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Article

Additive Results of Group Inverses in Banach Algebras

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Abstract: In this paper, we present new presentations of group inverses for the sum of two group invertible elements in a Banach algebra. We then apply these results to block complex matrices. The group invertibility of certain block complex matrices is thereby obtained.

Keywords: group inverse; block complex matrix; Banach algebra; Banach space

MSC: 15A09; 47L10

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1. Introduction

A Banach algebra is an algebra $\mathcal{A}$ over $\mathbb{C}$, with an identity e, which has a norm $\| \cdot \|$, making it into a Banach space and satisfying $\|e\| = 1$, $\|ab\| \leq \|a\|\|b\|$, where $a, b \in \mathcal{A}$. Let X be a Banach space, then $L(X)$, the algebra of all bounded linear operators on X, is a Banach algebra with respect to the usual operator norm. The identity operator I is its unit elements. $L(X)$ is noncommutative when $\dim(X) > 1$. An element a in a Banach algebra $\mathcal{A}$ has a group inverse, provided that there exists $b \in \mathcal{A}$ such that $a = aba, b = bab$ and $ab = ba$. Such a b is unique if exists, denoted by $a^{\#}$, and is called the group inverse of a. In view of $(aa^{\#})^2 = (aa^{\#}a)a^{\#} = aa^{\#}$, $aa^{\#}$ is an idempotent of $\mathcal{A}$. The symbol $\mathcal{A}^{\#}$ denotes the set of all group invertible elements in $\mathcal{A}$. For some examples related to the group inverse in Banach algebras see [1]. As is well known, a square complex matrix A has a group inverse if and only if $rank(A) = rank(A^2)$. The group invertibility in a ring is attractive. It has interesting applications of resistance distances to the bipartiteness of graphs (see [2,3]). Recently, the group inverse in a Banach algebra or a ring was extensively studied by many authors, e.g., [4–10]. In [11], Theorem 2.3, Liu et al. presented the group inverse of the combinations of two group invertible complex matrices P and Q under the condition $PQQ^{\#} = QPP^{\#}$. In Theorem 3.1 of [12], Zhou et al. investigated the group inverse of $a + b$ under the condition $abb^{\#} = baa^{\#}$ in a Dedekind finite ring in which 2 is invertible. Group inverse is very useful, for example, in solving singular differential and difference equations formulated over a Banach space X [13]. In fact, since the structure of the Banach space is mainly considered, we can regard the operators on Banach space X as an element of the Banach algebra $L(X)$ of all bounded linear operators on a complex Banach space X. The motivation of this paper is to extend the preceding results to a general setting for Banach algebras.

In Section 2, we present the group inverse for the sum of two group invertible elements in a Banach algebra. Let $a, b \in \mathcal{A}^{\#}$. If $abb^{\#} = \lambda baa^{\#}$, then $a + b \in \mathcal{A}^{\#}$. The representation of its group inverse is also given. In Section 3, we apply our results and investigate the group inverse of a block complex matrix

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

where $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times n}, D \in \mathbb{C}^{n\times n}$. This problem is quite complicated, and was extensively studied by many authors. As applications, the group invertibility of certain block complex matrices M is thereby obtained. Additionally, this paper extends the results obtained in Theorem 2.3 of [11], and Theorem 3.1 of [12].

Throughout the paper, all Banach algebras are complex with an identity. Let $\mathcal{A}$ be the Banach algebra. We use $\mathcal{A}^{-1}$ to denote the set of all invertible elements in $\mathcal{A}$. λ always stands for a complex number. $\mathbb{C}^{m\times n}$ stand for the set of all complex $m \times n$ matrices.

2. Main Results

Let $S = \{e_1, \cdots, e_n\}$ be a complete set of idempotents in $\mathcal{A}$, i.e., $e_ie_j = 0 (i \neq j)$, $e_i^2 = e_i (1 \leq i \leq n)$ and $\sum_{i=1}^{n} e_i = 1$. Then, we have $a = \sum_{i,j=1}^{n} e_iae_j$. We write a as the matrix form $a = (a_{ij})_S$, where $a_{ij} = e_iae_j \in e_i\mathcal{A}e_j$, and call it the Peirce matrix of a relatively to S. We shall use this new technique with relative Peirce matrices and generalize Theorem 2.3 of [11] and Theorem 3.1 of [12] as follows.

Theorem 1. *Let $a, b \in \mathcal{A}^{\#}, \lambda \in \mathbb{C}$. If $abb^{\#} = \lambda baa^{\#}$; then, $a + b \in \mathcal{A}^{\#}$. In this case,*

$$(a+b)^{\#} = \begin{cases} (a+b)(a^{\#}+b^{\#})^2 & , \quad \lambda = -1, \\ \frac{1}{1+\lambda}[a^{\#}+b^{\#}-a^{\#}bb^{\#}] + \frac{\lambda}{1+\lambda}[b^{\pi}a^{\#}+a^{\pi}b^{\#}] & , \quad \lambda \neq -1. \end{cases}$$

Proof. Let $p = aa^{\#}$. Write

$$b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}_p, bb^{\#} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}_p.$$

Then,

$$abb^{\#} = \begin{pmatrix} ax_1 & ax_2 \\ 0 & 0 \end{pmatrix}_p, baa^{\#} = \begin{pmatrix} b_1 & 0 \\ b_3 & 0 \end{pmatrix}_p.$$

Since $abb^{\#} = \lambda baa^{\#}$, we have

$$ax_1 = \lambda b_1, x_2 = 0, b_3 = 0.$$

Then,

$$b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}_p.$$

Moreover, we have

$$b^{\#} = \begin{pmatrix} b_1^{\#} & z \\ 0 & b_4^{\#} \end{pmatrix}_p$$

for some $z \in \mathcal{A}$. This implies that

$$x_1 = b_1b_1^{\#}, x_3 = 0, x_4 = b_4b_4^{\#}.$$

Therefore,

$$bb^{\#} = \begin{pmatrix} b_1b_1^{\#} & 0 \\ 0 & b_4b_4^{\#} \end{pmatrix}_p.$$

Since $b = (bb^{\#})b = b(bb^{\#})$, we see that

$$b_2 = (b_1b_1^{\#})b_2 = b_2(b_4b_4^{\#}).$$

Then,

$$b_2 = (b_1b_1^{\#})b_2(b_4b_4^{\#}).$$

Let $e_1 = b_1b_1^{\#}, e_2 = aa^{\#} - b_1b_1^{\#}, e_3 = b_4b_4^{\#}, e_4 = a^{\pi} - b_4b_4^{\#}$. Since $a, e_1 \in aa^{\#}\mathcal{A}aa^{\#}$, we write

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{e_1} \in aa^{\#}\mathcal{A}aa^{\#}.$$

Then,

$$\begin{aligned} a_1 &= x_1ax_1 = \lambda x_1b_1 = \lambda b_1b_1^{\#}b_1 = \lambda b_1, \\ a_3 &= (aa^{\#} - e_1)ax_1 = ax_1 - \lambda b_1 = 0. \end{aligned}$$

Moreover, we see that

$$a_1, a_4 \in (aa^{\#}\mathcal{A}aa^{\#})^{-1}.$$

Since $S = \{e_1, e_2, e_3, e_4\}$ is a complete set of idempotents in $\mathcal{A}$, we have two Peirce matrices of a and b relative to S:

$$a = \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S,$$

$$b = \begin{pmatrix} b_1 & 0 & b_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S.$$

Then,

$$a + b = \begin{pmatrix} (1+\lambda)b_1 & a_2 & b_2 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S.$$

One directly checks that

$$a^{\#} = \begin{pmatrix} a_1^{-1} & -a_1^{-1}a_2a_4^{-1} & 0 & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S,$$

$$b^{\#} = \begin{pmatrix} b_1^{-1} & 0 & -b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S.$$

Then,

$$aa^{\#} = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S, \quad bb^{\#} = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S.$$

Case 1. $\lambda = -1$. Then,

$$a + b = \begin{pmatrix} 0 & a_2 & b_2 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S,$$

$$(a+b)^{\#} = \begin{pmatrix} 0 & a_2(a_4)^{-2} & b_2(b_4)^{-2} & 0 \\ 0 & (a_4)^{-1} & 0 & 0 \\ 0 & 0 & (b_4)^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S .$$

Since $a_1^{-1} + b_1^{-1} = a^{-1}(b_1 + a_1)b_1^{-1} = a^{-1}(1+\lambda)b_1 b_1^{-1} = 0$, we see that

$$a^{\#} + b^{\#} = \begin{pmatrix} 0 & -a_1^{-1}a_2a_4^{-1} & -b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S .$$

Therefore,

$$\begin{aligned} &(a+b)(a^{\#}+b^{\#})^2 \\ &= \begin{pmatrix} 0 & a_2 & b_2 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S \begin{pmatrix} 0 & -a_1^{-1}a_2a_4^{-1} & -b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S^2 \\ &= (a+b)^{\#}, \end{aligned}$$

as desired.

Case 2. $\lambda \neq -1$. Then,

$$\begin{aligned} &(a+b)^{\#} \\ &= \begin{pmatrix} (1+\lambda)^{-1}b_1^{-1} & -(1+\lambda)^{-1}b_1^{-1}a_2a_4^{-1} & -(1+\lambda)^{-1}b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S \\ &= (1+\lambda)^{-1} \begin{pmatrix} b_1^{-1} & -b_1^{-1}a_2a_4^{-1} & -b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S \\ &\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S . \end{aligned}$$

We compute that

$$
\begin{aligned}
& a^{\#}+b^{\#}-a^{\#}bb^{\#} \\
&= \begin{pmatrix} a_1^{-1} & -a_1^{-1}a_2a_4^{-1} & 0 & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S + \begin{pmatrix} b_1^{-1} & 0 & -b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S \\
&\quad - \begin{pmatrix} a_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S \\
&= \begin{pmatrix} b_1^{-1} & -a_1^{-1}a_2a_4^{-1} & -b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S,
\end{aligned}
$$

$$
b^{\pi}a^{\#} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S, \quad a^{\pi}b^{\#} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S.
$$

Therefore, we have

$$
\begin{aligned}
& \frac{1}{1+\lambda}[a^{\#}+b^{\#}-a^{\#}bb^{\#}] + \frac{\lambda}{1+\lambda}[b^{\pi}a^{\#}+a^{\pi}b^{\#}] \\
&= (1+\lambda)^{-1} \begin{pmatrix} b_1^{-1} & -b_1^{-1}a_2a_4^{-1} & -b_1^{-1}b_2b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_4^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_S.
\end{aligned}
$$

Therefore, we have

$$
(a+b)^{\#} = \frac{1}{1+\lambda}[a^{\#}+b^{\#}-a^{\#}bb^{\#}] + \frac{\lambda}{1+\lambda}[b^{\pi}a^{\#}+a^{\pi}b^{\#}],
$$

as asserted. □

Corollary 1. *Let $a, b \in \mathcal{A}^{\#}, \lambda \in \mathbb{C}$. If $aa^{\#}b = \lambda bb^{\#}a$, then $a+b \in \mathcal{A}^{\#}$. In this case,*

$$
(a+b)^{\#} = \begin{cases} (a^{\#}+b^{\#})^2(a+b) & , \quad \lambda = -1, \\ \frac{1}{1+\lambda}[a^{\#}+b^{\#}-aa^{\#}b^{\#}] + \frac{\lambda}{1+\lambda}[a^{\#}b^{\pi}+b^{\#}a^{\pi}] & , \quad \lambda \neq -1. \end{cases}
$$

Proof. Let $(R, *)$ be the opposite ring of R. That is, it is a ring with the multiplication $a * b = b \cdot a$. Applying Theorem 1 to the opposite ring $(R, *)$ of R, we obtain the result. □

Corollary 2. *Let $a, b \in \mathcal{A}$ be idempotents, and let $\lambda \in \mathbb{C}$. If $ab = \lambda ba$, then $a + b \in \mathcal{A}^{\#}$. In this case,*

$$(a+b)^{\#} = \begin{cases} (a+b)^3 & , \quad \lambda = -1, \\ a + b - \dfrac{2+\lambda}{1+\lambda} ab & , \quad \lambda \neq -1. \end{cases}$$

Proof. Since a and b are idempotents, we have

$$aa^{\#}b = ab = \lambda ba = \lambda bb^{\#}a.$$

Therefore, we establish the result by Corollary 2.2 [8]. □

Theorem 2. *Let $a, b \in \mathcal{A}^{\#}, \lambda \in \mathbb{C}$. If $abb^{\#} = \lambda b (\lambda \neq -1)$, then $a + b \in \mathcal{A}^{\#}$. In this case,*

$$\begin{aligned}(a+b)^{\#} = {} & (1+\lambda)^{-1}b^{\#} + b^{\pi}a^{\#}b^{\pi} \\ & + \lambda(1+\lambda)^{-2}b^{\#}aa^{\#}b^{\pi} - (1+\lambda)^{-1}b^{\#}ab^{\pi}a^{\#}b^{\pi}.\end{aligned}$$

Proof. Let $p = bb^{\#}$. Write $a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_p, b = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}_p$. Then, $a_1 = bb^{\#}abb^{\#} = \lambda bb^{\#}b = \lambda b$ and $a_3 = (1 - bb^{\#})abb^{\#} = \lambda(1 - bb^{\#})b = 0$. Hence,

$$a = \begin{pmatrix} \lambda b & a_2 \\ 0 & a_4 \end{pmatrix}_p, \quad a + b = \begin{pmatrix} (1+\lambda)b & a_2 \\ 0 & a_4 \end{pmatrix}_p.$$

Obviously, $a_4 = (1 - bb^{\#})a(1 - bb^{\#}) = b^{\pi}a$. We easily check that $a_4^{\#} = b^{\pi}a^{\#}b^{\pi}$ and $a_4^{\pi} = 1 - b^{\pi}ab^{\pi}a^{\#}b^{\pi} = 1 - b^{\pi}aa^{\#}b^{\pi}$. Moreover, we have

$$\begin{aligned}[(1+\lambda)b]^{\pi}a_2a_4^{\pi} &= b^{\pi}bb^{\#}a_4^{\#} \\ &= b^{\pi}bb^{\#}b^{\pi}a^{\#}b^{\pi} \\ &= 0.\end{aligned}$$

According to Theorem 2.1 of [8], $a + b \in \mathcal{A}^{\#}$. Further, we have

$$(a+b)^{\#} = \begin{pmatrix} [(1+\lambda)b]^{\#} & z \\ 0 & a_4^{\#} \end{pmatrix}_p,$$

where $z = [(1+\lambda)^{-1}b^{\#}]^2 a_2 a_4^{\pi} - (1+\lambda)^{-1}b^{\#}a_2a_4^{\#}$. We compute that

$$\begin{aligned}a_2a_4^{\pi} &= bb^{\#}ab^{\pi}[1 - b^{\pi}aa^{\#}b^{\pi}] \\ &= bb^{\#}ab^{\pi}a^{\pi}b^{\pi}] \\ &= bb^{\#}abb^{\#}a^{\pi}b^{\pi}] \\ &= \lambda baa^{\#}b^{\pi}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(a+b)^{\#} &= (1+\lambda)^{-1}b^{\#} + b^{\pi}a^{\#}b^{\pi} \\ &\quad + [(1+\lambda)^{-1}b^{\#}]^2 a_2a_4^{\pi} - (1+\lambda)^{-1}b^{\#}a_2a_4^{\#} \\ &= (1+\lambda)^{-1}b^{\#} + b^{\pi}a^{\#}b^{\pi} \\ &\quad + \lambda(1+\lambda)^{-2}b^{\#}aa^{\#}b^{\pi} - (1+\lambda)^{-1}b^{\#}ab^{\pi}a^{\#}b^{\pi},\end{aligned}$$

as asserted. □

Corollary 3. *Let $a, b \in \mathcal{A}^{\#}, \lambda \in \mathbb{C}$. If $aa^{\#}b = \lambda a (\lambda \neq -1)$, then $a + b \in \mathcal{A}^{\#}$. In this case,*

$$\begin{aligned}(a+b)^{\#} &= (1+\lambda)^{-1}a^{\#} + a^{\pi}b^{\#}a^{\pi} \\ &\quad + \lambda(1+\lambda)^{-2}a^{\pi}bb^{\#}a^{\#} - (1+\lambda)^{-1}a^{\pi}b^{\#}a^{\pi}ba^{\#}.\end{aligned}$$

Proof. Let $(\mathcal{A}, *)$ be the opposite algebra of $\mathcal{A}$. By applying Theorem 2 to elements b, a in this opposite ring, we obtain the result. □

We demonstrate Theorem 2 by the following numerical example.

Example 1. *Let $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. Then, A and B have group inverses and $ABB^{\#} = -2B$. Since $A^{\#} = \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$ and $B^{\#} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. By using Theorem 2, we obtain*

$$\begin{aligned}(A+B)^{\#} &= -B^{\#} + B^{\pi}A^{\#}B^{\pi} - 2B^{\#}AA^{\#}B^{\pi} + B^{\#}AB^{\pi}A^{\#}B^{\pi} \\ &= \begin{pmatrix} -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.\end{aligned}$$

3. Applications

The aim of this section is to present the group invertibility of the block matrix M by using our main results. We are ready to prove the following.

Theorem 3. *Let A and D have group inverses. If $A^{\pi}B = 0, D^{\pi}C = 0, ACD^{\#} = \lambda C$ and $BCD^{\#} = \lambda D$, then M has a group inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & C \\ 0 & D \end{pmatrix}.$$

Since $A^{\pi}B = 0, D^{\pi}C = 0$, it follows by Theorem 3.4 of [4] that P and Q have group inverses. Moreover, we obtain

$$Q^{\#} = \begin{pmatrix} 0 & C(D^{\#})^2 \\ 0 & D^{\#} \end{pmatrix}.$$

We easily check that

$$\begin{aligned}PQQ^{\#} &= \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & C \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & C(D^{\#})^2 \\ 0 & D^{\#} \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & CD^{\#} \\ 0 & D^{\#} \end{pmatrix} \\ &= \begin{pmatrix} 0 & ACD^{\#} \\ 0 & BCD^{\#} \end{pmatrix} \\ &= \lambda\begin{pmatrix} 0 & C \\ 0 & D \end{pmatrix} \\ &= \lambda Q.\end{aligned}$$

In light of Theorem 2, $M = P + Q$ has a group inverse, as desired. □

Corollary 4. *Let A and D have group inverses. If $CD^{\pi} = 0, BA^{\pi} = 0, A^{\#}BD = \lambda B$ and $A^{\#}BC = \lambda A$, then M has a group inverse.*

Proof. Applying Theorem 3 to the block matrix

$$M^T = \begin{pmatrix} D^T & B^T \\ C^T & A^T \end{pmatrix},$$

we prove that M^T has a group inverse. Therefore, we easily check that $M = (M^T)^T$ has a group inverse, as asserted. □

Theorem 4. *Let A and D have group inverses. If $A^\pi C = 0, D^\pi B = 0, A^\# AB = \lambda A$ and $A^\# AD = \lambda C$, then M has a group inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ B & D \end{pmatrix}.$$

Since $A^\pi C = 0, D^\pi B = 0$, by using Theorem 3.4 of [4], we see that P and Q have group inverses. Moreover, we obtain

$$P^\# = \begin{pmatrix} A^\# & (A^\#)^2 A \\ 0 & 0 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} PP^\# Q &= \begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^\# & (A^\#)^2 A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & D \end{pmatrix} \\ &= \begin{pmatrix} AA^\# & A^\# A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & D \end{pmatrix} \\ &= \begin{pmatrix} A^\# AB & A^\# AD \\ 0 & 0 \end{pmatrix} \\ &= \lambda \begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix} \\ &= \lambda P. \end{aligned}$$

In light of Corollary 3, $M = P + Q$ has a group inverse, as desired. □

Corollary 5. *Let A and D have group inverses. If $BD^\pi = 0, CA^\pi = 0, CDD^\# = \lambda D$ and $ADD^\# = \lambda B$, then M has a group inverse.*

Proof. Applying Theorem 4 to the block matrix

$$M^T = \begin{pmatrix} D^T & B^T \\ C^T & A^T \end{pmatrix},$$

we easily obtain the result as in Corollary 4. □

It is convenient at this stage to prove the following.

Theorem 5. *Let $A \in \mathbb{C}^{m\times m}, D \in \mathbb{C}^{n\times n}$ be idempotents and $rank(B) = rank(C) = rank(BC) = rank(CB)$. If $AD = \lambda AC, A\,(I - CB) = 0$ and $DBA^\pi C = 0$, then M has a group inverse.*

Proof. Since $r(B) = r(C) = r(BC) = r(CB)$, it follows by Lemma 2.3 of [14] that BC and CB have group inverses. Let $K = \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}$. Then, $K^2 = \begin{pmatrix} CB & 0 \\ 0 & BC \end{pmatrix}$. By hypothesis, we have

$$\begin{aligned} rank(K^2) &= rank(CB) + rank(BC) \\ &= rank(C) + rank(B) \\ &= rank(K). \end{aligned}$$

Then, K has a group inverse.

Write $Q := \begin{pmatrix} 0 & A^\pi C \\ D^\pi B & 0 \end{pmatrix}$. Then, we have

$$Q = \begin{pmatrix} A^\pi & 0 \\ 0 & D^\pi \end{pmatrix} \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}.$$

By hypothesis, we see that

$$Q = \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} \begin{pmatrix} A^\pi & 0 \\ 0 & D^\pi \end{pmatrix}.$$

Therefore, N has a group inverse, and

$$Q^\# = \begin{pmatrix} A^\pi & 0 \\ 0 & D^\pi \end{pmatrix} \begin{pmatrix} 0 & C(BC)^\# \\ B(CB)^\# & 0 \end{pmatrix}.$$

Let $P = \begin{pmatrix} A & AC \\ DB & D \end{pmatrix}$. Then, $M = P + Q$. Clearly, $A^\# A(DB) = ADB = \lambda A$, $A^\# AD = AD = \lambda AC$, $A^\pi(AC) = 0$ and $D^\pi(DB) = 0$. In light of Theorem 4, P has a group inverse. Since $ACD^\pi B = 0, DBA^\pi C = 0$, we check that

$$\begin{aligned} PQ &= \begin{pmatrix} A & AC \\ DB & D \end{pmatrix} \begin{pmatrix} 0 & A^\pi C \\ D^\pi B & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

According to Theorem 2.1 of [4], M has a group inverse, as asserted. □

Corollary 6. *Let $A \in \mathbb{C}^{m\times m}, D \in \mathbb{C}^{n\times n}$ be idempotents and $rank(B) = rank(C) = rank(BC) = rank(CB)$. If $AD = \lambda BD$, $(I - CB)D = 0$ and $BD^\pi CA = 0$, then M has a group inverse.*

Proof. Applying Theorem 5 to the block matrix

$$M^T = \begin{pmatrix} D^T & B^T \\ C^T & A^T \end{pmatrix},$$

we complete the proof as in Corollary 5. □

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