. \end{aligned}$$

Theorem 1. *For given delay parameters μ and d_M, linear system (1) with delays satisfying boundary restriction condition (3) is stable if there exist some positive definite matrices, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{Q}_1$, $\mathcal{Q}_2$;*

symmetric matrices, $\mathcal{P}_0$, $\mathcal{P}_1$*, satisfying* $\mathcal{P}_0 + \ell(t)\mathcal{P}_1 > 0$*; and arbitrary matrices* $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{N}_1$, $\mathcal{N}_2$*, such that* (4) *and* (5) *are feasible:*

$$\begin{bmatrix} \Xi(0, \dot{\ell}(t)) & d_M \mathcal{N}_2^T \\ * & -d_M \hat{\mathcal{R}}_2 \end{bmatrix} < 0 \tag{4}$$

$$\begin{bmatrix} \Xi(d_M, \dot{\ell}(t)) & d_M \mathcal{N}_1^T \\ * & -d_M \hat{\mathcal{R}}_1 \end{bmatrix} < 0 \tag{5}$$

where

$$\begin{aligned}
\Xi(\ell(t), \dot{\ell}(t)) &= \Xi_0(\ell(t), \dot{\ell}(t)) + \Xi_1(\ell(t), \dot{\ell}(t)) \\
\Xi_0(\ell(t), \dot{\ell}(t)) &= Sym\{\mathcal{D}_1^T(\mathcal{P}_0 + \ell(t)\mathcal{P}_1)\lambda_1\} + \mathcal{D}_2^T \mathcal{Q}_1 \mathcal{D}_2 + \dot{\ell}(t)\mathcal{D}_1^T \mathcal{P}_1 \mathcal{D}_1 \\
&\quad - (1 - \dot{\ell}(t))\mathcal{D}_3^T(\mathcal{Q}_1 - \mathcal{Q}_2)\mathcal{D}_3 - \mathcal{D}_4^T \mathcal{Q}_2 \mathcal{D}_4 \\
&\quad + d_M g_0^T \mathcal{R}_1 g_0 - (1 - \dot{\ell}(t))(d_M - \ell(t))g_4^T(\mathcal{R}_1 - \mathcal{R}_2)g_4 \\
\Xi_1(\ell(t), \dot{\ell}(t)) &= Sym\{\mathcal{L}_1^T \mathcal{N}_1 + \mathcal{L}_2^T \mathcal{N}_2 + (\mathcal{W}_1 + \dot{\ell}(t)\mathcal{W}_2)\Psi(\ell(t))\}
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{D}_1 =& [g_1^T,\ g_2^T,\ g_3^T,\ g_{10}^T,\ g_{11}^T,\ g_{12}^T,\ g_{13}^T,\ g_{14}^T,\ g_{15}^T]^T \\
\lambda_1 =& [g_0^T,\ (1 - \dot{\ell}(t))g_4^T,\ g_5^T,\ g_1^T - (1 - \dot{\ell}(t))g_2^T,\ g_1^T - (1 - \dot{\ell}(t))g_6^T - \dot{\ell}(t)g_7^T, \\
& (1 - \dot{\ell}(t))g_2^T - g_3^T,\ (1 - \dot{\ell}(t))g_2^T - g_8^T + \dot{\ell}(t)g_9^T,\ g_{16}^T - (1 - \dot{\ell}(t))g_{10}^T \\
& (1 - \dot{\ell}(t))g_{17}^T - g_{12}^T]^T \\
\mathcal{D}_2 =& [g_1^T,\ g_0^T]^T,\ \mathcal{D}_3 = [g_2^T,\ g_4^T]^T,\ \mathcal{D}_4 = [g_3^T,\ g_5^T]^T \\
\mathcal{L}_1 =& [g_1^T - g_2^T,\ g_1^T + g_2^T - 2g_6^T,\ g_1^T - g_2^T + 6g_6^T - 12g_7^T]^T \\
\mathcal{L}_2 =& [g_2^T - g_3^T,\ g_2^T + g_3^T - 2g_8^T,\ g_2^T - g_3^T + 6g_8^T - 12g_9^T]^T \\
\Psi(\ell(t)) =& [\ell(t)g_6^T - g_{10}^T\ \ \ell(t)g_7^T - g_{11}^T\ \ (d_M - \ell(t))g_8^T - g_{12}^T\ \ (d_M - \ell(t))g_9^T - g_{13}^T \\
& \ell(t)g_{11}^T - g_{14}^T\ \ (d_M - \ell(t))g_{13}^T - g_{15}^T\ \ \ell(t)g_1^T - g_{16}^T\ \ (d_M - \ell(t))g_2^T - g_{17}^T]^T \\
\hat{\mathcal{R}}_i =& diag\{\mathcal{R}_i, 3\mathcal{R}_i, 5\mathcal{R}_i\},\ \ g_0 = Ag_1 + A_d g_2.
\end{aligned}$$

Proof. On the basis of augmented variables, the functional is selected as:

$$\begin{aligned}
V(t) =& \varphi_1^T(t)(\mathcal{P}_0 + \ell(t)\mathcal{P}_1)\varphi_1(t) + \int_{t-\ell(t)}^{t} \varphi_2^{\mathrm{T}}(\varsigma)\mathcal{Q}_1\varphi_2(\varsigma)d\varsigma + \int_{t-d_M}^{t-\ell(t)} \varphi_2^{\mathrm{T}}(\varsigma)\mathcal{Q}_2\varphi_2(\varsigma)d\varsigma \\
&+ \int_{t-\ell(t)}^{t} (d_M - t + \varsigma)\dot{x}^T(\varsigma)\mathcal{R}_1\dot{x}(\varsigma)d\varsigma + \int_{t-d_M}^{t-\ell(t)} (d_M - t + \varsigma)\dot{x}^T(\varsigma)\mathcal{R}_2\dot{x}(\varsigma)d\varsigma
\end{aligned} \tag{6}$$

where $\mathcal{P}_0 + \ell(t)\mathcal{P}_1 > 0$, and $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ are positive definite symmetric matrices.

Derive $V(t)$ to obtain:

$$\begin{aligned}
\dot{V}(t) =& 2\varphi_1^T(t)(\mathcal{P}_0 + \ell(t)\mathcal{P}_1)\dot{\varphi}_1(t) + \dot{\ell}(t)\varphi_1^T(t)\mathcal{P}_1\varphi_1(t) + \varphi_2^{\mathrm{T}}(t)\mathcal{Q}_1\varphi_2(t) \\
&- (1 - \dot{\ell}(t))\varphi_2^{\mathrm{T}}(t - \ell(t))(\mathcal{Q}_1 - \mathcal{Q}_2)\varphi_2(t - \ell(t)) - \varphi_2^{\mathrm{T}}(t - d_M)\mathcal{Q}_2\varphi_2(t - d_M) \\
&- (1 - \dot{\ell}(t))(d_M - \ell(t))\dot{x}^T(t - \ell(t))(\mathcal{R}_1 - \mathcal{R}_2)\dot{x}(t - \ell(t)) \\
&+ d_M\dot{x}^T(t)\mathcal{R}_1\dot{x}(t) + \mathcal{J}_1 + \mathcal{J}_2,
\end{aligned} \tag{7}$$

where

$$\mathcal{J}_1 = -\int_{t-\ell(t)}^{t} \dot{x}^T(\varsigma)\mathcal{R}_1\dot{x}(\varsigma)d\varsigma,\ \ \mathcal{J}_2 = -\int_{t-d_M}^{t-\ell(t)} \dot{x}^T(\varsigma)\mathcal{R}_2\dot{x}(\varsigma)d\varsigma$$

Employing the inequality in (2), the integral terms $\mathcal{J}_1$ and $\mathcal{J}_2$ can be estimated as follows:

$$\begin{aligned}\mathcal{J}_1+\mathcal{J}_2\le\zeta^T(t)\{Sym\{\mathcal{L}_1^T\mathcal{N}_1+\mathcal{L}_2^T\mathcal{N}_2\}+\ell(t)\mathcal{N}_1^T\hat{\mathcal{R}}_1^{-1}\mathcal{N}_1\\+(d_M-\ell(t))\mathcal{N}_2^T\hat{\mathcal{R}}_2^{-1}\mathcal{N}_2\}\zeta(t)\end{aligned}\tag{8}$$

where $\mathcal{L}_1,\ \mathcal{L}_2$ are listed in Theorem 1, $\mathcal{N}_1,\ \mathcal{N}_2$ are arbitrary matrices with suitable dimensions, and the new augmented vector

$$\begin{aligned}\zeta(t)=\Big[&x^T(t)\ \ x^T(t-\ell(t))\ \ x^T(t-d_M)\ \ \dot{x}^T(t-\ell(t))\ \ \dot{x}^T(t-d_M)\ \ \frac{1}{\ell(t)}\rho_1^T(t)\\&\frac{1}{d_M-\ell(t)}\rho_2^T(t)\ \ \rho_1^T(t)\ \ \rho_2^T(t)\ \ \rho_3^T(t)\ \ \ell(t)x^T(t)\ \ (d_M-\ell(t))x^T(t-\ell(t))\Big]^T.\end{aligned}\tag{9}$$

Define the following vector

$$\begin{aligned}\Psi(\ell(t))=\Big[&\ell(t)g_6^T-g_{10}^T\ \ \ell(t)g_7^T-g_{11}^T\ \ (d_M-\ell(t))g_8^T-g_{12}^T\ \ (d_M-\ell(t))g_9^T-g_{13}^T\\&\ell(t)g_{11}^T-g_{14}^T\ \ (d_M-\ell(t))g_{13}^T-g_{15}^T\ \ \ell(t)g_1^T-g_{16}^T\ \ (d_M-\ell(t))g_2^T-g_{17}^T\Big]^T\end{aligned}\tag{10}$$

where $g_\kappa=\begin{bmatrix}0_{n\times(\kappa-1)n} & I_n & 0_{n\times(17-\kappa)n}\end{bmatrix}$, $\kappa=1,2,\ldots,17$.

Based on the defined augmented variable (9), it is known that the existence of matrices $\mathcal{W}_1$ and $\mathcal{W}_2$ with any suitable dimensions can make the zero Equation (11) hold. The following zero equality can be derived:

$$0=2\big((\mathcal{W}_1+\dot{\ell}(t)\mathcal{W}_2)\Psi(\ell(t))\big)\zeta(t).\tag{11}$$

On the basis of (7), (8) and (11), we have

$$\dot{V}(t)\le\zeta^T(t)\big(\Xi(\ell(t),\dot{\ell}(t))+\ell(t)\mathcal{N}_1^T\bar{\mathcal{R}}_1^{-1}\mathcal{N}_1+(d_M-\ell(t))\mathcal{N}_2^T\bar{\mathcal{R}}_2^{-1}\mathcal{N}_2\big)\zeta(t)\tag{12}$$

where $\Xi(\ell(t),\dot{\ell}(t))$ is defined in Theorem 1.

By applying the Schur complement lemma, if LMIs (4) and (5) are satisfied, it can be verified that $\Xi(\ell(t),\dot{\ell}(t))+\ell(t)\mathcal{N}_1^T\bar{\mathcal{R}}_1^{-1}\mathcal{N}_1+(d_M-\ell(t))\mathcal{N}_2^T\bar{\mathcal{R}}_2^{-1}\mathcal{N}_2<0$ holds when the time delay $0\le\ell(t)\le d_M$. Then, there exists a sufficient $\varsigma>0$ such that $\dot{V}(t)<-\varsigma|x(t)|^2$, which verifies that system (1) is asymptotically stable. □

Remark 1. *On the basis of the traditional augmented functional, this paper adds double integral terms $\int_{t-\ell(t)}^{t}\int_{s}^{t}x^T(v)dvds$ and $\int_{t-d_M}^{t-\ell(t)}\int_{s}^{t-\ell(t)}x^T(v)dvds$ to $\varphi_1(t)$, and then the new information of double integral terms can be considered. At the same time, the existence of a delay product matrix $(\mathcal{P}_0+\ell(t)\mathcal{P}_1)$ allows more effective delay cross information to be captured by the Lyapunov functional. Due to the presence of the delay product matrix $(\mathcal{P}_0+\ell(t)\mathcal{P}_1)$, defining the conventional $\zeta(t)$ would lead to the derived Lyapunov functional derivative being in quadratic form with a delayed higher order. Therefore, to represent the Lyapunov functional derivative in the form of a linear quadratic matrix, novel augmented variables $[\rho_3(t)\ \ \ell(t)x(t)\ \ (d_M-\ell(t))x(t-\ell(t))]$ are introduced into $\zeta(t)$, as shown in (9). As a result, the derived $\dot{V}(t)$ is a linear quadratic form of $\ell(t)$. This method not only simplifies the analysis process of system problems, but also provides a more efficient and accurate means for analyzing and controlling time-delay systems. Especially when dealing with complex dynamic systems, this method can significantly reduce the computational complexity and improve the operability of mathematical models.*

Remark 2. *Inspired by the literature [35–37], this paper adopts a time-varying matrix $\mathcal{W}_1+\dot{\ell}(t)\mathcal{W}_2$ to link the new variables introduced in $\zeta(t)$, allowing for a more flexible connection of the zero equations generated by the newly augmented variables. This approach breaks away from the traditional framework that relies on fixed constant connection matrices $\mathcal{W}_1$, offering a more diversified and dynamic way of linking. Additionally, the introduction of the time-varying matrix $(\mathcal{W}_1+\dot{\ell}(t)\mathcal{W}_2)$*

provides the possibility of capturing more effective information for stability analysis, which can reveal more complex dynamic relationships in time-delay systems, thereby offering the potential to derive better stability criteria.

For the sake of facilitating the verification of the benefits of the augmented functional constructed in this article, another stability criterion can be easily obtained by removing double integral terms $\int_{t-\ell(t)}^{t}\int_{s}^{t} x^T(\varsigma)d\varsigma ds$ and $\int_{t-d_M}^{t-\ell(t)}\int_{s}^{t-\ell(t)} x^T(\varsigma)d\varsigma ds$ from $\varphi_1(t)$ and removing $[\rho_3^T(t) \;\; \ell(t)x^T(t) \;\; (d_M-\ell(t))x^T(t-\ell(t))]$ from $\zeta(t)$ accordingly.

Corollary 1. *For the given delay parameters μ and d_M, linear system (1) with delay-satisfying boundary restriction condition (3) is stable if some positive definite matrices $\bar{\mathcal{Q}}_1$, $\bar{\mathcal{R}}_1$, $\bar{\mathcal{R}}_2$, $\bar{\mathcal{Q}}_2$; symmetric matrices $\bar{\mathcal{P}}_0$, $\bar{\mathcal{P}}_1$ satisfying $\bar{\mathcal{P}}_0+\ell(t)\bar{\mathcal{P}}_1>0$; and arbitrary matrices $\bar{\mathcal{N}}_1$, $\bar{\mathcal{N}}_2$, $\bar{\mathcal{W}}_1$, $\bar{\mathcal{W}}_2$ exist such that (13) and (14) are feasible:*

$$\begin{bmatrix}\bar{\Xi}(0,\dot{\ell}(t)) & d_M\bar{\mathcal{N}}_2^T \\ * & -d_M\tilde{\bar{\mathcal{R}}}_2\end{bmatrix}<0 \tag{13}$$

$$\begin{bmatrix}\bar{\Xi}(d_M,\dot{\ell}(t)) & d_M\bar{\mathcal{N}}_1^T \\ * & -d_M\tilde{\bar{\mathcal{R}}}_1\end{bmatrix}<0 \tag{14}$$

where

$$\begin{aligned}
\bar{\Xi}(\ell(t),\dot{\ell}(t)) &= \bar{\Xi}_0(\ell(t),\dot{\ell}(t))+\bar{\Xi}_1(\ell(t),\dot{\ell}(t))\\
\bar{\Xi}_0(\ell(t),\dot{\ell}(t)) &= Sym\{\bar{\lambda}_1^T(\bar{\mathcal{P}}_0+\ell(t)\bar{\mathcal{P}}_1)\bar{\mathcal{D}}_1\}+\dot{\ell}(t)\bar{\mathcal{D}}_1^T\bar{\mathcal{P}}_1\bar{\mathcal{D}}_1+\bar{\mathcal{D}}_2^T\bar{\mathcal{Q}}_1\bar{\mathcal{D}}_2\\
&\quad+(1-\dot{\ell}(t))\bar{\mathcal{D}}_3^T(\bar{\mathcal{Q}}_2-\bar{\mathcal{Q}}_1)\bar{\mathcal{D}}_3-\bar{\mathcal{D}}_4^T\bar{\mathcal{Q}}_2\bar{\mathcal{D}}_4\\
&\quad+d_M\bar{g}_0^T\mathcal{R}_1\bar{g}_0-(1-\dot{\ell}(t))(d_M-\ell(t))\bar{g}_4^T(\bar{\mathcal{R}}_1-\bar{\mathcal{R}}_2)\bar{g}_4\\
\bar{\Xi}_1(\ell(t),\dot{\ell}(t)) &= Sym\{(\bar{\mathcal{W}}_1+\dot{\ell}(t)\bar{\mathcal{W}}_2)\Psi(\ell(t))+\bar{\mathcal{L}}_1^T\bar{\mathcal{N}}_1+\bar{\mathcal{L}}_2^T\bar{\mathcal{N}}_2\}
\end{aligned}$$

with

$$\begin{aligned}
\bar{\mathcal{D}}_1 &= [\bar{g}_1^T,\ \bar{g}_2^T,\ \bar{g}_3^T,\ \bar{g}_{10}^T,\ \bar{g}_{11}^T,\ \bar{g}_{12}^T,\ \bar{g}_{13}^T]^T\\
\bar{\lambda}_1 &= [\bar{g}_0^T,\ (1-\dot{\ell}(t))\bar{g}_4^T,\ \bar{g}_5^T,\ \bar{g}_1^T-(1-\dot{\ell}(t))\bar{g}_2^T,\ \bar{g}_1^T-(1-\dot{\ell}(t))\bar{g}_6^T-\dot{\ell}(t)\bar{g}_7^T,\\
&\quad (1-\dot{\ell}(t))\bar{g}_2^T-\bar{g}_3^T,\ (1-\dot{\ell}(t))\bar{g}_2^T-\bar{g}_8^T+\dot{\ell}(t)\bar{g}_9^T]^T\\
\bar{\mathcal{D}}_2 &= [\bar{g}_1^T,\ \bar{g}_0^T]^T,\ \bar{\mathcal{D}}_3=[\bar{g}_2^T,\ \bar{g}_4^T]^T,\ \bar{\mathcal{D}}_4=[\bar{g}_3^T,\ \bar{g}_5^T]^T\\
\bar{\mathcal{L}}_l &= [\bar{g}_1^T-\bar{g}_2^T,\ \bar{g}_1^T+\bar{g}_2^T-2\bar{g}_6^T,\ \bar{g}_1^T-\bar{g}_2^T+6\bar{g}_6^T-12\bar{g}_7^T]^T\\
\bar{\mathcal{L}}_l &= [\bar{g}_2^T-\bar{g}_3^T,\ \bar{g}_2^T+\bar{g}_3^T-2\bar{g}_8^T,\ \bar{g}_2^T-\bar{g}_3^T+6\bar{g}_8^T-12\bar{g}_9^T]^T\\
\bar{g}_0 &= A\bar{g}_1+A_d\bar{g}_2,\ \ \tilde{\mathcal{R}}_i=diag\{\mathcal{R}_i,3\mathcal{R}_i,5\mathcal{R}_i\}\\
\bar{g}_\kappa &= \begin{bmatrix}0_{n\times(\kappa-1)n} & I_n & 0_{n\times(13-\kappa)n}\end{bmatrix},\ \ \kappa=1,\ldots,13.
\end{aligned}$$

Proof. The proof process is consistent with Theorem 1. □

3.2. Periodic Time-Varying Delay

Similar to references [38,39], in this subsection, we study the stability of systems under periodic time-varying delays. It is assumed that the time-delay function $\ell(t)$ is monotone decreasing in the intervals $[t_{2p},t_{2p+1})$ and monotone increasing in the intervals $[t_{2p+1},t_{2(p+1)}]$, where $p\in\mathbb{N}$. Assume that the delay and its derivative boundary satisfy:

$$0\le\ell(t)\le d_M,\quad -\mu\le\dot{\ell}(t)\le\mu \tag{15}$$

Then, we have $\ell(t_{2p})=d_M$ and $\ell(t_{2p+1})=0$.

Considering the known monotonic increasing and decreasing intervals of the time delay, a Lyapunov function can be constructed separately for each of these intervals. Inspired by [38,39], based on the loop functional idea, two distinct Lyapunov functionals were constructed for the monotonic increasing and decreasing intervals, respectively.

$$\tilde{V}(t) = \begin{cases} V(t) + W_1(t), \ t \in [t_{2p}, t_{2p+1}) \\ V(t) + W_2(t), \ t \in [t_{2p+1}, t_{2(p+1)}] \end{cases} \tag{16}$$

where

$$\begin{aligned} W_i(t) =& 2\chi_1^T(t)\mathcal{X}_i\chi_2(t) + (\ell(t) - d_M)\int_{t-\ell(t)}^{t} \dot{x}^T(\varsigma)\mathcal{Z}_{1i}\dot{x}(\varsigma)\mathrm{d}\varsigma \\ &+ \ell(t)\int_{t-d_M}^{t-\ell(t)} \dot{x}^T(\varsigma)\mathcal{Z}_{2i}\dot{x}(\varsigma)\mathrm{d}\varsigma, \ \ i = \{1,2\} \end{aligned} \tag{17}$$

with

$$\chi_1(t) = \begin{bmatrix} \ell(t)\big(x^T(t-d_M) - x^T(t-\ell(t))\big) & (d_M - \ell(t))\big(x^T(t) - x^T(t-\ell(t))\big) \end{bmatrix}^T$$
$$\chi_2(t) = \rho_0(t).$$

In the monotonically decreasing interval $[t_{2p}, t_{2p+1})$, $\lim\limits_{t\to t_{2p}} W_1(t) = \lim\limits_{t\to t_{2p+1}} W_1(t) = 0$ holds, satisfying the looped function construction rule in [40,41]. For more information about the looped function, please refer to [40,41]. Based on the looped function defined in [40,41], it is not necessary to constrain the positive definiteness of $W_1(t)$. Consequently, the positive definiteness of $\tilde{V}(t) = V(t) + W_1(t)$ can be inferred from $V(t) > 0$. The same is true for the monotonically increasing interval $[t_{2p+1}, t_{2(p+1)})$. So, the coupling matrices in $W_1(t)$ and $W_2(t)$ do not need to be positive definite, which relaxes the positive definiteness restriction of the constructed Lyapunov functional.

Theorem 2. *For given delay parameters μ and d_M, linear system (1) with a periodic delay meeting boundary restriction (15) is stable if some symmetric matrices $\mathcal{Z}_{1i}$, $\mathcal{Z}_{2i}$; positive definite matrices $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{Q}_1$, $\mathcal{Q}_2$; symmetric matrices $\mathcal{P}_0$, $\mathcal{P}_1$ satisfying $\mathcal{P}_0 + \ell(t)\mathcal{P}_1 > 0$; and arbitrary matrices $\mathcal{X}_i$, $\mathcal{F}_{1i}$, $\mathcal{F}_{2i}$, $\mathcal{W}_{1i}$, $\mathcal{W}_{2i} (i = 1,2)$ exist such that (18)–(21) are feasible,*

$$\begin{bmatrix} \Upsilon_1(0, \dot{\ell}(t)) & d_M\mathcal{F}_{21}^T \\ * & -d_M\bar{\mathcal{R}}_{z21} \end{bmatrix}_{\dot{\ell}(t)\in[-\mu,\ 0]} < 0 \tag{18}$$

$$\begin{bmatrix} \Upsilon_1(d_M, \dot{\ell}(t)) & d_M\mathcal{F}_{11}^T \\ * & -d_M\bar{\mathcal{R}}_{z11} \end{bmatrix}_{\dot{\ell}(t)\in[-\mu,\ 0]} < 0 \tag{19}$$

$$\begin{bmatrix} \Upsilon_2(0, \dot{\ell}(t)) & d_M\mathcal{F}_{22}^T \\ * & -d_M\bar{\mathcal{R}}_{z22} \end{bmatrix}_{\dot{\ell}(t)\in[0,\ \mu]} < 0 \tag{20}$$

$$\begin{bmatrix} \Upsilon_2(d_M, \dot{\ell}(t)) & d_M\mathcal{F}_{12}^T \\ * & -d_M\bar{\mathcal{R}}_{z12} \end{bmatrix}_{\dot{\ell}(t)\in[0,\ \mu]} < 0 \tag{21}$$

where

$$\begin{aligned} \Upsilon_i(\ell(t), \dot{\ell}(t)) &= \Xi_0(\ell(t), \dot{\ell}(t)) + \Phi_i(\ell(t), \dot{\ell}(t)) \\ \Phi_i(\ell(t), \dot{\ell}(t)) &= Sym\{\bar{\lambda}_1^T\mathcal{X}_i\Pi_2 + \Pi_1^T\mathcal{X}_i\bar{\lambda}_2 + (\mathcal{W}_{1i} + \dot{\ell}(t)\mathcal{W}_{2i})\Psi(\ell(t)) + \mathcal{L}_1^T\mathcal{F}_{1i} \\ &\quad + \mathcal{L}_2^T\mathcal{F}_{2i}\} + (\ell(t) - d_M)\big(g_0^T\mathcal{Z}_{1i}g_0 - (1-\dot{\ell}(t))g_4^T\mathcal{Z}_{1i}g_4\big) \\ &\quad + \ell(t)\big((1-\dot{\ell}(t))g_4^T\mathcal{Z}_{2i}g_4 - g_5^T\mathcal{Z}_{2i}g_5\big) \end{aligned}$$

with

$$\begin{aligned}\Pi_1 =&[\ell(t)(g_3^T - g_2^T),\ (d_M - \ell(t))(g_1^T - g_2^T)]^T \\ \bar{\lambda}_1 =&[\dot{\ell}(t)(g_3^T - g_2^T) + \ell(t)(g_5^T - (1-\dot{\ell}(t))g_4^T), \\ &- \dot{\ell}(t)(g_1^T - g_2^T) + (d_M - \ell(t))(g_0^T - (1-\dot{\ell}(t))g_4^T)]^T \\ \Pi_2 =&[g_1^T,\ g_2^T,\ g_3^T]^T,\ \ \bar{\lambda}_2 = [g_0^T,\ (1-\dot{\ell}(t))g_4^T,\ g_5^T]^T \\ \bar{\mathcal{R}}_{zji} =&diag\{\mathcal{R}_{zji}, 3\mathcal{R}_{zji}, 5\mathcal{R}_{zji}\},\ \ \mathcal{R}_{zji} = \mathcal{R}_j - \dot{\ell}(t)\mathcal{Z}_{ji}.\end{aligned}$$

Proof. First, consider the decreasing subinterval $[t_{2p}, t_{2p+1})$. The derivative of $W_1(t)$ gives

$$\begin{aligned}\dot{W}_1(t) =&2\dot{\chi}_1^T(t)\mathcal{X}_1\chi_2(t) + 2\chi_1^T(t)\mathcal{X}_1\dot{\chi}_2(t) + (\ell(t) - d_M)\big(\dot{x}^T(t)\mathcal{Z}_{11}\dot{x}(t) \\ &- (1-\dot{\ell}(t))\dot{x}^T(t-\ell(t))\mathcal{Z}_{11}\dot{x}(t-\ell(t))\big) - \dot{x}^T(t-d_M)\mathcal{Z}_{21}\dot{x}(t-d_M)\big) \\ &+ \ell(t)((1-\dot{\ell}(t))\dot{x}^T(t-\ell(t))\mathcal{Z}_{21}\dot{x}(t-\ell(t)) + \mathcal{K}_1 + \mathcal{K}_2.\end{aligned} \tag{22}$$

where

$$\mathcal{K}_1 =\dot{\ell}(t)\int_{t-\ell(t)}^{t} \dot{x}^T(\varphi)\mathcal{Z}_{11}\dot{x}(\varphi)\mathrm{d}\varphi,\ \ \mathcal{K}_2 = \dot{\ell}(t)\int_{t-d_M}^{t-\ell(t)} \dot{x}^T(\varphi)\mathcal{Z}_{21}\dot{x}(\varphi)\mathrm{d}\varphi.$$

Considering the integral terms $\mathcal{J}_1$ and $\mathcal{J}_2$ in (7) and combining them with Lemma 1, the following integral estimation expressions can be obtained for any matrices $\mathcal{F}_{11}$ and $\mathcal{F}_{21}$.

$$\begin{aligned}\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{K}_1 + \mathcal{K}_2 \le \zeta^T(t)\{&Sym\{\mathcal{L}_1^T\mathcal{F}_{11} + \mathcal{L}_2^T\mathcal{F}_{21}\} + \ell(t)\mathcal{F}_{11}^T\bar{\mathcal{R}}_{z11}^{-1}\mathcal{F}_{11} \\ &+ (d_M - \ell(t))\mathcal{F}_{21}^T\bar{\mathcal{R}}_{z21}^{-1}\mathcal{F}_{21}\}\zeta(t)\end{aligned} \tag{23}$$

where $\bar{\mathcal{R}}_{zi1}$ is the delay derivative dependence matrix, specifically $\bar{\mathcal{R}}_{zi1} = diag\{\mathcal{R}_{zi1},\ 3\mathcal{R}_{zi1},\ 5\mathcal{R}_{zi1}\}$, and $\mathcal{R}_{zi1} = \mathcal{R}_i - \dot{\ell}(t)\mathcal{Z}_{i1}$.

Additionally, by introducing arbitrary matrices $\mathcal{W}_{11}$ and $\mathcal{W}_{21}$, the time-varying zero equation in (11) can be modified to the following time-varying zero equality:

$$0 =2\zeta^T(t)\big((\mathcal{W}_{11} + \dot{\ell}(t)\mathcal{W}_{21})\Psi(\ell(t))\big)\zeta(t). \tag{24}$$

Combining the derivative function $\dot{V}(t)$ in (7) with (22)–(24), the derivative function of the monotone decreasing interval can be derived:

$$\dot{V}(t) \le\zeta^T(t)\big(\Upsilon_1(\ell(t), \dot{\ell}(t)) + \ell(t)\mathcal{F}_{11}^T\bar{\mathcal{R}}_{z11}^{-1}\mathcal{F}_{11} + (d_M - \ell(t))\mathcal{F}_{21}^T\bar{\mathcal{R}}_{z21}^{-1}\mathcal{F}_{21}\big)\zeta(t) \tag{25}$$

where $\Upsilon_1(\ell(t), \dot{\ell}(t))$ is listed in Theorem 2. If LMIs (18) and (19) hold, then $\Upsilon_1(0, \dot{\ell}(t)) + d_M\mathcal{F}_{21}^T\bar{\mathcal{R}}_{z21}^{-1}\mathcal{F}_{21} < 0$ and $\Upsilon_1(d_M, \dot{\ell}(t)) + d_M\mathcal{F}_{11}^T\bar{\mathcal{R}}_{z11}^{-1}\mathcal{F}_{11} < 0$ hold for $\dot{\ell}(t) \in [-\mu, 0]$. Then, there exists a scalar $\sigma_1 > 0$ that meets $\dot{V}(t) < -\sigma_1|x(t)|^2$.

For the monotone increasing interval $[t_{2p+1}, t_{2(p+1)}]$, the Lyapunov functional derivative (26) can be obtained by using a process and method similar to those for the monotone decreasing interval $[t_{2p}, t_{2p+1})$.

$$\dot{V}(t) \le\zeta^T(t)\big(\Upsilon_2(\ell(t), \dot{\ell}(t)) + \ell(t)\mathcal{F}_{12}^T\bar{\mathcal{R}}_{z12}^{-1}\mathcal{F}_{12} + (d_M - \ell(t))\mathcal{F}_{22}^T\bar{\mathcal{R}}_{z22}^{-1}\mathcal{F}_{22}\big)\zeta(t). \tag{26}$$

Accordingly, there exists a scalar $\sigma_2 > 0$ such that $\dot{V}(t) < -\sigma_2|x(t)|^2$ if LMIs (20) and (21) hold.

Therefore, it is concluded that $\dot{V}(t) < -\sigma_m|x(t)|^2$ for $t \in [t_{2p}, t_{2(p+1)}]$, where $\sigma_m = min\{\sigma_1,\ \sigma_2\}$. Therefore, the system (1) is stable. □

Remark 3. *Like references [38,39], this paper constructs two different Lyapunov functionals for monotone increasing and decreasing intervals, respectively. This method effectively relaxes the* *traditional limitation of constructing only one Lyapunov functional. Thus, information on periodic time delays can be captured more accurately, improving the stability analysis accuracy. Compared to [38,39], the Lyapunov functional proposed in this paper incorporates more comprehensive and practical delay information. This achievement stems from using augmented variable methods, particularly integrating some delay-product-type augmented terms. These augmented terms enhance the functional's ability to capture the system's dynamic characteristics deeply and create conditions for obtaining more effective cross-term information. This approach can further optimize the precision of stability determination criteria. Research on periodic time-varying delays has yet to involve these enhanced techniques extensively. Therefore, based on these improved techniques, we can derive a less conservative stability condition for periodic time-varying delay systems, which, to some extent, extends and deepens the studies in the literature [38,39].*

Remark 4. *Considering that the defined $\zeta(t)$, $\Psi(\ell(t))\zeta(t)$ is always equal to 0, to further relax the conservatism of the stability determination condition in the case of time-varying periodic delays, different time-varying zero equations are introduced for these two different delay intervals, namely $\big((\mathcal{W}_{11}+\dot{\ell}(t)\mathcal{W}_{21})\Psi(\ell(t))\big)\zeta(t)=0$ for the monotone decreasing interval and $\big((\mathcal{W}_{12}+\dot{\ell}(t)\mathcal{W}_{22})\Psi(\ell(t))\big)\zeta(t)=0$ for the monotone increasing interval.*

4. Examples Simulation

This part will elucidate the advantages of the stability criteria derived from the methods presented in this paper through three detailed examples.

Example 1. *Consider linear system (1) with the following system matrices:*

$$A=\begin{bmatrix}-2.0 & 0.0\\ 0.0 & -0.9\end{bmatrix},\quad A_d=\begin{bmatrix}-1.0 & 0.0\\ -1.0 & -1.0\end{bmatrix}.$$

Time-delay systems with the aforementioned system parameters are typically used to evaluate the advantages and disadvantages of stability determination criteria. Based on Corollary 1 and Theorem 1 derived in this article, the maximum delay upper bound (DUB) under the given delay derivative boundary $\mu=\{0.1,\ 0.5,\ 0.8\}$ is calculated. The calculated maximum DUBs and the existing results are presented in Table 1. From this table, it can be observed that Theorem 1 can obtain a larger DUB, which indicates that the functional developed in this article plays a positive role in reducing the conservatism of stability determination criteria. By comparing the DUBs derived from Corollary 1 and Theorem 1, we observe that the results from Theorem 1 are superior. This finding indicates that the additional variables and functional terms introduced in this paper contribute to achieving better stability conditions. These added variables and functional items enhance the stability analysis results to capture the system's dynamic information, thereby leading to a more refined stability criterion. It is worth noting that in [18], it is assumed that the second-order derivatives of the system state, $\ddot{x}(t)$, can be obtained. Relatively better stability conditions are achieved by using the information of $\ddot{x}(t)$ to construct the Lyapunov functional. If the information of $\ddot{x}(t)$ can be obtained and included in the Lyapunov functional constructed in this paper, similarly good results can be achieved. Additionally, when the time-varying matrix $(\bar{\mathcal{W}}_1+\dot{\ell}(t)\bar{\mathcal{W}}_2)$ degenerates to $\bar{\mathcal{W}}_1$, the calculated DUBs significantly decrease. This change indicates that the approach of using time-varying matrices to link the zero equalities generated by augmented variables in this paper has successfully enhanced the effectiveness of the stability conditions.

The maximum DUBs of the stability criterion derived in this paper were calculated under a periodic time-varying delay, and the results are listed in Table 2. For the convenience of a comparison, the existing results under periodic time-varying delay are also given in Table 2. From Table 2, it can be observed that the maximum DUBs obtained by Theorem 2 are significantly larger than those in [38,39], indicating that the stability conditions derived using the techniques used in this article are superior in the case of known periodic time-varying delays.

From the results presented in Tables 1 and 2, the stability result obtained in this paper is only a sufficient condition, and there is still some conservatism. However, compared with the existing results [5,13,14,34,35,42–45], it is less conservative. Therefore, new methods must still be explored in the future to obtain the necessary and sufficient stability criteria for time-delay systems.

Table 1. Maximum DUBs d_M for Example 1 with a non-periodic time-varying delay.

μ	0.1	0.5	0.8
[13]	4.910	3.233	2.789
[34]	4.921	3.221	2.792
[42]	4.93	3.09	2.66
[45]	4.993	3.474	3.053
[5]	5.015	3.452	3.030
[14]	5.102	3.411	2.981
[43]	5.026	3.428	2.997
[44]	5.097	3.549	3.147
[35]	5.110	3.593	3.119
Theorem 1	5.122	3.598	3.1406
Corollary 1	5.095	3.485	3.0078
Corollary 1 with $\bar{\mathcal{W}}_2 = 0$	4.949	3.339	2.9258

Table 2. Maximum DUBs d_M for Example 1 with a periodic time-varying delay.

μ	0.1	0.2	0.5	0.8
[38]	5.10	4.57	3.78	3.38
[39]	5.44	5.00	4.18	3.66
Theorem 2	5.70	5.38	4.75	4.32

To verify the results presented in Table 2, we plot the state trajectory under periodic time-varying delay $\ell(t) = \frac{5.7}{2} + \frac{5.7}{2} sin(\frac{0.2t}{5.7})$ with $\mu = 0.1$ and $d_M = 5.7$, as shown in Figure 1. Here, the initial condition is set as $x_0(t) = [0.5, -1]$. Clearly, as time progresses, all states eventually converge to zero, indicating that the system remains stable under periodic time-varying delay with $\ell(t) = \frac{5.7}{2} + \frac{5.7}{2} sin(\frac{0.2t}{5.7})$.

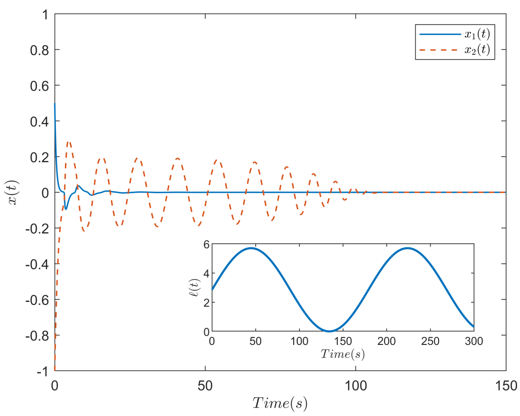

Figure 1. State trajectory for system (1) with $\ell(t) = \frac{5.7}{2} + \frac{5.7}{2} sin(\frac{0.2t}{5.7})$.

Example 2. *Consider linear system* (1) *with the following system matrices:*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}.$$

For the specified delay derivative boundaries $\mu = \{0.1, 0.2, 0.5, 0.8\}$, the maximum DUB has been calculated based on the stability criteria of Corollary 1 and Theorem 1 and is presented in Table 3 alongside the existing results. From Table 3, it is evident that the stability criteria provided by Theorem 1 offer larger DUBs. This further demonstrates the superiority of the methodology proposed in this paper.

Table 3. Maximum DUBs d_M for Example 2.

μ	0.1	0.2	0.5	0.8
[8]	7.176	4.543	2.496	1.922
[13]	7.230	4.556	2.509	1.940
[34]	7.308	4.670	2.664	2.072
[43]	7.651	4.936	2.764	2.114
[5]	7.656	4.992	2.868	2.172
[45]	7.677	4.996	2.815	2.146
[44]	7.730	5.034	2.841	2.176
[35]	7.741	5.054	2.858	2.200
Theorem 1	7.790	5.109	2.893	2.206
Corollary 1	7.721	5.017	2.822	2.158
Corollary 1 with $\bar{W}_2 = 0$	7.557	4.948	2.788	2.130

Example 3. *Consistent with [46], an example of single-area load frequency control is considered, and its model can be expressed as:*

$$\begin{aligned} x(t) &= \begin{bmatrix} \Delta f & \Delta P_m & \Delta P_v & \int ACE \end{bmatrix}^T, \\ A &= \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} & 0 & 0 \\ 0 & -\frac{1}{T_t} & \frac{1}{T_t} & 0 \\ -\frac{1}{T_g S} & 0 & -\frac{1}{T_g} & 0 \\ \rho & 0 & 0 & 0 \end{bmatrix}, \\ A_d &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\rho K_p}{T_g} & 0 & 0 & -\frac{K_i}{T_g} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{27}$$

where $\int ACE$, ΔP_v, ΔP_m, and Δf represent the integral of the area control error, valve position deviation, mechanical generator output and frequency deviation, respectively. In addition, $M = 10$ and $D = 1.0$ represent the moment of inertia and generator damping coefficient; $T_t = 0.3$ and $T_g = 0.1$ are the time constants for the turbine and governor. $\rho = 21$ and $S = 0.05$ denote the frequency bias factor and speed drop, while $K_i = 0.2$ and $K_p = 0.05$ represent the controller gain matrix. To compare with the existing results, we chose $\mu = \{0.1, 0.5, 0.9\}$ for simulations, and the obtained results are given in Table 4. It can be found that the maximum DUBs obtained by Theorem 1 are better than those in [8,34,44] and Corollary 1, which once again verifies the advantages of the technology used in this paper.

Table 4. Maximum DUBs d_M for Example 3.

μ	0.1	0.5	0.9
[8]	-	-	4.76
[34]	7.38	7.09	6.98
[44]	7.48	7.27	7.15
Theorem 1	7.4959	7.2950	7.1630
Corollary 1	7.4930	7.2859	7.1563
Corollary 1 with $\bar{W}_2 = 0$	7.4902	7.2681	7.1289

Based on the maximal DUBs obtained from Theorem 1, the state of system (27) is plotted in Figure 2. Here, the time delay $\ell(t)$ is considered as $\frac{7.49}{2} + \frac{7.49}{2} sin(\frac{0.2t}{7.49})$, and the initial condition $x_0(t)$ is set to $x_0(t) = [0.5, \ -1, \ 0.5, \ 1]$. It can also be observed that all states tend to stabilize.

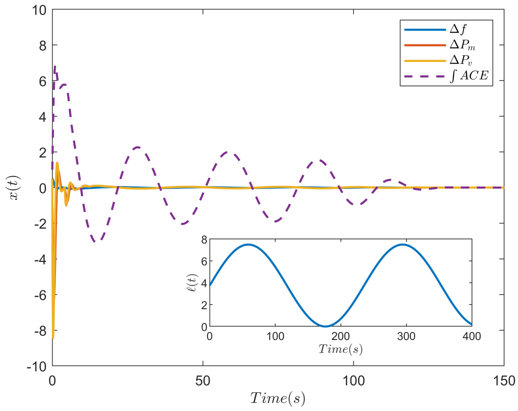

Figure 2. State trajectory for system (27) with $\ell(t) = \frac{7.49}{2} + \frac{7.49}{2} sin(\frac{0.2t}{7.49})$.

5. Conclusions

This paper has investigated the stability issues of a class of linear systems with bounded time-varying delays and periodic time-varying delays. Novel Lyapunov functionals have been constructed using the augmented variable method for these two time-varying delay cases. Based on the constructed Lyapunov functionals and time-varying matrix-dependent zero equations, some less conservative stability determination conditions have been obtained separately for these two delay scenarios. The benefits of the presented approach have been validated through three numerical examples. In future work, we will consider the control problem of time-delay systems with disturbances and uncertainties [47,48] and develop a new method with a lower computational complexity and less conservatism for studying time-delay system control problems.

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