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L^p-Boundedness of a Class of Bi-Parameter Pseudo-Differential Operators

Jinhua Cheng

School of Mathematical Sciences, Zhejiang University, Hangzhou 310058, China; chengjinhua@westlake.edu.cn

Abstract: In this paper, I explore a specific class of bi-parameter pseudo-differential operators characterized by symbols $\sigma(x_1, x_2, \xi_1, \xi_2)$ falling within the product-type Hörmander class $\mathbf{S}^m_{\rho,\delta}$. This classification imposes constraints on the behavior of partial derivatives of σ with respect to both spatial and frequency variables. Specifically, I demonstrate that for each multi-index α, β, the inequality $|\partial^\alpha_\xi \partial^\beta_x \sigma(x_1, x_2, \xi_1, \xi_2)| \leq C_{\alpha,\beta}(1 + |\xi|)^m \prod_{i=1}^2 (1 + |\xi_i|)^{-\rho|\alpha_i|+\delta|\beta_i|}$ is satisfied. My investigation culminates in a rigorous analysis of the L^p-boundedness of such pseudo-differential operators, thereby extending the seminal findings of C. Fefferman from 1973 concerning pseudo-differential operators within the Hörmander class.

Keywords: bi-parameter pseudo-differential operators; L^p-boundedness; cone decomposition; BMO space

MSC: 42B10; 42B20; 42B30

1. Introduction

Consider a Schwartz function denoted by f. We define a pseudo-differential operator T_σ as follows:

$$T_\sigma f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi)\sigma(x, \xi) d\xi, \tag{1}$$

where $\hat{f}(\xi)$ represents the Fourier transform of f, and $\sigma(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is referred to as the symbol. Of primary interest is the symbol class denoted $S^m_{\rho,\delta}$, commonly known as the Hörmander class. A symbol $\sigma(x, \xi)$ belongs to $S^m_{\rho,\delta}$ if it satisfies the following differential inequalities:

$$|\partial^\alpha_\xi \partial^\beta_x \sigma(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \tag{2}$$

for all multi-indices $\alpha,\ \beta$.

The L^p-boundedness of pseudo-differential operators, defined as in (1) and (2), has been a topic of extensive investigation in recent decades. Notably, the seminal works of Calderón and Vaillancourt [1,2] established the L^2-boundedness of T_σ for symbols $\sigma \in S^0_{0,0}$. Furthermore, Calderón and Vaillancourt showed that T_σ remains bounded on L^2 when the symbol σ belongs to $S^0_{\rho,\rho}$, $0 < \rho < 1$, a class known as the exotic symbol class. However, the boundedness results are not universal. For instance, consider the symbol $\sigma(\xi)$ given by the Fourier transform of the Riemann singularity distribution $R(x) = e^{\frac{i}{|x|}}|x|^{-\frac{3}{2}}$, then T_σ is not bounded on L^p for $p \neq 2$. More recently, Wang [3] investigated a subclass of the exotic symbol class and demonstrated that pseudo-differential operators belonging to this subclass are bounded on L^p for $0 < p < 1$.

The primary objective of this paper is to extend the following theorem originally established by C. Fefferman in 1973 [4].

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Theorem 1 (Fefferman). *Let $\sigma(x,\xi) \in S^{-\beta}_{1-a,\delta}$ with $0 \le \delta < 1-a < 1$ and $\beta < na/2$. Then, T_σ is bounded on L^p for*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \le \gamma = \frac{\beta}{n}\left[\frac{n/2+\lambda}{\beta+\lambda}\right], \quad \lambda = \frac{na/2-\beta}{1-a}.$$

To be more specific, we turn to the multi-parameter setting. Let $\sigma(x_1, x_2, \xi_1, \xi_2) \in C^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, where $n = n_1 + n_2$. We say $\sigma \in \mathbf{S}^m_{\rho,\delta}$ if it satisfies the estimates

$$|\partial^\alpha_\xi \partial^\beta_x \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^m \prod_{i=1}^{2}\left(\frac{1}{1+|\xi_i|}\right)^{\rho|\alpha_i|-\delta|\beta_i|} \tag{3}$$

for all multi-indices α, β. Moreover, we define the bi-parameter Hörmander class $BS^{m_1,m_2}_{\rho,\delta}$, we say $\sigma \in BS^{m_1,m_2}_{\rho,\delta}$ if it satisfies the estimates:

$$|\partial^\alpha_\xi \partial^\beta_x \sigma(x,\xi)| \le C_{\alpha,\beta} \prod_{i=1}^{2}\left(\frac{1}{1+|\xi_i|}\right)^{\rho|\alpha_i|-\delta|\beta_i|-m_i}. \tag{4}$$

Note that if $m = m_1 + m_2$ and $m_1, m_2 \le 0$, then we have

$$\mathbf{S}^m_{\rho,\delta} \subset BS^{m_1,m_2}_{\rho,\delta}.$$

The classical theory of harmonic analysis may be described as around the Hardy–Littlewood maximal operator and its relationship with certain singular integral operators which commute with the classical one-parameter family dilations $\delta : x \to \delta x = (\delta x_1, \ldots, \delta x_d)$, $\delta > 0$. The multi-parameter theory, sometimes called product theory corresponds to a range of questions which are concerned with issues of harmonic analysis that are invariant with respect to a family of dilations $\delta : x \to \delta x = (\delta_1 x_1, \ldots, \delta_d x_d)$, $\delta_i > 0$, $i = 1, \ldots, d$. Such multi-parameter symbol classes, associated with singular integral operators, pseudo-differential operators, and Fourier integral operators, have been the subject of extensive study by various authors. Notable contributions include works by Müller, Ricci, and Stein [5], Yamazaki [6], Wang [7], Chen, Ding, and Lu [8], Huang and Chen [9,10], Hong, Zhang, and Lu [11–14], Muscalu, Pipher, Tao, and Thiele [15,16], among others.

Main Results

Main Theorem:

(a) Let $\sigma(x_1, x_2, \xi_1, \xi_2) \in \mathbf{S}^{-\beta}_{1-a,\delta}$ with $0 \le \delta < 1-a < 1$, $\beta < \frac{na}{2}$, and $n_1, n_2 \ge 2$. Then, T_σ is bounded on L^p for

$$\left|\frac{1}{p} - \frac{1}{2}\right| \le \gamma = \frac{\beta}{n}\left[\frac{n/2+\lambda}{\beta+\lambda}\right], \quad \lambda = \frac{na/2-\beta}{1-a}. \tag{5}$$

(b) Let $\sigma(x_1, x_2, \xi_1, \xi_2) \in \mathbf{S}^{-na/2}_{1-a,\delta}$. The critical L^p space is L^1, while T_σ is unbounded on L^1, it is bounded on the Hardy space H^1.

Remark 1. *C. Fefferman originally proved the above theorem with symbols belonging to the classical Hörmander class $S^m_{\rho,\delta}$. Thus, the sharpness of the theorem follows from Fefferman's theorem, as $S^m_{\rho,\delta} \subset \mathbf{S}^m_{\rho,\delta}$.*

The results in [17] lead to the following propositions.

Proposition 1. *Let $\sigma(x,\xi) \in \mathbf{S}^0_{1-a,\delta}$ for $0 \le \delta < 1-a < 1$. Then, T_σ is a bounded operator from L^2 to L^2.*

Proposition 2. *Let $\sigma(x,\xi) \in \mathbf{S}^{-na/2}_{1-a,\delta}$ for $0 \le \delta < 1-a < 1$. Then, T_σ is a bounded operator from L^∞ to BMO.*

Here, $BMO(\mathbb{R}^n)$ denotes the class of functions of bounded mean oscillation defined by F. John and L. Nirenberg in [18]. A locally integrable function f on $\mathbb{R}^n$ belongs to BMO if

$$||f||_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where Q is an arbitrary cube in $\mathbb{R}^n$ and $f_Q = \frac{1}{|Q|} \int_Q f(x)dx$.

We will prove the L^2-boundedness of T_σ with σ of order 0 in Section 2, and T_σ is bounded from L^∞ to BMO with σ of order $-na/2$ in Section 3. We primarily follow the proofs in [19] and [4] to establish Propositions 1 and 2, respectively. However, a single Littlewood–Paley decomposition in the ξ-space is insufficient; we require a further cone decomposition to fully utilize the inequalities in (4).

2. L^2-Boundedness of T_σ of Order 0

Since $\mathbf{S}^0_{1-a,\delta} \subset BS^{0,0}_{1-a,\delta} \subset BS^{0,0}_{\delta,\delta}$, it suffices to prove

Lemma 1. *Suppose that $\sigma(x,\xi) \in BS^{0,0}_{\rho,\rho}$, where $0 \le \rho < 1$. Then the operator T_σ defined in (1) is bounded from $L^2(\mathbb{R}^n)$ to iteself.*

Proof. First we use the Cotlar–Stein lemma to show that the lemma is true in the case $\rho = 0$.

By Plancherel's theorem, we observe that it suffices to establish the L^2-boundedness of the operator S defined by

$$Sf(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(\xi)\sigma(x,\xi)d\xi.$$

Notice that, in view of the assumption of σ, the role of x and ξ in the above symbol class are perfectly symmetric. We choose a smooth non-negative function ϕ_i that is supported in the unit cube

$$Q^i_1 = \{x^i : |x^i_j| \le 1, \ \ j = 1, 2, \dots, n_i\}, \qquad i = 1, 2$$

and for which

$$\sum_{k^i \in \mathbb{Z}^{n_i}} \phi^i(x^i - k^i) = 1.$$

To construct such a ϕ^i, simply fix any smooth, non-negative ϕ^i_0 that equals 1 on the cube $Q^i_{1/2} = 1/2 \cdot Q^i_1$ and is supported in Q^i_1. Noting that $\sum_{k^i \in .\mathbb{Z}^{n_i}} \phi^i_0(x^i - k^i)$ converges and is bounded away from 0 for all $x \in \mathbb{R}^n$, we take

$$\phi^i(x^i) = \phi^i_0(x^i) \left[\sum_{l^i \in \mathbb{Z}^{n_i}} \phi^i_0(x^i - k^i) \right]^{-1}.$$

Next, let $\vec{k^i} = (k^i, k'^i) \in \mathbb{Z}^{2n_i} = \mathbb{Z}^{n_i} \times \mathbb{Z}^{n_i}$ denote an element of $\mathbb{Z}^{2n_i}$, and similarly write $\vec{j^i} = (j^i, j'^i)$ for another element of $\mathbb{Z}^{2n_i}$. We set $\vec{k} = (\vec{k}^1, \vec{k}^2)$ and

$$\sigma_{\vec{k}}(x,\xi) = \left[\prod_{i=1}^2 \phi^i(x^i - k^i) \right] \sigma(x,\xi) \left[\prod_{i=1}^2 \phi^i(\xi^i - k'^i) \right] = \phi(x-k)\sigma(x,\xi)\phi(\xi - k')$$

and

$$S_{\vec{k}} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(\xi)\sigma_{\vec{k}}(x,\xi)d\xi.$$

Therefore, we have the decomposition

$$Sf(x) = \sum_{\vec{k}\in\mathbb{Z}^{2n}} S_{\vec{k}}f(x).$$

The main point is then to verify the almost-orthogonality estimates as follows:

$$||S^*_{\vec{j}}S_{\vec{k}}|| \leq A\prod_{i=1}^{2}(1+|\vec{j}^i-\vec{k}^i|)^{-2N_i} \tag{6}$$

and

$$||S_{\vec{k}}S^*_{\vec{j}}|| \leq A\prod_{i=1}^{2}(1+|\vec{j}^i-\vec{k}^i|)^{-2N_i} \tag{7}$$

Here, $||\cdot||$ denotes the L^2 operator norm, N_i is sufficiently large, and the bound A is independent of $\vec{k}, \vec{j}$.

Now, we can write

$$S^*_{\vec{j}}S_{\vec{k}}f(\xi) = \int_{\mathbb{R}^n} f(\eta)K_{\vec{j},\vec{k}}(\xi,\eta)d\eta,$$

where

$$K_{\vec{j},\vec{k}}(\xi,\eta) = \int_{\mathbb{R}^n} \overline{\sigma}_{\vec{j}}(x,\xi)\sigma_{\vec{k}}(x,\xi)e^{2\pi ix\cdot(\eta-\xi)}dx.$$

In the above integral, we integrate by parts, using the identities

$$\prod_{i=1}^{2}(I-\Delta_{x^i})^{N_i}e^{2\pi ix\cdot(\eta-\xi)} = \prod_{i=1}^{2}(1+4\pi^2|\eta^i-\xi^i|^2)^{N_i}e^{2\pi ix\cdot(\eta-\xi)}.$$

We also note that $\sigma_{\vec{k}}(x,\xi)$ and $\sigma_{\vec{j}}(x,\eta)$ are given by

$$\sigma_{\vec{k}}(x,\xi) = \phi(x-k)\sigma(x,\xi)\phi(\xi-k'), \quad \sigma_{\vec{j}}(x,\xi) = \phi(x-j)\sigma(x,\eta)\phi(\xi-j')$$

respectively, and so have disjoint x-support unless $\vec{j}^i-\vec{k}^i \in Q^i_1$. These observations lead to the bounds

$$|K_{\vec{j},\vec{k}}(\xi,\eta)| \leq \prod_{i=1}^{d}\frac{A_{N_i}\phi^i(\xi^i-\vec{j}'^i)\phi^i(\eta^i-\vec{k}'^i)}{(1+|\xi_i-\eta_i|)^{2N_i}}, \quad \text{if } \vec{j}^i-\vec{k}^i\in Q^i_1, \quad i=1,2,$$

$$|K_{\vec{j},\vec{k}}(\xi,\eta)| = 0, \qquad \text{otherwise.}$$

Therefore, we have

$$\sup_{\xi}\int_{\mathbb{R}^n}|K_{\vec{j},\vec{k}}(\xi,\eta)|d\eta < A\prod_{i=1}^{2}(1+|\vec{j}^i-\vec{k}^i|)^{-2N_i},$$

and

$$\sup_{\eta}\int_{\mathbb{R}^n}|K_{\vec{j},\vec{k}}(\xi,\eta)|d\xi < A\prod_{i=1}^{2}(1+|\vec{j}^i-\vec{k}^i|)^{-2N_i},$$

which implies our desired estimate (6). Moreover, as we have noted, the situation is symmetric in x and ξ, the same proof also shows the estimate (7). Now, it is only a matter of applying the Cotlar–Stein lemma; setting N_i sufficiently large, we see

$$\sum_{\vec{k}\in\mathbb{Z}^n}\prod_{i=1}^{2}(1+|\vec{k}^i|)^{-2N_i} < \infty,$$

and as a result, $S = \sum_{\vec{k}} S_{\vec{k}}$ is bounded from $L^2(\mathbb{R}^n)$ to itself.

Now, we prove our Lemma 1.

We start by defining a C^∞ function φ with compact support on $\mathbb{R}$, satisfying $\varphi(t) = 1$ for $|t| \leq 1$ and $\varphi(t) = 0$ for $|t| \geq 2$. For each $i = 1, 2$, we set $\phi_0(\xi_i) = \varphi(|\xi_i|)$ and

$$\phi_{j_i}(\xi_i) = \varphi(2^{-j}|\xi_i|) - \varphi(2^{-j+1}|\xi_i|), \ \ j_i \in \mathbb{Z}, \ \ j_i > 0, \ \ i = 1, 2,$$

and

$$\phi_j(\xi) = \prod_{i=1}^{2} \phi_{j_i}(\xi_i), \quad j \in \mathbb{Z}^2.$$

Then, we define the partial operators

$$T_j f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi) \sigma_j(x, \xi) d\xi, \quad \sigma_j(x, \xi) = \sigma(x, \xi) \phi_j(\xi).$$

Let $\widehat{S_j f}(\xi) = \phi_j(\xi) \hat{f}(\xi)$, and we have the decomposition of T

$$T = \sum_{j \geq 0} T_j = \sum_{j \geq 0} T S_j, \qquad \sum_{j \geq 0} = \prod_{i=1}^{2} \sum_{j_i \geq 0}. \tag{8}$$

It will be convenient to break the sum (8) into two parts

$$T = \sum_{j\ even} T_j + \sum_{j\ odd}, \qquad \sum_{j\ even} = \prod_{i=1}^{2} \sum_{j_i \geq 0\ even},$$

so that the summands in each parts have disjoint ξ-support; it suffices to prove the boundedness of each sum separately.

Let us consider the sum taken over the odd j. Note that

$$T_j T_k^* = T S_j (T S_k)^* = T S_j S_k^* T = 0, \qquad j \neq k,$$

because the supports of the multipliers corresponding to S_j and S_k are disjoint. Next, we estimate $T_j^* T_k$, and we write

$$T_j^* T_k f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

with

$$K(x, y) = \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \overline{\sigma}_k(z, \eta) \sigma_j(z, \xi) e^{2\pi i [\xi \cdot (z-y) - \eta \cdot (z-x)]} dz d\eta d\xi.$$

First, one carries integration by parts with respect to z-variable by writing

$$\prod_{i=1}^{2} \frac{(I - \Delta_{z_i})^{N_i}}{(1 + 4\pi^2 |\xi_i - \eta_i|^2)^{N_i}} e^{2\pi(\xi - \eta) \cdot z} = e^{2\pi i (\xi - \eta) \cdot z}.$$

Next, one performs a similar process on the η-variable, beginning with

$$\prod_{i=1}^{2} \frac{(I - \Delta_{\eta_i})^{N_i}}{(1 + 4\pi^2 |x_i - z_i|^2)^{N_i}} e^{2\pi i \eta \cdot (x - z)} = e^{2\pi i \eta \cdot (x - z)}.$$

Finally, an analogous step is carried our for ξ-variable. If we take into account the differential inequalities for the symbols σ_j, and the restrictions on their supports, we see that each order of differentiation in the z_i-variable gives us a factor of order

$$(1 + |\xi_i - \eta_i|)^{-1} \sim 2^{-\max\{k_i, j_i\}}$$

for every factor of order

$$(1+|\xi_i|+|\eta_i|)^\rho \sim 2^{\rho\max\{k-i,j_i\}}$$

that may lose. As a result, the kernel K is dominated by a constant multiple of

$$\prod_{i=1}^{2} 2^{\max\{k_i,j_i\}(2\rho N_i-2N_i+2n_i)}\int_{\mathbb{R}^{n_i}} Q_i(x_i-z_i)Q_i(z_i-y_i)dz_i.$$

Now, if we let $K_i(x_i,y_i)=\int_{\mathbb{R}^{n_i}} Q_i(x_i-z_i)Q_i(z_i-y_i)dz_i$, then

$$\int_{\mathbb{R}^{n_i}} K_i(x_i,y_i)dy_i=\int_{\mathbb{R}^{n_i}} K_i(x_i,y_i)dx_i=\left(\int_{\mathbb{R}^{n_i}}(1+|z_i|)^{-2N_i}\right)^2<\infty,$$

if $2N_i>n_i$. Thus, we obtain

$$||T_j^*T_k||\le A\prod_{i=1}^{2} 2^{\max\{k_i,j_i\}(2\rho N_i-2N_i+2n_i)},\quad j\ne k,$$

which implies that

$$||T_j^*T_k||\le\prod_{i=1}^{2}\gamma_i(j_i)\gamma_i(k_i),\qquad j\ne k,$$

with $\gamma_i(j_i)=A\cdot 2^{-\epsilon j_i},\epsilon>0$, if we choose N_i so large that $N_i>n_i(1-\rho)$.

In order to apply the Cotlar–Stein lemma, we need to show that the partial operators T_j are uniformly bounded in the norm. To prove this, we set

$$\tilde{\sigma}_j(x,\xi)=\sigma_j(2^{-j\rho}x,2^{j\rho}\xi),\quad 2^{-j\rho}x=(2^{j_1\rho}x_1,2^{j_2\rho}x_2),\ \ 0\le\rho<1.$$

Thus, $\tilde{\sigma}_j(x,\xi)\in S^{0,0}_{\rho,\rho}$ for $m_i=0,\ \rho_i=0$ for each $i=1,2,\ldots,d$ uniformly in j . Therefore, the operator

$$\tilde{T}_jf(x)=\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{f}(\xi)\tilde{\sigma}_j(x,\xi)d\xi$$

is bounded on $L^2(\mathbb{R}^n)$. Next, define the scaling operators given by

$$\Lambda_jf(x)=f(2^{j\rho}x)=f(2^{j_1\rho}x_1,2^{j_2\rho}x_2),$$

then, as is easily verified,

$$T_j=\Lambda_j\tilde{T}_j\Lambda_j^{-1}.$$

Now, $||\Lambda_jf||_{L^2}=\prod_{i=1}^{2}2^{n_ij_i\rho/2}||f||_{L^2}$ and $||\Lambda_j^{-1}f||_{L^2}=\prod_{i=1}^{d}2^{-n_ij_i\rho/2}||f||_{L^2}$; so together with the L^2-boundedness of $\tilde{T}_j$, we have

$$||T_j||\le A,\quad \text{uniformly in}\quad j.$$

We may therefore conclude that $\sum_{j\ odd}T_j$ is bounded from $L^2(\mathbb{R}^n)$ to itself, the sum $\sum_{j\ even}$ is treated similarly, and our Lemma 1 is proved. □

3. L^p-Boundedness of T_σ

We make a further decomposition, let $\xi=(\xi_1,\xi_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Recalling the definition of φ in the above section, define

$$\delta_\ell(\xi)=\varphi\left(2^{-\ell}\frac{|\xi_2|}{|\xi_1|}\right)-\varphi\left(2^{-\ell+1}\frac{|\xi_2|}{|\xi_1|}\right),\qquad \ell\in\mathbb{Z}. \tag{9}$$

Note that $\delta_\ell(\xi)$ has a support in the cone region,

$$\Lambda_\ell = \{(\xi_1, \xi_2) : 2^{\ell-1} \leq \frac{|\xi_2|}{|\xi_1|} \leq 2^{\ell+1}\}. \tag{10}$$

By symmetry, we can always assume ℓ is a non-negative integer. Now for fixed j, we make a cone decomposition in the frequency space, define partial operators

$$T_{\ell j} f(x) = T_{\sigma_{\ell j}} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi)\sigma_{\ell j}(x,\xi)d\xi, \quad \sigma_{\ell j}(x,\xi) = \sigma(x,\xi)\phi_j(\xi)\delta_\ell(\xi). \tag{11}$$

Furthermore, we define

$$T_j^\flat f(x) = T_{\sigma_j^\flat} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi)\sigma_j^\flat(x,\xi)d\xi, \quad \sigma_j^\flat(x,\xi) = \sum_{\ell=j}^{\infty} \sigma(x,\xi)\phi_j(\xi)\delta_\ell(\xi), \tag{12}$$

and

$$T_j^\sharp f(x) = T_{\sigma_j^\sharp} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi)\sigma_j^\sharp(x,\xi)d\xi, \quad \sigma_j^\sharp(x,\xi) = \sum_{\ell=0}^{j} \sigma(x,\xi)\phi_j(\xi)\delta_\ell(\xi). \tag{13}$$

3.1. A Key Lemma

Let a symbol $\sigma(x,\xi) \in \mathbf{S}^m_{\rho,\delta}$, then we define its norm as

$$\|\sigma\|_{\mathbf{S}} = \sup_{|\alpha|\leq k, |\beta|\leq N} |\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)|(1+|\xi|)^{-m} \prod_{i=1}^{2} (1+|\xi_i|)^{\rho|\alpha_i| - \delta|\beta_i|}, \ \ k, N > n/2. \tag{14}$$

Let $r > 0$ be a real number, recall the definitions of $\phi_j(\xi)$ and $\delta_\ell(\xi)$, define the partial operators

$$T_{\ell j}^r f(x) = T_{\sigma_{\ell j}^r} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi)\sigma_{\ell j}^r(x,\xi)d\xi, \quad \sigma_{\ell j}^r(x,\xi) = \sigma(x,\xi)\delta_\ell(\xi)\phi_j(r\xi), \tag{15}$$

and

$$T_j^r f(x) = T_{\sigma_j^r} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi)\sigma_j^r(x,\xi)d\xi, \quad \sigma_j^r(x,\xi) = \sum_{\ell\geq 0} \sigma_{\ell j}^r(x,\xi). \tag{16}$$

Lemma 2. *Let the symbol $\sigma(x,\xi)$ be defined as (4), and $T_{\ell j}^r$, T_j^r defined as above, then we have*

$$\begin{aligned} \left\| T_j^r f \right\|_{L^\infty} &\leq C||\sigma||_{\mathbf{S}}||f||_{L^\infty}, \\ \left\| T_{\ell j}^r f \right\|_{L^\infty} &\leq C 2^{-n_1\ell/2}||\sigma||_{\mathbf{S}}||f||_{L^\infty}. \end{aligned} \tag{17}$$

Moreover, let $2^k \leq r^{-1} < 2^{k+1}$, if $\sigma_0(x,\xi) = \sum_{j\leq -k} \sigma_j^r(x,\xi)$, we have

$$\|T_{\sigma_0} f\|_{L^\infty} \leq C||\sigma||_{\mathbf{S}}||f||_{L^\infty}. \tag{18}$$

Proof. We denote $\hat{\sigma}(x,y) = \int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi}\sigma(x,\xi)d\xi$ throughout this paper. Now write

$$T_j^r f(x) = \int_{\mathbb{R}^n} f(y)\hat{\sigma}_j^r(x, y-x)dy. \tag{19}$$

We see that $|T_j^r f| \leq ||\hat{\sigma}_j^r(x,\cdot)||_{L^1}||f||_{L^\infty}$, where $||\hat{\sigma}_j^r(x,\cdot)||_{L^1} = \int_{\mathbb{R}^n} |\hat{\sigma}_j^r(x,y)|dy$. Therefore, it suffices to show that $||\hat{\sigma}_j^r(x,\cdot)||_{L^1} \leq C||\sigma||_{\mathbf{S}}$, $||\hat{\sigma}_{\ell j}^r(x,\cdot)||_{L^1} \leq C2^{-n_1\ell/2}||\sigma||_{\mathbf{S}}$ and

$||\hat{\sigma}_0(x,\cdot)||_{L^1} \leq \mathcal{C}||\sigma||_{\mathbf{S}}$. Let us consider T_j^r, let $b = (2^j r^{-1})^{a-1}$. Applying the Cauchy–Schwartz inequality and Plancherel theorem we see

$$\begin{aligned} \int_{|y|<b} |\hat{\sigma_j^r}(x,y)|dy &\leq \mathcal{C}b^{n/2}\left(\int_{|y|<b} |\hat{\sigma_j^r}(x,y)|^2 dy\right)^{\frac{1}{2}} \leq \mathcal{C}b^{n/2}\left(\int_{\mathbb{R}^n} |\sigma_j^r(x,\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &\leq \mathcal{C}||\sigma||_{\mathbf{S}} \qquad (\textit{since } \sigma_j^r \textit{ lives in } |\xi| \sim 2^j r^{-1}), \end{aligned} \tag{20}$$

and

$$\begin{aligned} \int_{|y|\geq b} |\hat{\sigma}_j^r(x,y)|dy &\leq \mathcal{C}b^{n/2-k}\left(\int_{|y|\geq b} |y|^{2k}|\hat{\sigma}_j^r(x,y)|^2 dy\right)^{\frac{1}{2}} \\ &\leq \mathcal{C}b^{n/2-k}\left(\int_{\mathbb{R}^n} |\nabla_\xi^k \sigma_j^r(x,\xi)|^2 d\xi\right)^{\frac{1}{2}} \leq \mathcal{C}||\sigma||_{\mathbf{S}} \\ &(\textit{since } \sigma \textit{ lives in } |\xi| \sim 2^j r^{-1}, k > n/2). \end{aligned} \tag{21}$$

Thus, $||\hat{\sigma_j^r}(x,\cdot)||_{L^1} \leq \mathcal{C}||\sigma||_{\mathbf{S}}$.

Similarly, we can prove $||\hat{\sigma}_{\ell j}^r(x,\cdot)||_{L^1} \leq \mathcal{C}2^{-n_1\ell/2}||\sigma||_{\mathbf{S}}$ and $||\hat{\sigma}_0(x,\cdot)||_{L^1} \leq \mathcal{C}||\sigma||_{\mathbf{S}}$ once we note that $\sigma_{\ell j}^r$ supported in the region $|\xi_1| \sim 2^{j-\ell}$, $|\xi_2| \sim 2^j$ and σ_0 supported in the region $|\xi| \leq 1$. Therefore, we have proven Lemma 2. □

3.2. Proof of the Main Theorem

Now, we can prove our proposition, fix $f \in L^\infty$ and $Q \subset \mathbb{R}^n$ having side r and center x_0. We have to show that

$$\frac{1}{|Q|}\int_{\mathbb{R}^n} |T_\sigma f(x) - (T_\sigma f)_Q|dx \leq \mathcal{C}||f||_{L^\infty}. \tag{22}$$

Case one $r < 1$. Now let $k \geq 0$ be an integer such that

$$2^k < r^{-1} \leq 2^{k+1}. \tag{23}$$

A direct computation shows $\partial_{x_i} T_{\sigma_j^r} f(x) = T_{\sigma_j'^r} f(x)$, where $\sigma_j'^r(x,\xi) = \partial_{x_i}\sigma_j^r(x,\xi) + 2\pi i\xi_i\sigma_j^r(x,\xi)$. Since $\sigma_j'^r(x,\xi)$ is supported in $|\xi| \sim 2^j r^{-1}$, then an elementary computation gives that $||\sigma_j'^r||_{\mathbf{S}} \leq \mathcal{C}r^{-1}2^{-jc}$ for $-k < j \leq 0$. By the Lemma 2,

$$\begin{aligned} \left\|\partial_{x_i}\sum_{j=-k}^{0} T_{\sigma_j^r} f\right\|_{L^\infty} &\leq \sum_{j\leq 0} ||T_{\sigma_j'^r} f||_{L^\infty} \\ &\leq \mathcal{C}r^{-1}\sum_{j\leq 0} 2^{-jc}||\sigma||_{\mathbf{S}}||f||_{L^\infty} \leq \mathbb{C}r^{-1}||\sigma||_{\mathbf{S}}||f||_{L^\infty}. \end{aligned} \tag{24}$$

Therefore, $|\sum_{j\leq 0} T_{\sigma_j^r} f(x) - a_Q|$ remains bounded in Q for some constant a_Q, so that

$$\frac{1}{|Q|}\int_Q \left|\sum_{j\leq 0} T_{\sigma_j^r} f(x) - a_Q\right| dx \leq \mathcal{C}||\sigma||_{\mathbf{S}}||f||_{L^\infty}. \tag{25}$$

Now, let us turn to $j > 0$ and let $\sigma^r(x,\xi) = \sum_{j>0}\sigma_j^r(x,\xi)$. Fix a bump function λ on $\mathbb{R}^n$, with $0 \leq \lambda \leq 10$, $\lambda \geq 1$ on Q and $\hat{\lambda}$ is supported in $|\xi| \leq r^{a-1}$. Then, we write

$$\lambda(x)T_{\sigma^r} f(x) = T_{\sigma^r}(\lambda f)(x) + [\lambda, T_{\sigma^r}]f(x) = I_1 + I_2. \tag{26}$$

In order to estimate I_1, we write

$$T_{\sigma^r}(\lambda f) = (T_{\sigma^r} \cdot G_{-na/4}) \cdot (G_{na/4}(\lambda f)), \quad \hat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha}.$$

Observe that $T_{\sigma^r} \cdot G_{-na/4}$ is a pseudo-differential operator with symbol $\sigma^r(x,\xi)$ $(1+|\xi|^2)^{na/4} \in S^0_{1-a,\delta}$, so that by Hörmander's L^2 result,

$$||T_{\sigma^r}(\lambda f)||^2_{L^2} \leq \mathcal{C}||\sigma||^2_{\mathbf{S}}||G_{na/4}(\lambda f)||^2_{L^2} \leq \mathcal{C}||\sigma||^2_{\mathbf{S}}||f||^2_{L^\infty}||G_{na/4}\lambda||^2_{L^2}, \tag{27}$$

now since for $\alpha > 0$, $G_\alpha \in L^1$, Young's inequality leads to $||G_{na/4}\lambda||^2_{L^2} \leq ||G_{na/4}||^2_{L^1}||\lambda||^2_{L^2} \leq \mathcal{C}|Q|$. Therefore,

$$\frac{1}{|Q|}\int_Q |T_{\sigma^r}(\lambda f)(x)|dx \leq \left(\frac{1}{|Q|}\int_Q |T_{\sigma^r}(\lambda f)(x)|^2 dx\right)^{\frac{1}{2}} \leq \mathcal{C}||\sigma||_{\mathbf{S}}||f||_{L^\infty}. \tag{28}$$

To estimate I_2, we set $\theta^r_{\ell j}(x,\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\eta}\hat{\lambda}(\eta)[\sigma^r_{\ell j}(x,\xi) - \sigma^r_{\ell j}(x,\xi+\eta)]d\eta$, then we can rewrite I_2 as $I_2 = \sum_{\ell\geq 0}\sum_{j>0} T_{\theta^r_{\ell j}} f(x)$. Notice that $\theta^r_{\ell j}$ is supported in $|\xi_1| \sim 2^{j-\ell}r^{-1}$, $|\xi_2| \sim 2^j r^{-1}$. Hence, simple calculations show that $||\theta^r_{\ell j}||_{\mathbf{S}} \leq \mathcal{C}2^{(\ell-j)(1-a)}||\sigma||_{\mathbf{S}}$. Applying the Lemma 2, we have

$$\begin{aligned} ||[\lambda, T_{\sigma^r}]f||_{L^\infty} &\leq \sum_{\ell\geq 0}\sum_{j>0}||T_{\theta^r_{\ell j}}f||_{L^\infty} \leq \sum_{\ell\geq 0}\sum_{j>0}\mathcal{C}2^{-n_1\ell/2}||\theta^r_{\ell j}||_{\mathbf{S}}||f||_{L^\infty} \\ &\leq \mathcal{C}\sum_{\ell\geq 0}\sum_{j>0}\mathcal{C}2^{-n_1\ell/2}2^{(\ell-j)(1-a)}||\sigma||_{\mathbf{S}}||f||_{L^\infty} \leq \mathcal{C}||\sigma||_{\mathbf{S}}||f||_{L^\infty}, \quad n_1 \geq 2. \end{aligned} \tag{29}$$

Now putting (28) and (29) into (26), we obtain

$$\frac{1}{|Q|}\int_Q |\lambda(x) \cdot T_{\sigma^r} f(x)|dx \leq \mathcal{C}||\sigma||_{\mathbf{S}}||f||_{L^\infty},$$

and since $|\lambda(x)| \geq 1$ on Q, we have

$$\frac{1}{|Q|}\int_Q |T_{\sigma^r} f(x)|dx \leq \mathcal{C}||\sigma||_{\mathbf{S}}||f||_{L^\infty},$$

and together with (25), this proves (22).

Case two $r > 1$. We make the decomposition by setting $r = 1$, then in the region $|\xi| < 1$, applying the Lemma 2 and for $|\xi| \geq 1$, repeating the proof above, we finally proved our main theorem.

Remark 2. *The above proof shows if $a > 1/2$, the main theorem holds for $n_1 = 1$ or $n_2 = 1$.*

Remark 3. *We posit that our results can be further extended to encompass the general bi-parameter Hörmander class $BS^{m_1,m_2}_{\rho,\delta}$. This broader class encompasses a wider range of operators and offers opportunities for deeper exploration and generalization of our findings.*

4. Conclusions

We extend Fefferman's results on pseudo-differential operators to the bi-parameter setting by introducing a novel decomposition on the phase space, known as the cone decomposition. This decomposition allows us to analyze pseudo-differential operators in a more intricate and nuanced manner, capturing the behavior of operators across multiple dimensions.

One key observation is that although there are infinitely many partial operators arising from the cone decomposition, the sum of their norms remains finite. This finiteness arises from the decay present in each partial operator norm, ensuring convergence in the overall operator analysis.

However, a challenge emerges when one subspace dimension is 1. In such cases, the decay of the partial operator norm may vanish, complicating the analysis and necessitating alternative approaches for handling these scenarios.

Overall, our paper contributes to advancing the understanding of pseudo-differential operators in bi-parameter settings, laying the groundwork for future research in this area. Through our novel decomposition approach and careful analysis, we uncover insights that extend Fefferman's seminal work and pave the way for broader applications in harmonic analysis and operator theory.

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