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Robust State-Feedback Control and Convergence Analysis for Uncertain LPV Systems Using State and Parameter Estimation

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Abstract: This study introduces the design of a state-feedback controller for Linear Parameter Varying (LPV) systems in scenarios where exogenous parameters are not directly accessible, and the state vector is to be estimated. Instead of considering a static feedback gain, it proposes a method for estimating these parameters and synthesizing a parameter-dependent state-feedback gain that is robust against uncertainties in parameter estimation. The state vector used by the state-feedback controller, and some quantities required by the estimation law, are both obtained by a robust filter synthesized by LMI (Linear Matrix Inequalities). This paper outlines the estimation, filtering, and control laws, detailing the conditions necessary for ensuring convergence and stability. A numerical experiment and a 2 DoF torsional system application show the enhanced dynamic performance of the method when applied to uncertain dynamic systems. The findings highlight the effectiveness of the proposed approach in maintaining system stability and improving performance despite the inherent uncertainties in parameter estimation, offering a significant contribution to the field of robust control for LPV systems.

Keywords: LPV systems; parametric estimation; state-feedback control; linear matrix inequalities; robust filtering; estimation uncertainty

MSC: 93D21

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1. Introduction

Linear Parameter Varying (LPV) systems have received significant attention in control studies due to their capability to manage systems with dependencies on external adjustment variables and moderate nonlinearities. This approach is particularly advantageous, as it enables the application of linear control techniques by using the polytopic representation of these systems, which can then be addressed through Linear Matrix Inequalities (LMIs) [1–3].

However, the development of LPV control systems often requires the availability of tuning parameters. Unfortunately, this requirement is not always met in practical applications. A notable example is the use of a canonical observer, a classical structure used to provide state information, which aims to replicate the dynamics of the LPV system. This replication process demands access to the exogenous and general system parameter to accurately construct the state estimation dynamics [4]. Similarly, LPV filter-based techniques require either knowledge of the parameter or the adoption of a robust polytopic formulation. In such a formulation, the parameter is treated as an uncertainty, a method that can lead to conservatism or even be infeasible in some scenarios [5]. This challenge highlights the necessity for innovative approaches to control LPV systems without relying strictly on the availability of these parameters, thereby ensuring both robustness and feasibility in practical applications.

When the exogenous parameter is unavailable for reading and direct measurement, combining parametric estimation techniques with LPV control methodologies becomes a viable solution. For example, in [6], the challenge of state estimation in LPV systems is addressed using switched Luenberger observers. These observers are fine-tuned through a least-squares approach, offering a practical method for parameter estimation. Similarly, in [7], a modified version of the extended Kalman filter is employed to determine the tuning parameter for an LPV controller, specifically for the control of reaction engines. This method resembles the approach developed by [8], which focuses on controlling arterial pressure. Neural network estimators and machine learning approaches have also provided good results under the circumstances mentioned above, especially when combined with observers and robust control techniques, as can be seen in [9,10].

Additional notable contributions on this subject include the works of [11–13]. These studies further explore various techniques for integrating parameter estimation with LPV control. Despite these advancements, the issue of guaranteeing the convergence of the combined closed-loop controller–estimator system remains underexplored and is frequently disregarded. Ensuring such convergence is crucial for the reliable performance of these systems in practical applications, indicating a significant area for future research and development. This gap underscores the need for continued investigation into robust methodologies that ensure both stability and convergence in LPV control systems using parametric estimation—a problem to which this article aims to contribute.

The problem of convergence for the closed-loop estimator–controller set is addressed in [14]. In this research, a single Lyapunov function is constructed that incorporates the dynamics of both the controller and the estimator. This approach ensures the convergence of the entire set and provides a robust solution to the problem. Similarly, in [5], the gain of an LPV controller is defined in terms of a new adaptive estimation law, which also guarantees the stability of the system. This study employs a joint Lyapunov function to achieve its results, further solidifying the connection between estimation and control.

Building on these foundations, ref. [15] presents more recent advances in the field by developing adaptive controllers that are combined with high-order estimators. This approach involves estimating not only the unknown parameter but also its derivatives, thereby enhancing the adaptability and precision of the control system. These high-order estimators contribute to a more nuanced understanding and management of the system dynamics. By integrating estimation with control to ensure system stability and convergence, and addressing the convergence issue through sophisticated techniques such as unified Lyapunov functions and adaptive estimation laws, these studies represent significant contributions to the field of LPV systems.

The problem becomes more involved if the states are unavailable for feedback. Considering the objective of robustly controlling dynamical systems in this case, two approaches are usually applied: estimating the states through observers [16,17] or developing output feedback controllers [18–20]. To the best of the authors' knowledge, the utilization of a control framework depending only on the system outputs, which performs estimation of both the parameters and states, and using them to achieve a proper closed-loop stable system, is still an open problem.

In this context, this work proposes novel LPV control conditions considering the estimation of the exogenous parameters and the states, along with guarantees of closed-loop convergence. The approach taken in this study assumes that the adjustment parameter is partially accessible. This means that it comprises both a nominal component and an uncertain component, which represents the estimation error.

The main contribution of this paper is the proposition of a framework for the development of a reference tracking parameter-dependent control system, supposing that only the system outputs are available for feedback. Three elements are crucial for the methodology:

the controller, which is synthesized by considering that the parameters and the estimation errors are within a given polytope [18,21], and also being robust on errors in the states; the filter, responsible for robustly computing the estimated states from the outputs of the LPV uncertain system; and the adaptive estimation procedure, which uses the filter information to determine the parameters. It is worth mentioning that the synthesis conditions for the controller and filter are also novel contributions from this paper. Finally, in this paper, a series of conditions are developed and proposed to assure the convergence of the entire framework. In this sense, the paper's novelties and main contributions can be summarized as follows:

- Development of a state feedback control law considering estimated states and parameters, an approach not yet performed through LPV filters;
- Introduction of conditions to assure the stability of the controller fed by the estimated system states;
- Design of two novel LMI conditions: one aiming to synthesize $\mathcal{H}_\infty$ state-feedback gains supposing that the exogenous input matrix also depends on the controller gain, and a condition to determine filter dynamics minimizing a generalized $\mathcal{H}_2$ norm;
- A new convergence law, joining the parameter estimation procedure with the LPV control law, achieved by the set estimation error boundaries.

The paper is organized as follows. Section 2 presents the preliminary results, which are important for the development of the proposed approach, intended to solve the problem whose formulation is detailed in Section 3. The main results of the paper, which are the conditions for the synthesis of the controller, filter, and estimation procedure, as well a convergence analysis, are presented in Section 4. Two experimental results, one analytic and another stemming from a practical implementation, are presented in Section 5. Section 6 concludes the paper.

2. Preliminaries

In this paper, the following notation is used: $\mathbf{I}_x$ is the identity matrix with dimension x, $\mathbf{0}_x$ is a matrix of zeros of dimension x, $\text{He}\{x\} = x + x^T$, and the symbol $\star$ replaces symmetric conjugate symmetric blocks, such that $x_{ij} = x_{ji}^T$. The following preliminary concepts are also used throughout this paper.

2.1. $\mathcal{H}_\infty$ Norm [22]

Let a dynamic system be

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \\ z(t) &= Cx(t) + Dw(t), \end{aligned} \tag{1}$$

with matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q}, C \in \mathbb{R}^{q\times n}, D \in \mathbb{R}^{q\times q}$, and transfer function from $w(t)$ to $z(t)$ given by $G(s) = C(s\mathbf{I}_n - A)^{-1}B + D$. Then, the norm $\mathcal{H}_\infty$ of system (1), from the input $w(t)$ to the output $z(t)$, is given by

$$\mathcal{H}_\infty : ||\mathcal{G}_{z,w}(s)||_\infty = \sup_{||w||\neq 0} \frac{||z||_2}{||w||_2}. \tag{2}$$

2.2. Bounded Real Lemma

Suppose the continuous system presented in (1), and consider $P = P^T \in \mathbb{R}^{n\times n}$. The following conditions are equivalent:

$$||\mathcal{G}_{z,w}(s)||_\infty < \gamma, \tag{3}$$

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ \star & -\gamma^2\mathbf{I}_q & D^T \\ \star & \star & -\mathbf{I}_q \end{bmatrix} < 0, \;\; P > 0, \tag{4}$$

$$\begin{bmatrix} AP + PA^T & PC^T & B \\ \star & -\gamma^2 \mathbf{I}_q & D \\ \star & \star & -\mathbf{I}_q \end{bmatrix} < 0, \;\; P > 0 \tag{5}$$

More specifically, Conditions (4) and (5) are, respectively, the primal and dual formulations for the Bounded Real Lemma.

2.3. Schur Complement

Consider any set of matrices (R, S, U) of appropriate dimensions, where $R = R^T$ and $U = U^T$. If U is invertible, the following conditions are equivalent:

$$\begin{bmatrix} R & S \\ S^T & U \end{bmatrix} < 0,$$

$$U < 0, \quad R - SU^{-1}S^T < 0$$

2.4. Generalized $\mathcal{H}_2$ Norm

The generalized $\mathcal{H}_2$ norm of system (1), from $w(t)$ to $z(t)$, is defined as [23,24]

$$||\mathcal{G}_{z,w}(s)||_2 = \sup_{i=1,\ldots,n_z} \left\{ ||z_i(t)|| \; : \; x(0) = 0, \; \int_0^\infty w(\tau)^T w(\tau) d\tau \leq 1 \right\}. \tag{6}$$

Using Lemma 1, the bounds for the $\mathcal{H}_2$ norm can also be obtained.

Lemma 1. *The generalized $\mathcal{H}_2$ norm $||\mathcal{G}_{zw}||_2$ is upper bounded by $\mu \in \mathbb{R}$ if there exists a Lyapunov function $V(x)$ satisfying*

$$\mu^2 V(x) - z(t)^T z(t) > 0, \tag{7}$$

$$\dot{V}(x) < w(t)^T w(t). \tag{8}$$

3. Problem Formulation

Consider a parameter-dependent linear system given by

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B_u(\theta)(t) + B_w(\theta)w(t), \\ y(t) &= C_y(\theta)x(t) + D_{yw}(\theta)w(t), \end{aligned} \tag{9}$$

in which $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector (control signal), $y(t) \in \mathbb{R}^{n_y}$ is the output vector, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input to the system—representing disturbances and noise, and $\theta \in \mathbb{R}^{n_\theta}$ is the vector that contains the system parameters, such that $\theta = \begin{bmatrix} \theta_1, & \theta_2, & \ldots, & \theta_{n_\theta} \end{bmatrix}^T$ presents magnitude limited by $\theta_i \in \begin{bmatrix} \underline{\theta}_i, & \bar{\theta}_i \end{bmatrix}$. The matrices of (9) present adequate dimensions and are related to θ in an affine way:

$$M(\theta) = M_0 + \sum_{i=1}^{n_\theta} \theta_i M_i. \tag{10}$$

In this paper, the polytopic representation of the matrices in (9) is used, facilitating its manipulation at later stages. Thus, the dependence of the matrices in relation to each parameter is rewritten as a function of vertices of the polytope $(\alpha_{i1}, \alpha_{i2})$, $i = 1, \ldots, n_\theta$, using (11):

$$\alpha_{i1} = \frac{\theta_i - \underline{\theta}_i}{\bar{\theta}_i - \underline{\theta}_i}, \; \alpha_{i2} = 1 - \alpha_{i1}, \; \alpha_i = (\alpha_{i1}, \alpha_{i2}) \in \Lambda_2, \tag{11}$$

in which Λ_r is the unit simplex presented in (12), with $r = 2$. The composition of several polytopic elements leads to the multi-simplex representation, presented in Definition 1.

$$\Lambda_r = \left\{ \alpha \in \mathbb{R}^r : \; \sum_{i=1}^{r} \alpha_i = 1, \; \alpha_i \geq 0, \; i = 1, \ldots, r \right\}. \tag{12}$$

Definition 1 (Multi-simplex [25])**.** *A multi-simplex Λ is the Cartesian product of a finite number m of simplexes $\Lambda_{N_1}, ..., \Lambda_{N_m}$, $i = 1, ..., m$, so that $\Lambda = \Lambda_{N_1} \times ... \times \Lambda_{N_m}$.*

For system (9), the output track of a reference $r(t)$, the tracking problem, can be achieved by defining the auxiliary state $\dot{q}(t) = r(t) - y(t)$, composing the extended state vector $\bar{x}(t) = \begin{bmatrix} x^T(t) & q^T(t) \end{bmatrix}^T$, where $\bar{x}(t) \in \mathbb{R}^{\bar{n}}$. The extended system is presented in Equation (13) and will be summarized in this paper as a general state feedback control problem $(\bar{A}(\theta) - \bar{B}(\theta)K(\theta))$,

$$\mathcal{G}(\theta)\begin{cases} \dot{\bar{x}}(t) = \left(\underbrace{\begin{bmatrix} A(\theta) & \mathbf{0}_{n\times n_y} \\ -C_y(\theta) & \mathbf{0}_{n_y} \end{bmatrix}}_{\bar{A}(\theta)} - \underbrace{\begin{bmatrix} B_u(\theta) \\ \mathbf{0}_{n\times n_y} \end{bmatrix}}_{\bar{B}(\theta)} K(\theta)\right)\bar{x}(t) + \underbrace{\begin{bmatrix} B_w(\theta) \\ \mathbf{0}_{n_w\times n_y} \end{bmatrix}}_{\bar{B}_w(\theta)} w(t) + \underbrace{\begin{bmatrix} \mathbf{0}_{n\times n_y} \\ \mathbf{I}_{n_y} \end{bmatrix}}_{B_r} r(t), \\ z(t) = \underbrace{\begin{bmatrix} C_z(\theta) & \mathbf{0}_{n_y} \end{bmatrix}}_{\bar{C}_z(\theta)} \bar{x}(t) + D_{zw}w(t), \\ y(t) = \underbrace{\begin{bmatrix} C_y(\theta) & \mathbf{0}_{n_y} \end{bmatrix}}_{\bar{C}_y(\theta)} \bar{x} + D_{yw}w(t), \end{cases} \tag{13}$$

with $z(t) \in \mathbb{R}^{n_z}$ being a virtual output vector.

As previously described, one of the objectives of this paper is to precisely determine a control signal that stabilizes the system, in the format presented in (14), where $K(\theta)$ is the scaled feedback gain and $\tilde{\bar{x}}(t)$ is the estimated state vector.

$$u(t) = -K(\theta)\tilde{\bar{x}}(t). \tag{14}$$

The feedback law shown in (14) conveys two main problems in the control area. The first problem is related to the synthesis of the gain $K(\theta)$, considering its parametric dependence, while the second problem is related to the use of an estimated state vector in state feedback control. In the following subsection, these two problems are disclosed.

3.1. Parametric Dependent Feedback Gain

When considering a completely available state vector, the synthesis of state feedback gain can be obtained by several robust control techniques consolidated in the literature, for which two approaches are commonly found:

1. The premise that the vector θ is precisely known [1,26,27];
2. The assumption that θ is not available, defining a static gain K valid for the entire polytope [28,29].

However, these approaches can have restrictive characteristics: the first approach has limitations in its practical implementation, while the second can generate conservative results, especially in cases related to performance parameters, such as controllers of type $\mathcal{H}_2$ and $\mathcal{H}_\infty$.

A reasonable alternative is to consider the case in which θ is partially available—that is, composed of an estimated nominal component—added to the reading and/or estimation error. Thus, the controller synthesis condition can be relaxed compared to a static feedback gain. Let $\tilde{\theta}_i$ be the estimated parameters, which can be described as

$$\tilde{\theta}_i = \theta_i + \delta_i, \quad i = 1, \ldots, n_\theta, \tag{15}$$

in which θ_i is the nominal parameter and δ_i is an additive estimation error limited by $|\delta_i| \le \bar{\delta}_i$, $\bar{\delta}_i \in \mathbb{R}^+$. The affine representation of the matrices in (10) can be generalized to cover the case of an estimated parameter:

$$M(\tilde{\theta}) = M_0 + \sum_{i=1}^{n_\theta} (\theta_i + \delta_i) M_i, \tag{16}$$

with the parameterization of the estimation error δ_i being obtained in a polytope such that [18]

$$\beta_{i1} = \frac{\delta_i + \bar{\delta}_i}{2\,\bar{\delta}_i},\ \beta_{i2} = 1 - \beta_{i1},\ \beta_i = (\beta_{i1}, \beta_{i2}) \in \Lambda_2. \tag{17}$$

Thus, combining the pair (θ_i, δ_i) in (16), and their respective polytopic representations presented in (11) and (17), the following is obtained:

$$M(\alpha, \beta) = M_0 + \sum_{i=1}^{n_\theta} \left[\alpha_{i1}\,\bar{\theta}_i + \alpha_{i2}\,\underline{\theta}_i\right] M_i + \sum_{i=1}^{n_\theta} \left[(\beta_{i1} - \beta_{i2})\bar{\delta}_i\right] M_i. \tag{18}$$

Applying the homogenization process to the matrix $M(\alpha, \beta)$, the representation of the LPV matrix is obtained in terms of all vertices of the multi-simplex Λ, available in (19) and (20) [18].

$$M(\alpha, \beta) = \sum_{i_1=1}^{2} \cdots \sum_{i_{n_\theta}=1}^{2} \sum_{j_1=1}^{2} \cdots \sum_{j_{n_\theta}=1}^{2} \alpha_{1i_1} \ldots \alpha_{n_\theta i_{n_\theta}} \beta_{1i_1} \ldots \beta_{n_\theta i_{n_\theta}} T_{i_1 \ldots i_{n_\theta} j_1 \ldots j_{n_\theta}}, \tag{19}$$

$$T_{i_1 \ldots i_{n_\theta} j_1 \ldots j_{n_\theta}} = M_0 + \sum_{\ell=1}^{n_\theta} \Big[(i_\ell - 1)\,\bar{\theta}_\ell + (2 - i_\ell)\,\underline{\theta}_\ell + (-1)^{j_\ell}\,\bar{\delta}_\ell\Big] M_\ell. \tag{20}$$

Therefore, the system matrices (9) can be described by the composition of two LPV elements—with α representing nominal components and β representing uncertainties and errors in the parameter adjustment process. In the context of systems control, an estimation law for θ_i can be proposed, which will allow removing the restrictive condition of considering θ as unavailable, and replacing such a condition with one in which only δ_i is not precisely known.

3.2. Estimated State Vector

The standard solutions for the state feedback gain, as well as the papers mentioned in Section 3.1, consider that all states are available. This condition is also conservative, as in some cases, these quantities cannot be easily obtained in a real-world environment. In this instance, the use of an estimated state vector is of interest.

Although a common approach to obtain an estimated version of the state vector is the use of LPV robust observers [16,17,30], the use of Luenberger-based observers is not viable in this case—in the sense of the problem presented in Section 3.1, due to the non-precise knowledge of the LPV parameters necessary to reconstruct the system dynamics. To overcome this problem, we propose the use of robust filters in this work. Consider the robust filter given by

$$\mathcal{F} \begin{cases} \dot{x}_f(t) = A_f x_f(t) + B_{fy} y(t) + B_{fr} r(t) \\ z_f(t) = C_f x_f(t) + D_{fy} y(t), \end{cases} \tag{21}$$

where $x_f(t) \in \mathbb{R}^{\tilde{n}}$ is the filter states and $z_f(t) \in \mathbb{R}^{n_z}$ the output satisfying $\lim_{t \to +\infty} e(t) \triangleq z(t) - z_f(t) = 0$. The dynamics of the augmented system, composed by the states from red (13) and from the filter (21), are given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{x}_f(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} \bar{A}(\alpha) & 0 \\ B_{fy}\bar{C}_y(\theta) & A_f \end{bmatrix}}_{\tilde{A}(\alpha)} \underbrace{\begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix}}_{\tilde{x}(t)} + \underbrace{\begin{bmatrix} \bar{B}_w \\ B_{fy}D_{yw} \end{bmatrix}}_{\tilde{B}_w} w(t) + \underbrace{\begin{bmatrix} B_r \\ B_{fr} \end{bmatrix}}_{\tilde{B}_r} r(t) \\
e(t) &= \underbrace{\begin{bmatrix} C_z - D_{fy}\bar{C}_y & -C_f \end{bmatrix}}_{\tilde{C}_z} \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} + \underbrace{\begin{bmatrix} D_{zw} - D_{fy}D_{yw} \end{bmatrix}}_{\tilde{D}_w} w(t).
\end{aligned} \tag{22}$$

So, the problem of estimating the state can be achieved by obtaining the matrix set $(A_f, B_{fy}, B_{fr}, C_f, D_{fy})$, composing and stabilizing system (21), which will be presented in the next section of this paper.

3.3. Summarized Problem Definition

In this context, the stability of the equivalent closed-loop system will depend on three factors: the guarantee of convergence of the parametric estimation, the guarantee of convergence of the filtered states, and the guarantee of stability of the feedback system, considering the polytope composed by the parameter and its estimation uncertainty (α, β). Assuming the conditions previously presented, the problems to be solved in this paper are as follows:

1. Find a feedback gain $K(\alpha, \beta)$ that stabilizes the system, for a case in which $\tilde{\theta}$ and the states $\tilde{\tilde{x}}(t)$ are the estimated quantities;
2. Determine the filter matrices $(A_f, B_{fy}, B_{fr}, C_f, D_{fy})$ that estimate the states $\tilde{\tilde{x}}(t)$.
3. Develop the estimation law for $\tilde{\theta}$ and its attraction condition.

To the best of the authors' knowledge, the combined problem of state feedback with estimated states and parameters is still an open problem in the control area that this papers aiming to contribute.

4. Main Results

In the present section, the proposed control system based on the estimated parameters and states is developed, with the results divided into two parts. Section 4.1 presents the conditions for the synthesis of a state-feedback gain $K(\tilde{\theta})$ robust to uncertainties over the parameters and depending on the estimated states (Theorem 1). The proposed robust filter (Theorem 2) is also detailed in Section 4.1, whose condition is defined upon the system controlled by the previously computed state-feedback gain, along with the procedure for estimation of exogenous parameters $\tilde{\theta}$ (Theorem 3). The conditions for the convergence of the entire control system, which consists of the state-feedback gain, the filter, and the estimation procedure, are then presented in Section 4.2. Two main results are detailed therein: the requirements to assure that the estimation procedure indeed yields a parameter whose uncertainties are contained within the prescribed bounds (Theorem 4), and the conditions to the state-feedback gain, depending on the estimated states, guarantee the stability of the entire system (Theorem 5).

4.1. Controller and Filter Synthesis

Consider the following state-feedback gain affinely dependent on the estimated parameters $\tilde{\theta}$

$$K(\tilde{\theta}) = K_0 + \sum_{i=1}^{n_\theta} (\theta_i + \delta_i) K_i, \tag{23}$$

which is similar to the formulation in (16). Considering the homogenized formulation in (19), it is possible to describe the state-feedback gain in (14) as $K(\alpha, \beta)$, i.e., depending on the vertices of the multi-simplex Λ. Therefore, the desired control signal to be computed is described as

$$u(t) = -K(\alpha, \beta)\tilde{\tilde{x}}(t), \;\; \forall (\alpha, \beta) \in \Lambda. \tag{24}$$

The application of such control into system (13), considering $D_{zw} = 0$, results in

$$\begin{cases} \dot{\bar{x}}(t) = (\bar{A}(\alpha) - \bar{B}(\alpha)K(\alpha,\beta))\bar{x}(t) + \begin{bmatrix} \bar{B}(\alpha)K(\alpha,\beta) & \bar{B}_w(\alpha) \end{bmatrix} \begin{bmatrix} e(t) \\ w(t) \end{bmatrix} + B_r r(t) \\ z(t) = \bar{C}_z \bar{x}(t), \\ y(t) = \bar{C}_y(\alpha)\bar{x}(t) + D_{yw}w(t), \end{cases} \tag{25}$$

where $e(t) = \bar{x}(t) - \tilde{\bar{x}}(t)$ is the state estimation error. Note that only the feedback gain depends on the variable β connected to the estimation error, since the system dynamics matrices $\bar{A}(\alpha)$ and $\bar{B}(\alpha)$ are inherent to the system.

The following theorem presents a set of conditions to generate the desired state-feedback gain $K(\alpha,\beta)$.

Theorem 1. *If there exists a definite positive symmetric matrix $P(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$; matrices $G \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $Z(\alpha,\beta) \in \mathbb{R}^{n_y\times\bar{n}}$; and positive predefined scalars ξ_1, ξ_2, and ξ_3 such that*

$$\begin{bmatrix} -G - G^T & \star & \star & \star \\ P(\alpha) - \xi_1 G^T + \bar{A}(\alpha)G - \bar{B}(\alpha)Z(\alpha,\beta) & \Psi & \star & \star \\ -\xi_2 G^T + \bar{C}_z(\alpha)G & \xi_2(G^T\bar{A}(\alpha)^T - Z(\alpha,\beta)^T\bar{B}(\alpha)^T) + \xi_1\bar{C}_z(\alpha)G & \xi_2(\bar{C}_z(\alpha)G + G^T\bar{C}_z(\alpha)^T) - \gamma^2\mathbf{I}_{\bar{n}} & \star \\ \mathbf{0}_{\bar{n}} & \xi_3 Z(\alpha,\beta)^T\bar{B}(\alpha)^T & \mathbf{0}_{\bar{n}} & \xi_3(G + G^T) - \mathbf{I}_{\bar{n}} \end{bmatrix} < 0 \tag{26}$$

is valid, where

$$\Psi = \xi_1(\bar{A}(\alpha)G + G^T\bar{A}(\alpha)^T - \bar{B}(\alpha)Z(\alpha,\beta) - Z(\alpha,\beta)^T\bar{B}(\alpha)^T) + \bar{B}_w(\alpha)\bar{B}_w(\alpha)^T,$$

then $K(\alpha,\beta) = Z(\alpha,\beta)G^{-1}$ is a state-feedback gain capable of stabilizing system (25) with $\mathcal{H}_\infty$ norm from $[e(t)^T w(t)^T]^T$ to $z(t)$ bounded by γ.

Proof. Replacing $Z(\alpha,\beta) = K(\alpha,\beta)G$ on condition (26) and multiplying it by M on the left and M^T on the right, where $A_{cl}(\alpha,\beta) = \bar{A}(\alpha) - \bar{B}(\alpha)K(\alpha,\beta)$ and

$$M = \begin{bmatrix} A_{cl}(\alpha,\beta) & \mathbf{I}_{\bar{n}} & \mathbf{0}_{\bar{n}\times n_y} \\ \bar{C}_z(\alpha) & \mathbf{0}_{n_y\times\bar{n}} & \mathbf{I}_{n_y} \end{bmatrix},$$

results in

$$\begin{bmatrix} P(\alpha)A_{cl}(\alpha,\beta)^T + A_{cl}(\alpha,\beta)P(\alpha) + \bar{B}(\alpha)K(\alpha,\beta)K(\alpha,\beta)^T\bar{B}(\alpha)^T + \bar{B}_w(\alpha)\bar{B}_w(\alpha)^T & \star \\ \bar{C}_z(\alpha)P(\alpha) & -\gamma^2\mathbf{I}_{n_y} \end{bmatrix} < 0.$$

The application of the Schur complement [22] yields

$$\begin{bmatrix} P(\alpha)A_{cl}(\alpha)^T + A_{cl}(\alpha)P(\alpha) & P(\alpha)\bar{C}_z(\alpha)^T & \bar{B}(\alpha)K(\alpha) & \bar{B}_w(\alpha) \\ \star & -\gamma^2\mathbf{I}_{n_y} & \mathbf{0}_{n_y\times\bar{n}} & \mathbf{0}_{n_y\times n_w} \\ \star & \star & -\mathbf{I}_{\bar{n}} & \mathbf{0}_{\bar{n}\times n_w} \\ \star & \star & \star & -\mathbf{I}_{n_w} \end{bmatrix} < 0,$$

which is equivalent to the dual formulation for the Bounded Real Lemma, as described in (5). □

Remark 1. *The scalars ξ_i are slack variables intended to improve the feasibility region of the LMI conditions, often used in similar applications [31–33]. In order to avoid convexity problems, such* *variables need to be predefined. A search procedure can be performed to determine the values that yield the best results, but such a procedure is beyond the scope of the paper.*

With an appropriate state-feedback gain, it is now possible to develop the proposed robust filter. For this, define the variable $z_2(t)$ as

$$z_2(t) = \tilde{C}_z \tilde{x}(t), \tag{27}$$

which consists of the variable $e(t)$ without considering the terms with $w(t)$, which is necessary to assure that the generalized $\mathcal{H}_2$ norm is finite [23]. The following theorem states the conditions for synthesizing the desired robust filter (21).

Theorem 2. *Suppose that $r(t)$ is such that $||r(t)||^2 \leq \rho^2 ||x(t)||^2$. If there exist definite positive symmetric matrices $P_{11}(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $P_{22}(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, and $H(\alpha) \in \mathbb{R}^{n_z\times n_z}$; matrices $P_{12}(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $K_{11}(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $K_{21}(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $G_{11}(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $G_{21}(\alpha) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $F_{11}(\alpha) \in \mathbb{R}^{n_w\times\bar{n}}$, $R_{11}(\alpha) \in \mathbb{R}^{n_y\times\bar{n}}$, $\widehat{K} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $M_1 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $M_2 \in \mathbb{R}^{\bar{n}\times n_y}$, $M_3 \in \mathbb{R}^{\bar{n}\times n_y}$, $D_{fy} \in \mathbb{R}^{n_z\times n_y}$, $D_{fr} \in \mathbb{R}^{n_z\times n_y}$, and $C_f \in \mathbb{R}^{n_z\times\bar{n}}$; and scalars λ_1, λ_2, λ_3, λ_4, τ, and μ solving the optimization problem*

$$\min \mu \qquad s.t. \tag{28}$$

$$Tr(H(\alpha)) \leq \mu^2, \tag{29}$$

$$\begin{bmatrix} H(\alpha) & \tilde{C}_z - D_{fy}\bar{C}_y(\alpha) & -C_f \\ \star & P_{11}(\alpha) & P_{12}(\alpha) \\ \star & \star & P_{22}(\alpha) \end{bmatrix} > 0, \tag{30}$$

$$\begin{bmatrix} \tau\rho^2\mathbf{I}_{\bar{n}} - \bar{C}_y(\alpha)^T\bar{C}_y(\alpha) & \mathbf{0}_{\bar{n}} & P_{11}(\alpha) & P_{12}(\alpha) & \bar{C}_y(\alpha)^T D_{yw} & \mathbf{0}_{\bar{n}\times n_y} \\ \star & \mathbf{0}_{\bar{n}} & P_{12}(\alpha)^T & P_{22}(\alpha) & \mathbf{0}_{\bar{n}\times n_w} & \mathbf{0}_{\bar{n}\times n_y} \\ \star & \star & \mathbf{0}_{\bar{n}} & \mathbf{0}_{\bar{n}} & \mathbf{0}_{\bar{n}\times n_w} & \mathbf{0}_{\bar{n}\times n_y} \\ \star & \star & \star & \mathbf{0}_{\bar{n}} & \mathbf{0}_{\bar{n}\times n_w} & \mathbf{0}_{\bar{n}\times n_y} \\ \star & \star & \star & \star & -\mathbf{I}_{n_w} - D_{yw}^T D_{yw} & \mathbf{0}_{n_w\times n_y} \\ \star & \star & \star & \star & \star & -\tau\mathbf{I}_{n_y} \end{bmatrix} + \mathcal{Y}\mathcal{F} + \mathcal{F}^T\mathcal{Y}^T < 0, \tag{31}$$

where

$$\mathcal{Y} = \begin{bmatrix} K_{11}(\alpha) & \lambda_1\mathbf{I}_{\bar{n}} \\ K_{21}(\alpha) & \lambda_2\mathbf{I}_{\bar{n}} \\ G_{11}(\alpha) & \lambda_3\mathbf{I}_{\bar{n}} \\ G_{21}(\alpha) & \lambda_4\mathbf{I}_{\bar{n}} \\ F_{11}(\alpha) & \mathbf{0}_{n_w\times\bar{n}} \\ R_{11}(\alpha) & \mathbf{0}_{n_y\times\bar{n}} \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} A_{cl}(\alpha) & \mathbf{0}_{\bar{n}} & -\mathbf{I}_{\bar{n}} & \mathbf{0}_{\bar{n}} & \bar{B}_w & B_r \\ M_2\bar{C}_y & M_1 & \mathbf{0}_{\bar{n}} & -\hat{K} & M_2 D_{yw} & M_3 \end{bmatrix} \tag{32}$$

then

$$A_f = \widehat{K}^{-1}M_1, \quad B_{fy} = \widehat{K}^{-1}M_2, \quad B_{fr} = \widehat{K}^{-1}M_3, \quad C_f \quad \text{and} \quad D_{fy}$$

are the matrices of the robust filter (21) that minimize the bound μ for the generalized $\mathcal{H}_2$ from $[w(t)^T \;\; y(t)^T]^T$ to $z_2(t)$.

Proof. Consider the augmented system (22) and the Lyapunov function $V(\tilde{x})$ given by

$$V(\tilde{x}) = \tilde{x}(t)^T P(\alpha)\tilde{x}(t), \quad P(\alpha) = \begin{bmatrix} P_{11}(\alpha) & P_{12}(\alpha) \\ P_{12}(\alpha)^T & P_{22}(\alpha) \end{bmatrix}.$$

First, since conditions (29) and (30) are valid, then it is also true that

$$\begin{bmatrix} \mu^2\mathbf{I}_{n_z} & \tilde{C}_z \\ \tilde{C}_z^T & P(\alpha) \end{bmatrix} > 0.$$

The application of the Schur complement [22] on the latter condition yields

$$P(\alpha) - \mu^{-2}\tilde{C}_z^T\tilde{C}_z > 0.$$

Thus,

$$\tilde{x}(t)^T\left(P(\alpha) - \mu^{-2}\tilde{C}_z^T\tilde{C}_z\right)\tilde{x}(t) > 0 \;\Rightarrow\; \mu^2 V(\tilde{x}) - z_2(t)^T z_2(t) > 0,$$

which is equivalent to (7).

Therefore, according to Theorem 1, the generalized $\mathcal{H}_2$ norm from $[w(t)^T \;\; y(t)^T]^T$ to $z_2(t)$ of the robust filter (21) is lower than μ, provided that $w(t) \in \mathcal{L}_2$ and $||r(t)||^2 \leq \rho^2||x(t)||^2$, if the following condition holds:

$$\dot{V}(\tilde{x}) < w(t)^T w(t) + y(t)^T y(t). \tag{33}$$

The condition $w(t) \in \mathcal{L}_2$ is valid by hypothesis. However, it is necessary to incorporate the bound

$$||r(t)||^2 \leq \rho^2||\bar{x}(t)||^2 \;\Rightarrow\; r(t)^T r(t) - \rho^2\bar{x}(t)^T\bar{x}(t) \leq 0.$$

Applying the $\mathcal{S}$-procedure [22], condition (33) is valid whenever the prior bound is satisfied if there exists a scalar $\tau > 0$ such that

$$\dot{V}(\tilde{x}) - w(t)^T w(t) - y(t)^T y(t) + \tau\rho^2\bar{x}(t)^T\bar{x}(t) - \tau r(t)^T r(t) \leq 0. \tag{34}$$

Using the system matrices from (22), inequality (34) can be rewritten as

$$\begin{bmatrix}\tilde{x}(t)\\ w(t)\\ r(t)\end{bmatrix}^T \begin{bmatrix} \tilde{A}(\alpha)^T P(\alpha) + P(\alpha)\tilde{A}(\alpha) + \tau\rho^2 E - E^T\bar{C}_y(\alpha)^T\bar{C}_y(\alpha)E & P(\alpha)\tilde{B}_w + E^T\bar{C}_y(\alpha)^T D_{yw} & P\tilde{B}_r \\ \star & -\mathbf{I}_{n_w} - D_{yw}^T D_{yw} & \mathbf{0}_{n_w\times n_y} \\ \star & \star & -\tau\mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix}\tilde{x}(t)\\ w(t)\\ r(t)\end{bmatrix}, \leq 0,$$

where

$$E = \begin{bmatrix}\mathbf{I}_{\bar{n}} & \mathbf{0}_{\bar{n}}\\ \mathbf{0}_{\bar{n}} & \mathbf{0}_{\bar{n}}\end{bmatrix}.$$

The utilization of Finsler's Lemma [22], with

$$\mathcal{Q} = \begin{bmatrix} \tau\rho^2 E - E^T\bar{C}_y(\alpha)^T\bar{C}_y(\alpha)E & P(\alpha) & E^T\bar{C}_y(\alpha)^T D_{yw} & \mathbf{0}_{\bar{n}\times n_y} \\ \star & \mathbf{0}_{\bar{n}} & \mathbf{0}_{\bar{n}\times n_w} & \mathbf{0}_{\bar{n}\times n_y} \\ \star & \star & -\mathbf{I}_{n_w} - D_{yw}^T D_{yw} & \mathbf{0}_{n_w\times n_y} \\ \star & \star & \star & -\tau\mathbf{I}_{n_y} \end{bmatrix}, \quad \mathcal{B}^T = \begin{bmatrix}\tilde{A}(\alpha)^T\\ -\mathbf{I}_{\bar{n}}\\ \tilde{B}_w^T\\ \tilde{B}_r^T\end{bmatrix}$$

$$\mathcal{B}^\perp = \begin{bmatrix} \mathbf{I}_{\bar{n}} & \mathbf{0}_{\bar{n}\times n_w} & \mathbf{0}_{\bar{n}\times n_y} \\ \tilde{A}(\alpha) & \tilde{B}_w & \tilde{B}_r \\ \mathbf{0}_{n_w\times\bar{n}} & \mathbf{I}_{n_w} & \mathbf{0}_{n_w\times n_y} \\ \mathbf{0}_{n_y\times\bar{n}} & \mathbf{0}_{n_y\times n_w} & \mathbf{I}_{n_y} \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix}K(\alpha)\\ G(\alpha)\\ F(\alpha)\\ R(\alpha)\end{bmatrix}$$

with the application of the following structures for the slack variables

$$K(\alpha) = \begin{bmatrix}K_{11}(\alpha) & \lambda_1\widehat{K}\\ K_{21}(\alpha) & \lambda_2\widehat{K}\end{bmatrix}, \quad G(\alpha) = \begin{bmatrix}G_{11}(\alpha) & \lambda_3\widehat{K}\\ G_{21}(\alpha) & \lambda_4\widehat{K}\end{bmatrix},$$

$$F(\alpha) = \begin{bmatrix}F_{11}(\alpha) & \mathbf{0}_{n_w\times\bar{n}}\end{bmatrix}, \quad R(\alpha) = \begin{bmatrix}R_{11}(\alpha) & \mathbf{0}_{n_y\times\bar{n}}\end{bmatrix}$$

and the following change of variables

$$M_1 = \widehat{K}A_f, \quad M_2 = \widehat{K}B_{fy}, \quad M_3 = \widehat{K}B_{fr}$$

result in Condition (31). □

The LMI conditions in Theorem 2 depend on predefined scalars λ_i. Similarly to the ξ_i slack variables in Theorem 1, they can be handled as described in Remark 1.

In order to determine the parametric estimation law, consider the closed-loop system (25). The parameter-independent elements, which compose the dynamics $\phi(t)$, are separated from the matrices affinely dependent on the parameters to be estimated, resulting in the dynamics $\Phi(t)$. As a consequence, the equivalent system can be rewritten as

$$\dot{x}(t) = \underbrace{A_{cl,0}x(t) + \bar{B}_r r(t)}_{\phi(t)} + \underbrace{\begin{bmatrix} A_{cl,1}x(t) & \dots & A_{cl,n_\theta}x(t) \end{bmatrix}}_{\Phi(t)} \underbrace{\begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{n_\theta} \end{bmatrix}}_{\theta} + \bar{B}_w \bar{w}(t), \tag{35}$$

with $A_{cl,i} = \bar{A}_i - \bar{B}_i K_i$. Using the dynamics of the elements $\dot{x}(t)$, $\phi(t)$, and $\Phi(t)$, the parametric estimation procedure applied in the present paper is developed. It is important to highlight that the dynamics $\dot{x}(t)$, $\phi(t)$, and $\Phi(t)$ can be reconstructed based on the knowledge of matrices $A_{cl,0}, \dots A_{cl,n_\theta}$, and the filtered state vector obtained with $\tilde{C}_z = \mathbf{I}_{\bar{n}}$.

Theorem 3 ([33])**.** *Consider $\dot{x}(t)$, $\phi(t)$, and $\Phi(t)$ as depicted in (35), supposing that $\Phi(t)$ is a persistently excited function [34], and consider also the solutions $P(t)$ and $Q(t)$ of the respective differential equations (36) and (37):*

$$\dot{P}(t) = -\ell P(t) + \Phi^T(t)\Phi(t), \; P(0) = 0, \tag{36}$$

$$\dot{Q}(t) = -\ell Q(t) + \Phi^T(t)(\dot{x}(t) - \phi(t)), \; Q(0) = 0. \tag{37}$$

Therefore, the estimation update law

$$\dot{\tilde{\theta}} = -\Gamma(P(t)\tilde{\theta}(t) - Q(t)), \tag{38}$$

with $\Gamma > 0$, assures that the estimation error $\hat{\theta} = \theta - \tilde{\theta}(t)$ uniformly ultimately converges to the compact set

$$\Theta = \left\{ \hat{\theta}(t) : ||\hat{\theta}(t)||_2 \le \frac{\epsilon_\psi}{\sigma} \right\}, \tag{39}$$

where ϵ_ψ is an upper bound for

$$\psi(t) = \int_0^t e^{-\ell(t-\tau)}\Phi^T(\tau)\bar{B}_w\bar{w}(\tau)d\tau$$

and σ a scalar such that $P(t) > \sigma I$.

Proof. The solution of Equation (36) is given by

$$P(t) = \int_0^t \exp^{-\ell(t-\tau)} \Phi^T(\tau)\Phi(\tau)d\tau. \tag{40}$$

Using $\dot{x}(t) - \phi(t)$ from (35), the solution of Equation (37) can be rewritten as

$$\begin{aligned} Q(t) = & \int_0^t e^{-\ell(t-\tau)}\Phi^T(\tau)\Phi(\tau)d\tau\theta \\ & + \underbrace{\left(\int_0^t e^{-\ell(t-\tau)}\Phi^T(\tau)\bar{B}_w\bar{w}(\tau)d\tau \right)}_{\psi(t)} = P(t)\theta - \psi(t), \end{aligned} \tag{41}$$

with $||\psi||_\infty < \epsilon_\psi$. On the other hand, the convergence of the estimation error $\hat{\theta}(t) = \theta - \tilde{\theta}(t)$ can be obtained through the analysis of the following Lyapunov function:

$$V(t) = \frac{1}{2}\hat{\theta}^T(t)\Gamma^{-1}\hat{\theta}(t),\ \dot{V}(t) < 0. \tag{42}$$

Since $\dot{\hat{\theta}} = -\dot{\tilde{\theta}}$, the derivative $\dot{V}(t)$ is given by

$$\begin{aligned}\dot{V}(t) =& \frac{1}{2}\dot{\hat{\theta}}^T(t)\Gamma^{-1}\hat{\theta}(t) + \frac{1}{2}\hat{\theta}^T(t)\Gamma^{-1}\dot{\hat{\theta}}(t)\\ =& -\frac{1}{2}\dot{\tilde{\theta}}^T(t)\Gamma^{-1}\hat{\theta}(t) - \frac{1}{2}\hat{\theta}^T(t)\Gamma^{-1}\dot{\tilde{\theta}}(t).\end{aligned} \tag{43}$$

Replacing $\dot{\tilde{\theta}}$ by (38) and $Q(t)$ by (41), one has

$$\begin{aligned}\dot{V}(t) =& \frac{1}{2}\Big(\tilde{\theta}^T(t)P(t)^T - Q(t)^T\Big)\hat{\theta}(t)+\\ & \frac{1}{2}\hat{\theta}^T(t)\big(P(t)\tilde{\theta}(t) - Q(t)\big) = \hat{\theta}^T\psi(t) - \hat{\theta}^T P(t)\hat{\theta}(t).\end{aligned} \tag{44}$$

Considering that $P(t) > \sigma I$ since Φ is supposed to be persistently excited, and using the upper bound for $\phi(t)$, the derivative of the Lyapunov function is bounded by

$$\begin{aligned}\dot{V}(t) \leq& -\sigma||\hat{\theta}(t)||_2^2 + ||\hat{\theta}||_2||\psi||_\infty,\\ \leq& -||\hat{\theta}(t)||_2(\sigma||\hat{\theta}(t)||_2 - \epsilon_\psi).\end{aligned} \tag{45}$$

Therefore, the ultimate bounding set (39) can be obtained. More details of the proof can be found in [33,34]. □

The scalar parameter ℓ, sometimes known as a leakage factor, assures that the matrices $P(t)$ and $Q(t)$ are bounded, and the appropriate value for ℓ depends on each application. Some guidelines can be found, for instance, in [35]. Theorems 1, 2, and 3 present the synthesis conditions for, respectively, the controller, the filter, and the estimation procedure. However, it is necessary to guarantee that connecting all pieces together results in a convergent control system. Such guarantees are proposed in the following section.

4.2. Convergence Guarantees

Theorem 3 describes the procedure for computing the estimated parameter $\tilde{\theta}$, which is then used on the gain-scheduling controller synthesized from Theorem 1. However, it is necessary to guarantee that the estimation error $\hat{\theta}_i = \delta_i$ satisfies the stability requirements. The condition proposed in this paper to assure the stability is described in Theorem 4.

Theorem 4. *Let $K(\alpha,\beta)$ be the gain-scheduling state-feedback gain robust to uncertainties on the estimated parameters within the intervals*

$$-\eta_i \leq \delta_i \leq \eta_i, \quad i = 1,\ldots,n_\theta \tag{46}$$

with η_i being positive scalars given by

$$\eta_i = \sqrt{\frac{\sqrt{\bar{\delta}_i}}{d}}, \tag{47}$$

with

$$d = \left(\sum_{i=1}^{n_\theta}(\bar{\delta}_i)^{3/2}\right)^{-1}. \tag{48}$$

Therefore, the estimation procedure presented in Theorem 3, considering

$$\Gamma = diag\left(\frac{\sqrt{\bar{\delta}_1}}{d}, \dots, \frac{\sqrt{\bar{\delta}_{n_\theta}}}{d}\right), \tag{49}$$

assures that the controller $K(\alpha, \beta)$ stabilizes the system for

$$-\bar{\delta}_i \leq \delta_i \leq \bar{\delta}_i.$$

Proof. According to the proof of Theorem 3, the update law (38) assures that the derivative of the Lyapunov function $V(t) = \frac{1}{2}\hat{\theta}(t)^T\Gamma^{-1}\hat{\theta}(t)$ is negative for all $\hat{\theta}$. On the other hand, the ellipsoid Ω defined as

$$\Omega = \left\{\hat{\theta} \in \mathbb{R}^{n_\theta} \ : \ V(t) = \frac{1}{2}\right\},$$

with Γ given by Equation (49), contains the set Ξ defined by

$$\Xi = [-\bar{\delta}_1, \bar{\delta}_1] \times \dots \times [-\bar{\delta}_{n_\theta}, \bar{\delta}_{n_\theta}].$$

The latter statement is valid since the equation defining the ellipsoid Ω, also given by

$$\sum_{i=1}^{n_\theta} \delta_i^2 \frac{d}{\sqrt{\bar{\delta}_i}} = 1, \tag{50}$$

is satisfied for d resulting from Equation (48). Figure 1 illustrates the described sets.

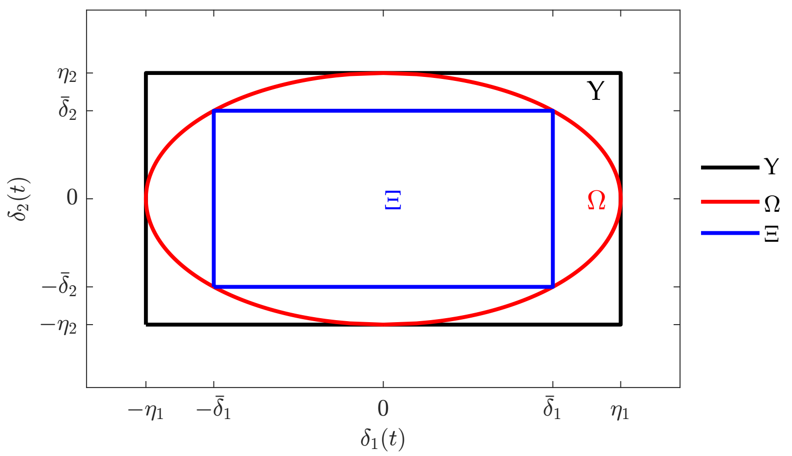


Figure 1. Representation of the sets described in the proof of Theorem 4.

The set Ω describes a positively invariant set in $\hat{\theta}$, since $\dot{V}(t) < 0$, $\forall\hat{\theta} \in \Omega$ [34]. Note, however, that it is possible to conceive a situation where $\hat{\theta}$ temporarily exits Ξ; thus, it is not sufficient that the controller assure the stability only for the interval described in Equation (46). Consequently, in order to guarantee that $K(\alpha, \beta)$ stabilizes the system $\forall\hat{\theta} \in \Omega$, it is necessary to design the controller to stabilize the system $\forall\hat{\theta} \in \Upsilon$, as illustrated in Figure 1. From Equation (50), one has that

$$\Upsilon = [-\eta_1, \eta_1] \times \dots \times [-\eta_{n_\theta}, \eta_{n_\theta}],$$

with η_i given by Equation (47), finishing the proof. □

The following theorem states the conditions to ensure the stability of the gain-scheduling controller $K(\tilde{\theta})$ under the feedback action of the estimated states $\tilde{\bar{x}}(t)$.

Theorem 5. *The control signal $u(t) = -K(\theta)\tilde{\bar{x}}(t)$, where $\tilde{\bar{x}}(t) = z_f(t)$ is the output of the filter (21), stabilizes the LPV system (13) if $\gamma\mu \leq 1$, where γ is the bound for the $\mathcal{H}_\infty$ norm from $[e(t)^T w(t)^T]^T$ to $y(t)$ determined from Theorem 1 and μ is the generalized $\mathcal{H}_2$ norm from $[y(t)^t w(t)^T]$ to $e(t)$ resultant from Theorem 2.*

Proof. According to the Small Gain Theorem [22], the robustness of the closed-loop system $\mathcal{G}(\theta)$, defined in (13), with the perturbed states stemming from the filter $\mathcal{F}$ from (21), is achieved through the following condition:

$$||\mathcal{G}_{y,e}(s,\theta)||_\infty < \gamma^2 \quad \text{if, and only if,} \quad ||\mathcal{F}_{e,y}(s)||_\infty \leq \frac{1}{\gamma^2}, \tag{51}$$

where $||\cdot||_\infty$ is the $\mathcal{H}_\infty$ norm, as defined in Section 2.1. Assuming that the state-feedback controller $K(\theta)$ assures that the bound for the $\mathcal{H}_\infty$ norm from $[e(t)^T w(t)^T]^T$ to $y(t)$ is lower than γ, the first condition of (51) is valid, since

$$\gamma^2 > ||\mathcal{G}_{y,ew}(s,\theta)||_\infty = \sup_{w(t),e(t)} \frac{||y(t)||^2}{||e(t)||^2 + ||w(t)||^2} > \sup_{e(t)} \frac{||y(t)||^2}{||e(t)||^2} = ||\mathcal{G}_{y,e}(s,\theta)||_\infty^2$$

Similarly, if the conditions of Theorem 2 are satisfied, then according to Theorem 1, the filtered variables $z_f(t)$ verify

$$\mu^2 > \sup ||e(t)||^2 \ : \ \int_0^\infty w(\tau)^T w(\tau) + y(\tau)^T y(\tau) \leq 1,$$

with the last expression being equivalent to

$$\sup \frac{||e(t)||^2}{||w(t)||^2 + ||y(t)||^2}.$$

Therefore,

$$\mu^2 > \sup_{w(t),y(t)} \frac{||e(t)||^2}{||w(t)||^2 + ||y(t)||^2} > \sup_{y(t)} \frac{||e(t)||^2}{||y(t)||^2} = ||\mathcal{F}_{e,y}(s)||_\infty^2.$$

Thus, if $\mu\gamma < 1$, the second condition of (51) is also valid, finishing the proof. □

In order to summarize the proposed methodology, Algorithm 1 shows the steps necessary to obtain the parameter-dependent state-feedback controller and the filter matrices. Figure 2 then presents a diagram that illustrates the filtering and control structures, combined with the parametric estimation.

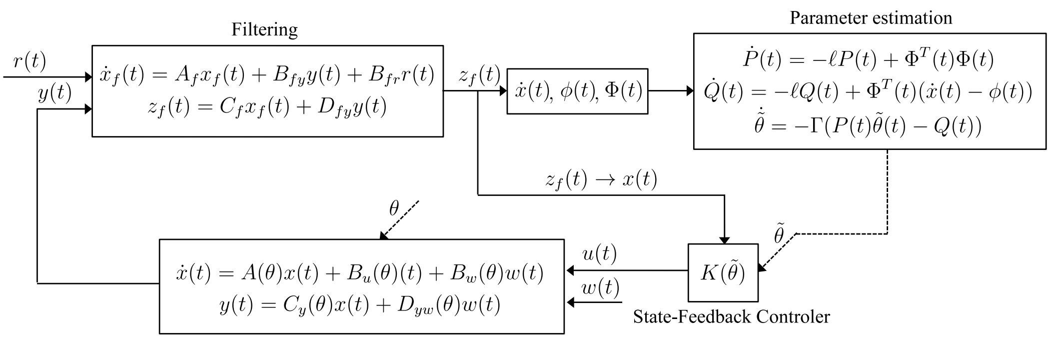


Figure 2. State-feedback control structure and its operation with the state and parameter estimation procedures.

Algorithm 1 $(K(\theta), A_f, B_{fy}, B_{fr}, C_f, D_{fy}, \Gamma)$ = Synthesis $(\xi_1, \xi_2, \xi_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \rho)$

$(\bar{A}(\alpha), \bar{B}(\alpha), \bar{B}_w, B_r, \bar{C}_y, D_{yw}, \bar{C}_z(\alpha)) \leftarrow$ Obtain system matrices
$\gamma \leftarrow$ Maximum value allowed for the considered system;
$\delta_i \leftarrow$ bounds for the additive parameters noises
$(\eta_i, \Gamma) \leftarrow$ Theorem 4 (δ_i)
finished $\leftarrow$ False
while Not finished **do**
  $K(\alpha,\beta) \leftarrow$ Theorem 1 $(\bar{A}(\alpha), \bar{B}(\alpha), \bar{C}_z(\alpha), \xi_1, \xi_2, \xi_3, \gamma)$
  if Theorem 1 is feasible **then**
    $A_{cl}(\alpha,\beta) \leftarrow \bar{A}(\alpha) - \bar{B}(\alpha)K(\alpha,\beta)$
    $(A_f, B_{fy}, B_{fr}, C_f, D_{fy}, \mu) \leftarrow$ Theorem 2 $(A_{cl}(\alpha,\beta), \bar{B}_w, B_r, \bar{C}_y, D_{yw}, \tilde{C}_z, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \rho)$
    if $\gamma\mu \leq 1$ **then**
      finished $\leftarrow$ True
    else
      Reduce γ
    end if
  else
    return Infeasible problem
  end if
end while
Convert $K(\alpha,\beta)$ to $K(\theta)$
return $K(\theta), A_f, B_{fy}, B_{fr}, C_f, D_{fy}, \Gamma$

5. Experimental Results

In this section, two experiments are presented to illustrate the operation and validity of the proposed methodology. In Experiment 1, a third-order system dependent on two parameters is considered, allowing a deep analysis on the obtained results. Experiment 2 depicts the implementation of the control approach in a physical system, which is important to verify the implementability of the technique in a real-time situation. The routines are implemented in Matlab R2017a, using the packages YALMIP [36] and ROLMIP [37], alongside the solver Mosek [38] with an Academic License.

5.1. Experiment 1

Consider the parameter-dependent system given by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2+\theta_1 & -3+\theta_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + 0.1w(t) \end{aligned}, \tag{52}$$

with θ_1 and θ_2 varying within $[2,6]$. System (52) represents a typical linear system in the controllable canonical form [39,40], chosen to better depict the proposed methodology. The outputs are the first state and its derivative, which is a common occurrence in physical systems, for instance. Note that the open loop system is unstable for all θ_1 and θ_2 in the given interval, thus being nontrivial to properly estimating the necessary variables in order to determine a stabilizing control law $u(t)$.

For gain-scheduling state-feedback synthesis, it is desired in this experiment that the state $x_1(t)$ follows an exogenous reference signal $r(t)$, using a controller robust to noises in the interval $\bar{\delta}_i \in [-1,1]$, for both parameters. Thus, according to Theorem 4, the synthesis condition must consider the parameters noises $\eta_i \in [-\sqrt{2}, \sqrt{2}]$. Theorem 1 is, therefore, applied with $\xi_1 = 100$ and $\xi_2 = \xi_3 = 0$, considering the augmented matrix resultant from the inclusion of the integrator $\dot{q}(t) = r(t) - x_1(t)$ and $\gamma = 0.3$, resulting in the stabilizing state-feedback gain given by

$$K(\theta) = \begin{bmatrix} 147.05 & 127.60 & 23.62 & -51.38 \end{bmatrix} + \theta_1 \begin{bmatrix} 33.36 & 29.69 & 6.22 & -11.80 \end{bmatrix} + \theta_2 \begin{bmatrix} 26.87 & 23.36 & 4.61 & -9.52 \end{bmatrix}. \quad (53)$$

For comparison purposes, the following robust static gain K_{st} is also synthesized:

$$K_{st} = \begin{bmatrix} 81.76 & 75.88 & 16.71 & -26.80 \end{bmatrix}. \quad (54)$$

The filter (21) is then synthesized through the application of Theorem 2, considering $\lambda_i = 1$ and $\rho = 10$. In order to satisfy the conditions of Theorem 5, assuring a proper state estimation from the filter, the generalized $\mathcal{H}_2$ norm is set to $\mu = 3.33$.

The controlled system is then simulated along with the adaptive estimation procedure depicted in Theorem 3, with $\Gamma = 2I$, as determined through Theorem 4 and $\ell = 10$. The initial states for the system and the filter are set as zero, and the initial estimated parameter values $\tilde{\theta}_i$ are initially set as 4.

Figure 3 presents the controlled state $x_1(t)$ resulting from considering both the gain-scheduling controller (53) and the static one (54), obtained by setting $\theta_1 = 3$ and $\theta_2 = 5$. Note that the gain-scheduling controller presented slightly improved behavior when compared to the static controller, being capable of arriving faster to the reference value, illustrating the capability of achieving better results due to its parameter-varying structure. Specifically to the results obtained from $K(\theta)$, Figures 4 and 5 show, respectively, the estimation error of the states and of the parameters. Note that the estimation procedure was successful in computing approximate values for both the states and the parameters.

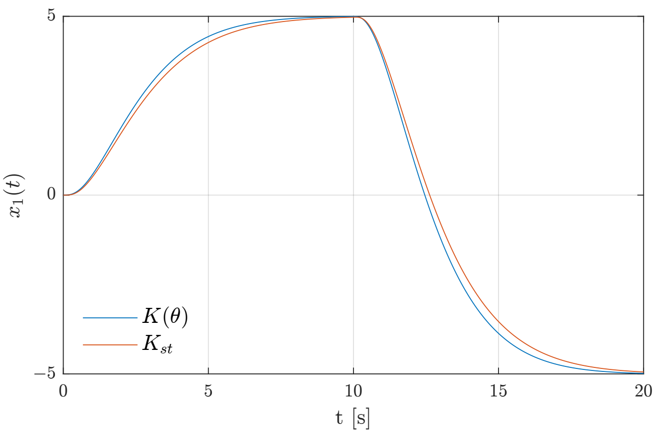


Figure 3. Trajectories $x_1(t)$ from Experiment 1 obtained using the gain-scheduling controller $K(\theta)$ (blue) and the static gain K_{st} (red), aiming to follow the reference signal $r(t)$ (black).

In order to illustrate the validity of the estimation procedure in a global situation, Figure 6 depicts the estimation errors for several different simulations, with each one considering different values for the parameters and their initial estimated values. A total of 20 distinct initial values for θ_1 and θ_2 have been simulated, and t_n for $n = 0, \ldots, 20$, represent each respective trajectory. The sets Ξ, Ω, and Υ from Theorem 4 are also included in the figure. Note that the errors are always within the positively invariant ellipsoid set Ω, as stated in the theorem, justifying the necessity of designing the state-feedback controllers to be robust to estimation errors in Υ.

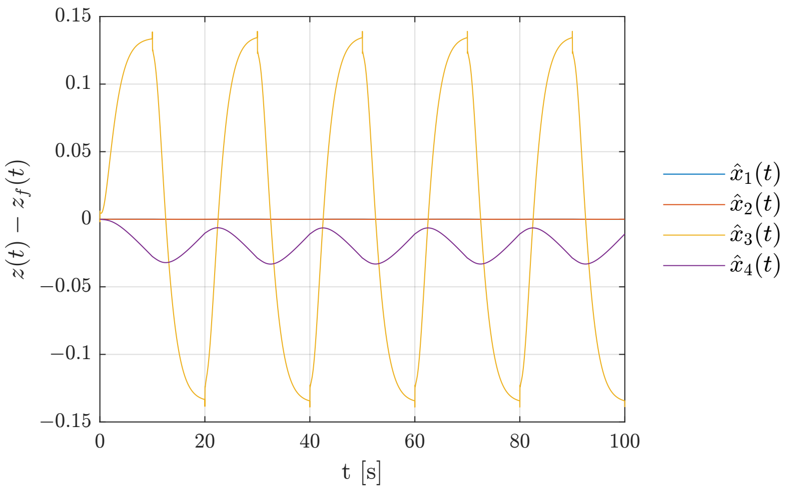


Figure 4. Estimation errors resultant from the filtering procedure in Experiment 1.

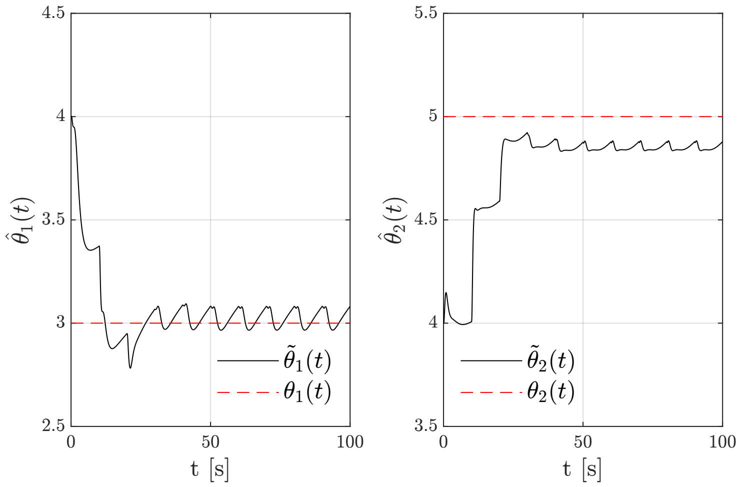


Figure 5. Parameter estimation errors in Experiment 1.

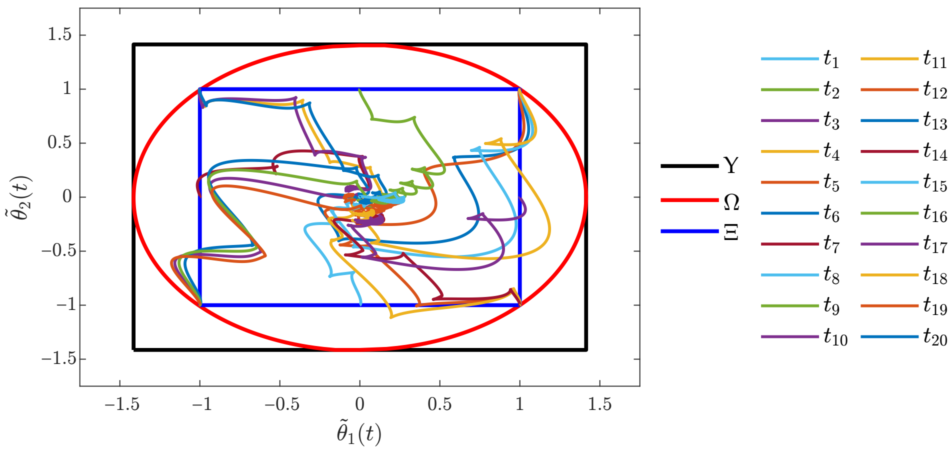


Figure 6. Evolution of $\tilde{\theta}(t)$ for Experiment 1, considering different values for θ.

Finally, Figure 7 shows the parameter estimation when the actual values of θ_1 and θ_2 change during the simulation. Although the technique supposes that such parameters are time-invariant, the results indicate that the proposed technique can be further improved, in future works, to tackle temporal variations.

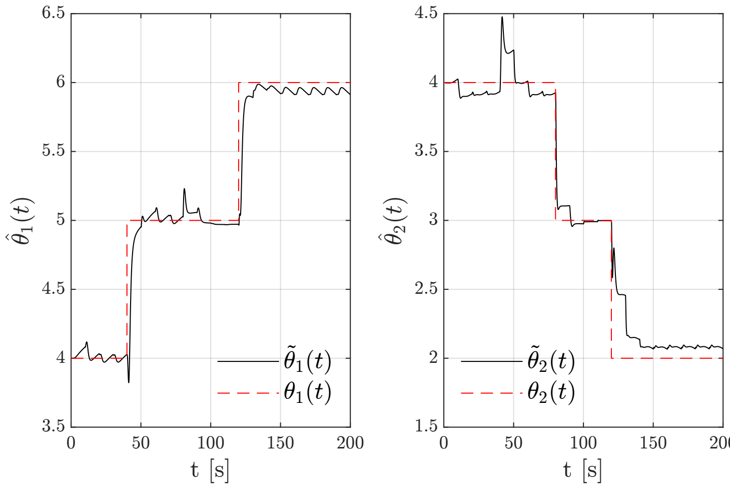
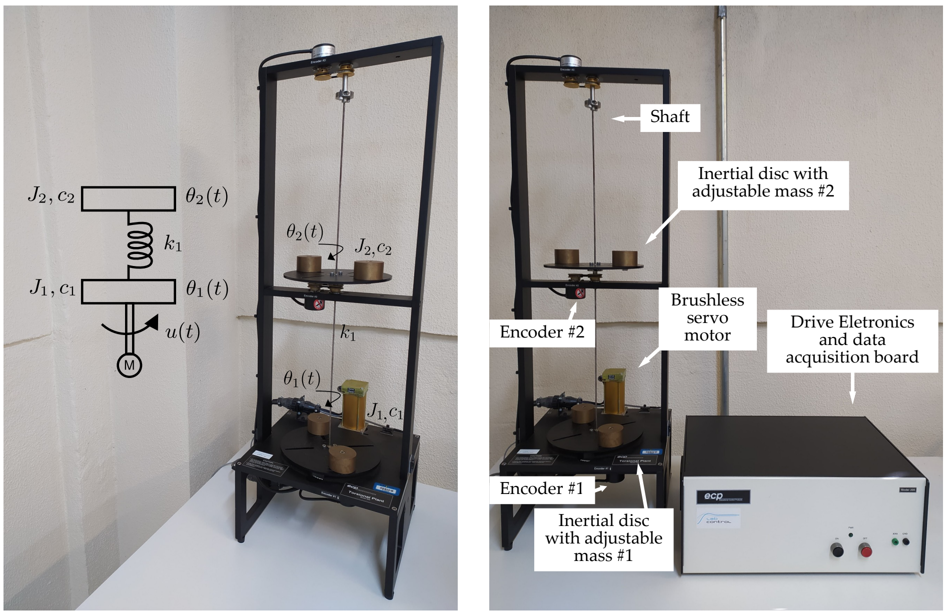


Figure 7. Parameter estimation errors in Experiment 1, when both parameters change their actual values.

5.2. Experiment 2

The present experiment consists of the implementation of the proposed control technique in the 2 DoF Torsional Control System plant produced by ECP Systems [41], model 205a, depicted in Figure 8. This plant consists of two disks connected by a flexible beam, a DC motor actuating on the lower one, and two encoders for reading the angular position of both disks. Two masses of 0.5 Kg each are positioned over the two disks, changing their moment of inertia. In this experiment, the masses of the lower disk are placed at 9 cm of distance from the center, and for the upper disk, the distance of the masses can vary from 7 cm to 9 cm, with such a distance being the parameter to be estimated from the adaptive procedure. Also, the angular speeds are supposed to be the estimated state variables from the robust filter (21). Figure 9 represents the experimental configuration, which includes the integration of the torsional model (Equation (55)), Algorithm 1, and the physical torsional system (Figure 8).



(**a**) (**b**)

Figure 8. Torsional system used in Experiment 2. The inertial moment of the upper disk can be changed by moving the masses. (**a**) Physical torsional system used in the experiment and its conceptual representation. (**b**) Description of the elements composing the torsional system.

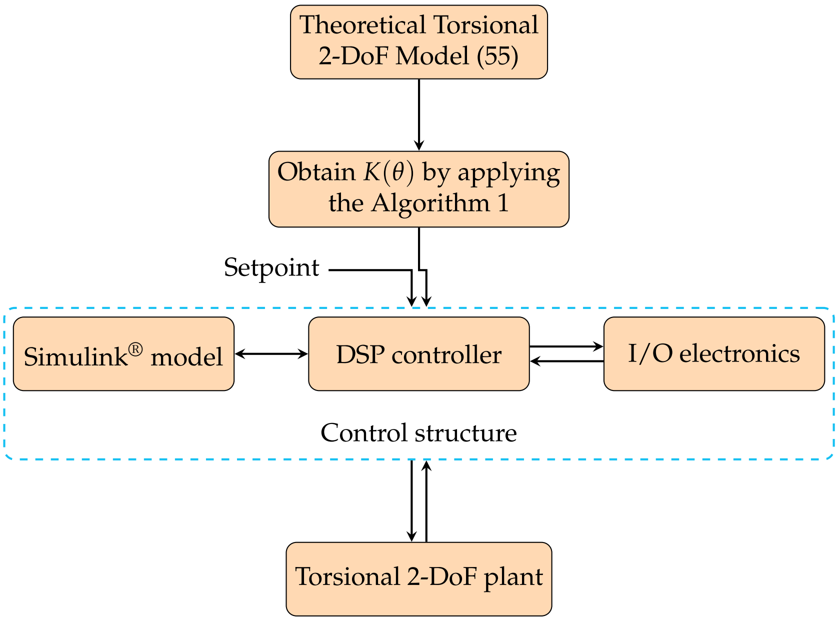


Figure 9. Configuration diagram considered for Experiment 2.

The mathematical model of the torsional system is given by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_1} & -\frac{c_1}{J_1} & \frac{k}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_2} & 0 & -\frac{k}{J_2} & -\frac{c_2}{J_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t), \end{aligned} \tag{55}$$

where $x_1(t)$ and $x_3(t)$ are the angular positions of, respectively, the lower and upper disks, and $x_2(t)$ and $x_4(t)$ are their respective angular speeds. The values of the elastic constant k of the flexible beam connecting both disks, the viscous friction coefficients c_1 and c_2, and the inertia moments J_1 and J_2 are detailed in Table 1.

Table 1. Torsional system coefficients.

Coefficient	Value
k	1.37
c_1	0.007
J_1	0.0108 $\text{Kg} \cdot \text{m}^2$
c_2	0.001
J_2	$[0.0071 \;\; 0.0103]$ $\text{Kg} \cdot \text{m}^2$

The uncertainty on J_2, caused by the variation of the masses on the upper disk, is modeled as

$$\frac{1}{J_2} = 140.8451 - 43.7577\theta, \tag{56}$$

where $\theta = 1$ refers to the masses at a distance $d = 7$ cm from the center and $\theta = 0$ maps to $d = 9$ cm. The objective of the control system is to guarantee that the angular position of the upper disk follows an external reference signal.

For the controller synthesis, the parameter estimation error is considered to be $\bar{\delta} = 1$. The controller $u(t) = -K(\theta)x(t)$ is obtained from the application of Theorem 1, with

$\xi_1 = 100$, $\xi_2 = \xi_3 = 0$, and $\gamma = 0.3$, on the system augmented with an integrator to assure reference tracking properties, resulting in

$$\begin{aligned} K(\theta) = &\begin{bmatrix} -0.7884 & 0.1453 & 1.1496 & 0.0463 & -0.1757 \end{bmatrix} \\ &+ \theta \begin{bmatrix} -0.0310 & -0.0006 & 0.0320 & 0.0009 & -0.0001 \end{bmatrix}. \end{aligned} \tag{57}$$

Note that, since one parameter is considered, $\Gamma = 1$ is yielded from Theorem 4. Since $\gamma = 0.3$, the filter from Theorem 2 is computed by assuring that $\mu \leq 3.33$. The scalars $\lambda_i = 1$ and $\rho = 10$ are also considered for the filter synthesis.

The proposed approach is then applied to the torsional physical system, considering $\ell = 20$ for the adaptive estimation procedure. Three executions are performed, setting $d = 7$ cm, $d = 8$ cm, and $d = 9$ cm. Figure 10 depicts the controlled angular position of the upper disk for the three cases. Note that the control system is capable of following the reference signal with similar performances, independent of the masses position. Figures 11 and 12 present, respectively, the control signal $u(t)$ and the actual angular speed of the disks, obtained from the derivation of the angular position, as well as the corresponding variable resulting from the filter. Only the signals for $d = 7$ cm are shown in both figures, but the results for other configurations are similar. Figure 12 also shows the errors between the filtered and actual variables. One can see that the filter is capable of reconstructing the angular speeds without resorting to any direct derivation procedure, which is important for real-time applications since derivatives tend to increase noisy artifacts.

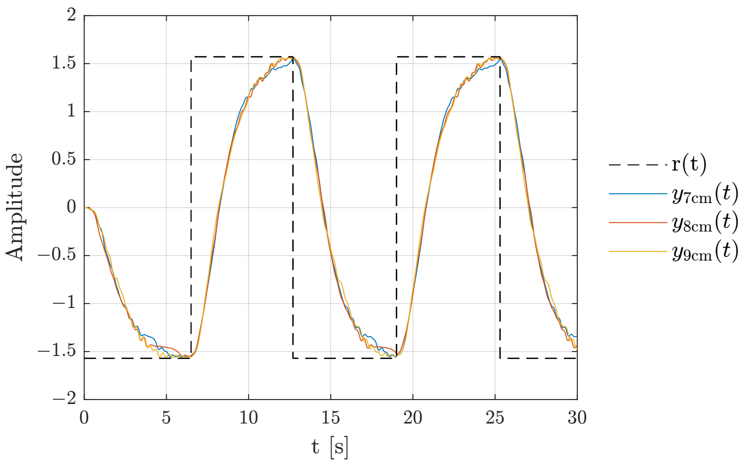


Figure 10. Controlled output of the torsional system in Experiment 2, obtained by setting $d \in \{7, 8, 9\}$ cm, aiming to follow the reference signal $r(t)$ (black).

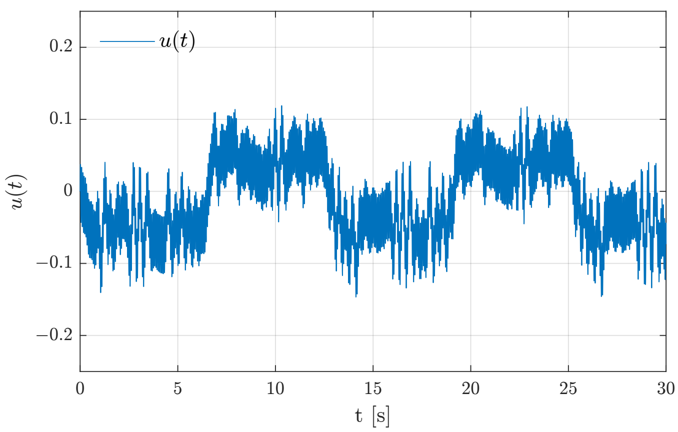


Figure 11. Control signal applied to the torsional system in Experiment 2 for the case $d = 7$ cm.

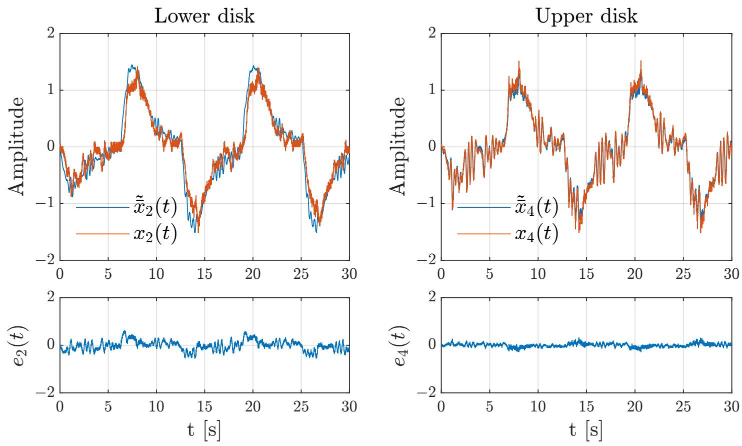


Figure 12. The top figures show the filtered (blue) and measured (red) angular speeds resulting from the filter applied in Experiment 2, obtained by setting $d = 7$ cm. The errors between the filtered and measured speeds are detailed in the bottom figures.

Finally, Figure 13 shows the estimated parameters $\tilde{\theta}$ for the three considered configurations. Although the curves present a different behavior from the expected results, one can see that, after some time, the estimated parameters are considerably distinct for each case. Nevertheless, the control system is successful in tracking the desired reference for all configurations.

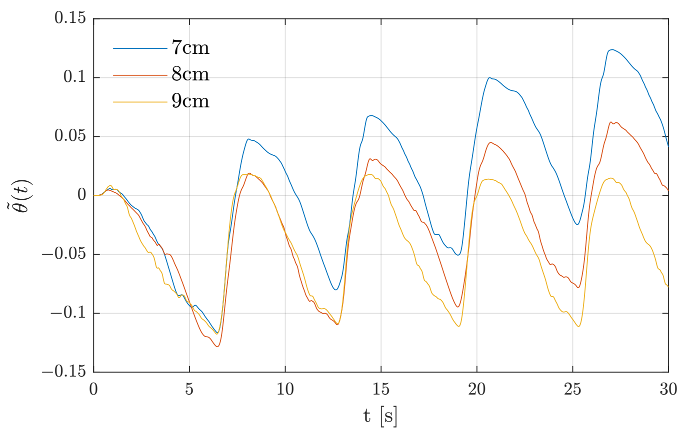


Figure 13. Estimated values $\tilde{\theta}$ for $d = 7$ cm (blue), $d = 8$ cm (red), and $d = 9$ cm (yellow).

5.3. Comparison of Results with Previous Works

In order to emphasize the advantages of the proposed method and its main characteristics, a comprehensive overview and comparison is provided. The comparison is summarized in Table 2 and provides the closest recent works related to this paper's proposal. The comparison is developed and commented on in terms of each method's technical characteristics and does not intend to compare performance and/or numerical behavior.

In Table 2, the related LPV robust control techniques and their applications are presented, where N/A stands for "not apply". It is possible to notice that the standard procedure found in literature—and mentioned in this comparison—considers the state vector to be usually available, while this work provides results using an estimated state vector with guaranteed convergence. Some works, such as Hanif et al. [11], perform a deterministic calculation of the required parameters, while this work uses a least mean squares procedure, adequate for more general cases and dynamics. Considering the main characteristics and differences between the methods, one can notice that the main advantage of this paper is the joint use of both state and parameter estimation procedures, with an innovative convergence analysis, ensuring its performance.

Table 2. Comprehensive overview and comparison of related works.

	Control Method	Robustness Analysis	θ Estim.	θ Estim. Method	$x(t)$ Estim.	$x(t)$ Estim. Method
de Souza et al. [1]	Gain-scheduled state-feedback gain	Parameter-dependent Lyapunov–Krasovskii function	✗	N/A	✗	N/A
Campos et al. [5]	Adaptive gain scheduling	Parameter-dependent Lyapunov function	✓	Adaptive Lyapunov-based	✗	N/A
Tasoujian et al. [8]	Time-delay LPV control	Parameter-dependent Lyapunov–Krasovskii function	✓	Bayesian-based multiple-model square-root cubature Kalman filtering	✗	N/A
Hanif et al. [11]	LPV-based observer + controller	Parameter-dependent Lyapunov function	N/A	Deterministic	✓	Robust LPV observer
This work	State-feedback gain	$\mathcal{H}_2$ and $\mathcal{H}_\infty$ to exogenous input $w(t)$	✓	Least Mean Squares	✓	Robust filtering ($\mathcal{H}_2$)

The θ symbol relates to parameter estimation procedures, while $x(t)$ refers to the method to perform the state estimation. The symbols ✓ and ✗ indicate, respectively, the presence or not of the respective procedure, and N/A stands for "Not Apply".

6. Conclusions

In this paper, a framework for the synthesis of parameter-dependent controllers for LPV systems is proposed, considering that both the states and exogenous parameters are unavailable and need to be estimated. The control system is composed by a gain-scheduling controller, which depends on the estimated states yielded by a robust filter, developed in this paper, and on the estimated exogenous parameters, resultant from an estimation procedure, which are also a contribution of the present work. Each of the three elements (controller, filter, and estimator) is synthesized separately, and the complete framework converges to the desired behavior if a series of conditions are valid. Two experimental results, one numerical and one implemented in a physical system, are presented to illustrate and validate the methodology. The obtained results show that the proposed framework is capable of stabilizing complex LPV systems based only on the available outputs, through the estimation of both the states and the parameters. Such a result, to the authors' knowledge, is a novel and important contribution to the area.

The presented techniques open some possibilities for future improvements. The incorporation of performance criteria into the control gain synthesis is of main interest, as it could improve the system performance and allow the implementation of the proposed framework even in restricted cases. Concerning the estimation procedure, the simulations presented satisfactory results; however, the implemented solution yielded a slower convergence rate than desired. Nevertheless, the different test situations yielded coherent behaviors for the estimated parameters, indicating the validity of the procedure, but improvements on the convergence rate, mainly through enhancing the estimation law, are also important to be considered in future works. An important improvement for further researches is the possibility to consider time-varying parameters. The current methodology supposes that the parameters are time-invariant, with such a requirement being necessary for the applied estimation procedure. The experiments show that even time-varying parameters could be properly estimated, thus indicating the potential for improving the techniques to formally deal with such cases. Further investigation of time-varying parameters and estimation performance indexes are suggested to be performed in future researches.

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Abbreviations

The following abbreviations are used in this manuscript:

DoF	Degree of Freedom
LMI	Linear Matrix Inequalities
LPV	Linear Parameter Varying

References

1. de Souza, L.T.; Peixoto, M.L.; Palhares, R.M. New gain-scheduling control conditions for time-varying delayed LPV systems. *J. Frankl. Inst.* **2022**, *359*, 719–742. [CrossRef]
2. Tan, J.; Zheng, H.; Meng, D.; Wang, X.; Liang, B. Active input design for simultaneous fault estimation and fault-tolerant control of LPV systems. *Automatica* **2023**, *151*, 110903. [CrossRef]
3. Saeed, A.; Bhatti, A.I.; Malik, F.M. LMIs-Based LPV Control of Quadrotor with Time-Varying Payload. *Appl. Sci.* **2023**, *13*, 6553. [CrossRef]
4. Marx, B.; Ichalal, D.; Ragot, J.; Maquin, D.; Mammar, S. Unknown input observer for LPV systems. *Automatica* **2019**, *100*, 67–74. [CrossRef]
5. Campos, V.C.S.; Nguyen, A.T.; Martínez Palhares, R. Adaptive gain-scheduling control for continuous-time systems with polytopic uncertainties: An LMI-based approach. *Automatica* **2021**, *133*, 109856. [CrossRef]
6. Ríos, H.; Mincarelli, D.; Efimov, D.; Perruquetti, W.; Davila, J. Continuous and discrete state estimation for switched LPV systems using parameter identification. *Automatica* **2015**, *62*, 139–147. [CrossRef]
7. Liu, Z.; Huang, Y.; Gou, L.; Fan, D. A robust adaptive linear parameter-varying gain-scheduling controller for aeroengines. *Aerosp. Sci. Technol.* **2023**, *138*, 108319. [CrossRef]
8. Tasoujian, S.; Salavati, S.; Franchek, M.A.; Grigoriadis, K.M. Robust delay-dependent LPV synthesis for blood pressure control with real-time Bayesian parameter estimation. *IET Control. Theory Appl.* **2020**, *14*, 1334–1345. [CrossRef]
9. Liu, K.; Yang, P.; Wang, R.; Jiao, L.; Li, T.; Zhang, J. Observer-Based Adaptive Fuzzy Finite-Time Attitude Control for Quadrotor UAVs. *IEEE Trans. Aerosp. Electron. Syst.* **2023**, *59*, 8637–8654. [CrossRef]
10. Liu, K.; Yang, P.; Jiao, L.; Wang, R.; Yuan, Z.; Li, T. Observer-Based Adaptive Finite-Time Neural Control for Constrained Nonlinear Systems With Actuator Saturation Compensation. *IEEE Trans. Instrum. Meas.* **2024**, *73*, 7502516. [CrossRef]
11. Hanif, A.; Bhatti, A.I.; Ahmed, Q. Managing Thermally Derated Torque of an Electrified Powertrain Through LPV Control. *IEEE/ASME Trans. Mechatronics* **2018**, *23*, 364–376. [CrossRef]
12. Liu, Z.; Yuan, C.; Zhang, Y. Active Fault-Tolerant Control of Unmanned Quadrotor Helicopter Using Linear Parameter Varying Technique. *J. Intell. Robot. Syst.* **2017**, *88*, 415–436. [CrossRef]
13. Sloth, C.; Esbensen, T.; Stoustrup, J. Robust and fault-tolerant linear parameter-varying control of wind turbines. *Mechatronics* **2011**, *21*, 645–659. [CrossRef]
14. Pan, Y.; Sun, T.; Yu, H. On parameter convergence in least squares identification and adaptive control. *Int. J. Robust Nonlinear Control.* **2019**, *29*, 2898–2911. [CrossRef]
15. Costa, R.R. Least-squares model-reference adaptive control with high-order parameter tuners. *Automatica* **2024**, *163*, 111544. [CrossRef]
16. de Oliveira, M.S.; Pereira, R.L. Improved LMI Conditions for Unknown Input Observer Design of Discrete-time LPV Systems. *Int. J. Control. Autom. Syst.* **2020**, *18*, 2543–2551. [CrossRef]
17. Wang, Y.; Bevly, D.M.; Rajamani, R. Interval observer design for LPV systems with parametric uncertainty. *Automatica* **2015**, *60*, 79–85. [CrossRef]

18. Agulhari, C.M.; Tognetti, E.S.; Oliveira, R.C.L.F.; Peres, P.L.D. $\mathcal{H}_\infty$ Dynamic Output Feedback for LPV Systems Subject to Inexactly Measured Scheduling Parameters. In Proceedings of the 2013 American Control Conference, Washington, DC, USA, 17–19 June 2013; pp. 6060–6065. [CrossRef]
19. Tan, K.; Grigoriadis, K.M. Output-feedback control of LPV sampled-data systems. In Proceedings of the American Control Conference, Chicago, IL, USA, 28–30 June 2000; pp. 3687–3691. [CrossRef]
20. Sadabadi, M.S.; Peaucelle, D. From static output feedback to structured robust static output feedback: A survey. *Annu. Rev. Control.* **2016**, *42*, 11–26. [CrossRef]
21. Lacerda, M.J.; Tognetti, E.S.; Oliveira, R.C.L.F.; Peres, P.L.D. A new approach to handle additive and multiplicative uncertainties in the measurement for LPV filtering. *Int. J. Syst. Sci.* **2016**, *47*, 1042–1053. [CrossRef]
22. Boyd, S.; El Ghaoui, L.; Feron, E.; Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory*; SIAM: Philadelphia, PA, USA, 1994.
23. Rotea, M.A. The generalized H_2 control problem. *Automatica* **1993**, *28*, 373–385. [CrossRef]
24. Du, D.; Zhou, S.; Zhang, B. Generalized H_2 output feedback controller design for uncertain discrete-time switched systems via switched Lyapunov functions. *Nonlinear Anal.* **2006**, *65*, 2135–2146. [CrossRef]
25. Oliveira, R.C.L.F.; Bliman, P.A.; Peres, P.L.D. Robust LMIs with parameters in multi-simplex: Existence of solutions and applications. In Proceedings of the 2008 47th IEEE Conference on Decision and Control, Cancun, Mexico, 9–11 December 2008; pp. 2226–2231. [CrossRef]
26. Montagner, V.F.; Oliveira, R.C.L.F.; Leite, V.J.S.; Peres, P.L.D. LMI approach for $\mathcal{H}_\infty$ linear parameter-varying state feedback control. *IEE Proc. Control. Theory Appl.* **2005**, *152*, 195–201. [CrossRef]
27. Rotondo, D.; Nejjari, F.; Puig, V. Robust state-feedback control of uncertain LPV systems: An LMI-based approach. *J. Frankl. Inst.* **2014**, *351*, 2781–2803. [CrossRef]
28. Sadeghzadeh, A. LMI relaxations for robust gain-scheduled control of uncertain linear parameter varying systems. *IET Control. Theory Appl.* **2019**, *13*, 486–495. [CrossRef]
29. Felipe, A.; Oliveira, R.C.L.F. An LMI-Based Algorithm to Compute Robust Stabilizing Feedback Gains Directly as Optimization Variables. *IEEE Trans. Autom. Control.* **2021**, *66*, 4365–4370. [CrossRef]
30. Do, M.H.; Koenig, D.; Theilliol, D. Robust $\mathcal{H}_\infty$ proportional-integral observer-based controller for uncertain LPV system. *J. Frankl. Inst.* **2020**, *357*, 2099–2130. [CrossRef]
31. Lacerda, M.J.; Oliveira, R.C.L.F.; Peres, P.L.D. Robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filter design for uncertain linear systems via LMIs and polynomial matrices. *Signal Process.* **2011**, *91*, 1115–1122. [CrossRef]
32. Agulhari, C.M.; Oliveira, R.C.L.F.; Peres, P.L.D. LMI relaxations for reduced-order robust $\mathcal{H}_\infty$ control of continuous-time uncertain linear systems. *IEEE Trans. Autom. Control.* **2012**, *57*, 1532–1537. [CrossRef]
33. Agulhari, C.M.; Neto, J.M.M.; Lacerda, M.J.; de Souza, R.P.P.; Castoldi, M.F.; Goedtel, A. Robust Real-Time Parameter Estimation for Linear Systems Affected by External Noises and Uncertainties. *Int. J. Adapt. Control. Signal Process.* **2021**, *35*, 203–220. [CrossRef]
34. Khalil, H.K. *Nonlinear Systems*, 3rd ed.; Prentice Hall: Upper Saddle River, NJ, USA, 2002.
35. Astrom, K.J.; Wittenmark, B. *Adaptive Control*, 2nd ed.; Dover Books on Electrical Engineering; Dover Publications: Mineola, NY, USA, 2008.
36. Löfberg, J. YALMIP: A Toolbox for Modeling and Optimization in MATLAB. In Proceedings of the IEEE International Conference on Robotics and Automation, Taipei, Taiwan, 2–4 September 2004. [CrossRef]
37. Agulhari, C.M.; Felipe, A.; Oliveira, R.C.L.F.; Peres, P.L.D. Algorithm 998: The Robust LMI Parser—A toolbox to construct LMI conditions for uncertain systems. *ACM Trans. Math. Softw.* **2019**, *45*, 1–25. [CrossRef]
38. Andersen, E.D.; Andersen, K.D. The MOSEK interior point optimizer for linear programming: An implementation of the homogeneous algorithm. In *High Performance Optimization*; Frenk, H., Roos, K., Terlaky, T., Zhang, S., Eds.; Springer: New York, NY, USA, 2000; Volume 33, pp. 197–232. Available online: http://www.mosek.com (accessed on 20 May 2024).
39. de Souza, C.E.; Trofino, A.; de Oliveira, J. Parametric Lyapunov function approach to $\mathcal{H}_2$ analysis and control of linear parameter-dependent systems. *IEE Proc. Control. Theory Appl.* **2003**, *150*, 501–508. [CrossRef]
40. Lathi, B.P. *Linear Systems and Signals*, 2nd ed.; Oxford University Press: Oxford, UK, 2004.
41. Parks, T.R. *Manual for Model 205/205a—Torsional Control System*, Instructor's ed.; ECP—Educational Control Products: Bell Canyon, CA, USA, 1999.