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The Role of Risk Factors in System Performance: A Comprehensive Study with Adaptive Progressive Type-II Censoring

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Abstract: The quality performance of many vital systems depends on how long the units are performing; hence, research works started focusing on increasing the reliability of systems while taking into consideration that many factors may cause the failures of operating systems. In this study, the combination of a parametric generalized linear failure rate distribution model and an adaptive progressive Type-II censoring scheme for practical purposes is explored. A comprehensive investigation is performed on the risk factors that cause failure and determines which of the factors has a more harmful effect on the units. A lifetime experiment is performed under the condition of an adaptive progressive Type-II censoring scheme to obtain observations as a result of the competing factors of failures. The obtained observations are assumed to follow a three-parameter generalized linear failure rate distribution and are assumed to be competing to cause failure. Two statistical inference methods are employed for estimating this model's parameters: the frequentist maximum likelihood method and the Bayesian approach. Our model's validity is demonstrated through extensive simulations and real data applications in the medical and electrical engineering fields.

Keywords: generalized linear failure rate; adaptive progressive censoring; competing risk models; Markov chain Monte Carlo; survival analysis; hazard rate; simulation

MSC: 62E10; 62F15; 62N05; 60E05; 62P30



Citation: Ahmad, H.H.; Aboshady, M.; Mansour, M. The Role of Risk Factors in System Performance: A Comprehensive Study with Adaptive Progressive Type-II Censoring. *Mathematics* **2024**, *12*, 1763. <https://doi.org/10.3390/math12111763>

Academic Editors: Ioannis S. Triantafyllou and Alex Karagrigoriou

Received: 10 April 2024

Revised: 30 May 2024

Accepted: 1 June 2024

Published: 5 June 2024



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1. Introduction

The competing risk model represents a critical framework for analyzing scenarios where multiple potential events of interest might prevent the observation of each other. This model is especially pivotal in biomedical research, reliability engineering, and demographic studies, where it is crucial to understand when events occur and which of several possible outcomes happens first. The inception of competing risk models dates back to the mid-20th century, emerging from the need to address the complexities in survival analysis and time-to-event data that traditional models could not adequately explain. Early work in this area sought to account for the fact that, in many real-world situations, the occurrence of one type of event (e.g., death due to a specific cause) precludes the occurrence of another (e.g., death due to an alternative cause), thus necessitating a more nuanced analytical approach. For more details, see [1–5].

In many technical and technological applications, such as power transmission, energy storage systems, and electronic devices, electrodes are essential, and the stability and longevity of these electrodes are important research issues. Researchers are testing

electrodes' voltage endurance life to understand their behavior and enhance performance to address this issue. Investigating how various electrode materials and designs fare during extended exposure to a certain voltage is the goal of this research.

Researchers may imitate real-world situations where such electrodes are employed, such as in power energy systems, by subjecting electrodes to extremely harsh electrical conditions. The goal is to pinpoint the constraints, failure sources, and ideal design adjustments required to improve the electrodes' overall performance and dependability. Engineers hope to increase electrical device safety, decrease downtime due to electrode failure, and improve the system efficiency. The resolution of this scientific issue will significantly impact numerous businesses that rely on electrodes, resulting in technical improvements and sustainable energy solutions.

The application in medicine relates to Multiple Myeloma, a type of blood cancer that affects plasma cells, which are important components of the immune system in the body. The need for novel medical applications to manage, treat, and possibly cure Multiple Myeloma is highlighted by the fact that there are now several therapy choices available but the disease is still incurable. Researchers are looking into many medical applications to address this significant scientific topic. Finding trustworthy Multiple Myeloma diagnostic and prognostic markers, developing customized treatment regimens, and investigating state-of-the-art therapies are all necessary to achieve this.

It is necessary to compare the causes of risk model failure to understand the suitable elements, influence risk management, and make an appropriate decision. A pair comprising a failure cause and a failure time must be expressed in competing risk models for each observed failure. In addition to different causes of failures, reliability tests often require filtering systems due to various factors such as time and budget constraints.

Because filtering samples significantly affects reliability analysis results, analysis of various censoring methods under competing risks has gained popularity in recent years. Censoring schemes have been introduced to solve the lack of information in lifetime experiments, saving time and cost. Type-I censoring has a predetermined time, while Type-II censoring has predetermined failure units. The progressive censoring scheme has been studied by many researchers; see [6–9]. The estimation problem of lifetime models under progressively censored data has also been investigated [10,11]. To control the total experimental time, an adaptive progressive censoring scheme (APCS) was explored by Ng et al. [12], which allows the test time to continue beyond the initial time determined earlier. The adaptive progressive Type-II scheme allows the experimenter to control the termination time of the experiment as much as possible to obtain more observations at a suitable time with a minimum experimental cost.

The APCS assumes that, during the life test, when the i th failure is obtained, R_i units are randomly eliminated from the experiment. Let $X_{i:m:n}$, $i = 1, 2, \dots, m$ represent the lifetime of the m failed observations. If the m th time of failure happens earlier than time T ($X_{m:m:n} < T$), then the experiment stops at time $X_{m:m:n}$, with progressive censored items described as (R_1, R_2, \dots, R_m) such that $R_m = n - m - \sum_{i=1}^{m-1} R_i$. If the J th failure time happens before time T , i.e., $X_{J:m:n} < T < X_{J+1:m:n}$, ($1 \leq J \leq m-1$), where $X_{0:m:n} = 0$ and $X_{m+1:m:n} = \infty$, then we adjust the number of progressively withdrawn items of the experiment at the time of failures by setting $R_{J+1} = R_{J+2} = \dots = R_{m-1} = 0$, whereas, at the time $X_{m:m:n}$, all the operating items R_m are discarded, where $R_m = n - m - \sum_{i=1}^J R_i$.

So, at this stage, an APCS is described as $(R_1, R_2, \dots, R_J, 0, \dots, 0, n - m - \sum_{i=1}^J R_i)$, $J = \max j : X_{j:m:n} < T$. This suggests that the initial failure time recorded is overlying the prefixed overall duration T . In other words, as long as the failures occur earlier than time T , the required progressive scheme is considered. After going through time T , no unit is eliminated, while, at the time of m th failure, the remaining operating units are discarded. This decision terminates the experiment whenever the failure $(J+1)$ th time exceeds T and the total experimental time is near the time T . In the rare event that $T = 0$, the system reduces to the conventional Type-II censoring scheme; otherwise, a basic

progressive Type-II censoring scheme is seen if $T \rightarrow \infty$. This illustrates how to conduct a well-controlled experiment.

One of the limitations of the adaptive progressive Type-II censoring system is the sample size. The plan might call for a greater number of samples than other censoring methods, which would increase the cost and resources required for data collection. Furthermore, assessing data from an adaptive progressive Type-II censoring scheme may be computationally demanding, requiring the use of advanced statistical methods. Although the adaptive progressive Type-II censoring scheme is flexible and efficient, it is also complex. This complexity might lead to difficulties in ensuring the correct application of the method across different scenarios.

Recent dialogues regarding the APCS, as depicted in Figure 1, have investigated its application across different statistical models such as the exponentiated Weibull, inverse Weibull, and log-logistic distributions, to name a few. In these investigations, the Bayesian and the maximum likelihood estimation methods are applied to deduce the model's parameters, reliability function, and hazard rates. The Weibull distribution's ability to take on the characteristics of other distributions makes it exceptionally flexible for practical applications. It can effectively model the lifetime of mechanical components, biological organisms, and complex systems under many conditions. This adaptability is useful in failure data analysis. The GLFR distribution, a generalization of the exponential and Weibull distributions, offers even greater flexibility by including an additional parameter to adjust the failure rate over time. For more details, refer to [13–18].

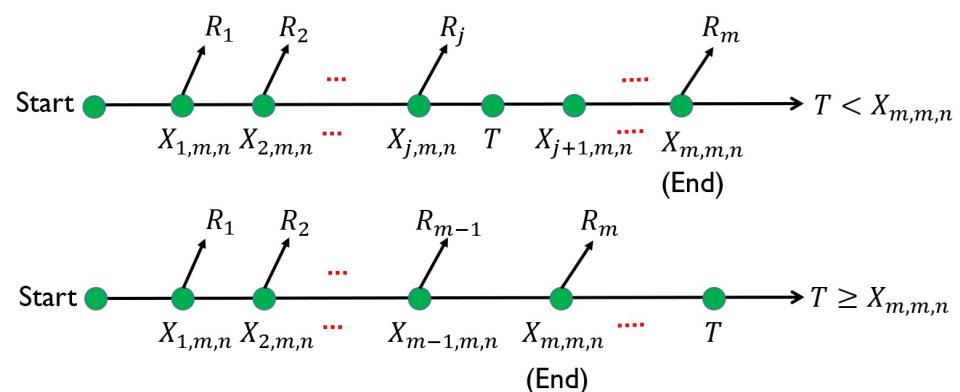


Figure 1. Illustration of APCS.

One of the foundational aspects of the competing risk model is its reliance on a censoring scheme, which is critical for handling incomplete observations. In the context of competing risks, censoring occurs when the event of interest is not observed either because a different event precludes it or due to the end of the study period. This necessitates sophisticated statistical techniques to estimate the probability of each competing event accurately. The progressively Type-II censoring technique in the context of competing risks has been the subject of several studies, such as [19–32].

We provide an adaptive progressive Type-II censoring scheme that operates within the framework of independent competing risk variables. This method increases the accuracy and dependability of event time forecasts while also improving the efficiency and cost effectiveness of data collection. Our method, in contrast to current models, dynamically modifies the observed data to enable more accurate and fast interventions. The APCS technique explored in this study has two independent competing risk variables, and component lifespans follow a three-parameter generalized linear failure rate (GLFR) model. Known for its flexible hazard function, the GLFR model extends several key lifetime models, including the Rayleigh, generalized Rayleigh, exponential, and generalized exponential distributions, and can handle a broad variety of real-world data. This technique works especially well when examining data that have been gradually filtered, such as in reliability studies, electri-

cal engineering, and other domains where test volunteers may be dropped mid-experiment. It is also assumed here that different independent risk factors follow the GLFR model. The reliability function is estimated using numerical techniques. This work's key objective is to investigate an adaptive progressive Type-II censoring strategy that improves the precision and adaptability of competing risk models. By enabling dynamic modifications based on real-time data, this approach seeks to increase the effectiveness of statistical models in domains that demand precise time analysis. Evaluating the effectiveness of the recently created filtering scheme in various contexts, such as reliability engineering and medical applications, is another important goal. In order to make sure the scheme can effectively handle the complexities of the actual world, it must be tested under a variety of scenarios in order to evaluate its robustness, dependability, and applicability.

We adopt an adaptive progressive Type-II censoring strategy under the GLFR distribution in the context of a competing risks model, which makes our study novel and a considerable contribution. In comparison to other uses (electrical and medicinal), this is comparatively rarely researched in the literature. Using the observed data, the APCS allows for a dynamic modification of the experiment that improves both the quality and efficiency of data gathering when gathering data on the lifetime of patients or components. In addition to saving money and time and enhancing reliability estimates, this approach is especially novel since it can change according to the data as it happens. Significant value is added to this research by the application and implementation of the GLFR model in a new censoring scheme. Because of its adaptability to representing a variety of data, the GLFR model is well suited for complicated data types that may not fit well to other conventional models. Our study adds to the collection of knowledge by shedding light on the practical applications of this model inside an APCS framework.

The paper is structured as follows: In Section 2, we define the model. The maximum likelihood estimation (MLE) of the unknown parameters is covered in Section 3. Section 4 describes and illustrates the procedure for finding the parameters' Bayes estimators. To emphasize each proposed inference technique, we provide a numerical real data example in Section 5. Simulation analysis is performed in Section 6. Some resulting remarks are presented in Section 7.

2. Model Description

The GLFR is a continuous lifetime distribution that generalizes some well-known distributions such as the linear failure rate, the exponential, the generalized exponential, and the Rayleigh and generalized Rayleigh distributions. The GLFR distribution is sometimes a better fit for statistical inference than the Weibull distribution for several reasons. One reason in its favor is that it is capable of modeling non-monotonic failure rates, which are outside the scope of the Weibull distribution. In certain cases, this flexibility in modeling various failure behaviors can result in more precise and trustworthy statistical inference outcomes. It was first studied by Sarhan and Kundu [33]. This new generalized distribution overlaps the shortage and limitation of other sub-models. In [33], it was shown that GLFR distribution can be decreasing or unimodal, and the hazard function can be increasing, decreasing, or bathtub shaped. This indicates the flexibility of this distribution and its ability to fit different kinds of real-life data in many fields of science.

The following are the assumptions of the proposed model: The lifetime experiment is performed with n identically distributed and independent random variables X_1, \dots, X_n . Two independent failure factors cause units to fail, with X_{ki} , $k = 1, 2$ denoting the latent failure times of the i th unit due to the k th failure reason. The model assumes also a third unknown factor competing with the other two factors.

Let $X_i = \min\{X_{1i}, X_{2i}\}$, $i = 1, \dots, n$. Assume X_{ki} , $k = 1, 2$ follow the GLFR distribution. Then, the cumulative distribution function (CDF) and the probability density function (pdf) are

$$F(x, \alpha, \beta, \theta) = \left[1 - e^{-\left(\alpha x + \frac{\beta}{2} x^2\right)} \right]^\theta, \quad x > 0, \alpha, \beta, \theta > 0, \quad (1)$$

and

$$f(x, \alpha, \beta, \theta) = \theta(\alpha + \beta x) \left[1 - e^{-\left(\alpha x + \frac{\beta}{2} x^2\right)} \right]^{\theta-1} e^{-\left(\alpha x + \frac{\beta}{2} x^2\right)}, \quad x > 0, \alpha, \beta, \theta > 0, \quad (2)$$

respectively, where θ_k is a shape parameter.

In competing risk situations with an APCS, the data are described as $(X_{1:m:n}, \delta_1, R_1), \dots, (X_{J:m:n}, \delta_J, R_J), (X_{J+1:m:n}, \delta_{J+1}, 0), \dots, (X_{m-1:m:n}, \delta_{m-1}, 0), (X_{m:m:n}, \delta_m, R_m)$,

where $J = \max\{j : X_{j:m:n} < T\}$, $R_m = n - m - \sum_{i=1}^m R_i$, and $\delta_i \in \{1, 2, *\}$. If $\delta_i = 1$ or 2, it means that the unit fails at time $X_{i:m:n}$ according to 1 or 2, respectively, while if $\delta_i = *$, it means failure happened with unknown cause.

Let $I(\varphi)$ be an indicator of φ . Then, $m_1 = \sum_{i=1}^m I(\delta_i = 1)$ and $m_2 = \sum_{i=1}^m I(\delta_i = 2)$ are the number of failures caused, respectively, by the first and the second reasons of failure. Additionally, $m_3 = \sum_{i=1}^m I(\delta_i = *)$ represents the number of failures with unknown causes; hence, $m_1 + m_2 + m_3 = m$.

Classical and Bayesian estimation methods are constructed in this work to estimate the unknown parameters for the GLFR distribution with an APCS of two competing risk factors. Furthermore, we aim to assess the reliability of this model and its suitability to fit and describe the trend of medical and electrical data.

3. Maximum Likelihood Estimation

Referring to the APCS with two competing risk factors, the following is the expression of the likelihood function for the data that were observed $(x_1, \delta_1), \dots, (x_m, \delta_m)$:

$$L(\Phi; x) = A_j \prod_{i=1}^m \{ [f_1(x_i) \bar{F}_2(x_i)]^{I_{(\delta_i=1)}} [f_2(x_i) \bar{F}_1(x_i)]^{I_{(\delta_i=2)}} [f_1(x_i) \bar{F}_2(x_i) + f_2(x_i) \bar{F}_1(x_i)]^{I_{(\delta_i=*)}} \} \\ \times \prod_{i=1}^j \{ [\bar{F}_1(x_i) \bar{F}_2(x_i)]^{R_i} \} \times [\bar{F}_1(x_m) \bar{F}_2(x_m)]^{R^*}, \quad (3)$$

where $A_j = \prod_{i=1}^m [n - i + 1 - \sum_{j=1}^{\max\{i-1, j\}} R_j]$, $R^* = n - m - \sum_{i=1}^j R_i$, $\Phi = (\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2)$.

Let $u_i(\alpha, \beta) = 1 - e^{-(\alpha x_i + \frac{\beta}{2} x_i^2)}$. Then, by using the CDF and pdf of the GLFR distribution in Equations (1) and (2), and substituting in the likelihood Equation (3), we obtain

$$L(\Phi; x) = A_j \prod_{i=1}^m \left[\theta_1 (\alpha_1 + \beta_1 x_i) [u_i(\alpha_1, \beta_1)]^{\theta_1-1} (1 - u_i(\alpha_1, \beta_1)) \left(1 - [u_i(\alpha_2, \beta_2)]^{\theta_2} \right) \right]^{I_{(\delta_i=1)}} \\ \times \left[\theta_2 (\alpha_2 + \beta_2 x_i) [u_i(\alpha_2, \beta_2)]^{\theta_2-1} (1 - u_i(\alpha_2, \beta_2)) (1 - [u_i(\alpha_1, \beta_1)]^{\theta_1}) \right]^{I_{(\delta_i=2)}} \\ \times \left[(\alpha_1 + \beta_1 x_i) (1 - u_i(\alpha_1, \beta_1)) \theta_1 [u_i(\alpha_1, \beta_1)]^{\theta_1-1} (1 - [u_i(\alpha_2, \beta_2)]^{\theta_2}) \right. \\ \left. + (\alpha_2 + \beta_2 x_i) (1 - u_i(\alpha_2, \beta_2)) \theta_2 [u_i(\alpha_2, \beta_2)]^{\theta_2-1} [1 - [u_i(\alpha_1, \beta_1)]^{\theta_1}] \right]^{I_{(\delta_i=*)}} \\ \times \prod_{i=1}^j \left[\left((1 - [u_i(\alpha_1, \beta_1)]^{\theta_1}) \right) \left(1 - [u_i(\alpha_2, \beta_2)]^{\theta_2} \right) \right]^{R_i} \prod \left[(1 - [u_m(\alpha_1, \beta_1)]^{\theta_1}) (1 - [u_m(\alpha_2, \beta_2)]^{\theta_2}) \right]^{R^*}. \quad (4)$$

The log-likelihood function is then

$$\begin{aligned}
\ell(\Phi; x) &\propto m_1 \log(\theta_1) + m_2 \log(\theta_2) \\
&+ \sum_{i=1}^m I_{(\delta_i=1)} \log(\alpha_1 + \beta_1 x_i) + \sum_{i=1}^m I_{(\delta_i=2)} \log(\alpha_2 + \beta_2 x_i) \\
&+ \sum_{i=1}^m I_{(\delta_i=1)} \left[(\theta_1 - 1) \log[u_i(\alpha_1, \beta_1)] + \log[1 - u_i(\alpha_1, \beta_1)] + \log \left(1 - [u_i(\alpha_2, \beta_2)]^{\theta_2} \right) \right] \\
&+ \sum_{i=1}^m I_{(\delta_i=2)} \left[(\theta_2 - 1) \log[u_i(\alpha_2, \beta_2)] + \log[1 - u_i(\alpha_2, \beta_2)] + \log \left(1 - [u_i(\alpha_1, \beta_1)]^{\theta_1} \right) \right] \\
&+ \sum_{i=1}^m I_{(\delta_i=*)} \left[\log \left[\theta_1(\alpha_1 + \beta_1 x_i) [u_i(\alpha_1, \beta_1)]^{\theta_1-1} (1 - [u_i(\alpha_1, \beta_1)]) (1 - [u_i(\alpha_2, \beta_2)]^{\theta_2}) \right. \right. \\
&\quad \left. \left. + \theta_2(\alpha_2 + \beta_2 x_i) [u_i(\alpha_2, \beta_2)]^{\theta_2-1} (1 - [u_i(\alpha_2, \beta_2)]) (1 - [u_i(\alpha_1, \beta_1)]^{\theta_1}) \right] \right] \\
&+ \sum_{i=1}^j R_i \left[\log \left(1 - [u_i(\alpha_1, \beta_1)]^{\theta_1} \right) + \log \left(1 - [u_i(\alpha_2, \beta_2)]^{\theta_2} \right) \right] \\
&+ R^* \left[\log \left(1 - [u_m(\alpha_1, \beta_1)]^{\theta_1} \right) + \log \left(1 - [u_m(\alpha_2, \beta_2)]^{\theta_2} \right) \right],
\end{aligned}$$

The partial derivatives with respect to $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$, and θ_2 are as follows:

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha_k} &= \sum_{i=1}^m I_{(\delta_i=k)} \left[\frac{1}{\alpha_k + \beta_k x_i} + x_i \left(\frac{\theta_k - 1}{u_i(\alpha_k, \beta_k)} - \theta_k \right) \right] - \sum_{i=1}^m I_{(\delta_i=3-k)} \frac{\theta_k x_i u_i^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_i(\alpha_k, \beta_k))}{1 - u_i^{\theta_k}(\alpha_k, \beta_k)} \\
&+ \sum_{i=1}^m \frac{I_{(\delta_i=*)}}{W(\Phi)} \theta_k (1 - u_i(\alpha_k, \beta_k)) \left[-x_i(\alpha_k + \beta_k x_i) u_i^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_i^{\theta_{3-k}}(\alpha_{3-k}, \beta_{3-k})) \right. \\
&+ (1 - u_i^{\theta_{3-k}}(\alpha_{3-k}, \beta_{3-k})) \left((\theta_k - 1)(\alpha_k + \beta_k x_i) x_i u_i^{\theta_k-2}(\alpha_k, \beta_k) (1 - u_i(\alpha_k, \beta_k)) + u_i^{\theta_k-1}(\alpha_k, \beta_k) \right) \\
&\left. - u_i^{\theta_k-1}(\alpha_k, \beta_k) x_i \theta_{3-k}(\alpha_{3-k} + \beta_{3-k} x_i) u_i^{\theta_{3-k}-1}(\alpha_{3-k}, \beta_{3-k}) (1 - u_i(\alpha_{3-k}, \beta_{3-k})) \right] \\
&- \sum_{i=1}^j R_i \frac{\theta_k x_i u_i^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_i(\alpha_k, \beta_k))}{1 - u_i^{\theta_k}(\alpha_k, \beta_k)} - R^* \frac{\theta_k x_m u_m^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_m(\alpha_k, \beta_k))}{1 - u_m^{\theta_k}(\alpha_k, \beta_k)}. \quad (5)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta_k} &= \sum_{i=1}^m I_{(\delta_i=k)} \left[\frac{x_i}{\alpha_k + \beta_k x_i} + \frac{x_i^2}{2} \left(\frac{\theta_k - 1}{u_i(\alpha_k, \beta_k)} - \theta_k \right) \right] \\
&- \sum_{i=1}^m I_{(\delta_i=3-k)} \frac{x_i^2}{2} \frac{\theta_k u_i^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_i(\alpha_k, \beta_k))}{1 - u_i^{\theta_k}(\alpha_k, \beta_k)} \\
&+ \sum_{i=1}^m \frac{I_{(\delta_i=*)}}{W(\Phi)} \theta_k (1 - u_i(\alpha_k, \beta_k)) \left[-\frac{x_i^2}{2} (\alpha_k + \beta_k x_i) u_i^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_i^{\theta_{3-k}}(\alpha_{3-k}, \beta_{3-k})) \right. \\
&+ (1 - u_i^{\theta_{3-k}}(\alpha_{3-k}, \beta_{3-k})) \left((\theta_k - 1)(\alpha_k + \beta_k x_i) \frac{x_i^2}{2} u_i^{\theta_k-2}(\alpha_k, \beta_k) (1 - u_i(\alpha_k, \beta_k)) + x_i u_i^{\theta_k-1}(\alpha_k, \beta_k) \right) \\
&\left. - u_i^{\theta_k-1}(\alpha_k, \beta_k) \frac{x_i^2}{2} \theta_{3-k}(\alpha_{3-k} + \beta_{3-k} x_i) u_i^{\theta_{3-k}-1}(\alpha_{3-k}, \beta_{3-k}) (1 - u_i(\alpha_{3-k}, \beta_{3-k})) \right] \\
&- \sum_{i=1}^j R_i \frac{x_i^2}{2} \frac{\theta_k u_i^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_i(\alpha_k, \beta_k))}{1 - u_i^{\theta_k}(\alpha_k, \beta_k)} - R^* \frac{x_m^2}{2} \frac{\theta_k u_m^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_m(\alpha_k, \beta_k))}{1 - u_m^{\theta_k}(\alpha_k, \beta_k)} \quad (6)
\end{aligned}$$

$$\frac{\partial \ell}{\partial \theta_k} = \frac{m_k}{\theta_k} + \sum_{i=1}^{m_k} \log[u_i(\alpha, \beta)] - \frac{[u_i(\alpha, \beta)]^{\theta_k} \log[u_i(\alpha, \beta)]}{1 - [u_i(\alpha, \beta)]^{\theta_k}} - \sum_{i=1}^j R_i \frac{[u_i(\alpha, \beta)]^{\theta_k} \log[u_i(\alpha, \beta)]}{1 - [u_i(\alpha, \beta)]^{\theta_k}} + R^* \frac{[u_m(\alpha, \beta)]^{\theta_k} \log[u_m(\alpha, \beta)]}{1 - [u_m(\alpha, \beta)]^{\theta_k}}, \quad (7)$$

where

$$W(\Phi) = \theta_k(\alpha_k + \beta_k x_i) u_i^{\theta_k-1}(\alpha_k, \beta_k) (1 - u_i(\alpha_k, \beta_k)) (1 - u_i^{\theta_{3-k}}(\alpha_{3-k}, \beta_{3-k})) + \theta_{3-k}(\alpha_{3-k} + \beta_{3-k} x_i) u_i^{\theta_{(3-k)}-1}(\alpha_{3-k}, \beta_{3-k}) (1 - u_i(\alpha_{3-k}, \beta_{3-k})) (1 - u_i^{\theta_k}(\alpha_k, \beta_k)),$$

and $k = 1, 2$.

To address the system of nonlinear equations presented in Equations (5)–(7), numerical approaches are essential. Various numerical methods have been applied in the existing research. In this instance, we employ the Newton–Raphson method. The outcomes of this application are detailed in Section 6.

Confidence interval estimation is a fundamental statistical method used to indicate the reliability of an estimate. The concept is central in inferential statistics and serves numerous applications across various fields such as engineering, economics, medicine, and social sciences. Among its key properties, the asymptotic interval is notable for its reliance on large sample sizes, where the distribution of the estimate approaches a normal distribution, making it increasingly accurate as the sample size grows. For constructing asymptotic confidence intervals (ACI), we must first define Fisher's information matrix.

The MLEs $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta}_1, \hat{\theta}_2)$ are approximately normal with a mean $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta}_1, \hat{\theta}_2)$ and a variance–covariance matrix $I^{-1}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta}_1, \hat{\theta}_2)$. Here, the Fisher information matrix $I(\Phi)$ can be obtained based on the second partial derivatives of the likelihood function for the parameters. Consequently, the estimated asymptotic variance–covariance matrix for the MLEs is obtained by taking the inverse of the observed information matrix; hence, its elements can be described as diagonal elements representing the variances of the estimated parameters as $\widehat{Var}(\hat{\alpha}_1)$, $\widehat{Var}(\hat{\alpha}_2)$, $\widehat{Var}(\hat{\beta}_1)$, $\widehat{Var}(\hat{\beta}_2)$, $\widehat{Var}(\hat{\theta}_1)$, and $\widehat{Var}(\hat{\theta}_2)$. In contrast, the off-diagonal elements represent the covariance between the estimated parameters.

The $100(1 - \zeta)\%$ two-sided confidence interval can be written as

$$\hat{\alpha}_k \pm Z_{\frac{\zeta}{2}} \sqrt{\widehat{Var}(\hat{\alpha}_k)}, \quad \hat{\beta}_k \pm Z_{\frac{\zeta}{2}} \sqrt{\widehat{Var}(\hat{\beta}_k)}, \text{ and } \hat{\theta}_k \pm Z_{\frac{\zeta}{2}} \sqrt{\widehat{Var}(\hat{\theta}_k)}. \quad (8)$$

The standard normal distribution's percentile, denoted by $Z_{\frac{\zeta}{2}}$, has a right-side probability of $\frac{\zeta}{2}$, and $k = 1, 2$.

In addition to estimating the unknown parameters of the proposed model, we aim to give estimates for the survival and the hazard rate functions. At this point, we use a statistical approach called the Delta method. This method is used to approximate the distribution of a function of a random variable, particularly when the variance of that random variable is known or can be estimated. This method is especially useful in constructing confidence intervals for complex functions of parameters, such as the hazard rate function and reliability function in survival analysis or reliability engineering, where direct analytical solutions may not be straightforward. The following summarizes the Delta algorithm:

- Find the MLE for the parameters $(\hat{\Phi})$;
- Linearize the hazard rate and the survival function around the estimated parameter using a first-order Taylor series expansion;
- Estimate the variance of the linearized hazard and survival function. The variance can be approximated by using the gradient of the function and the covariance matrix of the parameter estimates;

- Construct the confidence interval for the function by using its estimated value and variance.

Hence, the $100(1 - \zeta)\%$ confidence intervals for the hazard function $h(t)$ and the survival function $S(t)$ can be written, respectively, as

$$h(\hat{\Phi}) \pm Z_{\frac{\zeta}{2}} \sqrt{\text{Var}(h(\hat{\Phi}))} \quad \text{and} \quad S(\hat{\Phi}) \pm Z_{\frac{\zeta}{2}} \sqrt{\text{Var}(S(\hat{\Phi}))} \quad (9)$$

4. Bayesian Estimation

Bayesian estimation plays an important role in statistical inference; it permits the incorporation for integration of prior knowledge or beliefs into the estimation process. This is a useful feature when there is a lack of data or when predictions need to be made in the face of uncertainty. By using Bayesian statistics, researchers can update their assumptions in response to the latest data, producing estimates that are more reliable and accurate. Moreover, the Bayesian approach provides an organized means of measuring uncertainty, a crucial component of well-informed decision-making. Overall, Bayesian estimation presents a more adaptable and resilient approach to statistical inference than traditional frequentist methods; see [34–36]. The Bayesian technique assumes that the parameters of the model are random variables following a distribution known as the prior distribution. In this study, we opt for an exponential conjugate prior distribution for $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$, and θ_2 as follows:

$$\begin{aligned} \alpha_1 &\sim \text{Exp}(a_1) \Rightarrow \pi_1(\alpha_1) = a_1 e^{-\alpha_1 a_1}, & \alpha_1 &> 0, \\ \alpha_2 &\sim \text{Exp}(a_2) \Rightarrow \pi_2(\alpha_2) = a_2 e^{-\alpha_2 a_2}, & \alpha_2 &> 0, \\ \beta_1 &\sim \text{Exp}(a_3) \Rightarrow \pi_3(\beta_1) = a_3 e^{-\beta_1 a_3}, & \beta_1 &> 0, \\ \beta_2 &\sim \text{Exp}(a_4) \Rightarrow \pi_4(\beta_2) = a_4 e^{-\beta_2 a_4}, & \beta_2 &> 0, \\ \theta_1 &\sim \text{Exp}(a_5) \Rightarrow \pi_5(\theta_1) = a_5 e^{-\theta_1 a_5}, & \theta_1 &> 0, \\ \theta_2 &\sim \text{Exp}(a_6) \Rightarrow \pi_6(\theta_2) = a_6 e^{-\theta_2 a_6}, & \theta_2 &> 0. \end{aligned} \quad (10)$$

Our application of Bayesian inference using exponential priors is necessary in the competing risks model focusing on specific applications in electrical engineering and medical research. While Bayesian methods are well known in the statistical literature, their application in the censored reliability data involving competing risks represents a significant extension of existing methodologies. This results in a more robust and reliable statistical inference in complex real-world scenarios where traditional methods may fall short.

The joint prior distribution is then

$$p(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2) \propto e^{-(\alpha_1 a_1 + \alpha_2 a_2 + \beta_1 a_3 + \beta_2 a_4 + \theta_1 a_5 + \theta_2 a_6)},$$

and the joint posterior density is given by

$$p^*(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2 \mid \text{data}) = \frac{L(\Phi; x) p(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2)}{\int_{\alpha_1} \int_{\alpha_2} \int_{\beta_1} \int_{\beta_2} \int_{\theta_1} \int_{\theta_2} L(\Phi; x) p(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2) d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 d\theta_1 d\theta_2}, \quad (11)$$

or simply $p^*(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2 \mid \text{data}) = \frac{1}{k} L(\Phi; x) p(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2)$, where $L(\Phi; x)$ is the likelihood function of the GLFR distribution under an APCS in Equation (4). The full conditional posterior densities of $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2$, and the data is given by

$$\begin{aligned}
\pi_{\alpha_k}^*(\alpha_k \mid \alpha_{3-k}, \beta_1, \beta_2, \theta_1, \theta_2; \text{data}) &\propto e^{-\alpha_k a_1} \prod_{i=1}^m \left[(\alpha_k + \beta_k x_i) [u_i(\alpha_k, \beta_k)]^{\theta_k-1} e^{-\alpha_k x_i} \right]^{I_{(\delta_i=k)}} \\
&\times \left[1 - [u_i(\alpha_k, \beta_k)]^{\theta_k} \right]^{I_{(\delta_i=3-k)}} \left[\theta_k (\alpha_k + \beta_k x_i) (1 - u_i(\alpha_k, \beta_k)) [u_i(\alpha_k, \beta_k)]^{\theta_k-1} (1 - [u_i(\alpha_{3-k}, \beta_{3-k})]^{\theta_{3-k}}) \right. \\
&\left. + \theta_{3-k} (\alpha_{3-k} + \beta_{3-k} x_i) (1 - u_i(\alpha_{3-k}, \beta_{3-k})) [u_i(\alpha_{3-k}, \beta_{3-k})]^{\theta_{3-k}-1} [1 - [u_i(\alpha_k, \beta_k)]^{\theta_k}] \right]^{I_{(\delta_i=*)}} \\
&\times \prod_{i=1}^j \left(1 - [u_i(\alpha_k, \beta_k)]^{\theta_k} \right)^{R_i} \left(1 - [u_m(\alpha_k, \beta_k)]^{\theta_k} \right)^{R^*} \\
\pi_{\beta_k}^*(\beta_k \mid \alpha_1, \alpha_2, \beta_{3-k}, \theta_1, \theta_2; \text{data}) &\propto e^{-\beta_k a_3} \prod_{i=1}^m \left[(\alpha_k + \beta_k x_i) [u_i(\alpha_k, \beta_k)]^{\theta_k-1} e^{-\frac{\beta_k}{2} x_i^2} \right]^{I_{(\delta_i=k)}} \\
&\times \left[1 - [u_i(\alpha_k, \beta_k)]^{\theta_k} \right]^{I_{(\delta_i=3-k)}} \left[\theta_k (\alpha_k + \beta_k x_i) (1 - u_i(\alpha_k, \beta_k)) [u_i(\alpha_k, \beta_k)]^{\theta_k-1} [1 - [u_i(\alpha_{3-k}, \beta_{3-k})]^{\theta_{3-k}}] \right. \\
&\left. + \theta_{3-k} (\alpha_{3-k} + \beta_{3-k} x_i) (1 - u_i(\alpha_{3-k}, \beta_{3-k})) [u_i(\alpha_{3-k}, \beta_{3-k})]^{\theta_{3-k}-1} [1 - [u_i(\alpha_k, \beta_k)]^{\theta_k}] \right]^{I_{(\delta_i=*)}} \\
&\times \prod_{i=1}^j \left(1 - [u_i(\alpha_k, \beta_k)]^{\theta_k} \right)^{R_i} \left(1 - [u_m(\alpha_k, \beta_k)]^{\theta_k} \right)^{R^*} \\
\pi_{\theta_k}^*(\theta_k \mid \alpha_1, \alpha_2, \beta_1, \beta_2, \theta_{3-k}; \text{data}) &\propto e^{-\theta_k a_5} \prod_{i=1}^m \left[\theta_k [u_i(\alpha_k, \beta_k)]^{\theta_k} \right]^{I_{(\delta_i=k)}} \left[1 - [u_i(\alpha_k, \beta_k)]^{\theta_k} \right]^{I_{(\delta_i=3-k)}} \\
&\times \left[\theta_k (\alpha_k + \beta_k x_i) (1 - u_i(\alpha_k, \beta_k)) [u_i(\alpha_k, \beta_k)]^{\theta_k-1} [1 - [u_i(\alpha_{3-k}, \beta_{3-k})]^{\theta_{3-k}}] \right. \\
&\left. + \theta_{3-k} (\alpha_{3-k} + \beta_{3-k} x_i) (1 - u_i(\alpha_{3-k}, \beta_{3-k})) [u_i(\alpha_{3-k}, \beta_{3-k})]^{\theta_{3-k}-1} [1 - [u_i(\alpha_k, \beta_k)]^{\theta_k}] \right]^{I_{(\delta_i=*)}} \\
&\times \prod_{i=1}^j \left(1 - [u_i(\alpha_k, \beta_k)]^{\theta_k} \right)^{R_i} \left(1 - [u_m(\alpha_k, \beta_k)]^{\theta_k} \right)^{R^*},
\end{aligned}$$

where $k = 1, 2$.

Under various symmetric and asymmetric loss functions, the Bayes estimators of any function of $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$, and θ_2 , that is, $g(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2)$, can be obtained. Two distinct loss functions are used, namely, the general entropy loss (GEL) and the squared error loss (SEL) functions.

The conditional density function for $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$, and θ_2 is hard to obtain. Thus, the normal proposal distribution is used to apply the Metropolis–Hasting (M-H) method, which was proposed by Metropolis et al. [37], to generate random samples from the posterior density of $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$, and θ_2 . The description of the Gibbs sampling algorithm is explored in detail in [24,38–40].

5. Application to Real-Life Data

In this section, the proposed methods are applied to two real-life data sets, one related to medical application and the other to an engineering application, to demonstrate the applicability of the proposed inference methods to actual phenomena. The two data sets' observations include failure times caused by three failure factors, which may result in

certain anomalies. The GLFR distribution, which is a parametric model, is fitted to the two data sets; therefore, the impact of anomalies disappears.

5.1. Data Related to Multiple Myeloma

We analyze a genuine competing risks data set utilizing APCS to represent the effectiveness of parameter estimation under the underlying GLFR distribution. The data set includes 35 patients treated at University Hospital Hamburg-Effendorf, Hamburg, at Germany's Clinic for Stem Cell Transplantation. The patients exhibit ten instances of transplant-related mortality as a competing risk and 19 cases of Multiple Myeloma recurrence. Information on recipients of transplants from donors with type AA, type AB, or type BB haplotypes was supplied by Donoghoe and Gebiski [41]. While applying the competing risks approach, we focus on deriving statistical implications from these data. For (x_i, d_i, R_i) with $i = 1, 2, \dots, 35$, Table 1 lists the times along with the censoring and event pattern. The times with $d_i = 1$ represent the time to relapse, $d_i = 2$ represents transplant-related mortality, and $d_i = 0$ represents the censoring time with the removal $R_i = 1$. For $d_i = 1$ and $d_i = 2$, the model efficacy is analyzed using some measures of fitness such as the Kolmogorov–Smirnov (KS), Anderson–Darling, and Cramér–von Mises tests. Both distance (statistics) and the p -value are evaluated to assess the suitability of our model to describe these data. It can be noticed from Table 2 that the p -values of all measures are greater than 0.05, ensuring the suitability of the proposed model to fit the Myeloma data. Figures 2 and 3 illustrate the distance between the CDF for the GLFR distribution and the empirical distribution for the failure data, in addition to the observed probability distribution against the expected P-P plot and the quantile plot Q-Q for the two cases $d_i = 1$ and $d_i = 2$, respectively.

Table 1. Survival times according to censoring, relapse, and transplant-related mortality.

$d_i = 0$	89.89	56.57	53.55	44.02	46.55	23.79				
$d_i = 1$	3.45	45.96	41.17	15.74	22.31	80.46	4.57	17.31	9.33	14.72
	12.35	5.03	41.17	3.58	9.92	3.81	28.29	4.14	10.68	
$d_i = 2$	14.82	3.91	0.66	6.21	0.26	1.97	1.81	3.55	6.7	1.94

Table 2. Measures of goodness-of-fit test for Multiple Myeloma data.

	Anderson–Darling		Cramér–von Mises		KS	
	p -Value	Statistics	p -Value	Statistics	p -Value	Statistics
$d_i = 1$	0.7210	0.5228	0.7207	0.0751	0.7624	0.1457
$d_i = 2$	0.9898	0.2043	0.9733	0.0308	0.9613	0.1468

The MLEs of parameters, reliability, and hazard functions based on APCS data are presented in Table 3. The Bayes estimates relative to the Markov chain Monte Carlo (MCMC) approach for the parameters α , β , and θ for the two risk factors, as well as the reliability and hazard functions, are also displayed in Table 3. Table 3 demonstrates that the survival function for the data related to the time to relapse is greater than that of the transplant-related mortality. Furthermore, compared to transplant-related mortality, the hazard rate function for the data related to the time to relapse is lower. This table also shows that the estimators perform well because the values of the estimates are closely spaced. Each of the parameters α , β , and θ have their 95% ACI and credible intervals (CR) calculated, and the results are shown in Table 4.

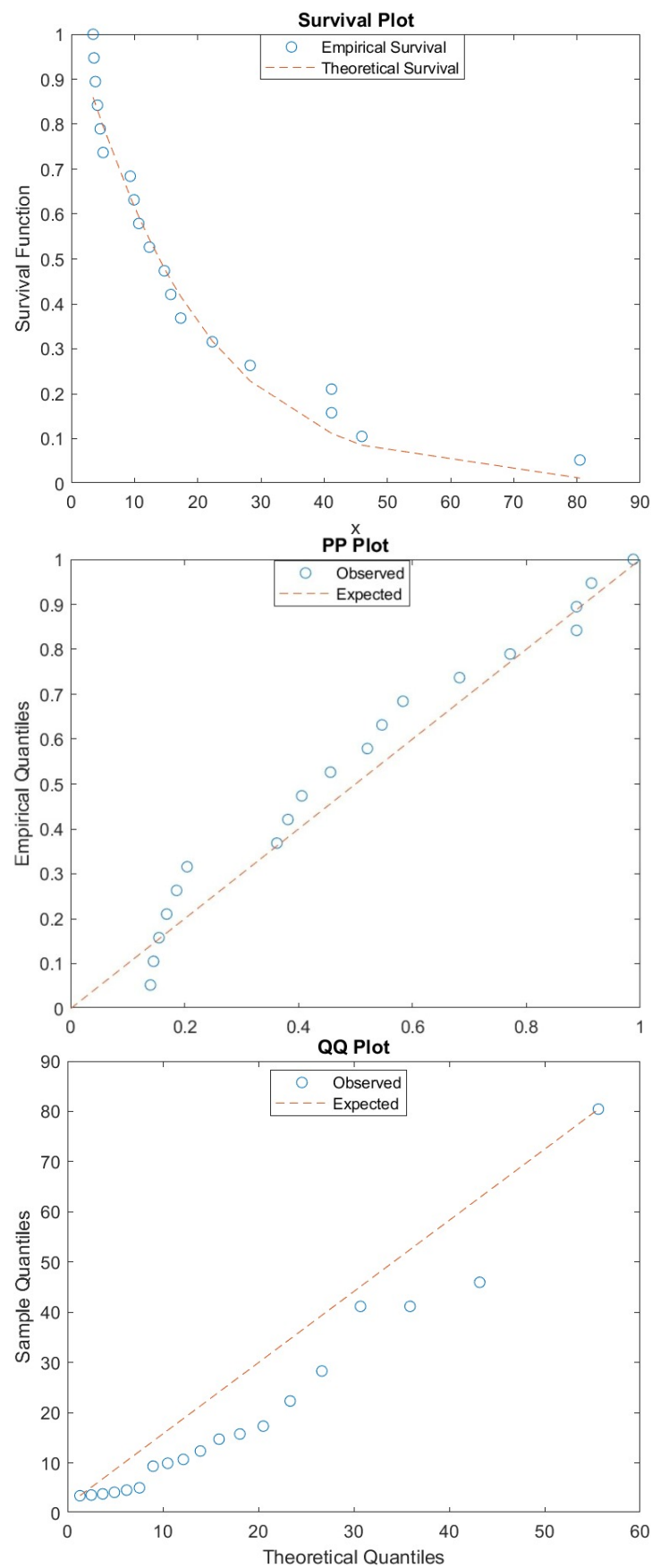


Figure 2. Empirical and fitted survival function, P-P, and Q-Q plots for the data set with $d_i = 1$ for Multiple Myeloma.

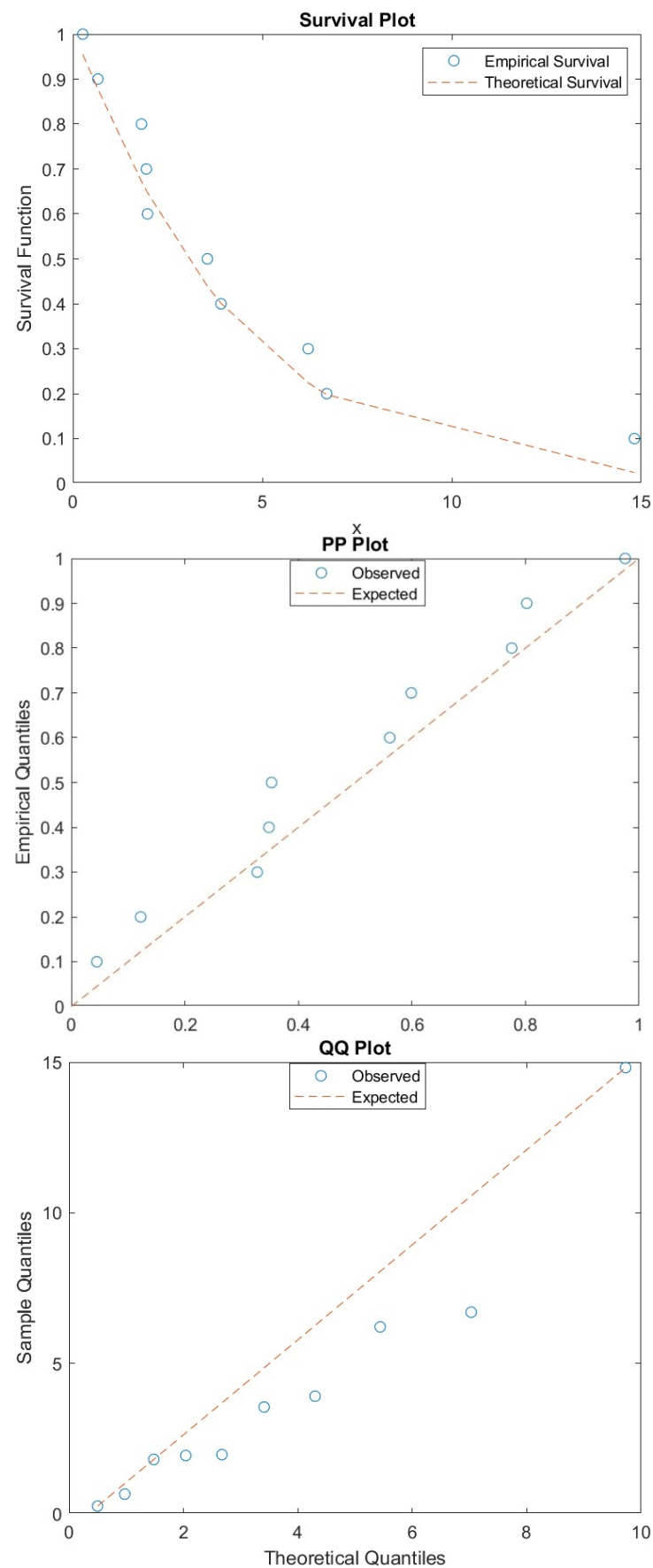


Figure 3. Empirical and fitted survival function, P-P, and Q-Q plots for the data set with $d_i = 2$ for Multiple Myeloma.

Table 3. Different point estimates for parameters and survival and hazard functions.

	$d_i = 1$				$d_i = 2$			
	MLE	MCMC			MLE	MCMC		
		SEL	$q_2 = -3$	$q_3 = 3$		SEL	$q_2 = -3$	$q_3 = 3$
α	0.0401	0.0403	0.0403	0.0402	0.0801	0.0869	0.0871	0.0865
β	0.002	0.0021	0.0022	0.0021	0.0031	0.0018	0.0018	0.0017
θ	1.6001	1.6072	1.6072	1.6072	0.7001	0.7506	0.7507	0.7504
$S(t)$	0.8079	0.8082	0.8065	0.8064	0.3262	0.3408	0.3089	0.3083
$h(t)$	0.0398	0.0398	0.0401	0.0402	0.1172	0.1111	0.1212	0.1211

Table 4. 95% ACI and CR for the parameters under the two risk factors.

	MLE		MCMC	
	Interval	Length	Interval	Length
α_1	{−0.0847,0.1647}	0.249452	{0.0394,0.0414}	0.00199836
α_2	{−0.9995,1.1595}	2.15892	{0.0797,0.0926}	0.0129493
β_1	{−0.006,0.1005}	0.106493	{0.0019,0.002}	0.000109413
β_2	{−1.7755,1.7815}	3.55697	{0.0014,0.0022}	0.00079682
θ_1	{−0.6058,3.8058}	4.41154	{1.5974,1.614}	0.0165445
θ_2	{−1.0785,2.4785}	3.55697	{0.7304,0.7638}	0.0334217

5.2. Data Related to Electrodes

We employ the real-life test data set published by Doganaksoy et al. [42] to demonstrate the applicability of the suggested inference processes to actual phenomena. Recently, Ahmed et al. [31] and Ren and Gui [25] have also examined this data set. The 58 electrodes that underwent a certain voltage endurance life test are included in the data set. One of two (modes) causes—Mode E, an insulating defect brought on by a processing issue that usually arises early in life, and Mode D, which is the deterioration of the organic material that generally happens later—was identified as the source of the failures. There were a total of 18 and 27 recorded failures attributed to Mode E and D causes, respectively. The remaining 13 vacant electrodes, indicated by the letter “+”, were still operating as a result of the missing cause. Every observed value in the original data set was divided by 1000 for computational ease. Table 5 displays the transformed failure times of the insulation voltage endurance test. In this application, we specifically focus on the observations that were fully observed from the entire competing risk samples, leaving the running observations alone. For Modes E and D, the model efficacy is tested using the same measures used in the previous example. Both distance (statistics) and the p -value are evaluated to assess the suitability of this model to fit the electrode data. It can be noticed from Table 6 that the p -values of all measures are greater than 0.05, indicating the suitability of the proposed model to fit the electrode data. Figures 4 and 5 show the distance between the CDF for the GLFR distribution and the empirical distribution for the failure data, in addition to the observed probability distribution, the expected P-P plot, and the quantile plot Q-Q for the two causes, Mode E and Mode D, respectively.

Table 5. Survival times according to censoring, relapse, and transplant-related mortality.

E	0.002	0.003	0.005	0.008	0.021	0.028	0.031	0.064	0.069	0.076
	0.104	0.119	0.144	0.160	0.221	0.236	0.282	0.303		
D	0.168	0.191	0.203	0.211	0.226	0.261	0.264	0.278	0.284	0.286
	0.298	0.314	0.317	0.318	0.320	0.327	0.328	0.328	0.348	0.350
	0.360	0.369	0.377	0.387	0.392	0.412	0.446			
$+$	0.013	0.031	0.052	0.053	0.067	0.078	0.113	0.135	0.157	0.179
	0.241	0.257	0.348							

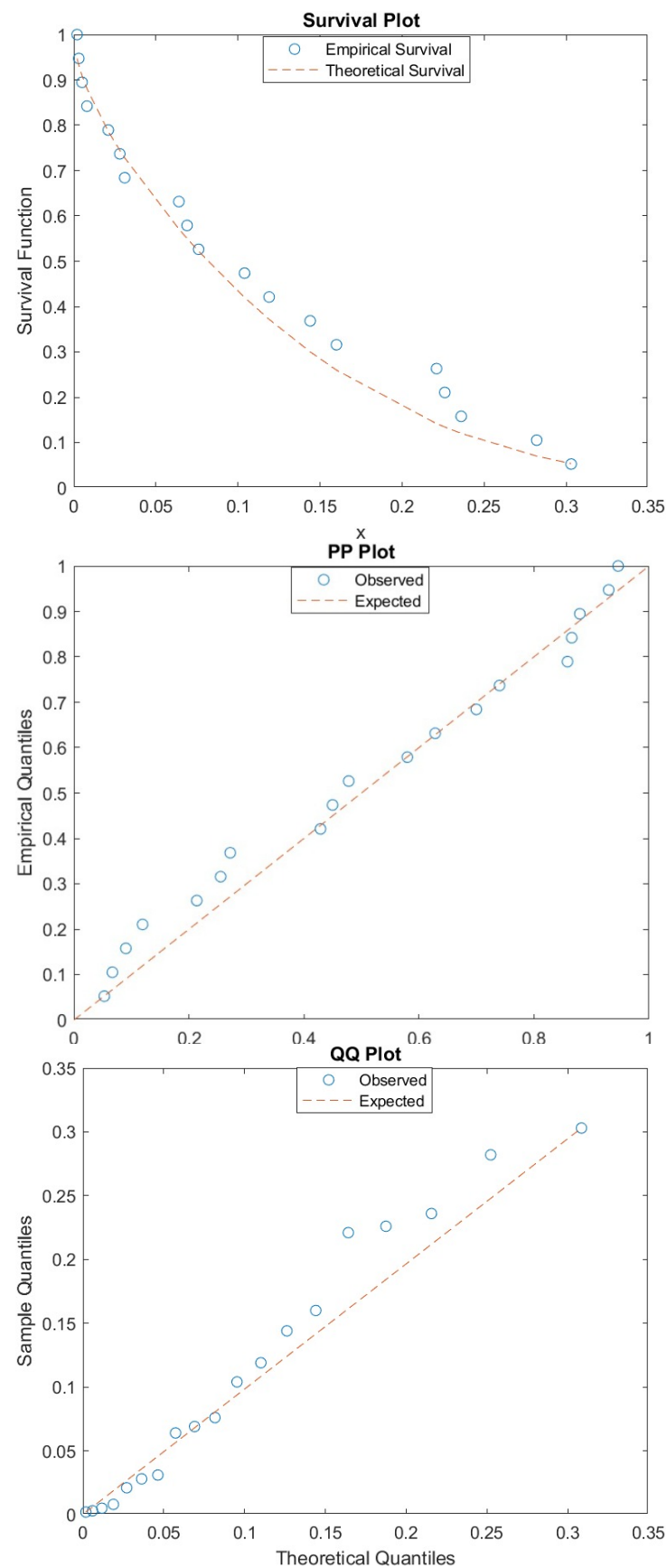


Figure 4. Empirical and fitted survival function, P-P, and Q-Q plots for the data set with failure cause Mode E for electrodes data.

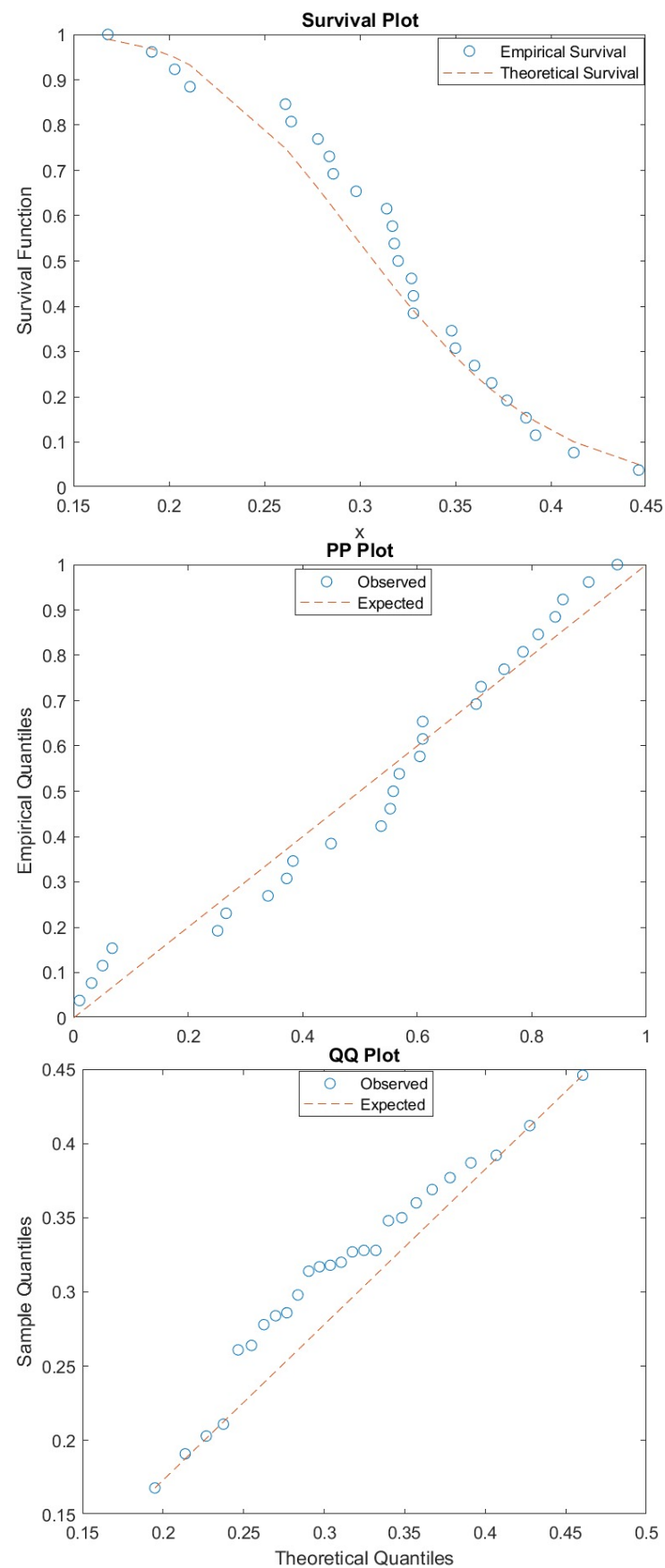


Figure 5. Empirical and fitted survival function, P-P, and Q-Q plots for the data set with failure cause Mode D for electrodes data.

Table 6. Measures of goodness-of-fit test for electrodes data.

Data	Anderson–Darling		Cramér–von Mises		KS	
	<i>p</i> -Value	Statistics	<i>p</i> -Value	Statistics	<i>p</i> -Value	Statistics
E	0.9518	0.2802	0.9532	0.0358	0.9108	0.1214
D	0.6813	0.5632	0.6184	0.0933	0.5327	0.1523

Table 7 displays the MLEs for the parameters, reliability, and hazard functions based on APCS data. It is evident that, while the hazard rate function for both modes is nearly equal, the survival function for the Mode D data is greater than for the Mode E data. Also, it shows the Bayesian estimates relative to the MCMC approach for the reliability and hazard functions, as well as the parameters α , β , and θ for the two risk factors. From Table 7, it can be seen that the estimators perform well since the values of the estimators are relatively close. The 95% ACI and CR for each of the parameters α , β , and θ are computed, and the results are displayed in Table 8.

Table 7. Different point estimates for parameters and survival and hazard functions.

	Mode E				Mode D			
	MLE	MCMC			MLE	MCMC		
		SEL	$q_2 = -3$	$q_3 = 3$		SEL	$q_2 = -3$	$q_3 = 3$
α	1.6905	1.6906	1.6906	1.6906	14.1293	13.9607	13.9608	13.9606
β	8.7303	8.7291	8.7291	8.7292	68.1379	66.9187	66.9191	66.9179
θ	0.6691	0.6691	0.6691	0.6691	86.0027	87.1915	87.1918	87.1905
$S(t)$	0.9341	0.9341	0.9341	0.9341	0.9997	0.9998	0.9998	0.9999
$h(t)$	0.0005	0.0001	0.0005	0.0005	0.0005	0.0001	0.0002	0.0001

Table 8. 95% ACI and CR for the parameters under the two risk factors.

	MLE		MCMC	
	Interval	Length	Interval	Length
α_1	{−2.8612, 6.2421}	9.10337	{1.6902, 1.6908}	0.00066367
α_2	{−847.576, 875.835}	1723.41	{13.9256, 14.0164}	0.090864
β_1	{−21.3068, 4652.29}	4673.59	{8.7275, 8.7317}	0.00419993
β_2	{−6885.18, 7021.46}	13906.6	{66.5187, 67.4028}	0.884154
θ_1	{−0.6058, 3.8058}	1.03315	{0.6691, 0.6692}	0.0000408301
θ_2	{−6867.32, 7039.32}	13906.6	{86.6302, 87.4632}	0.832986

6. Simulation Study

To exemplify the theoretical conclusions drawn in the earlier sections, we present the outcomes of simulation research in this section. This simulation is performed by considering different values of n , m , and T , and by assigning parameter values as $\alpha_1 = 0.4$, $\alpha_2 = 0.08$, $\theta_1 = 1.6$, and $\theta_2 = 0.7$, given in all the cases. While the initial values of β_1 and β_2 are chosen to be 2 and 3, respectively, the initial values of the parameters are selected to match the derived point estimates in each of the two case studies. Three distinct censoring schemes (CS) are employed as follows:

Scheme A : $R_1 = n - m$, $R_i = 0$ for $i \neq 1$.

Scheme B : $R_{\frac{m}{2}} = R_{\frac{m}{2}+1} = \frac{n-m}{2}$, $R_i = 0$ for $i \neq \frac{m}{2}$ and $i \neq \frac{m}{2} + 1$.

Scheme C : $R_m = n - m$, $R_i = 0$ for $i \neq m$.

The MLEs and the Bayes estimators are determined for every case using a thousand runs using *Mathematica* 12 software. By calculating their MSEs for $k = 1, 2$, along with

$(\Omega_1 = \beta_1, \Omega_2 = \beta_2)$, the various approaches to the derived estimators of β_1 and β_2 are compared.

$$MSE(\Omega_k) = \frac{1}{M} \sum_{i=1}^M \left(\hat{\Omega}_k^{(i)} - \Omega_k \right)^2,$$

where the total number of generated samples is $M = 1000$. Comparing confidence intervals is contrasted concerning the coverage probabilities (CP) and the length average of the confidence intervals (LACI).

The outcomes of the simulation study in Tables 9–12 clearly show that, for both β_1 and β_2 , scheme A performs better than both scheme B and scheme C for different T values. Furthermore, it is noted that the average values of estimates for both Bayesian and MLE improve as the sample size (m) increases. These results demonstrate not just scheme A's advantage over the other schemes but also the value of bigger sample numbers in improving estimation accuracy in statistical analysis.

Table 9. The average and MSE for the estimate of β_1 at $T = 0.5$.

(n, m)	CS	MLE		MCMC	
		SEL	GEL	$q = -1$	$q = 1$
(50, 20)	A	1.9946	1.9434	1.9036	2.0892
		(0.0054)	(0.0061)	(0.0019)	(0.0028)
	B	1.9373	1.8081	2.1998	1.8391
		(0.0013)	(0.0071)	(0.0013)	(0.0021)
	C	1.7396	2.1073	2.0753	1.7016
		(0.0020)	(0.0047)	(0.0038)	(0.0014)
(50, 30)	A	1.9214	1.9792	2.0840	2.0426
		(0.0045)	(0.0011)	(0.0021)	(0.0056)
	B	1.9044	2.1846	1.8402	2.1192
		(0.0040)	(0.0054)	(0.0086)	(0.0018)
	C	2.4018	2.2322	2.3569	1.9151
		(0.0065)	(0.0066)	(0.0019)	(0.0023)

Table 10. The average and MSE for the estimate of β_1 at $T = 0.9$.

(n, m)	CS	MLE		MCMC	
		SEL	GEL	$q = -1$	$q = 1$
(50, 20)	A	2.0053	2.0056	1.9557	2.0181
		(0.0058)	(0.0015)	(0.0073)	(0.0026)
	B	2.0847	1.9311	1.9463	1.9184
		(0.0054)	(0.0072)	(0.0075)	(0.0047)
	C	2.0817	1.8276	1.9855	2.1356
		(0.0031)	(0.0049)	(0.0023)	(0.0024)
(50, 30)	A	1.9732	1.9772	2.0277	2.0041
		(0.0027)	(0.0037)	(0.0081)	(0.0031)
	B	1.9306	2.0347	2.1433	2.0216
		(0.0089)	(0.0067)	(0.0070)	(0.0074)
	C	1.7968	1.9827	2.0810	2.1687
		(0.0085)	(0.0063)	(0.0024)	(0.0048)

Table 11. The average and MSE for the estimate of β_2 at $T = 0.5$.

(n, m)	CS	MLE		MCMC	
		SEL	GEL	$q = -1$	$q = 1$
(50, 20)	A	2.9885 (0.0082)	3.0720 (0.0012)	3.0175 (0.0085)	3.0398 (0.0049)
		3.1464 (0.0042)	2.8710 (0.0074)	2.8984 (0.0052)	3.1836 (0.0013)
	C	2.7598 (0.0036)	3.1583 (0.0016)	3.0394 (0.0068)	3.0782 (0.0068)
(50, 30)	A	3.0727 (0.0084)	2.9599 (0.0022)	3.0222 (0.0082)	3.0613 (0.0059)
		2.8585 (0.0083)	2.8615 (0.0080)	3.2856 (0.0047)	2.8614 (0.0074)
	C	2.9948 (0.0039)	2.9293 (0.0027)	3.1676 (0.0068)	2.8513 (0.0045)

Table 12. The average and MSE for the estimate of β_2 at $T = 0.9$.

(n, m)	CS	MLE		MCMC	
		SEL	GEL	$q = -1$	$q = 1$
(50, 20)	A	2.9801 (0.0087)	3.0420 (0.0063)	2.9731 (0.0029)	2.9654 (0.0087)
		2.9442 (0.0022)	3.0890 (0.0058)	3.0725 (0.0053)	3.1254 (0.0013)
	C	2.8590 (0.0043)	2.9725 (0.0051)	3.0470 (0.0014)	2.8974 (0.0083)
(50, 30)	A	2.9546 (0.0024)	3.0207 (0.0064)	3.0113 (0.0015)	3.0301 (0.0080)
		2.9361 (0.0038)	2.9051 (0.0015)	2.8528 (0.0054)	3.0586 (0.0027)
	C	3.0181 (0.0027)	2.9791 (0.0065)	2.8706 (0.0083)	3.0574 (0.0030)

The LACI and the associated CP are shown in Tables 13 and 14. In all cases, the confidence intervals have a confidence level of 0.95. We note that the confidence intervals in the MLE scenario are larger than those in the Bayesian case. Additionally, we note that, for various values of T and m , the CP for scheme A is superior to that for schemes B and C for both estimates β_1 and β_2 .

Table 13. The LACI and CP of 95% ACI for β_1 estimate.

(n, m)	CS	$T = 0.5$		$T = 0.9$	
		Non-Bayesian	Bayesian	Non-Bayesian	Bayesian
(50, 20)	A	0.8144 (0.9513)	0.9056 (0.9542)	1.0793 (0.9618)	0.5030 (0.9700)
		0.8950 (0.9648)	0.9062 (0.9363)	0.8737 (0.9256)	0.8399 (0.9652)
	C	0.8005 (0.9590)	0.5962 (0.9688)	0.9160 (0.9566)	0.7555 (0.9321)
(50, 30)	A	1.1484 (0.9457)	0.4820 (0.9335)	0.9443 (0.9491)	0.7025 (0.9507)
		1.1480 (0.9527)	0.7460 (0.9347)	0.9446 (0.9391)	0.8490 (0.9459)
	C	0.8431 (0.9279)	0.3609 (0.9595)	1.0624 (0.9636)	0.18140 (0.9483)

Table 14. The LACI and CP of 95% ACI for β_2 estimate.

(n, m)	CS	$T = 0.5$		$T = 0.9$	
		Non-Bayesian	Bayesian	Non-Bayesian	Bayesian
(50, 20)	A	1.1836	0.7253	1.0663	0.3521
		(0.9685)	(0.9254)	(0.9320)	(0.9369)
	B	0.9544	0.8411	1.0426	0.6931
		(0.9682)	(0.9475)	(0.9568)	(0.9566)
	C	0.9435	0.7559	1.1083	0.3583
		(0.9724)	(0.9368)	(0.9490)	(0.9704)
(50, 30)	A	0.9613	0.3819	1.1676	0.5692
		(0.9688)	(0.9461)	(0.9450)	(0.9476)
	B	1.0171	0.3602	1.1609	0.3558
		(0.9380)	(0.9470)	(0.9419)	(0.9428)
	C	1.2135	0.4140	1.1160	0.2151
		(0.9479)	(0.9496)	(0.9441)	(0.9491)

7. Conclusions

This study investigated the applicability of the adaptive progressive censoring scheme as an approach to collecting incomplete failure observations, in which the lifespan items of individual failure causes are independent and assumed to follow the GLFR distribution, highlighting its relevance and application in both electrical engineering and medical research, particularly in addressing the competing risk models. The proposed scheme was applied to some observed data, like medical and engineering data that were collected under competing factors of failure, as it was assumed that some of these factors are known and others are unknown. The statistically obtained results contributed to assigning the factor of failure most causal to the failure. We investigated the maximum likelihood and Bayesian estimate approaches to achieve our goal. The point and approximate confidence interval estimations for unknown parameters and the reliability and hazard rate functions were investigated. In the Bayesian methodology, the Metropolis–Hastings algorithm within the Gibbs sampler was used to provide Bayesian estimates under the squared error loss function, as well as the associated credible intervals. Our method's practicality was validated by real-world data analysis employing electrode and clinical data sets. Furthermore, it is clear from Monte Carlo simulation studies that the maximum likelihood estimate approach is not as good as the Bayesian estimation, which yields more satisfactory results. The findings suggest that the possibility of transplant-related mortality in medical case studies is higher than that of disease relapse, while, as demonstrated by the electrode case study, deterioration of the organic material is less likely to happen than early insulating defects. The study paves the way for medical researchers, stakeholders, and manufacturers to draw on the policies of maintenance by giving priority to dealing with the most common factor of failure. It is important to point out that, although this study focuses on the issue of two competing risk factors, similar inferential approaches may be readily extended to many failure factors and alternative censoring schemes.

Author Contributions: Conceptualization, M.M.; Methodology, H.H.A. and M.A.; Software, M.A. and M.M.; Validation, M.M.; Investigation, H.H.A.; Resources, H.H.A., M.A. and M.M.; Data curation, M.A.; Writing—original draft, H.H.A. and M.A.; Writing—review & editing, M.M.; Funding acquisition, H.H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia, Grant No. [GrantA323].

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

APCS	adaptive progressive censoring scheme
GLFR	generalized linear failure rate
MLE	maximum likelihood estimation
CDF	cumulative distribution function
pdf	probability density function
ACI	asymptotic confidence interval
MH	Metropolis–Hasting method
KS	Kolmogorov Smirnov distance
MCMC	Markov chain Monte Carlo
CR	credible interval
CP	coverage probability
LACI	length average of the confidence intervals
CS	censoring scheme
SEL	square error loss function
GEL	general entropy loss function

References

- Dutta, S.; Kayal, S. Inference of a competing risks model with partially observed failure causes under improved adaptive type-II progressive censoring. *Proc. Inst. Mech. Eng. Part J. Risk Reliab.* **2023**, *237*, 765–780. [\[CrossRef\]](#)
- Fine, J.; Lindqvist, B.H. Competing risks. *Lifetime Data Anal.* **2014**, *20*, 159–160. [\[CrossRef\]](#)
- Du, Y.; Zhang, C.; Gui, W. Accelerated life test for Pareto distribution under progressive type-II censored competing risks data with binomial removals and its application in electrode insulation system. *Commun. Stat. Simul. Comput.* **2023**, 1–25. [\[CrossRef\]](#)
- Ahmad, H.H.; Ramadan, D.A.; Almetwally, E.M. Tampered Random Variable Analysis in Step-Stress Testing: Modeling, Inference, and Applications. *Mathematics* **2024**, *12*, 1248. [\[CrossRef\]](#)
- Abushal, T.A.; AL-Zaydi, A.M. Statistical inference of inverted Nadarajah–Haghighi distribution under type-II generalized hybrid censoring competing risks data. *J. Eng. Math.* **2024**, *144*, 24. [\[CrossRef\]](#)
- Balakrishnan, N. Progressive censoring methodology: An appraisal. *Test* **2007**, *16*, 211–259. [\[CrossRef\]](#)
- Balakrishnan, N.; Cramer, E. *The Art of Progressive Censoring*; Birkhäuser: New York, NY, USA, 2014.
- Childs, A.; Chandrasekar, B.; Balakrishnan, N. Exact likelihood inference for an exponential parameter under progressive hybrid censoring schemes. In *Statistical Models and Methods for Biomedical and Technical Systems*; Vonta, F., Nikulin, M., Limnios, N., Huber-Carol, C., Eds.; Birkhäuser: Boston, MA, USA, 2008; pp. 323–334.
- Kundu, D.; Joarder, A. Analysis of Type-II progressively hybrid censored data. *Comput. Stat. Data Anal.* **2006**, *50*, 2509–2528. [\[CrossRef\]](#)
- Kundu, D. Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics* **2008**, *50*, 144–154. [\[CrossRef\]](#)
- Pradhan, B.; Kundu, D. On progressively censored generalized exponential distribution. *Test* **2009**, *18*, 497–515. [\[CrossRef\]](#)
- Ng, H.K.T.; Kundu, D.; Chan, P.S. Statistical analysis of exponential lifetimes under an adaptive Type-II progressively censoring scheme. *Nav. Res. Logist.* **2009**, *56*, 687–698. [\[CrossRef\]](#)
- Kazempoor, J.; Habibirad, A.; Nadi, A.A.; Borzadaran, G.R.M. Statistical inferences for the Weibull distribution under adaptive progressive type-II censoring plan and their application in wind speed data analysis. *Stat. Optim. Inf. Comput.* **2023**, *11*, 829–852. [\[CrossRef\]](#)
- Sobhi, M.M.A.; Soliman, A.A. Estimation for the exponentiated Weibull model with adaptive Type-II progressive censored schemes. *Appl. Math. Model.* **2016**, *40*, 1180–1192. [\[CrossRef\]](#)
- Nassar, M.; Abo-Kasem, O.E. Estimation of the inverse Weibull parameters under adaptive Type-II progressive hybrid censoring scheme. *J. Comput. Appl. Math.* **2017**, *315*, 228–239. [\[CrossRef\]](#)
- Sewailam, M.F.; Baklizi, A. Inference for the log-logistic distribution based on an adaptive progressive Type-II censoring scheme. *Cogent Math. Stat.* **2019**, *6*, 1684228. [\[CrossRef\]](#)
- Chen, S.; Gui, W. Estimation of Unknown Parameters of Truncated Normal Distribution under Adaptive Progressive Type II Censoring Scheme. *Mathematics* **2021**, *9*, 49. [\[CrossRef\]](#)
- Lv, Q.; Tian, Y.; Gui, W. Statistical inference for Gompertz distribution under adaptive type-II progressive hybrid censoring. *J. Appl. Stat.* **2024**, *51*, 451–480. [\[CrossRef\]](#) [\[PubMed\]](#)
- Kundu, D.; Kannan, N.; Balakrishnan, N. Analysis of progressively censored competing risks data. *Handb. Stat.* **2004**, *23*, 331–348.
- Pareek, B.; Kundu, D.; Kumar, S. On progressive censored competing risks data for Weibull distributions. *Comput. Stat. Data Anal.* **2009**, *53*, 4083–4094. [\[CrossRef\]](#)
- Cramer, E.; Schmiedt, A.B. Progressively Type-II censored competing risks data from Lomax distributions. *Comput. Stat. Data Anal.* **2011**, *55*, 1285–1303. [\[CrossRef\]](#)

22. Chacko, M.; Mohan, R. Bayesian analysis of Weibull distribution based on progressive Type-II censored competing risks data with binomial removals. *Comput. Stat.* **2019**, *34*, 233–252. [\[CrossRef\]](#)
23. Qin, X.; Gui, W. Statistical inference of Burr-XII distribution under progressive Type-II censored competing risks data with binomial removals. *J. Comput. Appl. Math.* **2020**, *378*, 112922. [\[CrossRef\]](#)
24. Ahmed, E.A.; Ali Alhussain, Z.; Salah, M.M.; Haj Ahmed, H.; Eliwa, M.S. Inference of progressively type-II censored competing risks data from Chen distribution with an application. *J. Appl. Stat.* **2020**, *47*, 2492–2524. [\[CrossRef\]](#) [\[PubMed\]](#)
25. Davies, K.F.; Volterman, W. Progressively Type-II censored competing risks data from the linear exponential distribution. *Commun. Stat. Theory Methods* **2022**, *51*, 1444–1460. [\[CrossRef\]](#)
26. Lodhi, C.; Tripathi, Y.M.; Bhattacharya, R. On a progressively censored competing risks data from Gompertz distribution. *Commun. Stat. Simul. Comput.* **2021**, *52*, 1278–1299. [\[CrossRef\]](#)
27. Almuqrin, M.A.; Salah, M.M.; A. Ahmed, E. Statistical Inference for Competing Risks Model with Adaptive Progressively Type-II Censored Gompertz Life Data Using Industrial and Medical Applications. *Mathematics* **2022**, *10*, 4274. [\[CrossRef\]](#)
28. Nassar, M.; Alotaibi, R.; Zhang, C. Estimation of Reliability Indices for Alpha Power Exponential Distribution Based on Progressively Censored Competing Risks Data. *Mathematics* **2022**, *10*, 2258. [\[CrossRef\]](#)
29. Elshahhat, A.; Nassar, M. Inference of improved adaptive progressively censored competing risks data for Weibull lifetime models. *Stat. Pap.* **2023**, 1–34. [\[CrossRef\]](#)
30. Lv, Q.; Hua, R.; Gui, W. Statistical inference of Gompertz distribution under general progressive type II censored competing risks sample. *Commun. Stat. Simul. Comput.* **2024**, *53*, 682–701. [\[CrossRef\]](#)
31. Salem, S.A.; Abo-Kasem, O.E.; Elassar, M.A. Inference for inverse weibull competing risks data under adaptive progressive hybrid censored with engineering application. *Pak. J. Stat.* **2023**, *39*, 125.
32. Salem, S.A.; Abo-Kasem, O.E.; Elassar, M.A. Analysis of Generalized Inverted Exponential Distribution under Adaptive Type-I Progressive Hybrid Censored Competing Risks Data. *J. Stat. Appl. Probab.* **2022**, *12*, 109–133.
33. Sarhan, A.M.; Kundu, D. Generalized Linear Failure Rate Distribution. *Commun. Stat. Theor. Methods* **2009**, *38*, 642–660. [\[CrossRef\]](#)
34. Tolba, A. Bayesian and non-Bayesian estimation methods for simulating the parameter of the Akshaya distribution. *Comput. J. Math. Stat. Sci.* **2022**, *1*, 13–25. [\[CrossRef\]](#)
35. Yamamura, K. Bayes estimates as an approximation to maximum likelihood estimates. *Popul. Ecol.* **2016**, *58*, 45–52. [\[CrossRef\]](#)
36. Mahmoudi, L.; Fallah, R.; Roshanaei, G.; Asghari-Jafarabadi, M. A bayesian approach to model the underlying predictors of early recurrence and postoperative death in patients with colorectal Cancer. *BMC Med. Res. Methodol.* **2022**, *22*, 269. [\[CrossRef\]](#) [\[PubMed\]](#)
37. Metropolis, N.; Rosenbluth, A.W.; Rosenbluth, M.N.; Teller, A.H.; Teller, E. Equations of state calculations by fast computing machines. *J. Chem. Phys.* **1953**, *21*, 1087–1092. [\[CrossRef\]](#)
38. Gelfand, A.E. Gibbs sampling. *J. Am. Stat. Assoc.* **2000**, *95*, 1300–1304. [\[CrossRef\]](#)
39. Gelman, A.; Carlin, J.B.; Stern, H.S.; Rubin, D.B. *Bayesian Data Analysis*; Chapman and Hall: London, UK, 1995.
40. Mahmoud, M.A.W.; Ramadan, D.A.; Mansour, M.M.M. Estimation of lifetime parameters of the modified extended exponential distribution with application to a mechanical model. *Commun. Stat. Simul. Comput.* **2022**, *51*, 7005–7018. [\[CrossRef\]](#)
41. Donoghoe, M.W.; Gebski, V. The importance of censoring in competing risks analysis of the subdistribution hazard. *BMC Med. Res. Methodol.* **2017**, *17*, 52. [\[CrossRef\]](#)
42. Doganaksoy, N.; Hahn, G.J.; Meeker, W.Q. Reliability analysis by failure mode. *Qual. Prog.* **2002**, *35*, 47–52.

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