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Systems of Hemivariational Inclusions with Competing Operators

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Abstract: This paper focuses on a system of differential inclusions expressing hemivariational inequalities driven by competing operators constructed with p-Laplacians that involve two real parameters. The existence of a generalized solution is shown by means of an approximation process through approximate solutions in finite dimensional spaces. When the parameters are negative, the generalized solutions become weak solutions. The main novelty of this work is the solvability of systems of differential inclusions for which the ellipticity condition may fail.

Keywords: system of differential inclusions; hemivariational inequalities; competing operators; p-Laplacian; Galerkin basis

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1. Introduction

Consider the following system of differential inclusions subject to the Dirichlet boundary condition:

$$\begin{cases} (-\Delta_{p_1}u_1 + \mu_1\Delta_{q_1}u_1, -\Delta_{p_2}u_2 + \mu_2\Delta_{q_2}u_2) \in \partial F(u_1, u_2) & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ for $N \geq 2$ with a Lipschitz boundary $\partial\Omega$. For a later use, $|\Omega|$ denotes the Lebesgue measure of Ω. In (1) we have, for $1 < q_1 < p_1 < +\infty$ and $1 < q_2 < p_2 < +\infty$, the p_1-Laplacian $\Delta_{p_1} : W_0^{1,p_1}(\Omega) \to W^{-1,p_1'}(\Omega)$, q_1-Laplacian $\Delta_{q_1} : W_0^{1,q_1}(\Omega) \to W^{-1,q_1'}(\Omega)$, p_2-Laplacian $\Delta_{p_2} : W_0^{1,p_2}(\Omega) \to W^{-1,p_2'}(\Omega)$, and q_2-Laplacian $\Delta_{q_2} : W_0^{1,q_2}(\Omega) \to W^{-1,q_2'}(\Omega)$. Throughout the paper, corresponding to any real number $r > 1$ we denote $r' = \frac{r}{r-1}$ (the Hölder conjugate of r). Furthermore, λ_{1,p_1} and λ_{1,p_2} denote the first eigenvalues of $-\Delta_{p_1}$ and $-\Delta_{p_2}$, respectively (see Section 2 for a brief review).

The multivalued term in the inclusion (1) is expressed as the generalized gradient ∂F of a locally Lipschitz function $F : \mathbb{R}^2 \to \mathbb{R}$, so pointwise $\partial F(u_1(x), u_2(x))$ is a subset of $\mathbb{R}^2$. We reference [1] for the subdifferentiation of locally Lipschitz functionals. Some basic elements are presented in Section 2. Any $\zeta \in \partial F(t,s)$ is a point of $\mathbb{R}^2$; thus, it has two components, i.e., $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$. Hence, (1) is a system of two differential inclusions that we call hemivariational inclusions because they involve generalized gradients. The inclusion problem (1) incorporates systems of equations with discontinuous nonlinearities. Differential equations with discontinuous nonlinearities via the generalized gradients were first studied in [2].

According to the definition of generalized gradient, it is apparent that each solution to system (1) solves the inequality problem.

$$\begin{aligned} &\langle -\Delta_{p_1}u_1, v_1\rangle + \mu_1\langle \Delta_{q_1}u_1, v_1\rangle + \langle -\Delta_{p_2}u_2, v_2\rangle + \mu_2\langle \Delta_{q_2}u_2, v_2\rangle \\ &\leq \int_\Omega F^0(u_1(x), u_2(x); v_1(x), v_2(x))dx \ \text{ for all } (v_1, v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega), \end{aligned} \tag{2}$$

where the notation F^0 stands for the generalized directional derivative of the locally Lipschitz function F on $\mathbb{R}^2$. Problem (2) is a hemivariational inequality in the product space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. The interest in hemivariational inequalities is that they allow nonconvex potentials. For the study of hemivariational inequalities, we refer to [3–7].

For the locally Lipschitz function $F : \mathbb{R}^2 \to \mathbb{R}$, we assume the following condition:

(H) There are positive constants c_0, c_1, c_2, d_0, d_1, d_2, r_1, r_2, with $c_1 < \lambda_{1,p_1}$ $d_2 < \lambda_{1,p_2}$, $1 < r_1 < p_1$, and $1 < r_2 < p_2$ such that

$$|\zeta_1| \leq c_0 + c_1|t|^{p_1-1} + c_2|s|^{\frac{p_2}{r_1'}}$$

and

$$|\zeta_2| \leq d_0 + d_1|t|^{\frac{p_1}{r_2'}} + d_2|s|^{p_2-1}$$

for all $(t,s) \in \mathbb{R}^2$ and $(\zeta_1, \zeta_2) \in \partial F(t,s)$.

In the statement of (1), there are two parameters $\mu_1 \in \mathbb{R}$ and $\mu_2 \in \mathbb{R}$. The leading operators are $-\Delta_{p_1} + \mu_1\Delta_{q_1}$ and $-\Delta_{p_2} + \mu_2\Delta_{q_2}$, for which the ellipticity condition fails when $\mu_1 > 0$ and $\mu_2 > 0$, which is the main point of our work (note that μ_1 and μ_2 are arbitrary real numbers). In this case, they become the so-called competing operators that were introduced in [8]. Precisely, a competing operator was defined in reference [8] as $-\Delta_p + \Delta_q$ versus $-\Delta_p - \Delta_q$ ((p,q)-Laplacian) for $1 < q < p < +\infty$. The essential feature of such an operator is that the ellipticity property is lost. For any $u \in W_0^{1,p}(\Omega)$ and any scalar $\lambda > 0$, the following expression does not have a constant sign when λ varies:

$$\langle -\Delta_p(\lambda u), \lambda u\rangle + \langle \Delta_q(\lambda u), \lambda u\rangle = \lambda^p\langle -\Delta_p u, u\rangle + \lambda^q\langle \Delta_q u, u\rangle.$$

Systems of differential equations with competing operators were investigated in [9].

Due to the possible loss of ellipticity for system (1), we introduce a new type of solution called a generalized solution. It is said that $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ is a generalized solution to problem (1) if there exists a sequence $\{(u_{1n}, u_{2n})\} \subset W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ such that

(i) $u_{in} \rightharpoonup u_i$ in $W_0^{1,p_i}(\Omega)$ as $n \to \infty$ for $i = 1,2$;

(ii) $-\Delta_{p_i}u_{in} + \mu_i\Delta_{q_i}u_{in} - z_{in} \rightharpoonup 0$ in $W^{-1,p_i'}(\Omega)$ as $n \to \infty$, with $z_{in} \in L^{p_i'}(\Omega)$ for $i = 1,2$, and $(z_{1n}(x), z_{2n}(x)) \in \partial F(u_{1n}(x), u_{2n}(x))$ a.e. on Ω;

(iii) $\lim_{n\to\infty}\langle -\Delta_{p_1}u_{1n} + \mu_1\Delta_{q_1}u_{1n}, u_{1n} - u_1\rangle = 0$ and $\lim_{n\to\infty}\langle -\Delta_{p_2}u_{2n} + \mu_2\Delta_{q_2}u_{2n}, u_{2n} - u_2\rangle = 0$.

The notion of a generalized solution was proposed in [10] for differential equations driven by competing operators and in [9] for systems of differential equations with competing operators. The notion of a generalized solution for hemivariational inequalities with competing operators was recently introduced in [7]. Here, for the first time, we define the generalized solution for a system of hemivariational inclusions exhibiting competing operators.

We also introduce the notion of a weak solution to system (1). By a weak solution to system (1), we understand any pair $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ for which the following holds:

$$\begin{aligned} &\langle -\Delta_{p_1}u_1, v_1\rangle + \mu_1\langle \Delta_{q_1}u_1, v_1\rangle + \langle -\Delta_{p_2}u_2, v_2\rangle + \mu_2\langle \Delta_{q_1}u_2, v_2\rangle \\ &= \int_\Omega (z_1(x)v_1(x) + z_2(x)v_2(x))dx \ \text{ for all } (v_1, v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega), \end{aligned} \tag{3}$$

with $(z_1, z_2) \in L^{p_1'}(\Omega) \times L^{p_2'}(\Omega)$ satisfying $(z_1, z_2) \in \partial F(u_1, u_2)$ a.e. on Ω. Equivalently, (3) can be written in the system form as follows:

$$\begin{aligned} -\Delta_{p_1}u_1 + \mu_1\Delta_{q_1}u_1 + z_1 = 0, \\ -\Delta_{p_2}u_2 + \mu_2\Delta_{q_2}u_2 + z_2 = 0, \end{aligned}$$

with (z_1, z_2) as in (3), where the equalities hold in dual spaces $W^{-1,p'_1}(\Omega)$ and $W^{-1,p'_2}(\Omega)$. Notice that any weak solution to system (1) is a generalized solution. Indeed, if $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ is a weak solution, it is sufficient to take $(u_{1n}, u_{2n}) = (u_1, u_2)$ and $(z_{1n}, z_{2n}) = (z_1, z_2)$ in the definition of a generalized solution.

Our main results read as follows.

Theorem 1. *Assume that condition (H) holds. Then, there exists a generalized solution to system* (1) *for every* $(\mu_1, \mu_2) \in \mathbb{R}^2$.

Theorem 2. *Assume that condition (H) holds. If* $\mu_1 \leq 0$ *and* $\mu_2 \leq 0$*, then each generalized solution to system* (1) *is a weak solution. In particular, if* $\mu_1 \leq 0$ *and* $\mu_2 \leq 0$*, system* (1) *possesses a weak solution.*

In the proof of Theorem 1, we make use of approximation through finite dimensional subspaces via a Galerkin basis combined with minimization and nonsmooth analysis. We obtain a priori estimates, which are of independent interest in the context of competing operators. The proof of Theorem 2 relies on properties of the underlying spaces and of operators of the p-Laplacian type. We end the paper with an example illustrating the applicability of our results.

The rest of the paper is organized as follows. Section 2 is devoted to the related mathematical background. Section 3 contains the needed minimization results and estimates. Section 4 sets forth the finite dimensional approximation approach. Section 4 presents the proofs of Theorems 1 and 2, as well as an example.

2. Mathematical Background

Given a Banach space X with the norm $\|\cdot\|$, X^* denotes the dual space of X, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. The norm convergence in X and X^* is denoted by $\to$, and the weak convergence is denoted by $\rightharpoonup$.

We outline basic elements of nonsmooth analysis. For a detailed treatment, we refer to [1]. A function $G : X \longrightarrow \mathbb{R}$ on a Banach space X is called locally Lipschitz if, for every point $u \in X$, there are an open neighborhood U of u and a constant $C > 0$ such that

$$|G(v) - G(w)| \leq C\|v - w\| \quad \text{for all } v, w \in U.$$

The generalized directional derivative of a locally Lipschitz function $G : X \longrightarrow \mathbb{R}$ at point $u \in X$ in direction $v \in X$ is defined by

$$G^\circ(u; v) := \limsup_{\substack{w \to u \\ t \downarrow 0}} \frac{G(w + tv) - G(w)}{t},$$

and the generalized gradient of G at $u \in X$ is the following set

$$\partial G(u) := \{\eta \in X^* : G^\circ(u; v) \geq \langle \eta, v\rangle \text{ for every } v \in X\}.$$

The following relation links the two notions:

$$G^\circ(u; v) = \max_{\eta \in \partial G(u)} \langle \eta, v\rangle \quad \text{for all } u, v \in V.$$

We illustrate these definitions in two significant situations. For a continuous and convex function $G : X \to \mathbb{R}$, the generalized gradient ∂G coincides with the subdifferential of G in the sense of convex analysis. If the function $G : X \to \mathbb{R}$ is continuously differentiable, the generalized gradient of G is just the differential of G.

We also mention a few things regarding the driving operators in system (1) (or hemivariational inequality (2)). Given any number $1 < r < +\infty$, the Sobolev space $W_0^{1,r}(\Omega)$ is endowed with the norm $\|\nabla u\|_r$, where $\|\cdot\|_r$ denotes the L^r norm. The dual space of $W_0^{1,r}(\Omega)$ is $W^{-1,r'}(\Omega)$. As usual, r^* denotes the Sobolev critical exponent, that is, $r^* = Nr/(N-r)$ if $N > r$ and $r^* = +\infty$ otherwise. The Rellich–Kondrachov embedding theorem ensures that $W_0^{1,r}(\Omega)$ is compactly embedded into $L^d(\Omega)$ for every $1 \le d < r^*$. In particular, there exists a positive constant $S_{d,r}$ such that

$$\|u\|_d \le S_{d,r}\|\nabla u\|_r, \quad \forall\, u \in W_0^{1,r}(\Omega). \tag{4}$$

For the background of Sobolev spaces, we refer to [11]. Here, we solely recall that a Banach space $W_0^{1,r}(\Omega)$ with $1 < r < +\infty$ is separable. This implies the existence of a Galerkin basis of space $W_0^{1,r}(\Omega)$, meaning a sequence $\{X_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W_0^{1,r}(\Omega)$ satisfying

(a) $\dim(X_n) < \infty, \quad \forall\, n;$
(b) $X_n \subset X_{n+1}, \quad \forall\, n;$
(c) $\overline{\bigcup_{n=1}^{\infty} X_n} = W_0^{1,r}(\Omega).$

We refer to [12] for background related to Galerkin bases.

The negative r-Laplacian $-\Delta_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is the operator (nonlinear if $r \neq 2$) given by

$$\langle -\Delta_r u, v\rangle = \int_\Omega |\nabla u(x)|^{r-2}\nabla u(x)\cdot\nabla v(x)dx, \quad \forall\, u, v \in W_0^{1,r}(\Omega).$$

The first eigenvalue of $-\Delta_r$ is given by

$$\lambda_{1,r} = \inf_{v\in W_0^{1,r}(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_r^r}{\|v\|_r^r}. \tag{5}$$

More details can be found, e.g., in [3]. Since $q_1 < p_1$ and $q_2 < p_2$, there are the continuous embeddings $W_0^{1,p_1}(\Omega) \hookrightarrow W_0^{1,q_1}(\Omega)$ and $W_0^{1,p_2}(\Omega) \hookrightarrow W_0^{1,q_2}(\Omega)$, which can be readily verified through Hölder's inequality. Therefore, the sums $-\Delta_{p_1} + \mu_1\Delta_{q_1} : W_0^{1,p_1}(\Omega) \to W^{-1,p_1'}(\Omega)$ and $-\Delta_{p_2} + \mu_2\Delta_{q_2} : W_0^{1,p_2}(\Omega) \to W^{-1,p_2'}(\Omega)$ entering system (1) are well defined.

3. Associated Euler Functional

We focus on nonsmooth function $F : \mathbb{R}^2 \to \mathbb{R}$, for which assumption (H) holds true.

Lemma 1. *Assume that condition (H) is satisfied. Then, for each $\varepsilon > 0$, there exist constants $c(\varepsilon) > 0$ and $d(\varepsilon) > 0$ such that*

$$\begin{aligned} |F(t,s)| \quad \le \quad & |F(0,0)| + c_0|t| + d_0|s| + \left(\frac{c_1}{p_1} + \varepsilon\right)|t|^{p_1} \\ & + \left(\frac{d_2}{p_2} + \varepsilon\right)|s|^{p_2} + c(\varepsilon)|t|^{r_1} + d(\varepsilon)|s|^{r_2}. \end{aligned} \tag{6}$$

Proof. Rademacher's theorem ensures that there exists a gradient $\nabla F(x_1, x_2) = (\frac{\partial F}{\partial x_1}(x_1, x_2), \frac{\partial F}{\partial x_2}(x_1, x_2))$ for almost all $(x_1, x_2) \in \mathbb{R}^2$. On the other hand, for every $(t,s) \in \mathbb{R}^2$, the function $\tau \mapsto F(\tau t, \tau s)$ belongs to space $W^{1,1}(I)$ on any bounded open interval I that contains $[0,1]$. Therefore, we may write

$$F(t,s) - F(0,0) = \int_0^1 \left(\frac{\partial F}{\partial x_1}(\tau t, \tau s)t + \frac{\partial F}{\partial x_2}(\tau t, \tau s)s\right)d\tau \text{ for all } (t,s) \in \mathbb{R}^2.$$

Then, taking into account that

$$\left(\frac{\partial F}{\partial x_1}(\tau t,\tau s),\frac{\partial F}{\partial x_2}(\tau t,\tau s)\right)\in\partial F(\tau t,\tau s)$$

(see [1], p. 32), hypothesis (H) implies

$$\begin{aligned}|F(t,s)| &\leq |F(0,0)| \\ &\quad + \int_0^1\left((c_0+c_1|\tau t|^{p_1-1}+c_2|\tau s|^{\frac{p_2}{r_1'}})|t|+(d_0+d_1|\tau t|^{\frac{p_1}{r_2'}}+d_2|\tau s|^{p_2-1})|s|\right)d\tau \\ &\leq |F(0,0)|+c_0|t|+d_0|s|+\frac{c_1}{p_1}|t|^{p_1}+\frac{d_2}{p_2}|s|^{p_2} \\ &\quad +\frac{c_2r_1'}{p_2+r_1'}|t||s|^{\frac{p_2}{r_1'}}+\frac{d_1r_2'}{p_1+r_2'}|t|^{\frac{p_1}{r_2'}}|s|.\end{aligned}$$

Now, using Young's inequality with ε, we arrive at (6), which completes the proof. □

Lemma 2. *Under assumption (H), the functional $\Phi : L^{p_1}(\Omega)\times L^{p_2}(\Omega)\to\mathbb{R}$ given by*

$$\Phi(v_1,v_2)=\int_\Omega F(v_1(x),v_2(x))\,\mathrm{d}x \text{ for all } (v_1,v_2)\in L^{p_1}(\Omega)\times L^{p_2}(\Omega) \tag{7}$$

is Lipschitz continuous on the bounded subsets of $L^{p_1}(\Omega)\times L^{p_2}(\Omega)$. The generalized gradient $\partial\Phi : L^{p_1}(\Omega)\times L^{p_2}(\Omega)\to 2^{L^{p_1'}(\Omega)\times L^{p_2'}(\Omega)}$ has the following property: if $(\zeta_1,\zeta_2)\in\partial\Phi(u_1,u_2)$, with $(u_1,u_2)\in L^{p_1}(\Omega)\times L^{p_2}(\Omega)$, then

$$(\zeta_1(x),\zeta_2(x))\in\partial F(u_1(x),u_2(x)) \text{ for a.e. } x\in\Omega. \tag{8}$$

Proof. The verification of the Lipschitz condition for the function Φ in (7) on the bounded subsets of the product space $L^{p_1}(\Omega)\times L^{p_2}(\Omega)$ is straightforward. The Aubin–Clarke theorem on the subdifferentiation under the integral sign (see [1], p. 83) can be shown to be valid under hypothesis (H). This readily leads to Formula (8), thus completing the proof. □

In view of Lemma 2, the compact embeddings $W_0^{1,p_i}(\Omega)\hookrightarrow L^{p}(\Omega)$, $i=1,2$ yield the multivalued mapping $\partial\Phi : W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)\to 2^{W^{-1,p_1'}(\Omega)\times W^{-1,p_2'}(\Omega)}$. On this basis, we introduce the functional $J : W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)\to\mathbb{R}$ as follows:

$$\begin{aligned}J(v_1,v_2) &= \frac{1}{p_1}\|\nabla v_1\|_{p_1}^{p_1}-\frac{\mu_1}{q_1}\|\nabla v_1\|_{q_1}^{q_1}+\frac{1}{p_2}\|\nabla v_2\|_{p_2}^{p_2}-\frac{\mu_2}{q_2}\|\nabla v_2\|_{q_2}^{q_2} \\ &\quad -\int_\Omega F(v_1(x),v_2(x))\,dx\end{aligned} \tag{9}$$

for all $(v_1,v_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$.

Proposition 1. *Assume condition (H). Then, the functional J in (9) is locally Lipschitz, with the generalized gradient expressed as*

$$\partial J(v_1,v_2)=(-\Delta_{p_1}v_1+\mu_1\Delta_{q_1}v_1,-\Delta_{p_2}v_2+\mu_2\Delta_{q_2}v_2)-\partial\Phi(v_1,v_2) \tag{10}$$

for all $(v_1,v_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$.

Proof. The functional J in (9) is the difference of a continuously differentiable function and Φ in (7), which is known from Lemma 2 to be locally Lipschitz. Therefore J is locally Lipschitz continuous, and its generalized gradient on the product space $L^{p_1}(\Omega)\times L^{p_2}(\Omega)$ has the expression in (10). □

Proposition 2. *Assume condition* (H)*. Then, the functional* J *in* (9) *is coercive on* $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$*, that is,* $J(v_1, v_2) \to \infty$ *as* $\|\nabla v_1\|_{p_1} + \|\nabla v_2\|_{p_2} \to \infty$*.*

Proof. From (9) and (6) in Lemma 1, we infer, for every $\varepsilon > 0$, that

$$\begin{aligned} J(v_1, v_2) \quad \geq \quad & \frac{1}{p_1}\|\nabla v_1\|_{p_1}^{p_1} + \frac{1}{p_2}\|\nabla v_2\|_{p_2}^{p_2} - \frac{|\mu_1|}{q_1}\|\nabla v_1\|_{q_1}^{q_1} - \frac{|\mu_2|}{q_2}\|\nabla v_2\|_{q_2}^{q_2} \\ & - \int_\Omega \Bigg(|F(0,0)| + c_0|v_1| + d_0|v_2| + \left(\frac{c_1}{p_1} + \varepsilon\right)|v_1|^{p_1} \\ & + \left(\frac{d_2}{p_2} + \varepsilon\right)|v_2|^{p_2} + c(\varepsilon)|v_1|^{r_1} + d(\varepsilon)|v_2|^{r_2} \Bigg) dx \end{aligned}$$

for all $(v_1, v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, with constants $c(\varepsilon)$ and $d(\varepsilon)$. Using (4), (5), and Hólder's inequality, the preceding estimate entails

$$\begin{aligned} J(v_1, v_2) \quad \geq \quad & \frac{1}{p_1}\left(1 - c_1\lambda_{1,p_1}^{-1} - p_1\lambda_{1,p_1}^{-1}\varepsilon\right)\|\nabla v_1\|_{p_1}^{p_1} + \frac{1}{p_2}\left(1 - d_2\lambda_{1,p_2}^{-1} - p_2\lambda_{1,p_2}^{-1}\varepsilon\right)\|\nabla v_2\|_{p_2}^{p_2} \\ & - \frac{|\mu_1|}{q_1}|\Omega|^{\frac{p_1-q_1}{p_1}}\|\nabla v_1\|_{p_1}^{q_1} - \frac{|\mu_2|}{q_2}|\Omega|^{\frac{p_2-q_2}{p_2}}\|\nabla v_2\|_{p_2}^{q_2} - c_0 S_{1,p_1}\|\nabla v_1\|_{p_1} \\ & - d_0 S_{1,p_2}\|\nabla v_2\|_{p_2} - c(\varepsilon)S_{r_1,p_1}^{r_1}\|\nabla v_1\|_{p_1}^{r_1} - d(\varepsilon)S_{r_2,p_2}^{r_2}\|\nabla v_2\|_{p_2}^{r_2} - |F(0,0)||\Omega|. \end{aligned}$$

It is known from assumption (H) that $c_1 < \lambda_{1,p_1}$ and $d_2 < \lambda_{1,p_2}$. A value of $\varepsilon > 0$ so small that $1 - c_1\lambda_{1,p_1}^{-1} - p_1\lambda_{1,p_1}^{-1}\varepsilon > 0$ and $1 - d_2\lambda_{1,p_2}^{-1} - p_2\lambda_{1,p_2}^{-1}\varepsilon > 0$ is selected. Since $1 < r_1 < p_1$, $1 < r_2 < p_2$, $1 < q_1 < p_1$, and $1 < q_2 < p_2$, we conclude that the functional J is coercive, which completes the proof. □

4. Finite Dimensional Approximations to Resolve System (1)

Let us fix a Galerkin basis $\{X_n\}$ of the space $W_0^{1,p_1}(\Omega)$ and a Galerkin basis $\{Y_n\}$ of the space $W_0^{1,p_2}(\Omega)$. It follows that $\{X_n \times Y_n\}$ is a Galerkin basis of the product space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. Minimization in the finite dimensional space $X_n \times Y_n$ will enable us to construct a generalized solution to system (1).

Proposition 3. *Assume condition (H). For each positive integer* n*, there exist* $(u_{1n}, u_{2n}) \in X_n \times Y_n$ *and* $(z_{1n}, z_{2n}) \in L^{p_1'}(\Omega) \times L^{p_2'}(\Omega)$ *with* $(z_{1n}(x), z_{2n}(x)) \in \partial F(u_{1n}(x), u_{2n}(x))$ *for a.e.* $x \in \Omega$ *such that*

$$\langle -\Delta_{p_1} u_{1n}, v_1 \rangle + \mu_1 \langle \Delta_{q_1} u_{1n}, v_1 \rangle - \int_\Omega z_{1n} v_1 dx = 0, \quad \forall v_1 \in X_n, \tag{11}$$

$$\langle -\Delta_{p_2} u_{2n}, v_2 \rangle + \mu_2 \langle \Delta_{q_2} u_{2n}, v_2 \rangle - \int_\Omega z_{2n} v_2 dx = 0, \quad \forall v_2 \in Y_n. \tag{12}$$

Proof. According to Proposition 1, the functional $J : W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \to \mathbb{R}$ in (9) is locally Lipschitz and, thus, continuous, while according to Proposition 2, J is coercive. Taking into account that the subspace $X_n \times Y_n$ of $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ is finite dimensional, there exists $(u_{1n}, u_{2n}) \in X_n \times Y_n$ satisfying

$$J(u_{1n}, u_{2n}) = \inf_{(v_1,v_2) \in X_n \times Y_n} J(v_1, v_2). \tag{13}$$

A necessary condition of optimality for (13) is that

$$(0,0) \in \partial(J|_{X_n \times Y_n})(u_{1n}, u_{2n}). \tag{14}$$

In view of (10), inclusion (14) provides $(z_{1n}, z_{2n}) \in \partial\Phi(u_{1n}, u_{2n})$ for which (11) and (12) hold. The fact that $(z_{1n}, z_{2n}) \in \partial F(u_{1n}, u_{2n})$ a.e. in Ω is the consequence of Lemma 2. □

Proposition 4. *Assume condition* (H)*. Then, the sequence* $\{(u_{1n}, u_{2n})\}$ *in Proposition 3 is bounded in* $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$*.*

Proof. Proposition 3 ensures that equalities (11) and (12) hold true. As $(u_{1n}, u_{2n}) \in X_n \times Y_n$, we are allowed to use $v_1 = u_{1n}$ in (11) and $v_2 = u_{2n}$ in (12) as test functions. In conjunction with Hölder's inequality, this gives

$$\|\nabla u_{1n}\|_{p_1}^{p_1} = \mu_1 \|\nabla u_{1n}\|_{q_1}^{q_1} + \int_\Omega z_{1n} u_{1n} dx \le |\mu_1| |\Omega|^{\frac{p_1 - q_1}{p_1}} \|\nabla u_{1n}\|_{p_1}^{q_1} + \int_\Omega |z_{1n}||u_{1n}| dx \tag{15}$$

and

$$\|\nabla u_{2n}\|_{p_2}^{p_2} = \mu_2 \|\nabla u_{2n}\|_{q_2}^{q_2} + \int_\Omega z_{2n} u_{2n} dx \le |\mu_2| |\Omega|^{\frac{p_2 - q_2}{p_2}} \|\nabla u_{2n}\|_{p_2}^{q_2} + \int_\Omega |z_{2n}||u_{2n}| dx, \tag{16}$$

with $(z_{1n}, z_{2n}) \in \partial F(u_{1n}, u_{2n})$ a.e. in Ω. We are entitled to invoke hypothesis (H) to obtain

$$\begin{aligned}
&\int_\Omega |z_{1n}||u_{1n}| dx \le \int_\Omega (c_0 + c_1 |u_{1n}(x)|^{p_1 - 1} + c_2 |u_{2n}(x)|^{\frac{p_2}{r_1'}}) |u_{1n}(x)| dx \\
&= c_0 \|u_{1n}\|_1 + c_1 \|u_{1n}\|_{p_1}^{p_1} + c_2 \int_\Omega |u_{2n}(x)|^{\frac{p_2}{r_1'}} |u_{1n}(x)| dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |z_{2n}||u_{2n}| dx \le \int_\Omega (d_0 + d_1 |u_{1n}(x)|^{\frac{p_1}{r_2'}} + d_2 |u_{2n}(x)|^{p_2 - 1}) |u_{2n}(x)| dx \\
&= d_0 \|u_{2n}\|_1 + d_1 \int_\Omega |u_{1n}(x)|^{\frac{p_1}{r_2'}} |u_{2n}(x)| dx + d_2 \|u_{2n}\|_{p_2}^{p_2}.
\end{aligned}$$

Through Young's inequality with any $\varepsilon > 0$, we find that

$$c_2 \int_\Omega |u_{2n}(x)|^{\frac{p_2}{r_1'}} |u_{1n}(x)| dx \le \varepsilon \|u_{2n}\|_{p_2}^{p_2} + c(\varepsilon) \|u_{1n}\|_{r_1}^{r_1}$$

and

$$d_1 \int_\Omega |u_{1n}(x)|^{\frac{p_1}{r_2'}} |u_{2n}(x)| dx \le \varepsilon \|u_{1n}\|_{p_1}^{p_1} + d(\varepsilon) \|u_{2n}\|_{r_2}^{r_2},$$

with positive constants $c(\varepsilon)$ and $d(\varepsilon)$. Take the sum of Inequalities (15) and (16) and insert the preceding estimates, also using (4) and (5), which result in

$$\begin{aligned}
&\left(1 - \lambda_{1,p_1}^{-1}(c_1 + \varepsilon)\right) \|\nabla u_{1n}\|_{p_1}^{p_1} + \left(1 - \lambda_{1,p_2}^{-1}(d_2 + \varepsilon)\right) \|\nabla u_{2n}\|_{p_2}^{p_2} \\
&\le |\mu_1| |\Omega|^{\frac{p_1 - q_1}{p_1}} \|\nabla u_{1n}\|_{p_1}^{q_1} + |\mu_2| |\Omega|^{\frac{p_2 - q_2}{p_2}} \|\nabla u_{2n}\|_{p_2}^{q_2} \\
&+ c_0 S_{1,p_1} \|\nabla u_{1n}\|_{p_1} + d_0 S_{1,p_2} \|\nabla u_{2n}\|_{p_2} + c(\varepsilon) S_{r_1,p_1}^{r_1} \|\nabla u_{1n}\|_{p_1}^{r_1} + d(\varepsilon) S_{r_2,p_2}^{r_2} \|\nabla u_{2n}\|_{p_2}^{r_2}.
\end{aligned}$$

Assumption (H) postulates that $c_1 < \lambda_{1,p_1}$ and $d_2 < \lambda_{1,p_2}$, so we may choose a value of $\varepsilon > 0$ so small so as to have $1 - \lambda_{1,p_1}^{-1}(c_1 + \varepsilon) > 0$ and $1 - \lambda_{1,p_2}^{-1}(d_2 + \varepsilon) > 0$. Because $1 < r_1 < p_1$, $1 < r_2 < p_2$, $1 < q_1 < p_1$, and $1 < q_2 < p_2$, we can conclude that the sequence $\{(u_{1n}, u_{2n})\}$ is bounded in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, thus completing the proof. □

Proposition 5. *Assume condition* (H)*. The sequence* $\{(u_{1n}, u_{2n})\} \subset W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ *given in Proposition 3 has the following property: there exists a constant* $M > 0$ *such that*

$$\| - \Delta_{p_1} u_{1n} + \mu_1 \Delta_{q_1} u_{1n} - z_{1n} \|_{W^{-1,p_1'}(\Omega)} \le M, \quad \forall n \tag{17}$$

and

$$\| -\Delta_{p_2}u_{2n} + \mu_2\Delta_{q_2}u_{2n} - z_{2n}\|_{W^{-1,p_2'}(\Omega)} \le M, \quad \forall\, n, \tag{18}$$

with z_{1n} *and* z_{2n} *as stated in* (11) *and* (12)*, respectively.*

Proof. According to Proposition 4 there is a constant $M_0 > 0$ such that

$$\max\{\|\nabla u_{1n}\|_{p_1}, \|\nabla u_{2n}\|_{p_2}\} \le M_0, \quad \forall\, n. \tag{19}$$

Notice that

$$(-\Delta_{p_1}u_{1n} + \mu_1\Delta_{q_1}u_{1n} - z_{1n}, -\Delta_{p_2}u_{1n} + \mu_2\Delta_{q_2}u_{2n} - z_{2n}) \in \partial J(u_{1n}, u_{2n}), \quad \forall\, n.$$

As the functional $J : W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \to \mathbb{R}$ is Lipschitz continuous on the bounded subsets of the space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, we directly infer from (19) the existence of a constant $M > 0$ for which (17) and (18) are fulfilled. The proof is achieved. □

5. Proofs of the Main Results and Example

Proof of Theorem 1. Consider the sequence $\{(u_{1n}, u_{2n})\} \subset W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, which is provided by Proposition 3 corresponding to the Galerkin basis $\{X_n \times Y_n\}$ of the space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. It is known from Proposition 4 that the sequence $\{(u_{1n}, u_{2n})\}$ is bounded in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. Precisely, the bound in (19) holds.

Thanks to the reflexivity of the space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, we may admit that along a subsequence, we have $u_{1n} \rightharpoonup u_1$ in $W_0^{1,p_1}(\Omega)$ and $u_{2n} \rightharpoonup u_2$ in $W_0^{1,p_2}(\Omega)$, as $n \to \infty$ for some $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. We will show that the weak limit (u_1, u_2) is a generalized solution to system (1).

It is clear that condition (i) is verified. For each positive integer n, Proposition 3 provides $(z_{1n}, z_{2n}) \in L^{p_1'}(\Omega) \times L^{p_2'}(\Omega)$ with $(z_{1n}, z_{2n}) \in \partial F(u_{1n}, u_{2n})$ a.e. in Ω such that (11) and (12) are satisfied. Proposition 5 ensures that the sequence $\{-\Delta_{p_1}u_{1n} + \mu_1\Delta_{q_1}u_{1n} - z_{1n}\}$ is bounded in $W^{-1,p_1'}(\Omega)$ and that the sequence $\{-\Delta_{p_2}u_{2n} + \mu_2\Delta_{q_2}u_{2n} - z_{2n}\}$ is bounded in $W^{-1,p_2'}(\Omega)$. Specifically, the bounds are expressed in (17) and (18).

The reflexivity of the spaces $W^{-1,p_1'}(\Omega)$ and $W^{-1,p_2'}(\Omega)$ implies that we can pass to relabeled subsequences satisfying $-\Delta_{p_1}u_{1n} + \mu_1\Delta_{q_1}u_{1n} - z_{1n} \rightharpoonup \eta_1$ in $W^{-1,p_1'}(\Omega)$ and $-\Delta_{p_2}u_{2n} + \mu_2\Delta_{q_2}u_{2n} - z_{2n} \rightharpoonup \eta_2$ in $W^{-1,p_2'}(\Omega)$ for some $(\eta_1, \eta_2) \in W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$.

We claim that $\eta_1 = 0$ and $\eta_2 = 0$, that is, $\langle \eta_1, v\rangle = 0$ for all $v \in W_0^{1,p_1}(\Omega)$ and $\langle \eta_2, v\rangle = 0$ for all $v \in W_0^{1,p_2}(\Omega)$. We only prove the first assertion because the second one can be checked analogously. Let $v \in W_0^{1,p_1}(\Omega)$ and suppose, first, that $v \in \bigcup_{n=1}^{\infty} X_n$. Fix some m with $v \in X_m$. Then, for each $n \ge m$, the element v can be used as a test function in (11), which gives

$$\langle -\Delta_{p_1}u_{1n}, v\rangle + \mu_1\langle \Delta_{q_1}u_{1n}, v\rangle - \int_\Omega z_{1n}v dx = 0.$$

In the limit, as $n \to \infty$, we obtain $\langle \eta_1, v\rangle = 0$. If $v \in W_0^{1,p_1}(\Omega)$ is arbitrary, we obtain $\langle \eta_1, v\rangle = 0$, owing to the density of $\bigcup_{n=1}^{\infty} X_n$ in $W_0^{1,p_1}(\Omega)$, as required by condition (c) of the Galerkin basis. Therefore, the claim is proven, which shows that condition (ii) in the definition of the generalized solution to system (1) is satisfied.

Now, we deal with condition (iii) in the definition of the generalized solution to (1). It is known from (11) and (12) that

$$\langle -\Delta_{p_1} u_{1n}, u_{1n} \rangle + \mu_1 \langle \Delta_{q_1} u_{1n}, u_{1n} \rangle - \int_\Omega z_{1n} u_{1n} dx = 0, \quad \forall n,$$
$$\langle -\Delta_{p_2} u_{2n}, u_{2n} \rangle + \mu_2 \langle \Delta_{q_2} u_{2n}, u_{2n} \rangle - \int_\Omega z_{2n} u_{2n} dx = 0, \quad \forall n.$$

On the other hand, according to assertion (ii), one has

$$\lim_{n\to\infty} \left[\langle -\Delta_{p_1} u_{1n}, u_1 \rangle + \mu_1 \langle \Delta_{q_1} u_{1n}, u_1 \rangle - \int_\Omega z_{1n} u_1 dx \right] = 0,$$
$$\lim_{n\to\infty} \left[\langle -\Delta_{p_2} u_{2n}, u_2 \rangle + \mu_2 \langle \Delta_{q_2} u_{2n}, u_2 \rangle - \int_\Omega z_{2n} u_2 dx \right] = 0.$$

Combining the preceding estimates renders

$$\lim_{n\to\infty} \left[\langle -\Delta_{p_1} u_{1n}, u_{1n} - u_1 \rangle + \mu_1 \langle \Delta_{q_1} u_{1n}, u_{1n} - u_1 \rangle - \int_\Omega z_{1n} (u_{1n} - u_1) dx \right] = 0, \quad (20)$$
$$\lim_{n\to\infty} \left[\langle -\Delta_{p_2} u_{2n}, u_{2n} - u_2 \rangle + \mu_2 \langle \Delta_{q_2} u_{2n}, u_{2n} - u_2 \rangle - \int_\Omega z_{2n} (u_{2n} - u_2) dx \right] = 0. \quad (21)$$

Lemma 2 guarantees that the functional $\Phi : L^{p_1}(\Omega) \times L^{p_2}(\Omega) \to \mathbb{R}$ given in (7) is Lipschitz continuous on the bounded subsets of $L^{p_1}(\Omega) \times L^{p_2}(\Omega)$; thus, its generalized gradient $\partial\Phi : L^{p_1}(\Omega) \times L^{p_2}(\Omega) \to 2^{L^{p_1'}(\Omega) \times L^{p_2'}(\Omega)}$ is a bounded multifunction, which means that the image of every bounded set is a bounded set. Hence, on the basis of the inclusion $(z_{1n}, z_{2n}) \in \partial\Phi(u_{1n}, u_{2n})$ and Proposition 4, we are led to the conclusion that the sequence $\{(z_{1n}, z_{2n})\}$ is bounded in $L^{p_1'}(\Omega) \times L^{p_2'}(\Omega)$. Recalling that $u_{1n} \rightharpoonup u_1$ in $W_0^{1,p_1}(\Omega)$ and $u_{2n} \rightharpoonup u_2$ in $W_0^{1,p_2}(\Omega)$, the Rellich–Kondrachov compact embedding theorem provides strong convergence $(u_{1n}, u_{2n}) \to (u_1, u_2)$ in $L^{p_1}(\Omega) \times L^{p_2}(\Omega)$. It turns out that

$$\lim_{n\to\infty} \int_\Omega z_{1n}(u_{1n} - u_1) dx = 0,$$
$$\lim_{n\to\infty} \int_\Omega z_{2n}(u_{2n} - u_2) dx = 0.$$

Inserting this into (20) and (21), we see that requirement (iii) in the definition of the generalized solution is fulfilled. Therefore, $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ is a generalized solution to system (1). The proof of Theorem 1 is complete. □

Proof of Theorem 2. Assume that $\mu_1 \le 0$ and $\mu_2 \le 0$. Let $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ be a generalized solution to system (1). Then, there exists a sequence $\{(u_{1n}, u_{2n})\} \subset W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ satisfying conditions (i), (ii), and (iii).

Using conditions (i) and (iii), as well as $\mu_1 \le 0$ and the monotonicity of $-\Delta_{q_1}$, we derive

$$\begin{aligned}
&\limsup_{n\to\infty} \langle -\Delta_{p_1} u_{1n}, u_{1n} - u_1 \rangle \\
&= \lim_{n\to\infty} [\langle -\Delta_{p_1} u_{1n}, u_{1n} - u_1 \rangle + \mu_1 \langle \Delta_{q_1} u_{1n}, u_{1n} - u_1 \rangle] - \mu_1 \limsup_{n\to\infty} \langle \Delta_{q_1} u_{1n}, u_{1n} - u_1 \rangle \\
&= \mu_1 \liminf_{n\to\infty} \langle -\Delta_{q_1} u_{1n}, u_{1n} - u_1 \rangle \\
&\le \mu_1 \liminf_{n\to\infty} \langle -\Delta_{q_1} u_{1n} + \Delta_{q_1} u_1, u_{1n} - u_1 \rangle + \mu_1 \lim_{n\to\infty} \langle -\Delta_{q_1} u_1, u_{1n} - u_1 \rangle \le 0.
\end{aligned}$$

This enables us to use the S_+ property of the operator $-\Delta_{p_1} : W_0^{1,p_1}(\Omega) \to W^{-1,p_1'}(\Omega)$, meaning that $u_{1n} \rightharpoonup u_1$ in $W_0^{1,p_1}(\Omega)$ and $\limsup_{n\to\infty} \langle -\Delta_{p_1} u_{1n}, u_{1n} - u_1 \rangle \le 0$ provide $u_{1n} \to u_1$ (refer to [3]). Therefore, the S_+ property of the operator $-\Delta_{p_1}$ implies the strong convergence $u_{1n} \to u_1$ in $W_0^{1,p_1}(\Omega)$. According to the continuity of the operators $-\Delta_{p_1}$ and Δ_{q_1} in the norm topologies, we have $-\Delta_{p_1} u_{1n} + \mu_1 \Delta_{q_1} u_{1n} \to -\Delta_{p_1} u_1 + \mu_1 \Delta_{q_1} u_1$ in

$W^{-1,p_1'}(\Omega)$. Similarly, we prove that $u_{2n} \to u_2$ in $W_0^{1,p_2}(\Omega)$ and $-\Delta_{p_2}u_{2n} + \mu_2\Delta_{q_2}u_{2n} \to -\Delta_{p_2}u_2 + \mu_2\Delta_{q_2}u_2$ in $W^{-1,p_2'}(\Omega)$.

Lemma 2 establishes that the functional $\Phi : L^{p_1}(\Omega) \times L^{p_2}(\Omega) \to \mathbb{R}$ in (7) is Lipschitz continuous on the bounded subsets of $L^{p_1}(\Omega) \times L^{p_2}(\Omega)$. Since $(z_{1n}, z_{2n}) \in \partial\Phi(u_{1n}, u_{2n})$, the sequence $\{(z_{1n}, z_{2n})\}$ is bounded in $L^{p_1'}(\Omega) \times L^{p_2'}(\Omega)$ up to subsequence $z_{1n} \rightharpoonup z_1$ in $L^{p_1'}(\Omega)$ and $z_{2n} \rightharpoonup z_2$ in $L^{p_2'}(\Omega)$ for some $(z_1, z_2) \in L^{p_1'}(\Omega) \times L^{p_2'}(\Omega)$. Taking into account the strong convergence $(u_{1n}, u_{2n}) \to (u_1, u_2)$ in $L^{p_1}(\Omega) \times L^{p_2}(\Omega)$, we find that $(z_1, z_2) \in \partial\Phi(u_1, u_2)$ due to the fact that the generalized gradient $\partial\Phi$ is strongly-weakly* closed.

At this point, it suffices to pass to the limit as $n \to \infty$ in condition (ii) in the definition of the generalized solution (u_1, u_2) of system (1) to deduce that in the dual space $W^{1,p_1'}(\Omega) \times W^{1,p_2'}(\Omega)$, the following equality holds:

$$(-\Delta_{p_1}u_1 + \mu_1\Delta_{q_1}u_1 - z_1, -\Delta_{p_2}u_2 + \mu_2\Delta_{q_2}u_2 - z_2) = (0, 0).$$

This is equivalent to (3). Since $(z_1, z_2) \in \partial\Phi(u_{1n}, u_{2n})$, hypothesis (H) and the Aubin–Clarke theorem (see [1]) confirm the validity of the pointwise inclusion $(z_1(x), z_2(x)) \in \partial F(u_1(x), u_2(x))$ for almost all $x \in \Omega$. We conclude that (u_1, u_2) is a weak solution to system (1).

The existence of a weak solution to system (1) when $\mu_1 \le 0$ and $\mu_2 \le 0$ follows from Theorem 1 and the first part of Theorem 2 that we have already proven. The proof is, thus, complete. □

Here is an example showing how our results can be applied.

Example 1. *Let $B : \mathbb{R} \to 2^R$ denote the generalized gradient of the absolute value function $|\cdot|$ on $\mathbb{R}$, that is, $B(t) = -1$ if $t < 0$, $B(t) = 1$ if $t > 0$, and $B(0) = [-1, 1]$. Given the numbers $p_1 \in (2, +\infty)$ and $p_2 \in (2, +\infty)$, consider on the bounded domain $\Omega \subset \mathbb{R}^N$ the following system of hemivariational inclusions:*

$$\begin{cases} -\Delta_{p_1}u_1 + \Delta u_1 \in u_2 B(u_1) + \cos(u_1 + |u_2|) & \text{in } \Omega \\ -\Delta_{p_2}u_1 + \Delta u_2 \in |u_1| + \cos(u_1 + |u_2|)B(u_2) & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where Δ stands for the ordinary Laplacian operator, i.e., $\Delta = \Delta_2$. This is system (1) for $\mu_1 = \mu_2 = 1$, $q_1 = q_2 = 2$, and $F : \mathbb{R}^2 \to R$ given by $F(t, s) = (s|t|) + \sin(t + |s|)$ for all $(t, s) \in \mathbb{R}^2$, since $\partial F(t, s) = (s\partial|t| + \cos(t + |s|), |t| + \cos(t + |s|)\partial|s|)$ for all $(t, s) \in \mathbb{R}^2$.

Setting $r_1 = r_2 = 2$ (so $r_1' = r_2' = 2$), it is seen that condition (H) is fulfilled. Indeed, for every $(\zeta_1, \zeta_2) \in \partial F(t, s)$, we have

$$|\zeta_1| \le |s| + 1 \le 2 + |s|^{\frac{p_2}{2}}$$

and

$$|\zeta_2| \le |t| + 1 \le 2 + |t|^{\frac{p_1}{2}}.$$

Theorem 1 guarantees that the system presented in (22) admits a generalized solution (u_1, u_2) in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$.

If in place of $-\Delta_{p_1} + \Delta$ and $-\Delta_{p_2} + \Delta$, we take $-\Delta_{p_1} - \Delta$ and $-\Delta_{p_2} - \Delta$, respectively. Theorem 2 ensures the existence of a weak solution (u_1, u_2) in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ for the obtained system.

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