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Ricci–Bourguignon Almost Solitons with Special Potential on Sasaki-Like Almost Contact Complex Riemannian Manifolds

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Abstract: Almost contact complex Riemannian manifolds, also known as almost contact B-metric manifolds, are equipped with a pair of pseudo-Riemannian metrics that are mutually associated with each other using the tensor structure. Here, we consider a special class of these manifolds, namely those of the Sasaki-like type. They have an interesting geometric interpretation: the complex cone of such a manifold is a holomorphic complex Riemannian manifold (also called a Kähler–Norden manifold). The basic metric on the considered manifold is specialized here as a soliton, i.e., has an additional curvature property such that the metric is a self-similar solution to an intrinsic geometric flow. Almost solitons are more general objects than solitons because they use functions rather than constants as coefficients in the defining condition. A β-Ricci–Bourguignon-like almost soliton (β is a real constant) is defined using the pair of metrics. The introduced soliton is a generalization of some well-known (almost) solitons (such as those of Ricci, Schouten, and Einstein) which, in principle, arise from a single metric rather than a pair of metrics. The soliton potential is chosen to be pointwise collinear to the Reeb vector field, or the Lie derivative of any B-metric along the potential to be the same metric multiplied by a function. The resulting manifolds equipped with the introduced almost solitons are characterized geometrically. Suitable examples for both types of almost solitons are constructed, and the properties obtained in the theoretical part are confirmed.

Keywords: Ricci–Bourguignon; almost contact B-metric manifold; almost contact complex Riemannian manifold; Sasaki-like manifold; vertical potential; conformal potential

MSC: 53C25; 53D15; 53C50; 53C44; 53D35; 70G45

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1. Introduction

The notion of *Ricci–Bourguignon flow* was introduced by J. P. Bourguignon in 1981 [1]. A time-dependent family of (pseudo-)Riemannian metrics $g(t)$ considered on a smooth manifold $\mathcal{M}$ is said to evolve through Ricci–Bourguignon flow if $g(t)$ satisfies the following evolution equation:

$$\frac{\partial}{\partial t}g = -2(\rho - \beta\tau g), \qquad g(0) = g_0,$$

where β is a real constant, $\rho(t)$ and $\tau(t)$ are the Ricci tensor and the scalar curvature regarding $g(t)$, respectively.

This flow is an intrinsic geometric flow on $\mathcal{M}$ whose fixed points or self-similar solutions are its solitons. The *Ricci–Bourguignon soliton* (RB soliton for short) is described by the following equation [2,3]:

$$\rho + \frac{1}{2}\mathcal{L}_{\vartheta}g + (\lambda + \beta\tau)g = 0, \tag{1}$$

where $\mathcal{L}_{\vartheta}g$ denotes the Lie derivative of g along the vector field ϑ, called the soliton potential, and λ is the soliton constant. Briefly, we denote this soliton by $(g, \beta; \vartheta, \lambda)$. In the case where λ is a differential function on $\mathcal{M}$, the solution is called the *RB almost soliton* [3]. The RB soliton is called expanding if $\lambda > 0$, steady if $\lambda = 0$, and shrinking if $\lambda < 0$. In the case where the soliton potential ϑ is a Killing vector field, i.e., $\mathcal{L}_\vartheta g = 0$, the RB soliton is called trivial.

This family of geometric flows contains the famous Ricci flow for $\beta = 0$, the Einstein flow for $\beta = \frac{1}{2}$, the traceless Ricci flow for $\beta = \frac{1}{m}$, and the Schouten flow for $\beta = \frac{1}{2(m-1)}$, where m is the dimension of the manifold [4,5]. For this reason, we consider it more correct to say *β-RB solitons* and *β-RB almost solitons*, respectively.

Many authors have recently investigated various aspects of these solitons and almost solitons considered on manifolds with different additional structures. For example, some results for conformal Ricci–Bourguignon flow were obtained in [6]. Also, for suitable values of the scalar parameter involved in these flows, the short-time existence is proven in [2], and curvature estimates are provided.

Other recent studies on (almost) solitons of this type have been performed in the last four years. In particular, some integral formulas were derived in [3] for compact gradient Ricci–Bourguignon (almost) solitons. The scalar curvatures of Ricci–Bourguignon solitons of various special types were studied in [7,8]. Ricci–Bourguignon solitons and almost solitons with a concurrent potential vector field on Riemannian manifolds were studied in [9], where such (almost) solitons were first classified and then their geometric properties were obtained. The geometrical characteristics of Ricci–Bourguignon almost solitons with a potential parallel to the Reeb vector field on Kenmotsu manifolds were given in [10]. In [11], some properties of Ricci–Bourguignon almost solitons with a gradient and a torse-forming potential vector field were first derived, then some of these results were generalized to η-Ricci–Bourguignon almost solitons using a structure 1-form. Furthermore, gradient Ricci–Bourguignon almost solitons on a doubly warped product manifold were investigated. In [12], properties of η-Ricci–Bourguignon solitons on a Riemannian manifold equipped with a semi-symmetric metric (or non-metric) connection are studied. Moreover, the potential vector field is torse-forming with respect to the corresponding metric. Almost η-Ricci–Bourguignon solitons in compact and non-compact cases were investigated in [13]. The object of study in [14] was (gradient) η-Ricci–Bourguignon almost solitons on almost Kenmotsu manifolds and (κ, μ) almost Kenmotsu manifolds.

These studies inspire us to study almost solitons of this type generated by the pair of metrics and the contact 1-form on almost contact complex Riemannian manifolds, which we have been working on for a long time.

2. Almost Contact Complex Riemannian Manifolds

Let us consider a $(2n+1)$-dimensional smooth manifold $\mathcal{M}$, equipped with an almost contact structure (φ, ξ, η) and a B-metric g. It is called an *almost contact B-metric manifold* or *almost contact complex Riemannian* (abbreviated *accR*) *manifold*, and is denoted by $(\mathcal{M}, \varphi, \xi, \eta, g)$. In detail, φ is an endomorphism of the tangent bundle $T\mathcal{M}$, ξ is a Reeb vector field, η is its dual contact 1-form, and g is a pseudo-Riemannian metric of signature $(n+1, n)$, such that

$$\begin{gathered}\varphi\xi = 0, \qquad \varphi^2 = -\iota + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1, \\ g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y),\end{gathered} \tag{2}$$

where ι denotes the identity on $\Gamma(T\mathcal{M})$ [15].

In the last equality and further, x, y, and z will denote arbitrary elements of $\Gamma(T\mathcal{M})$ or vectors in the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at an arbitrary point p in $\mathcal{M}$.

The following equations are immediate consequences of Equation (2):

$$g(\varphi x, y) = g(x, \varphi y), \qquad g(x, \xi) = \eta(x), \qquad g(\xi, \xi) = 1, \qquad \eta(\nabla_x \xi) = 0, \tag{3}$$

where ∇ denotes the Levi-Civita connection of g.

The associated metric $\tilde{g}$ of g on $\mathcal{M}$ is also a B-metric, and is defined by

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y). \tag{4}$$

The Ganchev–Mihova–Gribachev classification of the investigated manifolds, given in [15], consists of eleven basic classes $\mathcal{F}_i$, $i \in \{1, 2, \ldots, 11\}$, determined by conditions for the (0,3)-tensor F defined by

$$F(x, y, z) = g\big((\nabla_x \varphi)y, z\big). \tag{5}$$

It has the following basic properties:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi), \tag{6}$$

$$F(x, \varphi y, \xi) = (\nabla_x \eta)y = g(\nabla_x \xi, y). \tag{7}$$

2.1. Sasaki-Like accR Manifolds

An interesting class of accR manifolds was introduced in [16] by the condition that the complex cone of such a manifold is a Kähler–Norden manifold. They are called *Sasaki-like manifolds*, and are defined by the condition

$$F(x, y, z) = g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y). \tag{8}$$

The class of Sasaki-like manifolds is contained in the basic class $\mathcal{F}_4$ of the Ganchev–Mihova–Gribachev classification, not intersecting with the special class $\mathcal{F}_0$ of cosymplectic accR manifolds defined by $F = 0$.

Moreover, the following identities hold for this type of accR manifolds [16]:

$$\nabla_x \xi = -\varphi x, \qquad \rho(x, \xi) = 2n\, \eta(x), \qquad \rho(\xi, \xi) = 2n, \tag{9}$$

where ρ stands for the Ricci tensor for g.

Let τ and $\tilde{\tau}$ be the scalar curvatures with respect to g and $\tilde{g}$, respectively, and let τ^* be the associated quantity of τ regarding φ, defined by $\tau^* = g^{ij}\rho(e_i, \varphi e_j)$. Then, for a Sasaki-like manifold, we have

$$\tilde{\tau} = -\tau^* + 2n. \tag{10}$$

2.2. Einstein-Like accR Manifolds

The following concept was introduced in [17]. An accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ is said to be *Einstein-like* if its Ricci tensor ρ satisfies

$$\rho = a\, g + b\, \tilde{g} + c\, \eta \otimes \eta \tag{11}$$

for some triplet of constants (a, b, c). In particular, when $b = 0$ and $b = c = 0$, the manifold is called an *η-Einstein manifold* and an *Einstein manifold*, respectively. If a, b, c in Equation (11) are functions on $\mathcal{M}$, then the manifold is called *almost Einstein-like*, *almost η-Einstein*, and *almost Einstein*, respectively [17].

The consequences of Equation (11) are the following: $\tau = (2n+1)a + b + c$ and $\tau^* = -2nb$.

3. β-RB Almost Solitons

A generalization of the known RB soliton on a manifold with additional 1-form η is the *η-Ricci–Bourguignon soliton*, defined following Equation (1) by

$$\rho + \frac{1}{2}\mathcal{L}_\vartheta g + (\lambda + \beta\tau)g + \mu\eta \otimes \eta = 0, \tag{12}$$

where μ is also a constant [11]. Obviously, the η-Ricci–Bourguignon soliton with $\mu = 0$ is an RB soliton. Again, in the case where λ and μ are functions on the manifold, almost solitons of the corresponding kind are said to be given.

In the present paper, we study the accR manifold. The presence of two B-metrics, g and $\tilde{g}$, related to each other with respect to the structure of such a manifold gives us reason to introduce a more natural generalization of the β-RB soliton than Equation (12). In addition, we also have the structure 1-form η so that $\eta \otimes \eta$ is included in both B-metrics g and $\tilde{g}$ as their restriction on the vertical distribution span(ξ).

Definition 1. *An accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ is called a β-Ricci–Bourguignon-like soliton (briefly β-RB-like soliton) with potential vector field ϑ if its Ricci tensor ρ satisfies the following condition for a pair of constants $(\lambda, \tilde{\lambda})$:*

$$\rho + \frac{1}{2}\mathcal{L}_\vartheta g + \frac{1}{2}\mathcal{L}_\vartheta \tilde{g} + (\lambda + \beta\tau)g + (\tilde{\lambda} + \beta\tilde{\tau})\tilde{g} = 0, \tag{13}$$

where $\tilde{\tau}$ is the scalar curvature of the manifold with respect to $\tilde{g}$ and the corresponding Levi-Civita connection $\tilde{\nabla}$. If $(\lambda, \tilde{\lambda})$ is a pair of functions on $\mathcal{M}$ satisfying Equation (13), then $(\mathcal{M}, \varphi, \xi, \eta, g)$ is called a β-Ricci–Bourguignon-like almost soliton (briefly β-RB-like almost soliton).

Taking the trace in (13) with respect to g, we obtain

$$[1 + (2n+1)\beta]\tau + \beta\tilde{\tau} + \operatorname{div}_g \vartheta + \frac{1}{2}\operatorname{tr}_g(\mathcal{L}_\vartheta \tilde{g}) + (2n+1)\lambda + \tilde{\lambda} = 0$$

by means of the formula $\operatorname{div}_g \vartheta = \frac{1}{2}\operatorname{tr}_g(\mathcal{L}_\vartheta g) = g^{ij}g(\nabla_{e_i}\vartheta, e_j)$. The trace expression in the above equality is the following:

$$\frac{1}{2}\operatorname{tr}_g(\mathcal{L}_\vartheta \tilde{g}) = g^{ij}\tilde{g}(\tilde{\nabla}_{e_i}\vartheta, e_j).$$

3.1. The Potential Is a Conformal Vector Field

Recall that a vector field, e.g., the potential ϑ on $\mathcal{M}$, is called a *conformal vector field with respect to g* if there exists a function ψ on $\mathcal{M}$ such that [3]

$$\mathcal{L}_\vartheta g = 2\psi g.$$

The conformal vector field is nontrivial if $\psi \neq 0$. If $\psi = 0$, then ϑ is called a *Killing vector field* with respect to g.

Similarly, ϑ is called a *conformal vector field with respect to $\tilde{g}$* if there exists a function $\tilde{\psi}$ on $\mathcal{M}$ such that

$$\mathcal{L}_\vartheta \tilde{g} = 2\tilde{\psi}\tilde{g}.$$

Depending on whether $\tilde{\psi}$ is nonzero or zero, we have a vector field ϑ that is nontrivial scalar or Killing, respectively.

Theorem 1. *Let $(\mathcal{M}, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Sasaki-like accR manifold that is a β-RB-like almost soliton with a pair of soliton functions $(\lambda, \tilde{\lambda})$ and a conformal potential vector field ϑ with potential functions ψ and $\tilde{\psi}$ with respect to g and $\tilde{g}$, respectively. Then, the manifold is Einstein-like, and has the following Ricci tensor:*

$$\rho = \left(\frac{\tau}{2n} - 1\right)g + \left(\frac{\tilde{\tau}}{2n} - 1\right)\tilde{g} \tag{14}$$

and the following property is valid:

$$\tau + \tilde{\tau} = 4n(n+1). \tag{15}$$

If $\beta \neq -\frac{1}{2n}$, then the scalar curvatures with respect to g and $\tilde{g}$ can be expressed separately as

$$\tau = -\frac{2n}{1+2n\beta}(\psi + \lambda - 1), \qquad \tilde{\tau} = -\frac{2n}{1+2n\beta}(\tilde{\psi} + \tilde{\lambda} - 1) \tag{16}$$

and the following condition for the used functions is valid:

$$\psi + \lambda + \tilde{\psi} + \tilde{\lambda} + 2n[1 + 2(n+1)\beta] = 0. \tag{17}$$

If $\beta = -\frac{1}{2n}$, then the following properties hold:

$$\psi + \lambda = 1, \qquad \tilde{\psi} + \tilde{\lambda} = 1. \tag{18}$$

Proof. Under these circumstances, regarding the considered manifold, due to Equation (13), its Ricci tensor for g has the following form:

$$\rho = -(\psi + \lambda + \beta\tau)g - (\tilde{\psi} + \tilde{\lambda} + \beta\tilde{\tau})\tilde{g}. \tag{19}$$

Taking the trace of the last expression, we obtain that the scalar curvatures of $(\mathcal{M}, \varphi, \xi, \eta, g)$ with respect to g and $\tilde{g}$ are related as follows:

$$[1 + (2n+1)\beta]\tau + \beta\tilde{\tau} + (2n+1)(\psi + \lambda) + \tilde{\psi} + \tilde{\lambda} = 0. \tag{20}$$

Using Equation (19), we take the appropriate trace to obtain τ^* on the left-hand side, and the resulting relation to $\tilde{\tau}$ is

$$\tau^* = 2n(\tilde{\psi} + \tilde{\lambda} + \beta\tilde{\tau}). \tag{21}$$

By virtue of Equation (10) for a Sasaki-like accR manifold, and Equation (21), we obtain the expression of the scalar curvature with respect to $\tilde{g}$ as follows:

$$\tilde{\tau} = -2n\frac{\tilde{\psi} + \tilde{\lambda} - 1}{1 + 2n\beta}. \tag{22}$$

The last formula is true for the case $\beta \neq -\frac{1}{2n}$.

Combining Equations (20) and (22), we obtain the following form of the scalar curvature with respect to g:

$$\tau = -\frac{1}{1 + (2n+1)\beta}\left\{(2n+1)(\psi + \lambda) + 1 + \frac{\tilde{\psi} + \tilde{\lambda} - 1}{1 + 2n\beta}\right\}, \tag{23}$$

where $\beta \neq -\frac{1}{2n+1}$. Otherwise, for $\beta = -\frac{1}{2n+1}$, the identity in Equation (17) is valid for this value of β.

On the other hand, a consequence of Equation (19) and the value of $\rho(\xi, \xi)$ from Equation (9) for the Sasaki-like case gives the following relation:

$$\beta\tau + \beta\tilde{\tau} + \psi + \lambda + \tilde{\psi} + \tilde{\lambda} + 2n = 0. \tag{24}$$

Let us check what follows in the particular case where $\beta = 0$. The last relation implies

$$\psi + \lambda + \tilde{\psi} + \tilde{\lambda} + 2n = 0,$$

which we apply in Equations (22) and (23) to specialize them in the following form:

$$\tau = -2n(\psi + \lambda - 1), \qquad \tilde{\tau} = -2n(\tilde{\psi} + \tilde{\lambda} - 1). \tag{25}$$

Therefore, the identity in Equation (15) holds, as do the formulas in Equation (16). Thus, we find that no different results are obtained for $\beta = 0$ compared to the case $\beta \neq -\frac{1}{2n}$.

Let us return to Equation (24) and the solution of the system of equations in (20) and (24) in terms of τ and $\tilde{\tau}$ for $\beta \neq 0$ and $\beta \neq -\frac{1}{2n}$, which gives us

$$\tau = -2n\frac{\psi + \lambda - 1}{1 + 2n\beta}, \qquad \tilde{\tau} = -\frac{1}{\beta}\left\{\frac{\psi + \lambda - 1}{1 + 2n\beta} + \tilde{\psi} + \tilde{\lambda} + 2n + 1\right\}. \tag{26}$$

After comparing the equalities for τ in Equations (23) and (26), we obtain Equation (17). The same relation results from a similar comparison for $\tilde{\tau}$ in Equations (22) and (26). The identity in Equation (17) is a generalization of the corresponding result in the $\beta = 0$ case. Thus, we obtain the formulas in Equation (16).

Because of Equation (17), the sum of τ and $\tilde{\tau}$ is a constant that depends only on the dimension of the manifold, as in Equation (15).

For the case $\beta = -\frac{1}{2n}$, the equalities in Equations (15) and (17) imply the relations in Equation (18). Therefore, Equations (20) and (24) take the form in Equation (15).

In conclusion, using Equations (16) and (19) , we obtain the Ricci tensor expression in Equation (14), which shows that the manifold is Einstein-like, since Equation (11) is satisfied for $a = \frac{\tau}{2n} - 1$, $b = \frac{\tilde{\tau}}{2n} - 1$ and $c = 0$. □

Example of a β-RB Almost Soliton with a Conformal Potential

As in [16], let us consider a Sasaki-like accR manifold of arbitrary dimension with Einstein metric $\bar{g}$. The image of this manifold by contact homothetic transformation of the metric given by $g = p\bar{g} + q\tilde{\bar{g}} + (1-p-q)\eta \otimes \eta$ for $p, q \in \mathbb{R}$, $(p,q) \neq (0,0)$, is also a Sasaki-like accR manifold. The corresponding Ricci tensor has the form

$$\rho = \frac{2n}{p^2+q^2}\Big\{pg - q\tilde{g} + \Big(p^2+q^2-p\Big)\eta \otimes \eta\Big\}.$$

We now calculate the scalar curvatures with respect to B-metrics g and $\tilde{g}$ as follows:

$$\tau = 2n\Big\{1 + \frac{2np}{p^2+q^2}\Big\}, \qquad \tilde{\tau} = 2n\Big\{1 - \frac{2nq}{p^2+q^2}\Big\}. \tag{27}$$

Let ϑ be a conformal vector field with respect to the two B-metrics, g and $\tilde{g}$, with functions ψ and $\tilde{\psi}$, respectively. We construct a β-RB-like almost soliton on the transformed manifold with potential ϑ and a pair of functions $(\lambda, \tilde{\lambda})$ satisfying Equation (13).

By virtue of Equations (15) and (27), we obtain the following condition $p^2 + q^2 - p + q = 0$, which has a solution $p = \frac{1}{2}(1 + \sqrt{2}\cos t)$, $q = -\frac{1}{2}(1 - \sqrt{2}\sin t)$ for $t \in \mathbb{R}$. Then, the scalar curvatures from Equation (27) specialize into the following form for $t \neq (8l+3)\frac{\pi}{4}$, $l \in \mathbb{Z}$:

$$\begin{aligned} \tau &= 2n\frac{(n+1)\sqrt{2} + (2n+1)\cos t - \sin t}{\sqrt{2} + \cos t - \sin t}, \\ \tilde{\tau} &= 2n\frac{(n+1)\sqrt{2} + \cos t - (2n+1)\sin t}{\sqrt{2} + \cos t - \sin t}. \end{aligned} \tag{28}$$

We then determine the functions ψ, $\tilde{\psi}$, λ, $\tilde{\lambda}$ in the case $\beta \neq -\frac{1}{2n}$, as follows:

$$\begin{aligned} \psi + \lambda &= 1 - (1+2n\beta)\frac{(n+1)\sqrt{2} + (2n+1)\cos t - \sin t}{\sqrt{2} + \cos t - \sin t}, \\ \tilde{\psi} + \tilde{\lambda} &= 1 - (1+2n\beta)\frac{(n+1)\sqrt{2} + \cos t - (2n+1)\sin t}{\sqrt{2} + \cos t - \sin t}. \end{aligned} \tag{29}$$

In the case $\beta = -\frac{1}{2n}$, the equalities in Equation (29) obviously reduce to those in Equation (18).

We directly verify that Equations (28) and (29) satisfy the expressions in Equations (16) and (17).

In conclusion, we found that the constructed manifold satisfies the conditions of Theorem 1.

3.2. The Potential Is a Vertical Vector Field

Suppose that $(\mathcal{M}, \varphi, \xi, \eta, g)$ is a Sasaki-like accR manifold admitting a β-RB-like almost soliton, whose potential vector field ϑ is pointwise collinear with ξ, i.e., $\vartheta = k\xi$, where k is a differentiable function on $\mathcal{M}$. It is clear that $k = \eta(\vartheta)$ is true and, therefore, ϑ belongs to the vertical distribution $H^\perp = \text{span}(\xi)$, which is orthogonal to the contact distribution $H = \ker(\eta)$ with respect to both g and $\tilde{g}$.

We easily obtain the expression $(\mathcal{L}_\vartheta g)(x,y) = h(x,y) - 2kg(x, \varphi y)$ for the vertical potential ϑ, where the first equality of Equation (9) is used and the symmetric tensor $h(x,y) = \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x)$ is denoted for brevity. Then, by virtue of Equation (4), we obtain

$$\mathcal{L}_\vartheta g = h - 2k(\tilde{g} - \eta \otimes \eta). \tag{30}$$

Similarly, since one can obtain $\tilde{\nabla}_x \xi = -\varphi x$ for a Sasaki-like accR manifold, we have $(\mathcal{L}_\vartheta \tilde{g})(x,y) = h(x,y) - 2kg(\varphi x, \varphi y)$, and considering the last equality of Equation (2), the last expression takes the following form:

$$\mathcal{L}_\vartheta \tilde{g} = h + 2k(g - \eta \otimes \eta). \tag{31}$$

Theorem 2. *Let $(\mathcal{M}, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Sasaki-like accR manifold that is a β-RB-like almost soliton with a pair of soliton functions $(\lambda, \tilde{\lambda})$ and a vertical potential vector field ϑ with potential function k. Then, the manifold is Einstein-like, and has the following Ricci tensor:*

$$\rho = \left(\frac{\tau}{2n} - 1\right)g + \left(\frac{\tilde{\tau}}{2n} - 1\right)\tilde{g} - \left\{\frac{\tau + \tilde{\tau}}{2n} - 2(n+1)\right\}\eta \otimes \eta. \tag{32}$$

In the case of $\beta \neq -\frac{1}{2n}$, the scalar curvatures with respect to g and $\tilde{g}$ are determined by:

$$\tau = -\frac{2n}{1+2n\beta}(\lambda + k - 1), \qquad \tilde{\tau} = -\frac{2n}{1+2n\beta}(\tilde{\lambda} - k - 1). \tag{33}$$

Moreover, the following condition is true for the functions used:

$$\mathrm{d}k = \mathrm{d}k(\xi)\eta, \qquad \mathrm{d}k(\xi) = -\frac{\lambda + \tilde{\lambda} - 2}{2(1+2n\beta)} - n - 1. \tag{34}$$

In the case of $\beta = -\frac{1}{2n}$, the two scalar curvatures satisfy the following relation:

$$\tau + \tilde{\tau} = 4n\{\mathrm{d}k(\xi) + n + 1\} \tag{35}$$

and the soliton functions are expressed in terms of k as follows:

$$\lambda = 1 - k, \qquad \tilde{\lambda} = 1 + k. \tag{36}$$

Proof. Substituting Equations (30) and (31) into Equation (13), we obtain

$$\rho = -(\lambda + \beta\tau + k)g - (\tilde{\lambda} + \beta\tilde{\tau} - k)\tilde{g} - h. \tag{37}$$

For a Sasaki-like accR manifold, we know the expression of $\rho(x, \xi)$ from Equation (9). Then, we compare it with the corresponding consequence of Equation (37), and obtain $\mathrm{d}k = \mathrm{d}k(\xi)\eta$, where

$$\mathrm{d}k(\xi) = -\frac{1}{2}\{\lambda + \tilde{\lambda} + \beta(\tau + \tilde{\tau}) + 2n\}. \tag{38}$$

Therefore, k is a horizontal constant, and h takes the following form:

$$h = -\{\lambda + \tilde{\lambda} + \beta(\tau + \tilde{\tau}) + 2n\}\eta \otimes \eta. \tag{39}$$

Then, applying the last equality and Equation (38) in (37), for the Ricci tensor ρ, we obtain

$$\rho = -(\lambda + \beta\tau + k)g - (\tilde{\lambda} + \beta\tilde{\tau} - k)\tilde{g} + \{\lambda + \tilde{\lambda} + \beta(\tau + \tilde{\tau}) + 2n\}\eta \otimes \eta. \tag{40}$$

Taking the appropriate traces in Equation (40), we obtain for $\beta \neq -\frac{1}{2n}$ the first equality in Equation (33), as well as

$$\tau^* = 2n(\tilde{\lambda} + \beta\tilde{\tau} - k). \tag{41}$$

Then, bearing in mind Equation (10), the equality in Equation (41) implies, for $\beta \neq -\frac{1}{2n}$, the second equality in Equation (33). Substituting the expressions of τ and $\tilde{\tau}$ from Equation (33) into Equation (40), we obtain Equation (32).

As a consequence of Equations (33) and (38), we express $\tau + \tilde{\tau}$ in two ways. One is given in Equation (35), and the other is as follows:

$$\tau + \tilde{\tau} = -\frac{2n}{1+2n\beta}(\lambda + \tilde{\lambda} - 2). \tag{42}$$

Comparing two expressions implies the relation in Equation (34) between the functions used and the constant β.

Comparing Equation (40) with Equation (11), it follows that $(\mathcal{M}, \varphi, \xi, \eta, g)$ is Einstein-like, and the coefficients in (11) are

$$a = \frac{\tau}{2n} - 1, \qquad b = \frac{\tilde{\tau}}{2n} - 1, \qquad c = -\frac{\tau + \tilde{\tau}}{2n} + 2(n+1). \tag{43}$$

In the particular case where $\beta = -\frac{1}{2n}$, the dependencies in Equation (36) are valid due to Equation (33). Then, Equation (40) is specialized, as in Equation (32). In addition, according to Equation (38), the equality in Equation (35) also holds. Note that, in this case, the scalar curvatures with respect to each of the B-metrics cannot be expressed separately. □

Example of a β-RB Almost Soliton with a Vertical Potential

Let us consider an explicit example, given as Example 2 in [16]. It concerns a Sasaki-like accR manifold constructed on a Lie group G of dimension 5, i.e., for $n = 2$, with a basis of left-invariant vector fields $\{e_0, \dots, e_4\}$. The corresponding Lie algebra is defined by the commutators

$$\begin{array}{ll} [e_0, e_1] = pe_2 + e_3 + qe_4, & [e_0, e_2] = -pe_1 - qe_3 + e_4, \\ [e_0, e_3] = -e_1 - qe_2 + pe_4, & [e_0, e_4] = qe_1 - e_2 - pe_3, \qquad p, q \in \mathbb{R}. \end{array} \tag{44}$$

The introduced accR structure is defined as follows:

$$\begin{array}{l} g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = 1, \\ g(e_i, e_j) = 0, \quad i, j \in \{0, 1, \dots, 4\},\ i \neq j, \\ \xi = e_0, \quad \varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2. \end{array} \tag{45}$$

Then, in [17], the components of the curvature tensor $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ and those of the Ricci tensor $\rho_{ij} = \rho(e_i, e_j)$ are calculated. The non-zero ones are determined by the following equalities and the property $R_{ijkl} = -R_{jikl} = -R_{ijlk}$:

$$\begin{array}{l} R_{0110} = R_{0220} = -R_{0330} = -R_{0440} = 1, \\ R_{1234} = R_{1432} = R_{2341} = R_{3412} = 1, \\ R_{1331} = R_{2442} = 1, \qquad \rho_{00} = 4. \end{array} \tag{46}$$

Therefore, its Ricci tensor has the form $\rho = 4\eta \otimes \eta$, and the manifold is η-Einstein. Hence, the scalar curvature of g is $\tau = 4$, and the constructed manifold is $*$-scalar flat, i.e., $\tau^* = 0$. Then, due to Equation (10), we obtain the value $\tilde{\tau} = 4$ for the scalar curvature of $\tilde{g}$.

Using Equation (13), let us construct a β-RB-like almost soliton on $(G, \varphi, \xi, \eta, g)$ with vertical potential $\vartheta = k\xi$ for a constant β and a pair of functions $(\lambda, \tilde{\lambda})$.

From Equation (33), we determine the following conditions in the case where $\beta \neq -\frac{1}{2n}$:

$$\lambda + k = -4\beta, \qquad \tilde{\lambda} - k = -4\beta. \tag{47}$$

Then, Equation (34) implies $\mathrm{d}k(\xi) = -2$ and $\mathrm{d}k = -2\eta$. A solution to the last equation is, e.g., $k = -2t$, assuming that $\eta = \mathrm{d}t$. This form of k also satisfies the condition in Equation (35) for the case $\beta = -\frac{1}{2n}$. This allows us to determine functions $(\lambda, \tilde{\lambda})$ from Equation (36) and Equation (47) for all values of β by

$$\lambda = 2(t - 2\beta), \qquad \tilde{\lambda} = -2(t + 2\beta). \tag{48}$$

Hence, and from Equation (39), we obtain $h = -4\eta \otimes \eta$. Consequently, Equations (30) and (31) take the following form:

$$\mathcal{L}_\vartheta g = 4t\tilde{g} - 4(t+1)\eta \otimes \eta, \qquad \mathcal{L}_\vartheta \tilde{g} = -4tg + 4(t+1)\eta \otimes \eta.$$

Finally, we found that all the findings in Theorem 2 are true for the given example.

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