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Nash's Existence Theorem for Non-Compact Strategy Sets †

Xinyu Zhang [1], Chunyan Yang [2,*], Renjie Han [3] and Shiqing Zhang [4]

1 Beijing Normal University-Hong Kong Baptist University United International College, Zhuhai 519087, China; zhangxinyu@uic.edu.cn
2 Division of Mathematics, Sichuan University Jinjiang College, Pengshan 620860, China
3 School of Economics, Chongqing Technology and Business University, Chongqing 400067, China; chapman@ctbu.edu.cn
4 School of Mathematics, Sichuan University, Chengdu 610065, China; zhangshiqing@scu.edu.cn
* Correspondence: chunyanyang0125@foxmail.com
† In Memory of the Big Mathematician Professor Ky Fan for His 110th Birthday.

Abstract: In this paper, we apply the classical FKKM lemma to obtain the Ky Fan minimax inequality defined on nonempty non-compact convex subsets in reflexive Banach spaces, and then we apply it to game theory and obtain Nash's existence theorem for non-compact strategy sets, which can be regarded as a new, simple but interesting application of the FKKM lemma and the Ky Fan minimax inequality, and we can also present another proof about the famous John von Neumann's existence theorem in two-player zero-sum games. Due to the results of Li, Shi and Chang, the coerciveness in the conclusion can be replaced with the P.S. or G.P.S. conditions.

Keywords: game theory; Nash equilibrium; Ky Fan inequality; two-player zero-sum game

MSC: 54C40; 14E20; 46E25; 20C20

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1. Introduction

In a non-cooperative game, consider n players named $P_1, P_2, \cdots, P_n$, and each P_i has a set of strategies K_i, where $i = 1, 2, \cdots, n$ and K_i satisfy the following condition:

(O) Each K_i is a nonempty convex compact subset of a topological vector space E_i.

If each player P_i has chosen a strategy from K_i, let

$$f_i\colon K_1 \times \cdots \times K_n =: X \to \mathbb{R}$$

be the loss of player P_i. Equivalently, $-f_i$ gives P_i's payoff.

John Forbes Nash Jr., an American mathematician, introduced the following concept and proved its existence with a 28-page Ph.D. dissertation [1] in 1950.

Definition 1 (Nash). *The equilibrium is a point*

$$\tilde{q} := (\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_n) \in X$$

satisfying that

$$f_i(\tilde{q}) = \min_{p_i \in K_i} f_i(\tilde{q}_1, \cdots, \tilde{q}_{i-1}, p_i, \tilde{q}_{i+1} \cdots, \tilde{q}_n).$$

In other words, the Nash equilibrium is a state of a non-cooperative game such that no one can increase their expected return by changing their strategy while the others keep theirs unchanged.

Using Brouwer fixed point theorem, John Nash proved the following.

Theorem 1. *Assume that*

$$K_i = \{p_i = (p_{i1}, p_{i2}, \cdots, p_{in_i}) \in E_i = \mathbb{R}^{n_i}, 0 \le p_{ik} \le 1, k = 1, \cdots, n_i, \sum_{k=1}^{n_i} p_{ik} = 1\}$$

and the following:

(I) $f_i \colon K_1 \times \cdots \times K_n \to \mathbb{R}$ *is continuous for every* $i \in \{1, \cdots, n\}$.

(II) *Let* $p = (p_1, \cdots, p_n)$. *For* $i = 1, \cdots, n$, *fix all* p_j *when* $j \neq i$, *and assume* $p_i \mapsto f_i(p)$ *is convex on* K_i.

Then, Nash equilibrium does exist.

Nash's work on game theory shocked the economics community and won the John von Neumann Theory Prize in 1978 and the Nobel Memorial Prize in Economic Sciences (with John Harsanyi and Reinhard Selten) in 1994.

In 1961, the Chinese-born American mathematician Ky Fan extended the classical Knaster–Kuratowski–Mazurkiewicz (KKM) lemma to an infinite-dimensional result [2]. Later, in 1972, Ky Fan applied the FKKM lemma and obtained the Ky Fan minimax inequality [3].

Theorem 2 ([3]). *It is*

$$\min_{y \in X} \sup_{x \in X} f(x, y) \le \sup_{x \in X} f(x, x)$$

if the following three assumptions are satisfied:

(1) *The function* $f \colon X \times X \to \mathbb{R}$ *is given where* X *is a compact, convex and nonempty set in a topological vector space.*

(2) f *is quasi-concave in the first argument, i.e.,* $x \mapsto f(x, y)$ *is quasi-concave on* X *for every fixed* $y \in X$.

(3) f *is lower semicontinuous in the second argument, i.e.,* $y \mapsto f(x, y)$ *is lower semicontinuous on* X *for every fixed* $x \in X$.

The Ky Fan minimax inequality is equivalent to Brouwer's fixed-point theorem [4]; it is a powerful tool and has many applications [5], especially in mathematical economics and game theory (see Chapter 9 in [4]). One of its applications is to show Nash's existence theorem in a very concise way (see [6,7]). In this paper, we apply the classical FKKM lemma to obtain the Ky Fan minimax inequality defined on nonempty non-compact convex subsets in reflexive Banach spaces, and then we apply it to game theory and obtain Nash's existence theorem for non-compact strategy sets, which can be regarded as a new, simple but interesting application of the FKKM lemma and the Ky Fan minimax inequality, and we can also give another proof of the famous John von Neumann's existence theorem in two-player zero-sum games. Due to the results of Li [8], Shi and Chang [9], the coerciveness in the conclusion can be replaced with the P.S. or G.P.S. conditions.

The academic editor and the referee pointed out some important references on the FKKM lemma and the Ky Fan inequality [5,10–13]. Their generalizations are very complicated; our conditions and proofs are different from theirs.

2. Main Results and Proofs

We replace condition (O) with the following one:

(H) Each K_i is a nonempty convex subset of a reflexive Banach space E_i.

And we maintain the original definition of the Nash equilibrium. Then, we denote the set of strategy profiles as follows:

$$X := K_1 \times K_2 \times \cdots \times K_n \subseteq E_1 \times E_2 \times \cdots \times E_n =: E.$$

For a vector $p := (p_1, p_2, \cdots, p_n) \in E$, define its norm

$$\|p\|_E := \sqrt{\sum_{i=1}^{n} \|p_i\|_{E_i}^2}.$$

Since each E_i is a reflexive Banach space, it is not difficult to verify that E (equipped with its norm topology) is a reflexive Banach space either.

We cite the classical FKKM lemma as follows.

Lemma 1. *Let X be a nonempty subset of a topological vector space E and let the set-valued mapping*

$$T\colon X \to 2^E$$

satisfy the following conditions:

(1) For any fixed $x \in X$, $T(x)$ is a nonempty and closed subset of E.
(2) There exists a $x_0 \in X$ such that $T(x_0)$ is compact in E.
(3) For any finite set $\{x_1, x_2, \cdots, x_n\}$, the following holds

$$\mathrm{co}\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i).$$

Then,

$$\bigcap_{x \in X} \big(T(x) \cap T(x_0)\big) \neq \emptyset,$$

and especially,

$$\bigcap_{x \in X} T(x) \neq \emptyset.$$

For the convenience of the reader, here, we provide the proof, which is slightly different from that in [2].

Proof. Case 1. X is a set with finite points in it. The proof is similar to the that of the classic KKM lemma [14].

Case 2. X is an infinite set. From hypotheses (1) and (2), we have that

$$\tilde{T}(x) := T(x) \cap T(x_0)$$

is compact for any $x \in X$.

Next, we will prove by contradiction. If the conclusion in this case is not true, we will show that there exists the finite set $\{x_1, \cdots, x_m\} \subset X$ such that

$$\bigcap_{i=1}^{m} \tilde{T}(x_i) = \emptyset,$$

which is contradictory to Case 1.

Assume that

$$\bigcap_{x \in X} \tilde{T}(x) = \emptyset,$$

Then, by taking the complement of both sides, we have

$$\bigcup_{x \in X} \tilde{T}^c(x) = X.$$

For an arbitrary point $x_1 \in X$, the compact set is as follows:

$$\tilde{T}(x_1) = X \setminus \tilde{T}^c(x_1) \subset \bigcup_{x \neq x_1} \tilde{T}^c(x).$$

Notice that its right-hand side is an open covering of $\tilde{T}(x_1)$, so we can pick only a finite number of open subsets to cover $\tilde{T}(x_1)$, i.e., there exist some finite sets, which we denote as $\{x_2, x_3, \cdots, x_m\}$, that satisfy

$$\bigcup_{i=2}^{m} \tilde{T}^c(x_m) \supset \tilde{T}(x_1).$$

So, their complements satisfy

$$\bigcap_{i=2}^{m} \tilde{T}(x) \subset \tilde{T}^c(x_1).$$

Hence, we obtain that

$$\bigcap_{i=1}^{m} \tilde{T}(x) \subset \tilde{T}^c(x_1) \cap \tilde{T}(x_1) = \emptyset,$$

which contradicts Case 1. □

Applying Lemma 1, we have the following Ky Fan minimax inequality in the case that the topological vector space is especially a reflexive Banach space equipped with weak topology. Then, we can weaken the compactness assumption of the classical Ky Fan inequality.

Theorem 3. *Let X be a nonempty and convex subset of a reflexive Banach space E and let the functional*

$$f \colon X \times X \to \mathbb{R}$$

satisfy the following conditions:

(i) *For any fixed $y \in X$, the functional $x \mapsto f(x, y)$ is quasi-concave on X, i.e., for any $l \in \mathbb{R}$, the set*

$$\{x \in X | f(x, y) \geq l\}$$

is convex.

(ii) *For any fixed $x \in X$, $y \mapsto f(x, y)$ is weakly lower semicontinuous on X, i.e., for any $l \in \mathbb{R}$, the set*

$$\{(y, l) \in X \times \mathbb{R} | f(x, y) \leq l\}$$

is weakly closed.

(iii) $m := \sup_{x \in X} f(x, x) < +\infty$.

(iv) *There exists a $x_0 \in X$ such that the set*

$$T(x_0) := \{y \in X | f(x_0, y) \leq m\}$$

is bounded in X.

Then,

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x).$$

Remark 1. *In particular, condition (iv) in Theorem 3 is satisfied, while the functional*

$$y \mapsto f(x_0, y)$$

is coercive on X, i.e., $\|y_n\|_E \to +\infty$ implies that $f(x_0, y_n) \to +\infty$. In fact, if not, there must exist a sequence $\{y_n\} \subset T(x_0)$ satisfying $\|y_n\|_E \to +\infty$ so that $f(x_0, y_n) \to +\infty$, which contradicts the definition of $T(x_0)$.

Proof. Set

$$T(x) = \{y \in X | f(x, y) \leq m\}.$$

(1) From hypotheses (ii) and (iii), we have that for any $x \in X$, $T(x)$ is nonempty (since x must belong to $T(x)$) and weakly closed (by the definition of weak lower semicontinuity).

(2) By hypothesis (iv), we obtain the boundedness of $T(x_0)$. Using the Eberlein–Šmulian theorem (see Page 144 in [15]), $T(x_0)$ becomes weakly relatively compact. Then, combined with conclusion (1) above, $T(x_0)$ is weakly compact.

(3) We claim that

$$\mathrm{co}\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i)$$

for any finite set $\{x_1, x_2, \cdots, x_n\}$. Otherwise, there has to be a $\bar{y} \in \mathrm{co}\{x_1, x_2, \cdots, x_n\}$, but $\bar{y} \notin \bigcup_{i=1}^{n} T(x_i)$. Then,

$$f(x_i, \bar{y}) > m, \ \forall i = 1, 2, \cdots, n.$$

That is, $\exists \varepsilon > 0$ such that

$$f(x_i, \bar{y}) \geq m + \varepsilon.$$

From hypothesis (i), we have that the set $\{x \in X | f(x, \bar{y}) \geq m + \varepsilon\}$ is convex since for each x_i that belongs to it, $\bar{y}$ is also in it. Then,

$$f(\bar{y}, \bar{y}) \geq m + \varepsilon > m,$$

which is contradictory to the definition of m.

Using Lemma 1, we have that

$$\bigcap_{x \in X} T(x) \neq \emptyset,$$

i.e., there exists $\tilde{y} \in T(x)$ such that

$$f(x, \tilde{y}) \leq m, \ \forall x \in X.$$

Hence,

$$\inf_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, \tilde{y}) \leq \sup_{x \in X} f(x, x).$$

Next, we will show that the notation "inf" above can be replaced with "min". Notice hypothesis (ii): for any real number l, the set $\{(y, l) \in X \times \mathbb{R} | f(x, y) \leq l\}$ is weakly closed in E. Then, the set

$$\bigcap_{x \in X} \{(y, l) \in X \times \mathbb{R} | f(x, y) \leq l\}$$

is also weakly closed, which implies that the functional

$$y \mapsto \sup_{x \in X} f(x, y)$$

is weakly lower semicontinuous on X. Moreover, the set

$$\{y \in X | \sup_{x \in X} f(x, y) \leq m\} = \bigcap_{x \in X} T(x)$$

is weakly closed in E since

$$M := \bigcap_{x\in X} T(x) \subseteq T(x_0) = \{y \in X | f(x_0, y) \leq m\},$$

which implies that the set M is a weakly compact subset of $T(x_0)$. Since weakly lower semi-continuous functionals always have the minimum on weakly compact sets, the functional

$$y \mapsto \sup_{x\in X} f(x,y)$$

is able to reach its infimum in M. Then, we clearly have that

$$\inf_{y\in X} \sup_{x\in X} f(x,y) = \min_{y\in M} \sup_{x\in X} f(x,y) = \min_{y\in X} \sup_{x\in X} f(x,y).$$

Hence, we have

$$\min_{y\in X} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} f(x,x),$$

which completes the proof. □

Now, we can prove Nash's existence theorem for non-compact strategy sets in the following.

Theorem 4. *Assume the hypothesis*

(H) Each K_i is a nonempty convex closed subset of a reflexive Banach space E_i.

and the following assumptions:

(I) Let $p = (p_1, \cdots, p_n)$. For each $i = 1, \cdots, n$, fix all the components p_j when $j \neq i$ and the functional

$$p_i \mapsto f_i(p)$$

is convex on K_i.

(II) Each f_i is weakly continuous on $X := K_1 \times \cdots \times K_n$.

(III) There exists a $p_0 = (p_{10}, \cdots, p_{n0}) \in X$ such that $\forall i \in \{1, \cdots, n\}$, the set

$$T(p_0) := \{q \in X | \sum_{i=1}^{n} [f_i(q) - f_i(q_1, \cdots, q_{i-1}, p_{i0}, q_{i+1}, \cdots, q_n)] \leq 0\}$$

is bounded in E.

Then, there is at least one Nash equilibrium in X.

Proof. We set

$$\begin{aligned} f\colon X \times X &\to \mathbb{R} \\ (p,q) &\mapsto \sum_{i=1}^{n} [f_i(q) - f_i(q_1, \cdots, q_{i-1}, p_i, q_{i+1}, \cdots, q_n)]. \end{aligned}$$

and

$$\begin{aligned} T\colon X &\to 2^X \\ p &\mapsto \{q \in X | f(p,q) \leq 0\}. \end{aligned}$$

(i) From hypothesis (I), we obtain that for any fixed $q \in X$, the functional

$$p \mapsto f(p,q)$$

is concave on X.

(ii) By hypothesis (II), we obtain the weak continuity of f.

(iii) It is obvious that $f(p,p)=0$ for all $p \in X$.

(iv) Observing hypothesis (III), we have that $T(p_0)$ is bounded in E since for each $q \in T(p_0)$, we have that

$$\|q\|_E = \sqrt{\sum_{i=1}^{n} \|q_i\|_{E_i}^2} \le \sum_{i=1}^{n} \|q_i\|_{E_i} < +\infty.$$

Using Theorem 3, there exists a $\tilde{q} \in X$ such that

$$f(p,\tilde{q}) \le 0,\ \forall p \in X.$$

In particular, for any $i = 1, 2, \cdots, n$, we choose

$$p^i = (\tilde{q}_1, \cdots, \tilde{q}_{i-1}, p_i^i, \tilde{q}_{i+1} \cdots, \tilde{q}_n) \in X.$$

Then,

$$f_i(\tilde{q}) - f_i(p^i) \le 0,\ \forall i \in \{1, \cdots, n\},$$

that is,

$$f_i(\tilde{q}) \le f_i(p^i) = f_i(\tilde{q}_1, \cdots, \tilde{q}_{i-1}, p_i^i, \tilde{q}_{i+1} \cdots, \tilde{q}_n),\ \forall p_i^i \in K_i \text{ and } \forall i \in \{1, \cdots, n\}.$$

This means that $\tilde{q}$ is exactly the Nash equilibrium and that the proof is complete. □

Remark 2. *In Theorem 4, condition (III) is usually difficult to satisfy for $n \ge 3$; however, it holds for the following practical and concise situation for John von Neumann's two-person zero-sum game in unbounded strategy sets. Compared with the result of Zeidler (see Theorem 2.G. on Page 76 and Proposition 1 on Page 80 of [6]), our assumptions are weaker; in particular, here, we do not need the strict convexity of the space X.*

Theorem 5. *In a two-player zero-sum game, we denote the loss functional for player P_i as*

$$f_i\colon K_1 \times K_2 =: X \to \mathbb{R},$$

where K_i (i.e., the strategies of P_i) is a nonempty convex set in a suitable reflexive Banach space E_i. Then, it is clear that $f_2 = -f_1$. Assume the following:

- *(S1) For any fixed $p_2 \in K_2$, the functional $p_1 \mapsto f_1(p_1,p_2)$ is convex and lower semicontinuous on K_1.*
- *(S2) For any fixed $p_1 \in K_1$, the functional $p_2 \mapsto f_1(p_1,p_2)$ is concave and upper semicontinuous on K_2.*
- *(S3) There exists a $p_0 := (p_{10}, p_{20}) \in X$ such that the functional $\cdot \mapsto f_1(\cdot, p_{20})$ is coercive on K_1 and $\cdot \mapsto -f_1(p_{10}, \cdot)$ is coercive on K_2.*

Then, there is at least one Nash equilibrium in X and

$$\min_{p_1 \in K_1} \max_{p_2 \in K_2} f_1(p_1,p_2) = \max_{p_2 \in K_2} \min_{p_1 \in K_1} f_1(p_1,p_2).$$

Proof. We set

$$\begin{aligned} f\colon X \times X &\to \mathbb{R} \\ (p,q) &\mapsto -f_1(p_1,q_2) + f_1(q_1,p_2). \end{aligned}$$

(i) For any fixed $q \in X$, $p \mapsto f_1(q_1, q_2) - f_1(p_1, q_2) + f_2(q_1, q_2) - f_2(q_1, p_2) = f(p, q)$ is concave on X since $\forall x = (x_1, x_2), y = (y_1, y_2) \in X = K_1 \times K_2$ and $\forall \lambda \in [0, 1]$, in applying conditions (S1) and (S2), the following holds:

$$\begin{aligned} f(\lambda x + (1-\lambda)y, q) &= -f_1(\lambda x_1 + (1-\lambda)y_1, q_2) + f_1(q_1, \lambda x_2 + (1-\lambda)y_2) \\ &\geq -[\lambda f_1(x_1, q_2) + (1-\lambda)f_1(y_1, q_2)] \\ &\quad + [\lambda f_1(q_1, x_2) + (1-\lambda)f_1(q_1, y_2)] \\ &= \lambda[-f_1(x_1, q_2) + f_1(q_1, x_2)] + (1-\lambda)[-f_1(y_1, q_2) + f_1(q_1, y_2)] \\ &= \lambda f(x, q) + (1-\lambda)f(y, q). \end{aligned}$$

(ii) Similar to conclusion (i) above, the function $q \mapsto f(p, q)$ is convex on X (and of course quasi-convex). Then, we have that the set $T(p) := \{q \in X | f(p, q) \leq m\}$ is convex for every $m \in \mathbb{R}$ and for any $p \in X$. And clearly, both $q \mapsto f_1(q_1, p_2)$ and $q \mapsto -f_1(p_1, q_2)$ are lower semicontinuous on X; we have that $T(p)$ is closed on X. Since $T(p)$ is both closed and convex, using Mazur's lemma (see Page 6 in [16]), we obtain that $T(p)$ is weakly closed. Hence, the functional $q \mapsto f(p, q)$ is weakly lower semicontinuous on X for any fixed p by definition.

(iii) It is obvious that $f(p, p) = 0$ for all $p \in X$.

(iv) By applying condition (S3), we have that the functional

$$q \mapsto f(p_0, q) = -f_1(p_{10}, q_2) + f_1(q_1, p_{20})$$

is coercive on X and the set $\{q \in X | f(p_0, q) \leq 0\}$ is bounded in X directly from Remark 1.

Hence, using Theorem 3, we have that there exists a $\tilde{q} \in X$ such that

$$f(p, \tilde{q}) \leq 0, \ \forall p \in X.$$

Similar to Theorem 4, this $\tilde{q}$ is exactly the Nash equilibrium and

$$\min_{p_1 \in K_1} f_1(p_1, \tilde{q}_2) = f_1(\tilde{q}_1, \tilde{q}_2) = \max_{p_2 \in K_2} f_1(\tilde{q}_1, p_2),$$

that is,

$$\max_{p_2 \in K_2} \min_{p_1 \in K_1} f_1(p_1, p_2) = f_1(\tilde{q}_1, \tilde{q}_2) = \min_{p_1 \in K_1} \max_{p_2 \in K_2} f_1(p_1, p_2).$$

□

For the two-player zero-sum game, Shu-Zhong Shi and Kung-Chin Chang [9] proved the following theorem.

Theorem 6 (Theorem 3.1 of [9])**.** *Assume that the loss functional for the i-th player P_i is denoted as*

$$f_i \colon K_1 \times K_2 =: X \to \mathbb{R},$$

where K_i is not only nonempty and convex in some reflexive Banach space, but also weakly closed. Since the payoffs are zero-sum, $f_1 + f_2 = 0$, that is, $f_2 = -f_1$. Assume the following:

(S1)′ For any fixed $p_2 \in K_2$, the functional $p_1 \mapsto f_1(p_1, p_2)$ is quasi-convex and weakly lower semicontinuous on K_1.

(S2)′ For any fixed $p_1 \in K_1$, the functional $p_2 \mapsto f_1(p_1, p_2)$ is quasi-concave and weakly upper semicontinuous on K_2.

(S3)′ There exists a $p_0 := (p_{10}, p_{20}) \in X$ such that the functional $\cdot \mapsto f_1(\cdot, p_{20})$ is bounded below and coercive on K_1 and $\cdot \mapsto -f_1(p_{10}, \cdot)$ is bounded below and coercive on K_2.

Then, there is at least one Nash equilibrium in X and

$$\min_{p_1 \in K_1} \max_{p_2 \in K_2} f_1(p_1, p_2) = \max_{p_2 \in K_2} \min_{p_1 \in K_1} f_1(p_1, p_2).$$

The proof is directly from the Lop-sided Maximum Theorem (Page 213 of [9]). It only requires the quasi-convexity of $p_1 \mapsto f_1(p_1, p_2)$ and $p_2 \mapsto -f_1(p_1, p_2)$. But the cost is that K_1 and K_2 must be weakly closed, and it is also necessary that $\cdot \mapsto f_1(\cdot, p_{20})$ and $\cdot \mapsto -f_1(p_{10}, \cdot)$ have lower bounds. Due to the different proof methods, this theorem is difficult to generalize to the case of multiple people like Theorem 4.

Shi and Chang also noticed a result due to Li [8]: a function f must be coercive if it has a lower bound and satisfies the following Palais–Smale (P.S. in short) condition:

Definition 2 (Definition 5.3.1 in [7])**.** *For a closed set X in a Banach space and a functional $f \in C^1(X, \mathbb{R})$, f satisfies that*

$$\text{“}\forall \{x_n\} \subset X \text{ satisfying that } \exists M \in (0, +\infty) \text{ s.t. } |f(x_n)| \le M \text{ and } f'(x_n) \to 0\text{”}$$
$$\Rightarrow \text{“}\exists x_0 \in X \text{ and } \{x_k\} \subseteq \{x_n\} \text{ s.t. } x_k \to x_0\text{”}.$$

Thus, the coerciveness aspects in Theorem 5 and Theorem 6 can both be replaced with the P.S. condition.

Furthermore, Shi and Chang extended the P.S. condition to the G.P.S.′ condition (“G” stands for “generalized”) and Li's theorem for nonsmooth functionals; they then obtained the following theorem.

Theorem 7 (Theorem 3.2 in [9])**.** *Let X be a nonempty closed set in a Banach space. Assume that $f\colon X \to \mathbb{R}$ is lower semicontinuous and bounded below and satisfies the following G.P.S.′ condition:*

$$\text{“}\forall \{x_n\} \subset X \text{ satisfying that } \exists a \in \mathbb{R} \text{ and } b \in [0, +\infty) \text{ s.t.}$$
$$f(x_n) \to a \text{ and } \inf_{\gamma \in T_X(x_n)} f'_{+X}\left(x_n, \frac{\gamma}{\|\gamma\|}\right) \to b\text{”}$$
$$\Rightarrow \text{“}\exists x_0 \in X \text{ and } \{x_k\} \subseteq \{x_n\} \text{ s.t. } x_k \to x_0\text{”}.$$

Then, f is coercive. Here, the notation $T_X(\cdot)$ is the contingent cone of a nonempty subset X of a Banach space E; that is,

$$T_X(\cdot) := \{\gamma \in E | \exists t_n \searrow 0 \text{ and } \gamma_n \to \gamma \text{ s.t. } \cdot + t_n\gamma_n \in X\}.$$

And the notation $f'_{+X}(\cdot, \gamma)$ is the contingent derivative of f at $\cdot$, i.e.,

$$f'_{+X}(\cdot, \gamma) := \liminf_{\substack{\gamma' \to \gamma \\ \cdot + t\gamma' \in X \\ t \searrow 0}} \frac{f(\cdot + t\gamma') - f(\cdot)}{t}.$$

In the same way, we can also replace the coerciveness in Theorem 5 with the G.P.S. condition to make it more applicable.

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