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# Exploring Conformal Soliton Structures in Tangent Bundles with Ricci-Quarter Symmetric Metric Connections

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**Abstract:** In this study, we investigate the tangent bundle $TM$ of an $n$-dimensional (pseudo-)Riemannian manifold $M$ equipped with a Ricci-quarter symmetric metric connection $\widetilde{\nabla}$. Our primary goal is to establish the necessary and sufficient conditions for $TM$ to exhibit characteristics of various solitons, specifically conformal Yamabe solitons, gradient conformal Yamabe solitons, conformal Ricci solitons, and gradient conformal Ricci solitons. We determine that for $TM$ to be a conformal Yamabe soliton, the potential vector field must satisfy certain conditions when lifted vertically, horizontally, or completely from $M$ to $TM$, alongside specific constraints on the conformal factor $\lambda$ and the geometric properties of $M$. For gradient conformal Yamabe solitons, the conditions involve $\lambda$ and the Hessian of the potential function. Similarly, for $TM$ to be a conformal Ricci soliton, we identify conditions involving the lift of the potential vector field, the value of $\lambda$, and the curvature properties of $M$. For gradient conformal Ricci solitons, the criteria include the Hessian of the potential function and the Ricci curvature of $M$. These results enhance the understanding of the geometric properties of tangent bundles under Ricci-quarter symmetric metric connections and provide insights into their transition into various soliton states, contributing significantly to the field of differential geometry.

**Keywords:** complete lift metric; Ricci-quarter symmetric metric connection; tangent bundle; conformal Yamabe soliton; gradient conformal Yamabe soliton; conformal Ricci soliton; gradient conformal Ricci soliton

**MSC:** 53C07; 53A45



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## 1. Introduction

Golab [1] introduced the notion of a quarter-symmetric connection, defined by a linear connection $\nabla$ on a differentiable manifold $(M, g)$ of dimension $n$. The torsion tensor $T^{\nabla}$ associated with this connection must satisfy a certain criterion:

$$T^{\nabla}(\xi_1, \xi_2) = \eta(\xi_2)\phi\xi_1 - \eta(\xi_1)\phi\xi_2. \tag{1}$$

In this context, $\eta$ represents a non-zero 1-form, $\phi$ denotes a $(1,1)-$ tensor, and $\xi_i$ $(i = 1,2)$ represent vector fields. The most comprehensive form of quarter-symmetric metric connections on Riemannian, Hermitian and Kaehlerian manifolds was introduced by Yano and Imai [2]. In particular, when the $(1,1)$-tensor $\phi$ coincides with the identity tensor ($\phi = id$), the quarter-symmetric connection simplifies to a semi-symmetric connection. Friedmann and Schouten [3] were the pioneers in introducing the notion of a semi-symmetric linear connection on a differentiable manifold. The form of the semi-symmetric metric connection that is currently known was first proposed by Yano [4], who employed Hayden's method [5]. Consequently, one can perceive a quarter-symmetric connection as an expansion of the notion of a semi-symmetric connection. Fundamentally, a quarter-symmetric

metric connection aligns with a Hayden connection, distinguished by its torsion tensor properties (1).

When the tensor $\phi$ is specified as a $(1,1)$ type Ricci tensor, expressed as

$$g(\phi\xi_1, \xi_2) = R(\xi_1, \xi_2),$$

the resulting quarter-symmetric connection is termed a Ricci quarter-symmetric connection. Such a connection, denoted as $\nabla$, on a Riemannian manifold is classified as a Ricci quarter-symmetric metric connection (abbreviated as RQSMC) if it adheres to the condition:

$$(\nabla_{\xi_1} g)(\xi_2, \xi_3) = 0$$

for all vector fields $\xi_1$, $\xi_2$, $\xi_3$ on $M$. Kamilya and De introduced the concept of a RQSMC on a Riemannian manifold and determined the required conditions for the symmetry of the Ricci tensor of a RQSMC [6].

The Ricci flow transforms the metric $g$ (i.e., the shape of the manifold $M$) into a metric of appropriate constant curvature (i.e., spherical) proportional to the Ricci tensor. In differential geometry, self-similar solutions of the Ricci flow are known as Ricci soliton. On a Riemannian manifold $(M, g)$, a smooth vector field $V$ is considered to define a Ricci soliton if it satisfies the equation:

$$\frac{1}{2} L_V g + Ric = \lambda g,$$

where $L_V g$ denotes the Lie derivative of the Riemannian metric $g$ according to $V$ and $\lambda$ is a constant. A Ricci soliton is represented by the triple $(g, V, \lambda)$, with its classification as either shrinking, steady, or expanding whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

The Yamabe flow, introduced by Hamilton [7] shortly after the Ricci flow addresses the Yamabe problem—a quest to discover a metric on a given compact Riemannian manifold $(M, g)$ of dimension $n > 3$ that preserves the original metric g while upholding a constant scalar curvature. An intriguing aspect of this flow is the concept of a Yamabe soliton, elucidated by Barbosa and Ribeiro [8], which represents a solution showcasing self-similar behavior on the manifold $(M, g)$.

A Yamabe soliton, defined on a Riemannian or pseudo-Riemannian manifold $(M, g)$, is characterized by the equation

$$\frac{1}{2} L_V g = (r - \lambda) g,$$

where $r$ signifies the scalar curvature of $(M, g)$. The soliton is categorized as shrinking, steady, or expanding based on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

In 2005, Fischer [9] introduced a novel concept termed conformal Ricci flow, inspired by Hamilton's pioneering work on Ricci and Yamabe flows [7]. This approach, a modification of the traditional Ricci flow equation, replaces the unit volume restriction with a scalar curvature constraint. The resulting equations, known as the conformal Ricci flow equations, emerge from a blend of conformal flow and Ricci flow equations in terms of vector fields.

These equations, expressed as

$$\frac{\partial g}{\partial t} + 2\left(Ric(g) + \frac{1}{n} g\right) = -\rho g \quad \text{with } r(g) = 1,$$

where $Ric$ denotes the Ricci tensor of $M$, $r$ denotes the scalar curvature of the metric $g$ and $\rho$ is a scalar non-dynamical field, exhibit a notable resemblance to the Navier–Stokes equations from fluid mechanics:

$$\frac{\partial v}{\partial t} + \nabla_v v + v \Delta v = -grad\rho \quad divv = 0$$

is analogous to the Navier–Stokes equations in fluid mechanics. In this context, the time-dependent scalar field $\rho$, termed conformal pressure, plays a pivotal role, akin to real physical pressure, in ensuring the incompressibility of the fluid. Similarly, the conformal pressure serves as a Lagrange multiplier, conformally deforming the metric flow in order to maintain the scalar curvature constraint.

In their initial research contribution, Gezer and Karakas [10] introduced the innovative concept of an RQSMC $\widetilde{\nabla}$ on the tangent bundle $TM$, in conjunction with the complete lift metric ${}^{C}g$ defined on the Riemannian manifold $M$. This seminal work delved into an in-depth exploration of various curvature tensors and their associated properties under the framework of $\widetilde{\nabla}$, accompanied by the introduction of the mean connection to enhance the understanding of the geometric structures involved. The central focus of their research was the clarification of the fundamental conditions that govern the transformation of the tangent bundle $TM$ into a Ricci soliton, gradient Ricci soliton, and the manifestation of local conformal flatness. This elucidated the manifold's dynamic behavior in response to the RQSMC $\widetilde{\nabla}$.

Subsequently, in a follow-up research endeavor, Gezer and Karakas [11] undertook a classification of distinct special vector fields on $TM$ based on the RQSMC $\widetilde{\nabla}$. This encompassed a diverse range of vector fields, including incompressible vector fields, harmonic vector fields, concurrent vector fields, conformal vector fields, projective vector fields and $\widetilde{\varphi}(Ric)$ vector fields on $TM$. This comprehensive categorization provided the foundation for the delineation of the intricate interactions and behaviors of these vector fields within the tangent bundle structure. Furthermore, their investigation extended to defining the criteria under which the tangent bundle $TM$ can function as a Riemannian soliton and a generalized Ricci–Yamabe soliton under the influence of the connection $\widetilde{\nabla}$. This unlocked further insights into the manifold's soliton dynamics.

The overarching objective of their collective body of research work is to establish the fundamental conditions that dictate the transition of the tangent bundle $TM$ into a diverse array of solitons, including the manifestation of a conformal Yamabe soliton, gradient conformal Yamabe soliton, conformal Ricci soliton, and gradient conformal Ricci soliton relative to the RQSMC $\widetilde{\nabla}$. This comprehensive exploration not only expands the existing knowledge base concerning the intricate geometrical properties of the tangent bundle $TM$ but also provides insight into the transition of the tangent bundle into a diverse array of solitons, including the manifestation of a conformal Yamabe soliton, gradient conformal Yamabe soliton, conformal Ricci soliton, and gradient conformal Ricci soliton relative to the RQSMC $\widetilde{\nabla}$. While both this paper and the paper [11] deal with soliton theory and vector fields on $TM$ under a Ricci quarter-symmetric metric connection $\widetilde{\nabla}$, they differ in their primary focus: This paper focuses on conditions for specific types of solitons (conformal Yamabe solitons, gradient conformal Yamabe solitons, conformal Ricci solitons, and gradient conformal Ricci solitons) under $\widetilde{\nabla}$, while the paper [11] focuses primarily on specific types of vector fields (incompressible, harmonic, etc.) and establish conditions for solitons such as Riemannian solitons and generalized Ricci–Yamabe solitons within this framework. The motivation behind considering the article is to contribute new insights into soliton theory under Ricci quarter-symmetric metric connections, providing a rigorous framework to explore and understand the emergence and properties of solitons on tangent bundles $TM$ of (pseudo-)Riemannian manifolds $M$. By addressing these gaps, the article aimed to advance theoretical knowledge and potentially open avenues for future research and applications in mathematical and physical sciences.

## 2. Preliminaries

### *2.1. The Tangent Bundle*

Consider an $n$-dimensional differentiable manifold $M$. The tangent bundle $TM$ of the manifold $M$ is defined as:

$$TM = \underset{P \in M}{\cup} T_P M$$

where $T_P M$ denotes the tangent space of $M$ at $P$. Let us choose a local coordinate system $\{U, x^h\}$ within $M$, and use Cartesian coordinates $(y^h)$ in each tangent space $T_P M$ at a point $P \in M$. These Cartesian coordinates are established using the natural basis $\left\{\frac{\partial}{\partial x^h} \mid_P\right\}$. With this setup, we can define a local coordinate system in $TM$ denoted as $\{\pi^{-1}(U), x^h, y^h\}$. Here, $\pi$ represents the natural projection function defined as: $\pi : TM \longmapsto M$, and $P$ stands for an arbitrary point belonging to $U$. Moreover, the coordinate system $(x^h, y^h)$ is referred to as the induced coordinates on $\pi^{-1}(U)$, which comes from the original coordinate system $\{U, x^h\}$ within $M$.

Now, consider a vector field $\xi$ defined within the open subset $U$ of $M$. This can be locally expressed as $\xi = \xi^h \frac{\partial}{\partial x^h}$. Given that $\nabla$ is a torsion-free linear connection on $M$, we are able to provide the following geometric objects without delay.

1. The vertical lift ${}^V\xi$, the horizontal lift ${}^H\xi$ and the complete lift ${}^C\xi$ are, respectively, as follows:

$$ {}^V\xi = \xi^h \partial_{\overline{h}}, \ {}^H\xi = \xi^h \partial_h - y^s \Gamma^h_{sk} \xi^k \partial_{\overline{h}} \text{ and } {}^C\xi = \xi^h \partial_h + y^s \partial_s \xi^h \partial_{\overline{h}}. $$

2. The adapted frame $\{E_\beta\} = \left\{E_j, E_{\overline{j}}\right\}$ on the tangent bundle is given by:

$$ E_j = \partial_j - y^s \Gamma^h_{sj} \partial_{\overline{h}}, \ E_{\overline{j}} = \partial_{\overline{j}}. $$

   The expressions of the lifts of a vector field $\xi$ with regards to the adapted frame immediately follow:

$$ {}^V\xi = \xi^j E_{\overline{j}}, \ {}^H\xi = \xi^j E_j, \ {}^C\xi = \xi^j E_j + y^s \nabla_s \xi^j E_{\overline{j}}. \tag{2} $$

3. The following is the complete lift metric ${}^C g$ with respect to the adapted frame:

$$ {}^C g_{\alpha\beta} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}. $$

   For the fundamental concepts and details, we refer to [12,13].

### 2.2. The RQSMC on $(TM, {}^C g)$

The RQSMC $\widetilde{\nabla}$ functions within the tangent bundle $TM$ of a (pseudo-)Riemannian manifold $(M, g)$, where it is defined with respect to the complete lift metric ${}^C g$. This metric, derived from the underlying metric $g$ on $M$, serves as the basis for establishing geometric relationships and structures within the tangent space. In our previous investigation detailed in [10], we delve into the intricacies of the RQSMC $\widetilde{\nabla}$ and the associated curvature tensors. By representing the components of these entities as $\Gamma^k_{ij}$ and $R_{sij}{}^k$, respectively, we elucidate the behavior of the RQSMC $\widetilde{\nabla}$ and its associated curvature tensors in a structured manner.

1. The RQSMC $\widetilde{\nabla}$ on $TM$:

$$ \begin{cases} \widetilde{\nabla}_{E_i} E_j = \Gamma^k_{ij} E_k + \{y^s R_{sij}{}^k + y_j R_i{}^k - y^k R_{ij}\} E_{\overline{k}}, \\ \widetilde{\nabla}_{E_i} E_{\overline{j}} = \Gamma^k_{ij} E_{\overline{k}}, \\ \widetilde{\nabla}_{E_{\overline{i}}} E_j = 0, \ \widetilde{\nabla}_{E_{\overline{i}}} E_{\overline{j}} = 0 \end{cases} $$

2. The curvature tensor $\widetilde{R}$ of the RQSMC $\widetilde{\nabla}$:

$$\begin{aligned}
\widetilde{R}(E_i, E_j)E_k &= R_{ijk}{}^l E_l + \{y^s \nabla_s R_{ijk}{}^l\} E_{\bar{l}}, \\
\widetilde{R}(E_i, E_j)E_{\bar{k}} &= R_{ijk}{}^l E_{\bar{l}}, \\
\widetilde{R}(E_i, E_{\bar{j}})E_k &= \{R_{ijk}{}^l + R_{ik}\delta_j^l - g_{jk}R_i{}^l\} E_{\bar{l}}, \\
\widetilde{R}(E_{\bar{i}}, E_j)E_k &= \{R_{ijk}{}^l + g_{ik}R_j{}^l - R_{jk}\delta_i^l\} E_{\bar{l}}, \\
\widetilde{R}(E_{\bar{i}}, E_{\bar{j}})E_k &= 0, \ \widetilde{R}(E_{\bar{i}}, E_j)E_{\bar{k}} = 0, \\
\widetilde{R}(E_i, E_{\bar{j}})E_{\bar{k}} &= 0, \ \widetilde{R}(E_{\bar{i}}, E_{\bar{j}})E_{\bar{k}} = 0
\end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$. The Ricci tensor $\widetilde{K}$ of the RQSMC $\widetilde{\nabla}$:

$$\begin{aligned}
\widetilde{K}_{jk} &= (3-n)R_{jk}, \ \widetilde{K}_{j\bar{k}} = 0, \\
\widetilde{K}_{\bar{j}k} &= 0, \ \widetilde{K}_{\bar{j}\bar{k}} = 0.
\end{aligned} \tag{3}$$

The scalar curvature $\widetilde{r}$ of the RQSMC $\widetilde{\nabla}$ in relation to ${}^C g$ is established as zero.

A vector field $\widetilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on $TM$ within the adapted frame $\{E_\beta\}$ is identified as a fiber-preserving vector field if the components $v^h$ solely depend on the variables $\left(x^h\right)$. The Lie derivative of the fiber-preserving vector field $\widetilde{V}$ with respect to the RQSMC $\widetilde{\nabla}$ is expressed to be used in forthcoming proofs [11]:

$$\begin{cases}
(i)\ L_{\widetilde{V}}\widetilde{g}_{i\bar{j}} = \left(\nabla_i v^h\right) g_{hj} + \left(E_{\bar{j}} v^{\bar{h}}\right) g_{hi} \\
(ii)\ L_{\widetilde{V}}\widetilde{g}_{\bar{i}j} = \left(E_{\bar{i}} v^{\bar{h}}\right) g_{hj} + \left(\nabla_j v^h\right) g_{hi} \\
(iii)\quad \begin{aligned} L_{\widetilde{V}}\widetilde{g}_{ij} = & \left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right) v^a + \Gamma_{ia}^h v^{\bar{a}}\right] g_{hj} \\ & + \left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right) v^a + \Gamma_{ja}^h v^{\bar{a}}\right] g_{hi}. \end{aligned}
\end{cases} \tag{4}$$

## 3. Conformal Yamabe Soliton Structure on $(TM, {}^C g)$ in Relation to RQSMC $\widetilde{\nabla}$

In recent years, geometric flows and associated solitons have attracted the attention of many geometers. In 2021, Roy, Dey and Bhattacharyya [14] developed the idea of Conformal Yamabe solitons, defined on an $n$-dimensional Riemannian manifold as follows:

$$L_V g + \left[2\lambda - 2r - (\rho + \frac{2}{n})\right] g = 0. \tag{5}$$

The classification of a conformal Yamabe soliton into shrinking, steady, or expanding categories is determined by the value of the constant $\lambda$. Specifically, a soliton is considered shrinking if $\lambda > 0$, steady if $\lambda = 0$, and expanding if $\lambda < 0$. In this context, the scalar curvature $r$, a scalar non-dynamical field $\rho$ (which is time-dependent), and the Lie derivative $L_V g$ of the metric g along the vector field V are all key components.

**Theorem 1.** *When $TM$ is regarded as the tangent bundle of an $n$-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$, for the triplet $({}^C g, \widetilde{V}, \lambda)$ to be a conformal Yamabe soliton, it is necessary and sufficient that the specified conditions need to be satisfied:*

$$\begin{gathered}
(i)\ {}^V V = (v^a, v^{\bar{a}}) = (0, v^a), \\
(ii)\ \lambda = \tfrac{1}{2}(\rho + \tfrac{1}{n}), \\
(iii)\ V \text{ is a Killing vector field on } M.
\end{gathered}$$

*In here, the potential vector field is constructed as the vertical lift ${}^V V$ of a vector field $V$ defined on $M$ to the tangent bundle $TM$.*

**Proof.** Starting from the relation (5) we write

$$L_{\widetilde{V}}\widetilde{g}_{\varepsilon\beta} + [2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})]\widetilde{g}_{\varepsilon\beta} = 0. \tag{6}$$

Consider $(\varepsilon, \beta) = (i, \bar{j})$ from the previous equation:

$$L_{\widetilde{V}}\widetilde{g}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{i\bar{j}} = 0.$$

Based on the formulation of $L_{\widetilde{V}}\widetilde{g}$ in (4), we have

$$\left(\nabla_i v^h\right) g_{hj} + \left(E_{\bar{j}} v^{\bar{h}}\right) g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Consider the potential vector field $\widetilde{V}$ as the vertical lift ${}^V V = (v^a, v^{\bar{a}}) = (0, v^a)$ of a vector field $V$ on $M$, then

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

When we consider $(\varepsilon, \beta) = (i, j)$ in (6), we obtain

$$L_{\widetilde{V}}\widetilde{g}_{ij} + [2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})]\widetilde{g}_{ij} = 0.$$

From (4) we have

$$\begin{aligned} &\left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right) v^a + \Gamma_{ia}^{\ h} v^{\bar{a}}\right] g_{hj} \\ +&\left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right) v^a + \Gamma_{ja}^{\ h} v^{\bar{a}}\right] g_{hi} = 0 \end{aligned}$$

and if we choose the potential vector field $\widetilde{V}$ as the vertical lift ${}^V V = (v^a, v^{\bar{a}}) = (0, v^a)$ of a vector field $V$ on $M$, then

$$(\nabla_i v^h) g_{hj} + (\nabla_j v^h) g_{hi} = 0.$$

It follows

$$L_V g_{ij} = 0.$$

This shows $V$ is a Killing vector field on $M$ with respect to the Levi-Civita connection. □

A (pseudo-)Riemannian manifold $(M, g)$ is called an Einstein manifold if the relation

$$K_{jk} = \lambda g_{jk}$$

holds with a scalar function $\lambda$, where $K$ is the Ricci tensor of $(M, g)$.

**Theorem 2.** *Consider $TM$ as the tangent bundle of an $n$-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$. For the triplet $({}^C g, \widetilde{V}, \lambda)$ to be a conformal Yamabe soliton, it is necessary and sufficient that the specified conditions need to be satisfied:*

$$(i)\ {}^H V = (v^a, v^{\bar{a}}) = (v^a, 0),$$
$$(ii)\ \lambda = \tfrac{1}{2}(\rho + \tfrac{1}{n}) - \tfrac{\nabla_h v^h}{2n},$$
$$(iii)\ R_{sa} = \tfrac{r}{2} g_{sa},\ \textit{i.e., } M \textit{ is an Einstein manifold,}$$

*where the potential vector field is formed as the horizontal lift ${}^H V$ of a vector field $V$ defined on $M$ to the tangent bundle $TM$.*

**Proof.** Referring to the relation (5), the following equations are derived for a manifold $M$ in connection with a potential vector field $V$.

The equation

$$L_{\widetilde{V}}\widetilde{g}_{i\bar{j}} + [2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})]\widetilde{g}_{i\bar{j}} = 0$$

is obtained by considering specific values for $(\varepsilon, \beta) = (i, \bar{j})$. Using the expression of $L_{\widetilde{V}}\widetilde{g}$ in (4), the equation

$$\left(\nabla_i v^h\right)g_{hj} + \left(E_{\bar{j}} v^{\bar{h}}\right)g_{hi} + [2\lambda - 2\widetilde{r} - (\rho + \frac{1}{n})]g_{ij} = 0$$

is obtained.

For the potential vector field $\widetilde{V}$ as the horizontal lift ${}^H V = (v^a, v^{\bar{a}}) = (v^a, 0)$ with $\widetilde{r} = 0$, the resulting equation is

$$\left(\nabla_i v^h\right)g_{hj} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Contracting both sides of the previous equation with $g^{ij}$ yields

$$\nabla_h v^h + [2\lambda - (\rho + \frac{1}{n})]n = 0$$

and

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{2n}.$$

By considering the indices $(\varepsilon, \beta) = (i, j)$, the equation

$$L_{\widetilde{V}}\widetilde{g}_{ij} + [2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})]\widetilde{g}_{ij} = 0$$

is obtained. Referring to (4), we have

$$\begin{aligned} &\left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right)v^a + \Gamma_{ia}^{\ h} v^{\bar{a}}\right]g_{hj} \\ &+\left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right)v^a + \Gamma_{ja}^{\ h} v^{\bar{a}}\right]g_{hi} = 0. \end{aligned}$$

If we consider the potential vector field $\widetilde{V}$ as the horizontal lift ${}^H V = (v^a, v^{\bar{a}}) = (v^a, 0)$, then

$$\begin{aligned} &\left[\left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right)v^a\right]g_{hj} \\ &+\left[\left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right)v^a\right]g_{hi} = 0. \end{aligned}$$

By contracting both sides of the last equation with $g^{ij}$, we have

$$R_{sa} = \frac{r}{2}g_{sa}.$$

This shows $M$ is an Einstein manifold. Here, $r$ is the scalar curvature of $M$. □

**Theorem 3.** *When $TM$ is regarded as the tangent bundle of an n-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$, for the triplet $({}^C g, \widetilde{V}, \lambda)$ to be a conformal Yamabe soliton, it is necessary and sufficient that the specified conditions need to be satisfied:*

$$\begin{gathered} (i)\,{}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^a), \\ (ii)\ \lambda = \tfrac{1}{2}(\rho + \tfrac{1}{n}) - \tfrac{\nabla_h v^h}{n}, \\ (iii)\,\nabla_h \nabla_s v^h - (2R_{sa} - g_{sa} r)v^a = 0, \end{gathered}$$

*where the potential vector field is generated as the complete lift ${}^C V$ of a vector field $V$ defined on $M$ to the tangent bundle $TM$.*

**Proof.** Utilizing the concept from relation (5), we represent the equation as follows:

$$L_{\widetilde{V}}\widetilde{g}_{\varepsilon\beta} + [2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})]\widetilde{g}_{\varepsilon\beta} = 0. \tag{7}$$

When considering $(\varepsilon, \beta) = (i, \bar{j})$, this equation simplifies to:

$$L_{\widetilde{V}}\widetilde{g}_{i\bar{j}} + [2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})]\widetilde{g}_{i\bar{j}} = 0.$$

Expanding using the expression of $L_{\widetilde{V}}\widetilde{g}$ in (4), we obtain

$$\left(\nabla_i v^h\right)g_{hj} + \left(E_{\bar{j}}v^{\bar{h}}\right)g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

By considering the potential vector field $\widetilde{V}$ as the complete lift ${}^{C}V = (v^a, v^{\bar{a}}) = (v^a, y^s\nabla_s v^a)$ of a vector field $V$ on $M$, the equation simplifies to:

$$\left(\nabla_i v^h\right)g_{hj} + [\partial_{\bar{j}}(y^s\nabla_s v^h)]g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Further simplification yields:

$$\left(\nabla_i v^h\right)g_{hj} + (\nabla_j v^h)g_{hi} + \left[2\lambda - (\rho + \frac{1}{n})\right]g_{ij} = 0.$$

Contracting both sides of the last equation with $g^{ij}$, we obtain:

$$2\nabla_h v^h + \left[2\lambda - (\rho + \frac{1}{n})\right]n = 0,$$

leading to the relationship:

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{n}.$$

Upon analyzing $(\varepsilon, \beta) = (i, j)$ in (7), we obtain

$$L_{\widetilde{V}}\widetilde{g}_{ij} + [2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})]\widetilde{g}_{ij} = 0.$$

Referring back to (4), we express the equation as:

$$\begin{aligned}&\left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right)v^a + \Gamma_{ia}^{\ h} v^{\bar{a}}\right]g_{hj}\\ +&\left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right)v^a + \Gamma_{ja}^{\ h} v^{\bar{a}}\right]g_{hi} = 0.\end{aligned}$$

Opting for the potential vector field $\widetilde{V}$ as the complete lift ${}^{C}V = (v^a, v^{\bar{a}}) = (v^a, y^s\nabla_s v^a)$, then

$$\begin{aligned}&[(\partial_i - y^s\Gamma_{si}^{\ m}\partial_{\overline{m}})(y^t\nabla_t v^a) + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right)v^a + \Gamma_{ia}^{\ h}(y^s\nabla_s v^a)]g_{hj}\\ +&[\left(\partial_j - y^s\Gamma_{sj}^{\ m}\partial_{\overline{m}}\right)(y^t\nabla_t v^a) + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right)v^a + \Gamma_{ja}^{\ h}(y^s\nabla_s v^a)]g_{hi}\\ &= 0.\end{aligned}$$

This simplifies to:

$$\begin{aligned}&\left[y^s\nabla_i\nabla_s v^h + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right)v^a\right]g_{hj}\\ +&\left[y^s\nabla_j\nabla_s v^h + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right)v^a\right]g_{hi} = 0.\end{aligned}$$

Contracting both sides with $g^{ij}$ gives

$$2y^s\left[\nabla_h\nabla_s v^h - (2R_{sa} - g_{sa}r)v^a\right] = 0,$$

from which we conclude:

$$\nabla_h\nabla_s v^h - (2R_{sa} - g_{sa}r)v^a = 0.$$

Here, $r$ stands for the scalar curvature of $M$. This completes the proof. □

## 4. Gradient Conformal Yamabe Soliton Structure on $(TM, {}^C g)$ with Respect to RQSMC $\widetilde{\nabla}$

A Conformal Yamabe soliton $(g, V, \lambda)$ is labeled as a gradient Conformal Yamabe soliton when $V$ is equal to the gradient of a smooth function $f$, denoted as $\nabla f$. In this context, the smooth function $f$ is denoted as the potential function, and the Equation (5) is expressed as:

$$Hess f + \left[2\lambda - 2r - (\rho + \frac{2}{n})\right]g = 0, \tag{8}$$

where $\nabla f$ represents the gradient of $f$ and $Hess$ represents the Hessian. Typically, the Hessian of any function $f$ on $M$, in relation to the connection $\nabla$, is typically expressed as:

$$(Hess_\nabla f)(X, Y) = XYf - (\nabla_X Y)f,$$

for any vector fields $X$ and $Y$ on $M$.

**Lemma 1.** *The Hessian of the vertical lift of a smooth function $f$ on a Riemannian manifold $(M, g)$ can be described in terms of the RQSMC $\widetilde{\nabla}$ on the tangent bundle $(TM, {}^C g)$ as follows:*

$$\begin{aligned}
Hess_{\widetilde{\nabla}}\, {}^V f({}^H X, {}^H Y) &= {}^H X\, {}^H Y\, {}^V f - (\widetilde{\nabla}_{{}^H X}\, {}^H Y)^V f \\
Hess_{\widetilde{\nabla}}\, {}^V f(E_i, E_j) &= E_i\, E_j{}^V f - (\widetilde{\nabla}_{E_i} E_j)^V f \\
&= (\partial_i - y^s\Gamma^h_{si}\partial_{\bar{h}})(\partial_j - y^m\Gamma^l_{mj}\partial_{\bar{l}})f \\
&\quad -[\Gamma^h_{ij}E_h + (y^s R_{sij}{}^h + y_j R^h_i - y^h R_{ij})E_{\bar{h}}]^V f \\
&= (\partial_i - y^s\Gamma^h_{si}\partial_{\bar{h}})(\partial_j f) - \Gamma^h_{ij}(\partial_h - y^s\Gamma^m_{sh}\partial_{\overline{m}})f \\
&= \partial_i\partial_j f - \Gamma^h_{ij}\partial_h f \\
&= \nabla_i\nabla_j f,
\end{aligned} \tag{9}$$

$$\begin{aligned}
Hess_{\widetilde{\nabla}}\, {}^V f({}^V X, {}^V Y) &= {}^V X\, {}^V Y\, {}^V f - (\widetilde{\nabla}_{{}^V X}\, {}^V Y)^V f \\
Hess_{\widetilde{\nabla}}\, {}^V f(E_{\bar{i}}, E_{\bar{j}}) &= E_{\bar{i}}\, E_{\bar{j}}{}^V f - (\widetilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}})^V f \\
&= 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
Hess_{\widetilde{\nabla}}\, {}^V f({}^H X, {}^V Y) &= {}^H X\, {}^V Y\, {}^V f - (\widetilde{\nabla}_{{}^H X}\, {}^V Y)^V f \\
Hess_{\widetilde{\nabla}}\, {}^V f(E_i, E_{\bar{j}}) &= E_i\, E_{\bar{j}}{}^V f - (\widetilde{\nabla}_{E_i} E_{\bar{j}})^V f \\
&= -(\Gamma^h_{i\bar{j}}E_h + \Gamma^{\bar{h}}_{i\bar{j}}E_{\bar{h}})^V f \\
&= 0,
\end{aligned} \tag{11}$$

$$\begin{aligned} Hess_{\widetilde{\nabla}}\,{}^{V}f({}^{V}X,{}^{H}Y) &= {}^{V}X\,{}^{H}Y\,{}^{V}f - (\widetilde{\nabla}_{{}^{V}X}\,{}^{H}Y)^{V}f \\ Hess_{\widetilde{\nabla}}\,{}^{V}f(E_{\bar{i}},E_{j}) &= E_{\bar{i}}\,E_{j}\,{}^{V}f - (\widetilde{\nabla}_{E_{\bar{i}}}E_{j})^{V}f \\ &= \partial_{\bar{i}}(\partial_{j} - y^{s}\Gamma^{h}_{sj}\partial_{\bar{h}})^{V}f \\ &= \partial_{\bar{i}}\partial_{j}f \\ &= 0. \end{aligned} \tag{12}$$

Now, we focus on the gradient Conformal Yamabe soliton.

**Theorem 4.** *Consider $TM$ as the tangent bundle of an $n$-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$. The triplet $({}^{C}g, \widetilde{V}, \lambda)$ forms a conformal gradient Yamabe soliton if and only if the following conditions are met:*

$$(i)\ \lambda = \frac{1}{2}(\rho + \frac{1}{n}),$$
$$(ii)\ \nabla_{i}\nabla_{j}f = 0.$$

**Proof.** With help of the relation (8), we can represent the equation as:

$$(Hess f)_{\varepsilon\beta} + \left[2\lambda - 2\widetilde{r} - (\rho + \frac{2}{2n})\right]\widetilde{g}_{\varepsilon\beta} = 0. \tag{13}$$

Setting $(\varepsilon, \beta) = (i, \bar{j})$, we derive

$$(Hess f)_{i\bar{j}} + [2\lambda - 2\widetilde{r} - (\rho + \frac{1}{n})]\widetilde{g}_{i\bar{j}} = 0$$

from the previous equation. Referring to the expression for $Hess f$ in (11), we establish:

$$[2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Contracting both sides with $g^{ij}$ yields the relationship

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

Considering $(\varepsilon, \beta) = (i, j)$ in (13), we write

$$(Hess f)_{ij} + [2\lambda - 2\widetilde{r} - (\rho + \frac{1}{n})]\widetilde{g}_{ij} = 0.$$

Finally, the conclusion from (9) is that:

$$\nabla_{i}\nabla_{j}f = 0.$$

□

## 5. Conformal Ricci Soliton Structure on $(TM, {}^{C}g)$ in Relation to RQSMC $\widetilde{\nabla}$

In the context of an $n$-dimensional Riemannian manifold $M$, a conformal Ricci soliton is characterized by the equation:

$$L_{V}g + 2Ric = \left[2\lambda - (\rho + \frac{2}{n})\right]g = 0, \tag{14}$$

where $Ric$ denotes the Ricci tensor, $\lambda$ represents a constant, and $\rho$ is a scalar non-dynamical field (time-dependent scalar field) [15].

**Theorem 5.** *Consider $TM$ as the tangent bundle of an $n$-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$. The triplet $({}^{C}g, \widetilde{V}, \lambda)$ constitutes a conformal Ricci soliton if and only if the following conditions are met:*

$$(i)\ {}^{V}V = (v^{a}, v^{\bar{a}}) = (0, v^{a}),$$
$$(ii)\ \lambda = \frac{1}{2}(\rho + \frac{1}{n}),$$
$$(iii)\ \text{For } n \neq 3,\ \frac{1}{2(3-n)} L_V g_{ij} + R_{ij} = 0.$$

*Here, the potential vector field is obtained by vertically lifting $V$, a vector field defined on $M$, to the tangent bundle $TM$ as ${}^{V}V$.*

**Proof.** We commence by considering relation (14), which is represented as:

$$L_{\widetilde{V}}\widetilde{g}_{\varepsilon\beta} + 2\widetilde{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{\varepsilon\beta} = 0. \tag{15}$$

Substituting $(\varepsilon, \beta) = (i, \bar{j})$ into this equation, we derive:

$$L_{\widetilde{V}}\widetilde{g}_{i\bar{j}} + 2\widetilde{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{i\bar{j}} = 0.$$

Using the expression for $L_{\widetilde{V}}\widetilde{g}$ in (4), we have:

$$\left(\nabla_i v^{h}\right) g_{hj} + \left(E_{\bar{j}} v^{\bar{h}}\right) g_{hi} + [2\lambda - (\rho + \frac{1}{n})] g_{ij} = 0.$$

Choosing the potential vector field $\widetilde{V}$ as the vertical lift ${}^{V}V = (v^{a}, v^{\bar{a}}) = (0, v^{a})$ of a vector field $V$ on $M$, we simplify to:

$$[2\lambda - (\rho + \frac{1}{n})] g_{ij} = 0.$$

Contracting with $g^{ij}$ on both sides of the last equation we have

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

Next, we examine $(\varepsilon, \beta) = (i, j)$ in (15), which yields:

$$L_{\widetilde{V}}\widetilde{g}_{ij} + 2\widetilde{K}_{ij} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{ij} = 0$$

Utilizing (4), this can be expressed as:

$$\begin{gathered}\left[E_i v^{\bar{h}} + \left(y^{s} R_{sia}{}^{h} + y_a R_i^{h} - y^{h} R_{ia}\right) v^{a} + \Gamma_{ia}^{\ h} v^{\bar{a}}\right] g_{hj} \\ + \left[E_j v^{\bar{h}} + \left(y^{s} R_{sja}{}^{h} + y_a R_j^{h} - y^{h} R_{ja}\right) v^{a} + \Gamma_{ja}^{\ h} v^{\bar{a}}\right] g_{hi} \\ + 2(3-n) R_{ij} = 0\end{gathered}$$

and also step by step with further simplifications:

$$[E_i v^{h} + \Gamma_{ia}^{\ h} v^{a}] g_{hj} + [E_j v^{h} + \Gamma_{ja}^{\ h} v^{a}] g_{hi} + 2(3-n) R_{ij} = 0$$

$$\begin{gathered}[(\partial_i - y^{s}\Gamma_{si}^{\ m}\partial_{\overline{m}}) v^{h} + \Gamma_{ia}^{\ h} v^{a}] g_{hj} + [(\partial_j - y^{s}\Gamma_{sj}^{\ m}\partial_{\overline{m}}) v^{h} + \Gamma_{ja}^{\ h} v^{a}] g_{hi} \\ + 2(3-n) R_{ij} = 0\end{gathered}$$

$$\begin{gathered}[(\partial_i v^{h} + \Gamma_{ia}^{\ h} v^{a}] g_{hj} + [(\partial_j v^{h} + \Gamma_{ja}^{\ h} v^{a}] g_{hi} \\ + 2(3-n) R_{ij} = 0\end{gathered}$$

$$(\nabla_i v^h)g_{hj} + (\nabla_j v^h)g_{hi} + 2(3-n)R_{ij} = 0$$

and

$$L_v g_{ij} + 2(3-n)R_{ij} = 0.$$

This expression leads to:

$$\frac{1}{2(3-n)}L_V g_{ij} + R_{ij} = 0.$$

□

**Theorem 6.** *Consider $TM$ as the tangent bundle of an $n$-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$. The triplet $({}^{C}g, \widetilde{V}, \lambda)$ constitutes a conformal Ricci soliton if and only if the following conditions hold:*

$$(i)\ {}^{H}V = (v^a, v^{\bar{a}}) = (v^a, 0),$$
$$(ii)\ \lambda = \tfrac{1}{2}(\rho + \tfrac{1}{n}) - \tfrac{\nabla_h v^h}{2n},$$
$$(iii)\ r = 0 \text{ and } M \text{ is Ricci flat},$$

*where the potential vector field is determined by horizontally lifting the vector field $V$ from the base manifold $M$ to the tangent bundle $TM$.*

**Proof.** As defined in (14), we begin with the equation:

$$L_{\widetilde{V}}\widetilde{g}_{\varepsilon\beta} + 2\widetilde{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{\varepsilon\beta} = 0. \tag{16}$$

When applied to $(\varepsilon, \beta) = (i, \bar{j})$, the equation simplifies to:

$$L_{\widetilde{V}}\widetilde{g}_{i\bar{j}} + 2\widetilde{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{i\bar{j}} = 0.$$

Using the expression of $L_{\widetilde{V}}\widetilde{g}$ in (4), we derive:

$$\left(\nabla_i v^h\right)g_{hj} + \left(E_{\bar{j}} v^{\bar{h}}\right)g_{hi} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Assuming $\widetilde{V}$ as the horizontal lift vector field ${}^{H}V = (v^a, v^{\bar{a}}) = (v^a, 0)$, we can express:

$$\left(\nabla_i v^h\right)g_{hj} + [2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Upon contracting with $g^{ij}$, we determine:

$$\nabla_h v^h + [2\lambda - (\rho + \frac{1}{n})]n = 0$$

leading to:

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{2n}.$$

For $(\varepsilon, \beta) = (i, j)$ in (16), the equation simplifies to:

$$L_{\widetilde{V}}\widetilde{g}_{ij} + 2\widetilde{K}_{ij} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{ij} = 0.$$

From (4), the expression transforms into:

$$\begin{gathered}\left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right) v^a + \Gamma_{ia}^{\ h} v^{\bar{a}}\right] g_{hj} \\ + \left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right) v^a + \Gamma_{ja}^{\ h} v^{\bar{a}}\right] g_{hi} \\ + 2(3-n) R_{ij} = 0\end{gathered}$$

$$\begin{gathered}\left[\left(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\right) v^a\right] g_{hj} + \left[\left(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\right) v^a\right] g_{hi} \\ + 2(3-n) R_{ij} = 0.\end{gathered}$$

Contracting with $g^{ij}$, we obtain:

$$[-2y^s(2R_{sa} - g_{sa} r) v^a] + 2(3-n) r = 0.$$

This leads to the conclusion:

$$R_{sa} = \frac{r}{2} g_{sa} \text{ and } (3-n) r = 0,$$

implying that $r = 0$, signifying the Ricci flatness of $M$ with $r$ denoting the scalar curvature of $M$. □

**Theorem 7.** *If we consider $TM$ as the tangent bundle of an n-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$, the triplet $({}^C g, \widetilde{V}, \lambda)$ constitutes a conformal Ricci soliton if and only if the following conditions are met:*

$$\begin{gathered}(i)\, {}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^a), \\ (ii)\ \lambda = \tfrac{1}{2}(\rho + \tfrac{1}{n}) - \tfrac{\nabla_h v^h}{n}, \\ (iii)\ r = 0, \\ (iv)\ \nabla_h \nabla_s v^h - 2R_{sa} v^a = 0,\end{gathered}$$

*where the potential vector field is formed by completely lifting $V$, a vector field defined on $M$, to the tangent bundle $TM$ as ${}^C V$.*

**Proof.** Based on the relation (14), we write

$$L_{\widetilde{V}} \widetilde{g}_{\varepsilon\beta} + 2\widetilde{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})] \widetilde{g}_{\varepsilon\beta} = 0. \tag{17}$$

If we consider $(\varepsilon, \beta) = (i, \bar{j})$, from the previous equation,

$$L_{\widetilde{V}} \widetilde{g}_{i\bar{j}} + 2\widetilde{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{2}{2n})] \widetilde{g}_{i\bar{j}} = 0$$

and from the expression of $L_{\widetilde{V}} \widetilde{g}$ in (4) we have

$$\left(\nabla_i v^h\right) g_{hj} + \left(E_{\bar{j}} v^{\bar{h}}\right) g_{hi} + [2\lambda - (\rho + \frac{1}{n})] g_{ij} = 0.$$

Taking ${}^C V = (v^a, v^{\bar{a}}) = (v^a, y^s \nabla_s v^a)$, we obtain

$$\left(\nabla_i v^h\right) g_{hj} + [\partial_{\bar{j}}(y^s \nabla_s v^h)] g_{hi} + [2\lambda - (\rho + \frac{1}{n})] g_{ij} = 0$$

and

$$\left(\nabla_i v^h\right) g_{hj} + (\nabla_j v^h) g_{hi} + [2\lambda - (\rho + \frac{1}{n})] g_{ij} = 0.$$

Contracting with $g^{ij}$ on both sides of the last equation we have

$$2\nabla_h v^h + [2\lambda - (\rho + \frac{1}{n})]n = 0$$

and it follows

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}) - \frac{\nabla_h v^h}{n}.$$

Now we consider $(\varepsilon, \beta) = (i, j)$ in (17), we obtain

$$L_{\widetilde{V}} \widetilde{g}_{ij} + 2\widetilde{K}_{ij} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{ij} = 0$$

and from (4) we have

$$\begin{gathered}\Big[E_i v^{\bar{h}} + \Big(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\Big)v^a + \Gamma_{ia}^{\ h} v^{\bar{a}}\Big] g_{hj} \\ + \Big[E_j v^{\bar{h}} + \Big(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\Big)v^a + \Gamma_{ja}^{\ h} v^{\bar{a}}\Big] g_{hi} \\ +2(3-n)R_{ij} = 0\end{gathered}$$

and it follows

$$\begin{gathered}\Big[y^s \nabla_i \nabla_s v^h + \Big(y^s R_{sia}{}^h + y_a R_i^h - y^h R_{ia}\Big)v^a\Big] g_{hj} \\ + \Big[y^s \nabla_j \nabla_s v^h + \Big(y^s R_{sja}{}^h + y_a R_j^h - y^h R_{ja}\Big)v^a\Big] g_{hi} \\ +2(3-n)R_{ij} = 0.\end{gathered}$$

Contracting with $g^{ij}$ on both sides of the last equation we have

$$2y^s\Big[\nabla_h \nabla_s v^h - (2R_{sa} - g_{sa} r)v^a\Big] + 2(3-n)r = 0$$

and it follows

$$\nabla_h \nabla_s v^h - (2R_{sa} - g_{sa} r)v^a = 0 \text{ and } (3-n)r = 0.$$

This shows $r = 0$. Therefore, we obtain $\nabla_h \nabla_s v^h - 2R_{sa} v^a = 0$. Here, $r$ is the scalar curvature of $M$. □

## 6. Gradient Conformal Ricci Soliton Structure on $(TM, {}^C g)$ in Relation to RQSMC $\widetilde{\nabla}$

A gradient conformal Ricci soliton, defined by $(g, \nabla f, \lambda)$ with $V = \nabla f$, involves a smooth function $f$ known as the potential function. The Equation (14) takes the form:

$$Hess f + 2Ric = \Big[2\lambda - (\rho + \frac{2}{n})\Big] g = 0. \tag{18}$$

In this context, $\nabla f$ represents the gradient of $f$, and *Hess* denotes the Hessian. The Hessian of any function $f$ on $M$, with respect to the connection $\nabla$, is typically denoted as:

$$(Hess_{\nabla} f)(X, Y) = XYf - (\nabla_X Y) f,$$

for any vector fields $X$ and $Y$ on $M$.

**Theorem 8.** *If we consider $TM$ as the tangent bundle of an $n$-dimensional (pseudo-)Riemannian manifold $M$ equipped with the RQSMC $\widetilde{\nabla}$. The conformal gradient Ricci soliton can be identified by the triplet $({}^C g, \widetilde{V}, \lambda)$ if and only if the following conditions are satisfied:*

$$\begin{gathered}(i)\ \lambda = \tfrac{1}{2}(\rho + \tfrac{1}{n}), \\ (ii)\ \nabla_i \nabla_j f = 2(n-3)R_{ij}.\end{gathered}$$

**Proof.** In line with relation (18), we derive the equation:

$$(Hessf)_{\varepsilon\beta} + 2\overline{K}_{\varepsilon\beta} + [2\lambda - (\rho + \frac{2}{2n})]\widetilde{g}_{\varepsilon\beta} = 0. \tag{19}$$

By setting $(\varepsilon, \beta) = (i, \bar{j})$, we find:

$$(Hessf)_{i\bar{j}} + 2\overline{K}_{i\bar{j}} + [2\lambda - (\rho + \frac{1}{n})]\widetilde{g}_{i\bar{j}} = 0.$$

Using the expression for $Hessf$ from (11) and (3), we conclude:

$$[2\lambda - (\rho + \frac{1}{n})]g_{ij} = 0.$$

Contracting both sides with $g^{ij}$ leads us to:

$$\lambda = \frac{1}{2}(\rho + \frac{1}{n}).$$

When we substitute $(\varepsilon, \beta) = (i, j)$ in (19), we can rewrite the equation as:

$$(Hessf)_{ij} + 2\overline{K}_{ij} + [2\lambda - (\rho + \frac{1}{n})]\widetilde{g}_{ij} = 0.$$

Finally, (9) and (3) lead us to the outcome:

$$\nabla_i \nabla_j f = 2(n-3)R_{ij}.$$

□

## 7. Conclusions

This paper presents a comprehensive investigation into the geometric attributes of the tangent bundle $TM$ of a (pseudo-)Riemannian manifold $M$, equipped with a RQSMC $\widetilde{\nabla}$. The primary objective is to identify the essential conditions for $TM$ to exhibit different types of solitons and soliton-like structures under this connection. Through rigorous mathematical analysis, we have provided explicit conditions characterizing each type of soliton within the framework of the RQSMC $\widetilde{\nabla}$. By defining and scrutinizing the properties of potential vector fields lifted from $M$ to $TM$, we have elucidated the geometric and algebraic structures underlying these soliton formations. These solitons represent critical solutions of geometric flow equations and play pivotal roles in understanding the global and local geometry of (pseudo-)Riemannian manifolds.

The results presented here provide a comprehensive understanding of the interplay between geometric structures and connection properties within the tangent bundle $TM$. By delineating the precise conditions under which each type of soliton emerges, we contribute to the advancement of differential geometry and its applications in physics. These findings have significant implications for various fields, including general relativity, where soliton-like structures often arise as solutions to Einstein's field equations.

The contributions of this paper extend beyond pure mathematics, as the results obtained have implications in both differential geometry and physics. From a differential geometry perspective, our work enriches the understanding of soliton-like structures and their emergence in tangent bundles endowed with specific metric connections. Furthermore, our findings provide valuable insights into the geometric flows and curvature properties of tangent bundles, contributing to ongoing research in geometric analysis and mathematical physics.

In the context of physics, the elucidation of soliton formations in tangent bundles opens avenues for exploring the geometric aspects of physical theories. Solitons play a significant role in various physical phenomena, including nonlinear wave equations and field theories.

Therefore, our results pave the way for investigating the geometric underpinnings of soliton dynamics in the context of tangent bundles, offering new perspectives on the interplay between geometry and physics.

In conclusion, this paper represents a significant contribution to the fields of differential geometry and theoretical physics, providing valuable insights into the geometric properties of tangent bundles and their relevance to soliton theory. We anticipate that our findings will inspire further research and foster interdisciplinary collaborations aimed at unraveling the intricate geometrical structures of (pseudo-)Riemannian manifolds.

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