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Article

Characterizing Finite Groups through Equitable Graphs: A Graph-Theoretic Approach

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Abstract: This paper introduces equitable graphs of Type I associated with finite groups. We investigate the connectedness and some graph-theoretic properties of these graphs for various groups. Furthermore, we establish the novel concepts of the equitable square-free number and the equitable group. Our study includes an analysis of the equitable graphs for specific equitable groups. Additionally, we determine the first, second and forgotten Zagreb topological indices for the equitable graphs of Type I constructed from certain groups. Finally, we derive the adjacency matrix for this graph type built from cyclic p-groups.

Keywords: equitable graph; equitable group; topological indices

MSC: 05C62; 05C25

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1. Introduction

The connection between graphs and groups is an interesting field of research and has wide applications. Research on this subject leads to the investigation of the relationship between the group and the associated graph and explores theoretical properties from one to the other. The graph associated with a group can provide valuable information and offer a combinatorial approach to studying groups. This can give group theorists more tools to work with. Additionally, comparing groups with similar graph-theoretic properties can help classify these groups. The literature is rich with studies on this topic. This concept has been known since 1878, when Cayley graphs were presented [1]. Subsequently, several graphs have been constructed from groups, such as the commuting graph, which was introduced by Brauer and Fowler in 1955 [2]. Then, the prime graphs were defined by Gruenberg and Kegel in 1975 [3]. Later, in 2009, Chackrabarty, Gosh and Sen presented the power graph [4,5]. Many graphs have been introduced in the literature: for instance, the order-divisor graph, intersection graph and cyclic graph. All of these graphs have been thoroughly studied, including their characteristics and their relations with groups. For more details, we refer the reader to [6–11].

In light of the increasing significance of graphs linked to groups and their role in classifying both groups and graphs, as well as the importance of element orders in a finite group, we are inspired to introduce a new type of graph based on the distinctions between element orders within the group. Through this research, we study a graph associated with a finite group called *the equitable graph Type I* and denoted by $\mathcal{E}_1(G)$. The vertex set of this graph is a finite group G, and two distinct vertices x and y are adjacent if and only if $|o(x) - o(y)| \leq \min\{o(x), o(y)\}$.

In our research, we extensively studied important algebraic groups in order to create general formulaic representations of the resulting graphs. These representations were thoroughly analyzed to understand their theoretical properties and topological characteristics.

Moreover, our exploration of this innovative conceptual definition allowed us to establish new specialized terminology: specifically, the concepts of *the equitable square-free number* and *the equitable group*. These new concepts serve as valuable classifications within the respective domains of number theory and graph theory.

In this paper, G denotes a finite group, and e is the identity of G. For any element of G, say g, $o(g)$ is the order of g, and the number of elements of order m in a cyclic group is equal to $\phi(m)$, where ϕ is the *Euler's phi* function. For a real number x, the greatest integer $\leq x$ [or the least integer $\geq x$], called the floor [or ceiling] function and denoted by $\lfloor x \rfloor$ [or $\lceil x \rceil$], respectively.

Let Γ denote a graph with vertex set V and edge set E. Then, $m(\Gamma(V))$ denotes the size of the graph, and the number of edges incident to a single vertex $v \in V$ is called the degree of v, $d(v)$; the maximum and minimum degrees of the graph are denoted by $\Delta(\Gamma(V))$ and $\delta(\Gamma(V))$, respectively. The graph $\Gamma(V)$ is said to be connected if and only if there is a path between any two distinct vertices of V, while the graph is complete if and only if any two vertices are adjacent, and K_m denotes the complete graph on m vertices. The complete subgraph of $\Gamma(V)$ is called a clique, and the clique number, $\omega(\Gamma(V))$, is the cardinality of the maximum clique. The diameter, $\text{diam}(\Gamma(V))$, is defined as the maximum distance between two vertices, and the radius, $r(\Gamma(V))$, is the minimum eccentricity of the graph, where the eccentricity of any vertex v is defined as $e(v) = \max\{d(v, u) : u \in V\}$. The length of the shortest cycle in $\Gamma(V)$ is called the girth of the graph, and it is denoted by $gr(\Gamma(V))$. A set S of vertices is said to be a dominating set if every vertex v belong to $V \setminus S$ is adjacent to at least one vertex in S, and the cardinality of the minimum dominating set, $\gamma - set$, is called the domination number, $\gamma(\Gamma(V))$. The minimum number of colors needed to label the vertices such that no two adjacent vertices have the same color is called the chromatic number of the graph, $\chi(\Gamma(V))$. The adjacency matrix is an $(n \times n)$ matrix, where $|V| = n$, and is denoted by $A(\Gamma(V))$. Almost all of the definitions and notations can be found in [12–14] for group theory and graph theory.

Through this work, we deal with finite groups and simple graphs. We consider the vertex set as the elements of the group and introduce the first type of equitable graph, $\mathcal{E}_1(G)$. In this paper, we study the connectedness of the equitable graph Type I for some groups and investigate some of their theoretical properties in Section 2. In Section 3, we introduce the concepts of the equitable square-free number and the equitable group. Then, the graph of this group is studied. Next, we determine the first, second and forgotten Zagreb indices for the equitable graph Type I of some groups in Section 4. Finally, in Section 5, we obtain the adjacency matrix for $\mathcal{E}_1(G)$, where G is a cyclic p-group, and many examples are included. In this work, since the vertices are the elements of the group G, we use the words "elements" and "vertices" interchangeably. Also, for simplicity, we use $\delta(\mathcal{E}_1)$, for example, rather than $\delta(\mathcal{E}_1(G))$.

2. Equitable Graph Type I

The definition of the first type of an equitable graph from any finite group is introduced in this section. Later, we explore some theoretical properties of this graph from certain groups.

Definition 1. *Let G be a finite group. The equitable graph of Type I of G, denoted by $\mathcal{E}_1(G)$, is a graph with vertex set G in which any two distinct elements of G, x and y are adjacent if and only if*

$$| o(x) - o(y) | \leq \min\{o(x), o(y)\}.$$

Example 1. *Consider the special linear group $G = SL(2,3)$ that is the group of 2×2 matrices with determinant 1 over the field of three elements. Then, the list of the elements is as follows:*

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, v_4 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, v_6 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, v_7 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, v_8 = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$$v_9 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, v_{10} = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, v_{11} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, v_{12} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix},$$

$$v_{13} = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, v_{14} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, v_{15} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, v_{16} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

$$v_{17} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, v_{18} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, v_{19} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, v_{20} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$
$$v_{21} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, v_{22} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, v_{23} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, v_{24} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix},$$

where v_1 has order 1, v_2 has order 2, v_3 to v_{10} have order 3, v_{11} to v_{18} have order 6, and v_{19} to v_{24} have order 4. Then $\mathcal{E}_1(G)$ is depicted in Figure 1.

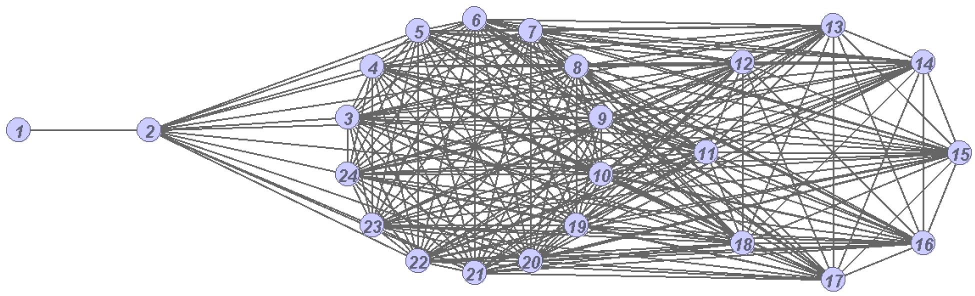

Figure 1. The equitable graph Type I of the group $SL(2,3)$.

- $\delta(\mathcal{E}_1(G)) = 1$, $\Delta(\mathcal{E}_1(G)) = 22$.
- $\chi(\mathcal{E}_1(G)) = \omega(\mathcal{E}_1(G)) = 22$.
- $gr(\mathcal{E}_1(G)) = 3$.
- $\gamma(\mathcal{E}_1(G)) = 2$.
- $\text{diam}(\mathcal{E}_1(G)) = 3$.
- $m(\mathcal{E}_1(G)) = 246$.

Lemma 1. $gr(\mathcal{E}_1(G)) = 3$ *for any finite group G with order greater than 3.*

Proof. Let G be a finite group of order n. Then, the result is clear for $n = 1$ or 2, and the only group of order 3 is a cyclic group in which the identity is isolated. Now, assume that $n = 4$; then there are only two possible cases for the group G. Either G is cyclic or G is isomorphic to the Klein four group $V_4 = \langle a, b : a^2 = b^2 = e, ab = ba \rangle$. In the first case, the element of order two is adjacent to the two elements of order four in $\mathcal{E}_1(G)$, forming a cycle with three edges. In the latter case, the graph is complete.

Now, if $n > 4$, it is clear that there exist at least three elements sharing the same order. Hence, $\mathcal{E}_1(G)$ contains K_3 as a subgraph. $\square$

The following lemma has been utilized in numerous proofs throughout this research; therefore, it is prudent to mention it here.

Lemma 2. *Let i be a positive integer. Then*

1. $2^{i-2} + 2^{i-1} + 2^i = (7)2^{i-2}$.

2. $2^{i-2} + 2^{i-1} = (3)2^{i-2}$.

Theorem 1. *Let G be a cyclic group of order 2^k; $k > 1$ is a positive integer. Then $\mathcal{E}_1(G)$ is connected.*

Proof. As G is a cyclic group, the orders of the elements are the divisors of $|G|$. Now, as is well known, $| 2^i - 2^{i+1} |= 2^i$ for all $0 \le i \le k-1$. Therefore, each element of order 2^i is adjacent to all elements of order 2^{i+1} (as vertices) for all $0 \le i \le k-1$. Thus, we conclude that there is a path between any two vertices, and the graph $\mathcal{E}_1(G)$ can be shown as in Figure 2 such that each circle forms a complete subgraph.

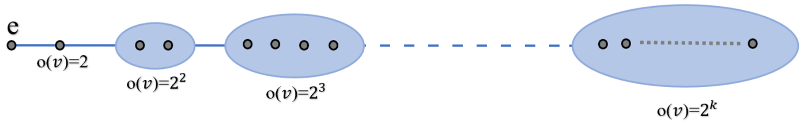

Figure 2. The equitable graph Type I of cyclic groups of order 2^k.
□

Theorem 2. *Let G be a cyclic group of order 2^k; $k > 1$ and is a positive integer. Then $\mathcal{E}_1(G)$ has the following properties:*

1. $\delta(\mathcal{E}_1) = 1$, *and* $\Delta(\mathcal{E}_1) = (7)2^{k-3} - 1$ *unless* $k = 2$, *in which case* $\Delta(\mathcal{E}_1) = 3$.
2. $\omega(\mathcal{E}_1) = (3)2^{k-2}$.
3. $\mathrm{diam}(\mathcal{E}_1) = k$.
4. $\mathcal{E}_1(G)$ *is a weakly perfect graph.*
5. $\gamma(\mathcal{E}_1) = \left\lceil \frac{k+1}{3} \right\rceil$.
6. $r(\mathcal{E}_1) = \begin{cases} \frac{k}{2}, & k \equiv 0(mod2); \\ \frac{k+1}{2}, & otherwise. \end{cases}$
7. $m(\mathcal{E}_1) = 1 + \sum_{i=1}^{k-1} 2^{i-1}(2^{i+1} - 1)$.

Proof. Let G be a cyclic group of order 2^k, where $k > 1$ is a positive integer.

1. In this case, the minimum degree and the maximum degree for $k = 2$ are obvious. Now, for $k > 2$, each element of order 2^i is adjacent to each element of order 2^{i-1} and 2^{i+1} for all $1 \le i \le k-1$, and since the number of elements of order 2^m is $\phi(2^m) = 2^{m-1}$ as G is cyclic, for all $1 \le m \le k$, we obtain the result.
2. According to the fact that $\phi(2) = 1 < \phi(2^2) < \ldots < \phi(2^k)$ and from the adjacency criteria, the result can be obtained using Lemma 2.
3. This follows from Theorem 1 and the adjacency method of the vertices.
4. Since for any graph Γ, obviously $\chi(\Gamma) \ge \omega(\Gamma)$, we obtain that $\chi(\mathcal{E}_1) \geqslant 3(2^{k-2})$. Then, according to the adjacency order, we can reuse these colors, and hence, $\chi(\mathcal{E}_1) \leqslant 3(2^{k-2})$. Therefore, the equality holds.
5. Through the adjacency method and by Figure 2, we deduce that for each of three consecutive cliques, one vertex of the middle one can be in a dominating set. So the cardinality of $\gamma - set \geqslant \frac{k+1}{3}$, and thus, from the definition of the domination number and the number of sequential cliques, we can obtain $\gamma(\mathcal{E}_1) \leqslant \frac{k+3}{3}$.

6. From Figure 2, we obtain that the eccentricity of the vertices ranges between k and $\frac{k}{2}$ (or $\left\lceil\frac{k}{2}\right\rceil$) if k is even (or odd). Therefore, $r(\mathcal{E}_1) = \left\lceil\frac{k}{2}\right\rceil$.
7. This follows from the adjacency method and the fact that the elements of the same order form a complete subgraph. Hence,

$$m(\mathcal{E}_1) = 1 + \sum_{i=2}^{k} \left[\frac{\phi(2^i)(\phi(2^i)-1)}{2} + \phi(2^{i-1})(\phi(2^i))\right]$$

□

Let n be a positive integer. Then the dihedral group of order $2n$ is defined as follows

$$D_{2n} = \langle a, b : a^n = b^2 = e, ab = ba^{-1}\rangle.$$

Example 2. *Consider the dihedral group of order 8, D_8. Then this group has one element of order 1, five elements of order 2, and two elements of order 4. Therefore, the equitable graph of D_8 is shown as in Figure 3, where v_1 denote the identity, $v_2 = a^2$, $v_3 = b$, $v_4 = ab$, $v_5 = a^2b$, $v_6 = a^3b$, $v_7 = a$, and $v_8 = a^3$.*

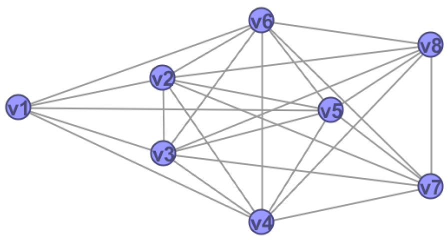

Figure 3. The equitable graph Type I of D_8.

Through the next two results, we explore the theoretical properties of the equitable graph of this group for special cases of n.

Theorem 3. *Consider the dihedral group $G \cong D_{2n}$; $n = 2^k$, $k > 1$. Then*

1. *$\mathcal{E}_1(G)$ is connected.*
2. *$\mathrm{diam}(\mathcal{E}_1(G)) = k$, and $\gamma(\mathcal{E}_1(G)) = \left\lceil\frac{k+1}{3}\right\rceil$.*
3. *$\chi(\mathcal{E}_1(G)) = \omega(\mathcal{E}_1(G)) = 2^k + 3$.*

Proof. Let $G \cong D_{2n}$; $n = 2^k$, $k > 1$. Then

1. The connectedness of this graph is satisfied since the order of the elements of D_{2n} in this case are clearly 2^i for each $1 \leq i \leq k$, which is the same as the cyclic group of order 2^k.
2. From the previous point, we obtain that the equitable graph Type I of this group and any cyclic group of order 2^k share the same *diameter and domination number*. Then by Theorem 2 we obtain the result.
3. The number of elements of order 2 in D_{2n} is equal to $n+1$, and for the remaining divisors of n, there are $\phi(2^m)$ elements for all $1 < m \leq k$. Hence, clearly, the maximum clique consists of the elements of order 2 in addition to the elements of order 2^2 by the connectedness. Therefore, we obtain the outcome.

□

Proposition 1. *Let G be the dihedral group D_{2n}; $n = 2^k$, $k > 1$. Then*

1. $\delta(\mathcal{E}_1(G)) = \begin{cases} 5, & k = 2 \text{ or } 3; \\ 11, & k = 4; \\ 13, & k \geq 5. \end{cases}$
2. $\Delta(\mathcal{E}_1(G)) = \begin{cases} 2^k + 6, & k \geq 3; \\ 7, & k = 2. \end{cases}$

Proof. Let G be the dihedral group D_{2n}; $n = 2^k$, $k > 1$. Then

1. From the adjacency method and according to the number of elements in each order in G, we attain the solution for $k = 2,\ 3$ *or* 4. Now, for all $k \geq 5$, we have that the degree of any element of order 2^3 is 13, which is the minimum among all others, and hence, we are done.
2. For the first case, since the elements of order 2^2 are adjacent to all elements of order 2, which include the maximum number of the elements, we obtain that

$$\Delta(\mathcal{E}_1(G)) = 2^k + 1 + \phi(2^2) - 1 + \phi(2^3) = 2^k + 6.$$

Now, when $k = 2$, let $v_{(j)}$ denote a vertex of order j. Then $d(v_{(1)}) = 5$, $d(v_{(2)}) = 7$, and $d(v_{(2^2)}) = 6$. Hence, we can conclude the result.

□

Theorem 4. *Let G be a cyclic group of order p^k, where $p > 2$ is a prime number and $k > 1$. Then $\mathcal{E}_1(G)$ is disconnected.*

Proof. Let G be a cyclic group of order p^k, where $p > 2$ is a prime number and $k > 1$. Then the graph $\mathcal{E}_1(G)$ is as shown in Figure 4.

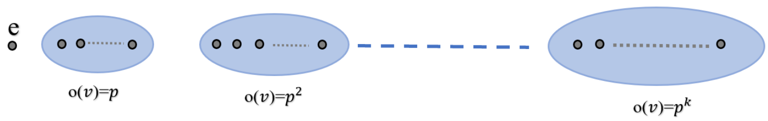

Figure 4. The equitable graph Type I of cyclic groups of order p^k.

Thus, for any $1 \leq i \leq k$, we have $|p^i - p^{i-1}| > \min\{p^i, p^{i-1}\}$. Hence, all elements of order p^i cannot be adjacent to any element of a different order. Therefore, the graph consists of disconnected cliques. □

Theorem 5. *Consider the cyclic group G of order p^k; $p > 2$ is a prime number, and $k > 1$. Then $\mathcal{E}_1(G)$ has the following properties*

1. $\delta(\mathcal{E}_1) = 0,\ \Delta(\mathcal{E}_1) = p^k - p^{k-1} - 1$.
2. *There are $k + 1$ components.*
3. $\gamma(\mathcal{E}_1) = k + 1$.
4. $\omega(\mathcal{E}_1) = p^{k-1}(p - 1)$.
5. $\chi(\mathcal{E}_1) = p^{k-1}(p - 1)$.
6. $m(\mathcal{E}_1) = \sum_{i=1}^{k} \frac{(p^i - p^{i-1})[p^i - p^{i-1} - 1]}{2}$.

Proof. Let G be a cyclic group of order p^k; $p > 2$ is a prime number, and $k > 1$. Then

1. The result is clear for the minimum degree. Now, for the maximum degree, the result follows as each element of the same order forms a complete subgraph and since $\phi(p) < \phi(p^2) < \ldots < \phi(p^k)$. Thus, $\Delta(\mathcal{E}_1)$ is equal to the degree of any element of order p^k.
2. Since the elements of the same order form a clique and by Theorem 4, we obtain that the number of the components is the number of the divisors of $|G|$.

3. According to Theorem 4 and the method of the adjacency, one vertex from each clique can be in the dominating set that includes the identity. Thus, the cardinality of the dominating set is at most $k+1$. Therefore, the dominating set, say S, contains the identity and one vertex of order p^i for all $1 \leq i \leq k$, and hence, $|S| = k+1$.
4. The result is direct as the number of vertices in each clique equals $\phi(p^i)$ for all $1 \leq i \leq k$.
5. By the previous point, we obtain that at least $p^{k-1}(p-1)$ colors are needed to label the vertices. Since the components are disjoint, these colors can be reused. Hence, $\chi(\mathcal{E}_1) = \omega(\mathcal{E}_1)$.
6. The result can be obtained through the adjacency method and from the fact that all of the elements of order p^i form a complete subgraph for all $1 \leq i \leq k$.

□

Theorem 6. *Let G be a cyclic group of order $2^k.q$; $q > 2$ is a prime number, and $k > 1$, such that $|2^i - q| \leq \min\{2^i, q\}$ for some $1 \leq i \leq k$. Then $\mathcal{E}_1(G)$ is connected.*

Proof. It is known that the divisors of n consist of $1,\ 2,\ 2^2,\ \ldots,\ 2^k,\ q,\ 2q,\ 2^2q,\ \ldots,\ 2^kq$. Then by Theorem 1, we obtain that the vertices of orders $1, 2, \ldots, 2^k$ are connected. Consequently, $|2^jq - 2^{j-1}q| = 2^{j-1}q = \min\{2^jq, 2^{j-1}q\}$ for all $1 \leq j \leq k$, and this is achieved by the connectedness of the vertices of orders q, 2q, ..., 2^kq. Therefore, by the condition $|2^i - q| \leq \min\{2^i, q\}$ for some $1 \leq i \leq k$, the connectedness of this graph holds. □

Proposition 2. *Let G be a cyclic group of order 2^kq; $q > 2$ is a prime number, and $k > 1$. Then $\mathcal{E}_1(G)$ has the following properties*

1. $\delta(\mathcal{E}_1) = 1$.
2. $\chi(\mathcal{E}_1) = \omega(\mathcal{E}_1) = \phi(n) + \phi(\frac{n}{2})$.
3. $\Delta(\mathcal{E}_1) = \begin{cases} \phi(n) + \phi(\frac{n}{2}) + \phi(\frac{n}{3}) + \phi(\frac{n}{4}) - 1, & if\ q = 3; \\ \phi(n) + \phi(\frac{n}{2}) + \phi(\frac{n}{4}) - 1, & if\ q > 3. \end{cases}$

Proof. Let G be a cyclic group of order 2^kq; $q > 2$ is a prime number, and $k > 1$. Then for the first point, the proof is followed, since $deg(e) = 1$, which is the minimum among all vertices. For (2), as the orders n and $\frac{n}{2}$ involve the largest number of elements, and since $|2^kq - 2^{k-1}q| \leq \min\{2^kq, 2^{k-1}q\}$, $|2^kq - 2^k| > \min\{2^kq, 2^k\}$, and $|2^kq - 2^{k-2}q| > \min\{2^kq, 2^{k-2}q\}$, we obtain that the vertices of orders n and $\frac{n}{2}$ form the maximum clique. It is clear that $\chi(\mathcal{E}_1) \geq \omega(\mathcal{E}_1)$. But from the relations above, we deduce that the colors of the vertices of order n can be reused. Thus, $\chi(\mathcal{E}_1) \leq \omega(\mathcal{E}_1)$.

The maximum degree of this graph is the degree of a vertex of order $\frac{n}{2}$; this follows from the previous points and according to the adjacency method. Now, if $q > 3$, we have $|2 - q| > \min\{2, q\}$. Consequently, $|2^k - 2^{k-1}q| > \min\{2^k, 2^{k-1}q\}$. Hence, by the arrangement of the order as in Theorem 6, we obtain the result. Otherwise, if $q = 3$, since $|3 - 2| \leq \min\{3, 2\}$, we obtain that $|2^{k-1}3 - 2^k| \leq \min\{2^{k-1}3, 2^k\}$. Also, as $|3 - 1| > \min\{3, 1\}$, then $|2^{k-1}3 - 2^{k-1}| > \min\{2^{k-1}3, 2^{k-1}\}$. Then

$$\Delta(\mathcal{E}_1) = \phi(2^k3) + \phi(2^{k-1}3) + \phi(2^k) + \phi(2^{k-2}3) - 1. \quad (1)$$

□

Theorem 7. *Let G be a cyclic group of order 2^kq; $q > 2$ is a prime number, and $k > 1$. Then $\mathcal{E}_1(G)$ has the following properties:*

1. *If $\mathcal{E}_1(G)$ is connected, then*
   (a) $\begin{cases} \gamma(\mathcal{E}_1) = \lfloor \frac{t-4}{5} \rfloor + 2, & if\ q = 3; \\ \lfloor \frac{t-4}{5} \rfloor + 2 \leq \gamma(\mathcal{E}_1) \leq \lceil \frac{t}{3} \rceil, & if\ q > 3. \end{cases}$

(b) $\operatorname{diam}(\mathcal{E}_1) = (t-1) - (k-i)$.

where i is a positive integer such that $2^i \leq q \leq 2^{i+1}$*, and t denotes the number of divisors of n, which in this case is equal to* $2(k+1)$.

2. *If* $\mathcal{E}_1(G)$ *is disconnected, then* $\gamma(\mathcal{E}_1) = 2\lceil \frac{k+1}{3} \rceil$.

Proof. Let G be a cyclic group of order $2^k.q$; $q > 2$ is a prime number, and $k > 1$. Then consider the connected case of the graph, and let $q = 3$; then the divisors of n will be in the following order:

$$1, 2, 3, 2^2, 2.3, 2^3, 2^2 3, 2^4, 2^3 3, \ldots, 2^k, 2^{k-1}3, 2^k 3.$$

Let $v_{(j)}$ denote a single vertex corresponding to an element of order j. Now, as the vertex that is associated with the element of order 2, say $v_{(2)}$, is adjacent to the identity and all vertices that are associated with elements of order 3 and 4, thus $v_{(2)}$ belongs to the dominating set S. For the remaining $t-4$ divisors, we have the following relation:

It is clear that any vertex associated with an element of order 2^i, say $v_{(2^i)}$, is adjacent to all symmetrical vertices and all $v_{(2^{i-1})}$ and $v_{(2^{i+1})}$ for all $2 \leq i \leq k-1$. This implies that any vertex $v_{(2^i 3)}$ is adjacent to all vertices of orders $2^{i+1}3$ and $2^{i-1}3$.

Also, since $|2-3| \leq \min\{2,3\}$, we have $|2^i - 2^{i-1}3| \leq \min\{2^i, 2^{i-1}3\}$. And as $|2^2 - 3| \leq \min\{2^2, 3\}$, we obtain $|2^i - 2^{i-2}3| \leq \min\{2^i, 2^{i-2}3\}$ for all $2 \leq i \leq k-2$. Therefore, $v_{(2^i 3)}$ is adjacent to all symmetrical vertices and all vertices $v_{(2^{i-1}3)}, v_{(2^{i+1}3)}, v_{(2^{i+1})}$ and $v_{(2^{i+2})}$. Also, each vertex $v_{(2^i)}$ is adjacent to all symmetrical vertices and all vertices $v_{(2^{i-1})}, v_{(2^{i+1})}, v_{(2^{i-2}3)}$ and $v_{(2^{i-1}3)}$. Thus, according to the order of elements mentioned at the beginning of the proof, we find that the dominating set contains $v_{(2^2 3)}$, $v_{(2^6)}$, $v_{(2^7 3)}$, $v_{(2^{11})}$, ..., $v_{(2^{k-1})}$ or $v_{(2^k)}$; so for every five consecutive divisors, one vertex can be in S, and so $\gamma(\mathcal{E}_1) > \lfloor \frac{t-4}{5} \rfloor + 1$. But since $|2^k 3 - 2^k| > \min\{2^k 3, 2^k\}$, one vertex of $v_{(n)}$ or $v_{(\frac{n}{2})}$ must be in S. Hence, we conclude the result. On the other hand, concerning the case of $q > 3$, $|2-q| > \min\{2, q\}$, where the minimum value for this occurs at q = 3, so the graph in this case is more interconnected based on the relationships mentioned previously. So $\gamma(\mathcal{E}_1) > \lfloor \frac{t-4}{5} \rfloor + 2$. But the equality is possible given that numerous examples achieve it. For instance, if $n = 2^4 5$, then, according to the order of the divisors, which is as follows:

$$1, 2, 2^2, 5, 2^3, 2(5), 2^4, 2^2(5), 2^3(5), 2^4(5),$$

we obtain that the minimum dominating set contains the vertices $v_{(2)}, v_{(2(5))}$ and $v_{(2^4(5))}$, where $v_{(j)}$ denotes a single vertex associated with an element of order j. Hence, $\gamma(\mathcal{E}_1) = 3 = \lfloor \frac{10-4}{5} \rfloor + 2$, and this yields the desired result. Also, it is clear that $\gamma(\mathcal{E}_1)$ cannot be more than $\lceil \frac{t}{3} \rceil$, whereas the occurrence of the maximum probability arises when for every set of three consecutive divisors (orders), a singular vertex having an order equal to the middle divisor is included in the dominating set.

The diameter of $\mathcal{E}_1(G)$ in this case is clearly equal to the distance between the identity and an element of order $2^k q$ (as each divisor of n corresponds to a clique in $\mathcal{E}_1(G)$). Thus, if $q > 2^k$, then $\operatorname{diam}(\mathcal{E}_1) = t - 1$. Otherwise, if $2^i < q < 2^{i+1}$ for some $1 \leq i \leq k-1$, then

$$\begin{aligned}
&2^{i+1} < 2q < 2^{i+2} \\
&2^{i+2} < 2^2 q < 2^{i+3} \\
&\vdots \\
&2^{k-1} < 2^{(k-1)-i} q < 2^k.
\end{aligned}$$

Then, the path, say P, that joined the identity with $v_{(2^k q)}$ will be as follows: $v_{(1)} - v_{(2)} - \ldots - v_{(2^i)} - v_{(2^{i+1})} - \ldots - v_{(2^k)} - v_{(2^{k-i})q} - \ldots - v_{(2^{k-1}q)} - v_{(2^k q)}$.

Now, as has been shown, all the vertices corresponding to the elements of orders $q, 2q, \ldots,$ and $2^{(k-1)-i}q$ have been excluded from P. This reduces the length by about $(k-1) - i + 1 = k - i$. Therefore, the result is obtained.

Finally, when the graph $\mathcal{E}_1(G)$ is disconnected, meaning $|2^k - q| > \min\{2^k, q\}$ and $q > 2^k$, there are two components by Theorem 6, and each component consists of $k+1$ cliques that are joined successively as in Theorem 1. Therefore, in this case, the domination number is twice the value of the domination number in Theorem 2. □

Theorem 8. *Let S_n and A_n be the symmetric and alternating groups, respectively, on a set of n elements. Then*

1. *$\mathcal{E}_1(S_n)$, where $n \geq 2$, is connected.*
2. *$\mathcal{E}_1(A_n)$ is connected for all $n > 3$.*

Proof. The proof is straightforward due to the nature of the orders of the elements in these groups. □

The following theorems have been referenced for their applications in verifying the *Eulerian* and planar properties of this graph.

Theorem 9 ([14] (Theorem 6.2.2))**.** *For nontrivial connected graph Γ, the following statements are equivalent:*

1. *Γ is Eulerian.*
2. *The degree of each vertex of Γ is an even positive integer.*
3. *Γ is an edge-disjoint union of cycles.*

Theorem 10 ([14] (Theorem 8.4.1))**.** *K_5 is nonplanar.*

Theorem 11. *Let G be a cyclic group of order n; n is a positive integer. Then*

1. *$\mathcal{E}_1(G)$ is not Eulerian for all $n \geq 3$.*
2. *$\mathcal{E}_1(G)$ is not Hamiltonian for all $n \geq 2$.*

Proof. Let G be a cyclic group of order n; n is a positive integer. Then the proof of the first point follows from Theorem 9 since whenever the graph $\mathcal{E}_1(G)$ is connected, the degree of the identity vertex is equal to one, which is an odd integer. Now, for the second point, according to the definition of the graph and since there is exactly one element of order 2 in this group, there is only one edge that is incident to the identity. Hence, it is impossible to have any Hamiltonian cycle in $\mathcal{E}_1(G)$. □

Theorem 12. *Let G be a cyclic group of order n; n is a positive integer. Then $\mathcal{E}_1(G)$ is planar for all $n \leq 6$ and nonplanar otherwise.*

Proof. Let G be a cyclic group of order n; n is a positive integer. Then for each $n > 6$, the graph $\mathcal{E}_1(G)$ contains an induced subgraph K_5, and this implies the nonplanarity of the graph. On the other hand, the proof is obvious for $n = 1, 2, 3$ and 4. Also, if $n = 5$, then $\mathcal{E}_1(G)$ consists of an isolated vertex, which is the identity, and the complete graph k_4, and hence, it is planar. Finally, the planarity of the graph when $n = 6$ is shown in Figure 5.

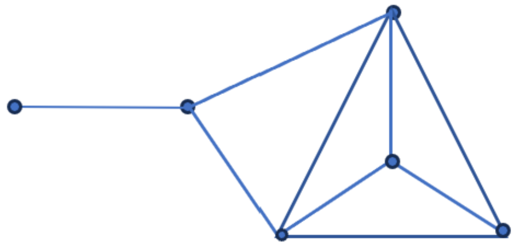

Figure 5. The plane embedding of a cyclic group of order 6.

□

3. Equitable Square-Free Number

This section endeavors to establish the conceptual frameworks of the equitable square-free number and the equitable group. Furthermore, it encompasses a comprehensive study of the connectedness properties inherent to the equitable graph Type I associated with such a group, and we analyze its characteristics in detail.

Definition 2. *Let $p_1 < p_2 < \cdots < p_k$ be distinct prime numbers. The square-free number $n = \prod_{i=1}^{k} p_i$ is called an equitable square-free number if and only if $p_{i+1} - p_i \leq p_i$ for all $i = 1, 2, \cdots, k-1$.*

Theorem 13. *Let n be an equitable square-free number and consider the cyclic group G of order n. Then*

1. *For $p_1 = 2$, $\mathcal{E}_1(G)$ is connected.*
2. *For $p_1 > 2$, $\mathcal{E}_1(G)$ is disconnected.*

Proof. Let G be a cyclic group of order n, where n is an equitable square-free number. Then the divisors of n will be arranged, in general, as follows:

$$1\,,\, p_1\,,\, p_2\,,\, \cdots\,,\, p_k\,,\, p_1p_2\,,\, p_1p_3\,,\, \cdots\,,\, p_1p_k\,,\, p_2p_3\,,\, \cdots\cdots\,,\, p_{k-1}p_k\,,\, p_1p_2p_3\,,\, p_1p_2p_4\,,\, \cdots\cdots\,,$$
$$p_{k-2}p_{k-1}p_k\,,\, p_1p_2p_3p_4\,,\, \cdots\cdots\,,\, p_1p_2\ldots p_{k-1}\,,\, \cdots\,,\, p_2p_3\ldots p_k\,,\, p_1p_2\ldots p_k = n$$

Since the order of the elements is the divisor of n, we first need to prove that any vertices that have an order equal to the product of the same number of primes form a component; that is, any two vertices of orders with the same number of primes have a path between them. This is clear for order 1 since there is only one vertex that has this order, which is the identity. Also, the vertices with order n clearly form a component.

Now we will prove this for the remaining divisors by using the mathematical induction on the number of primes in the prime factorization of the divisors, say m. The proof is clear for $m = 1$, $n = p_i$; $1 \leq i \leq k$ according to the choice of n.

The base case of $m = 2$:

Let $d_1 = p_ip_j$ and $d_2 = p_tp_s$ be any two divisors such that $j < i$, $t < s$, and $j \leq t$. By the definition of n, we have

$$|p_{i+1} - p_i| \leq \min\{p_{i+1}\,,\, p_i\}; \text{for all } i = 1, 2, \ldots, k-1.$$

Then

$$|p_jp_{i+1} - p_jp_i| \leq \min\{p_jp_{i+1}\,,\, p_jp_i\} \tag{2}$$

So if $t = j$, then this forms a path between the vertices of order d_1 and d_2. If $j < t$, then we have the following:

By inequality (2), we can find a path from the vertices of order p_jp_i to the vertices of order p_jp_k. Hence, from the ordering of the divisors, we obtain that:

$$\begin{aligned} p_{j+1}p_{j+2} - p_jp_k &= p_{j+1}p_{j+2} - p_jp_{j+2} + p_jp_{j+2} - p_jp_k && (3) \\ &\leq p_jp_{j+2} + p_jp_{j+2} - p_jp_k && (4) \\ &= p_j(2p_{j+2} - p_k) \leq p_jp_k && (5) \end{aligned}$$

Then

$$|p_{j+1}p_{j+2} - p_jp_k| \leq \min\{p_{j+1}p_{j+2}, p_jp_k\} \tag{6}$$

This forms an edge between the vertices of these orders. Then by using the same fact as in inequality (2), we obtain that there is a path from the vertices of order p_jp_i to the vertices of order $p_{j+1}p_k$. Continuing the process in the inequalities (2) and (6), we can find a path between vertices of orders d_1 and d_2. Therefore, for all $\alpha \in S_k$ such that $\alpha \neq e$ and $\alpha(j) < \alpha(i)$, there is a path from any vertex of order p_jp_i to any vertex of order $p_{\alpha(j)}p_{\alpha(i)}$, where $1 \leq j < i \leq k$.

The inductive hypothesis: Assume that this is true for all $m < k-1$. That is, for all $\alpha \in S_k$ such that $\alpha \neq 1$ and $\alpha(i_1) < \alpha(i_2) < \cdots < \alpha(i_m)$, there is a path between any two vertices of orders $\prod_{j=1}^{m} p_{i_j}$ and $\prod_{j=1}^{m} p_{\alpha(i_j)}$.

The inductive proof: Let $m = k-1$, and $d_i = \prod_{t=1}^{k-1} p_{i_t}$ and $d_j = \prod_{s=1}^{k-1} p_{j_s}$ are any two divisors such that $p_{i_t} < p_{i_{t+1}}$ and $p_{j_s} < p_{j_{s+1}}$ for all $1 \leq t, s \leq k-2$. By the inductive hypothesis, we have that there is a path between the vertices of orders $d'_i = \prod_{t=1}^{k-2} p_{i_t}$ and $d'_j = \prod_{s=1}^{k-2} p_{j_s}$. Now if $p_{i_{k-1}} = p_{j_{k-1}}$, we are done. So without loss of generality, let $p_{i_{k-1}} < p_{j_{k-1}}$. Then, similarly to the base case, we can find a path between the vertices of orders $d_i = d'_i p_{i_{k-1}}$ and $d_j = d'_j p_{j_{k-1}}$. Hence, for all $\alpha \in S_k$ such that $\alpha \neq e$ and $\alpha(i_1) < \alpha(i_2) < \cdots < \alpha(i_{k-1})$, there is a path from the vertices of order $\prod_{t=1}^{k-1} p_{i_t}$ to the vertices of order $\prod_{t=1}^{k-1} p_{\alpha(i_t)}$.

Now, assume that $p_1 = 2$. Then by the first part, we need to check the connectedness between the components, and this is clear from the fact that for any integer m,

$$|2m - m| = m = \min\{2m, m\}.$$

Thus, for any divisor $d = p_{i_1} p_{i_2} \ldots p_{i_t}$, where $2 \leq t \leq k-1$ and $2 < p_{i_1} < p_{i_2} < \cdots < p_{i_t}$, we have

$$|2d - d| = d \leq \min\{2d, d\} \tag{7}$$

Therefore, there is a path from any element of order $\prod_{t=1}^{m} p_{i_t}$ to any element of order $\prod_{j=1}^{m+1} p_{i_j}$, where $1 \leq m \leq k-1$. Therefore, there is a path between any two vertices in $\mathcal{E}_1(G)$. Otherwise, if $p_1 > 2$, since $|p_i - 1| > \min\{p_i, 1\}$ for all $i = 1, 2, \ldots, k$, we have that for any divisor d of $|G|$,

$$|d - 1| > \min\{d, 1\} \tag{8}$$

Then, there is no edge between the identity and any other vertex in the graph. Hence, the identity is an isolated vertex. □

For the disconnected case delineated in Theorem 13, the subsequent theorem examines the cardinality of its constituent components.

Theorem 14. *Let G be a cyclic group of order $n = \prod_{i=1}^{k} p_i$, where n is an equitable square-free number, and consider that $p_1 > 2$. Then*

1. *$\mathcal{E}_1(G)$ has 3 or 4 components for $k = 2$ or 3, respectively.*
2. *For $k > 3$, we have*
   - *If $|p_1 p_2 - p_t| \leq \min\{p_1 p_2, p_t\}$ for some $3 < t \leq k$. Then $\mathcal{E}_1(G)$ has 3 components.*
   - *If $|p_1 p_2 - p_i| > \min\{p_1 p_2, p_i\}$ for all $3 < i \leq k$, then the number of the components in $\mathcal{E}_1(G)$ will be as follows:*

$$\begin{cases} 5, & |p_1 p_2 p_3 - p_t p_l| \leq \{p_1 p_2 p_3, p_t p_l\} \ for\ some\ 1 \leq t < l \leq k; \\ k+1, & |p_1 p_2 p_3 - p_i p_j| > \{p_1 p_2 p_3, p_i p_j\} \ for\ all\ 1 \leq i < j \leq k. \end{cases}$$

Proof. Let G be a cyclic group of order $n = \prod_{i=1}^{k} p_i$, where n is an equitable square-free number. Now as $p_1 > 2$, we have $|p_i - 1| > \min\{p_i, 1\}$. Then

$$|n - \prod_{r=1}^{k-1} p_{i_r}| > \prod_{r=1}^{k-1} p_{i_r} = \min\{n, \prod_{r=1}^{k-1} p_{i_r}\} \tag{9}$$

Thus, there is no edge between the elements of order $\prod_{r=1}^{k-1} p_{i_r}$ and the elements of order equal to n. These two components are depicted in Figure 6.

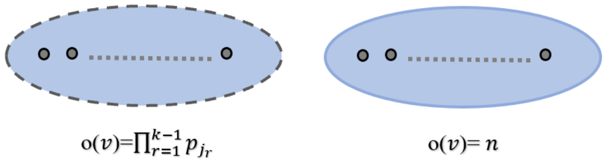

Figure 6. The disconnecting of the last components in $\mathcal{E}_1(G)$.

In the figure, the dotted line circle represents a connected (not complete) subgraph, and $o(v)$ denotes the order of the element v in the group G. Hence, when $k = 2$, $\mathcal{E}_1(G)$ obviously has three components, as is shown in Figure 7.

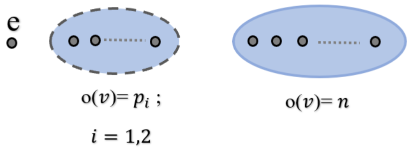

Figure 7. The equitable graph Type I of G with $k = 2$.

Let $k = 3$. Then according to the choice of the prime numbers, we have $|p_1p_2 - p_3| > \min\{p_1p_2, p_3\}$. So there is no edge between any element of order p_i and any element of order p_rp_s. Thus, the graph has 4 components, as shown in Figure 8.

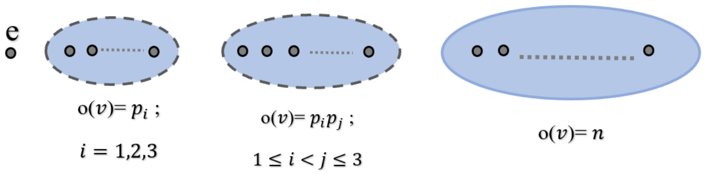

Figure 8. The equitable graph Type I of G with $k = 3$.

Now let $k > 3$ and assume that $|p_1p_2 - p_t| \leq \min\{p_1p_2, p_t\}$ for some $3 < t \leq k$. This implies that there is a path between any two elements of order p_i and p_rp_s for all $1 \leq i \leq k$ and $1 \leq r < s \leq k$, respectively. Also, by this assumption, we obtain that

$$|p_1p_2 \dots p_{t-1}p_{t+1} \dots p_k - p_3 \dots p_{t-1}p_tp_{t+1} \dots p_k| \leq \min\{p_1p_2 \dots p_{t-1}p_{t+1} \dots p_k, p_3 \dots p_{t-1}p_tp_{t+1} \dots p_k\} \quad (10)$$

Hence, there is a path between all elements of order $\prod_{r=1}^{k-1} p_{i_r}$ and $\prod_{s=1}^{k-2} p_{j_s}$. Also, by choosing any $2 < i \leq k$ such that $i \neq t$, we obtain that

$$|p_1p_2p_i - p_tp_i| \leq \min\{p_1p_2p_i, p_tp_i\} \quad (11)$$

And this forms a path between the elements of order $\prod_{r=1}^{3} p_{i_r}$ and $\prod_{s=1}^{2} p_{j_s}$. Continuing this process, we obtain that there is a path between any two elements of order $\prod_{r=1}^{m} p_{i_r}$ and $\prod_{s=1}^{m-1} p_{j_s}$ for all $3 \leq m \leq k-2$. Therefore, there is a path between any two elements of order $\prod_{r=1}^{m} p_{i_r}$ and $\prod_{s=1}^{t} p_{j_s}$, where $1 \leq m, t \leq k-1$, and hence, these vertices form a component. Thus, the graph in this case is expressed as in Figure 9.

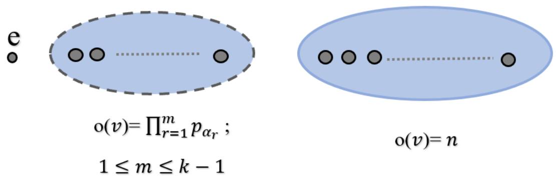

Figure 9. $\mathcal{E}_1(G)$ with 3 components, where $k > 3$.

On the other hand, if $|p_1p_2 - p_i| > \min\{p_1p_2, p_i\}$ for all $3 < i \leq k$. By the increasing of the primes, we have that $p_1p_2 < p_ip_j$ for all $1 \leq i \leq k-1, 2 < j \leq k$ and $i < j$. Then

$$|p_ip_j - p_r| > \min\{p_ip_j, p_r\}; \forall 1 \leq r \leq k \; and \; 1 \leq i < j \leq k \tag{12}$$

Therefore, there is no path between any two elements of order p_r and p_ip_j. Hence, the two components C_1 and C_2 are disjoint, where C_m denotes the components that consist of all elements of order $\prod_{j=1}^{m} p_{i_j}$ for all $1 \leq m \leq k$. Consequently, we have

$$|\prod_{r=1}^{k-1} p_{i_r} - \prod_{s=1}^{k-2} p_{j_s}| > \min\{\prod_{r=1}^{k-1} p_{i_r}, \prod_{s=1}^{k-2} p_{j_s}\} \tag{13}$$

Thus, the disjoint components are depicted in Figure 10.

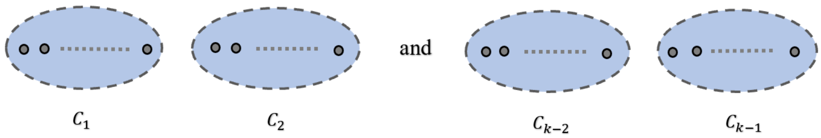

Figure 10. The disconnected components in $\mathcal{E}_1(G)$ with 5 components.

Now consider the case $|p_1p_2p_3 - p_tp_l| \leq \min\{p_1p_2p_3, p_tp_l\}$ for some $1 \leq t < l \leq k$. Hence, there is a path from any element of order p_ip_j to any element of order $\prod_{r=1}^{3} p_{i_r}$. Then, by choosing any $c \notin \{1,2,3,t,l\}$, we obtain that $|p_1p_2p_3p_c - p_tp_lp_c| \leq \min\{p_1p_2p_3p_c, p_tp_lp_c\}$. Thus, this forms a path from any element of order $\prod_{r=1}^{3} p_{i_r}$ to any element of order $\prod_{s=1}^{4} p_{j_s}$. Sustaining this procedure, we obtain that

$$|\prod_{r=1}^{m} p_{i_r} - \prod_{s=1}^{m-1} p_{j_s}| \leq \min\{\prod_{r=1}^{m} p_{i_r}, \prod_{s=1}^{m-1} p_{j_s}\}; \forall 4 \leq m \leq k-2 \tag{14}$$

Therefore, there is a path between any two elements of these orders, and hence, it forms a component such as that shown in Figure 11.

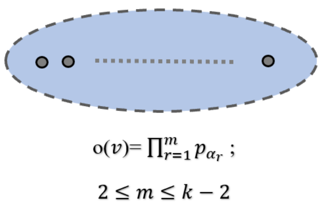

Figure 11. The middle component in $\mathcal{E}_1(G)$ with 5 components.

Otherwise, if $|p_1p_2p_3 - p_ip_j| > \min\{p_1p_2p_3, p_ip_j\}$ for all $1 \leq i < j \leq k$, then $p_1p_2p_3 > p_ip_j$ for all $1 \leq i < j \leq k$. Also, since $p_1p_2p_3 < p_ip_jp_r$ for all $1 \leq i < j < r \leq k$, we obtain that

$$|\prod_{r=1}^{3} p_{i_r} - \prod_{s=1}^{2} p_{j_s}| > \min\{\prod_{r=1}^{3} p_{i_r} - \prod_{s=1}^{2} p_{j_s} \tag{15}$$

Hence, there is no path between these components, as presented in Figure 12.

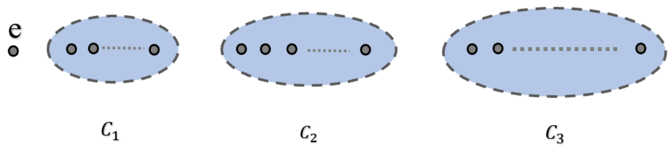

Figure 12. The disconnection of the first four components in $\mathcal{E}_1(G)$ with $k+1$ components.

From inequality (15), we obtain

$$|\prod_{r=1}^{k-2} p_{i_r} - \prod_{s=1}^{k-3} p_{j_s}| > \min\{\prod_{r=1}^{k-2} p_{i_r}, \prod_{s=1}^{k-3} p_{j_s}\} \tag{16}$$

Hence, these components are disjoint, as described in Figure 13.

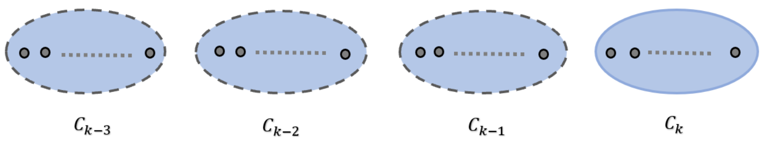

Figure 13. The disconnection of the last four components in $\mathcal{E}_1(G)$ with $k+1$ components.

Then, by the mathematical induction on the number of primes in the prime factorization of the divisors, say m, we will prove that the component containing elements of order $\prod_{r=1}^{m} p_{i_r}$ and the component consisting of elements of order $\prod_{r=1}^{m-1} p_{i_r}$ for all $4 \leq m \leq k-3$ are separated.

The base case, $m = 4$: First, claim $p_1p_2p_3p_4 > p_{k-2}p_{k-1}p_k$. Then

$$p_1p_2p_3p_4 > p_ip_jp_r; \text{ for all } 1 \leq i < j < r \leq k.$$

Now since $p_1p_2 > p_i$ for all $1 \leq i \leq k$ and $p_1p_2p_3 > p_ip_j$ for all $1 \leq i < j \leq k$, then $p_1p_2p_3 > p_{k-1}p_k$.

Moreover, as p_4 is greater than every prime on the left side of the inequality and p_{k-2} is smaller than every prime on the other side, according to the choice of the primes, we obtain that

$$p_1p_2p_3p_4 > p_{k-2}p_{k-1}p_k$$

Thus, by the increasing these numbers, we have

$p_1p_2p_3p_4 > p_ip_jp_r$ for all $1 \leq i < j < r \leq k$. From inequality (15) and for any $t \in \{1,2,\ldots,k\}$, such that $p_t > p_{i_r}$ and $p_t > p_{j_s}$ for all $r = 1,2,3$ and $s = 1,2$, respectively, then

$$|\prod_{r=1}^{3} p_{i_r}p_t - \prod_{s=1}^{2} p_{j_s}p_t| > \min\{\prod_{r=1}^{3} p_{i_r}p_t, \prod_{s=1}^{2} p_{j_s}p_t\} \tag{17}$$

Furthermore, $p_1p_2p_3 - p_{k-1}p_k > \min\{p_1p_2p_3, p_{k-1}p_k\} = p_{k-1}p_k$

$$p_1p_2p_3p_4 - p_4p_{k-1}p_k > p_4p_{k-1}p_k \tag{18}$$

and

$$p_1p_2p_3p_{k-2} - p_{k-2}p_{k-1}p_k > p_{k-2}p_{k-1}p_k \tag{19}$$

Then

$$p_1p_2p_3p_{k-2} - p_4p_{k-1}p_k > p_4p_{k-1}p_k \tag{20}$$

Then, adding the inequalities (18) and (19) gives

$$[p_1p_2p_3p_4 - p_{k-2}p_{k-1}p_k] + [p_1p_2p_3p_{k-2} - p_4p_{k-1}p_k] > p_{k-2}p_{k-1}p_k + p_4p_{k-1}p_k \tag{21}$$

Also, inequality (20) implies that

$$p_1p_2p_3p_4 - p_{k-2}p_{k-1}p_k > p_{k-2}p_{k-1}p_k$$

Then,

$$|\prod_{r=1}^{4} p_{i_r} - \prod_{s=1}^{3} p_{j_s}| > \min\{\prod_{r=1}^{4} p_{i_r}, \prod_{s=1}^{3} p_{j_s}\} \tag{22}$$

Thus, there is no path from any element of order $\prod_{s=1}^{3} p_{j_s}$ to any element of order $\prod_{r=1}^{4} p_{i_r}$.
The inductive hypothesis: Assume that this is true for all $m < k-3$, that is

$$|\prod_{r=1}^{m} p_{i_r} - \prod_{s=1}^{m-1} p_{j_s}| > \min\{\prod_{r=1}^{m} p_{i_r}, \prod_{s=1}^{m-1} p_{j_s}\} \tag{23}$$

Then the resulting components are depicted in Figure 14.

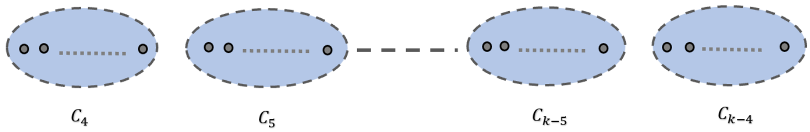

Figure 14. The disconnection of the middle components in $\mathcal{E}_1(G)$ with $k+1$ components.

The inductive proof: Claim that $|\prod_{r=1}^{k-3} p_{i_r} - \prod_{s=1}^{k-4} p_{j_s}| > \min\{\prod_{r=1}^{k-3} p_{i_r}, \prod_{s=1}^{k-4} p_{j_s}\}$. Now from the inductive hypothesis, we have for all $k-4 < s \le k$ and $p_s > p_{i_j}$ for all $1 \le j \le k$,

$$|p_1p_2\dots p_{k-4}p_s - \prod_{j=1}^{k-5} p_{i_j}p_s| > \min\{p_1p_2\dots p_{k-4}p_s, \prod_{j=1}^{k-5} p_{i_j}p_s\} \tag{24}$$

Then, similarly to the base case, we obtain

$$\begin{aligned} &p_1p_2\dots p_{k-3} > p_5p_6\dots p_k, \text{and} \\ &|p_1p_2\dots p_{k-3} - p_5\dots p_k| > \min\{p_1p_2\dots p_{k-3}, p_5\dots p_k\} \end{aligned}$$

Then

$$|p_1p_2\dots p_{k-3} - \prod_{j=1}^{k-4} p_{i_j}| > \min\{p_1p_2\dots p_{k-3}, \prod_{j=1}^{k-4} p_{i_j}\} \tag{25}$$

And hence, $p_1p_2\dots p_{k-3} > \prod_{j=1}^{k-4} p_{i_j}$.

Thus, the increase of the primes implies that

$$|\prod_{r=1}^{k-3} p_{i_r} - \prod_{s=1}^{k-4} p_{j_s}| > \min\{\prod_{r=1}^{k-3} p_{i_r}, \prod_{s=1}^{k-4} p_{j_s}\} \tag{26}$$

Therefore, there is no path between any two elements of orders $\prod_{r=1}^{k-3} p_{i_r}$ and $\prod_{s=1}^{k-4} p_{j_s}$. Hence, there is no path between any element of order $\prod_{r=1}^{m} p_{i_r}$ and any element of order $\prod_{s=1}^{m-1} p_{j_s}$ for all $4 \le m \le k-4$, and this complete the proof. $\square$

Definition 3. *Let G be a finite group. Then G is said to be an equitable group if the order of G is an equitable square-free number.*

Example 3. *The symmetric group S_3, the dihedral group D_{30} and the cyclic group of order* 1729 *are examples of equitable groups.*

Corollary 1. *Let G be a cyclic equitable group of order $n = \prod_{i=1}^{k} p_i$, where p_i are distinct primes for all $1 \leq i \leq k$. Then the only cases in which $\mathcal{E}_1(G)$ is connected to $k = 2,\ 3$ or 4 are $n = 6,\ 30$ or 210, respectively.*

Proposition 3. *Let G be a cyclic equitable group of order $n = \prod_{i=1}^{k} p_i$, where $k > 2$ is a positive integer and p_i are distinct primes for all $1 \leq i \leq k$. Consider the disconnected graph $\mathcal{E}_1(G)$. Then $\mathcal{E}_1(G)$ has the following properties:*

1. $\delta(\mathcal{E}_1) = 0$,
2. $\chi(\mathcal{E}_1) = \omega(\mathcal{E}_1) = \phi(n)$,
3. $\Delta(\mathcal{E}_1) = \phi(n) - 1$,
4. $2k - 2 \leq \gamma(\mathcal{E}_1) \leq \lceil \frac{2^k - 2}{3} \rceil + 2$.

Proof. Let G be a cyclic equitable group of order $n = \prod_{i=1}^{k} p_i$, where $k > 2$ is a positive integer and p_i are distinct primes, and consider the disconnected graph $\mathcal{E}_1(G)$. Then, the identity is isolated, and hence, we obtain the result of the minimum degree. Also, since all the vertices that are associated with the elements of order n in G, which occupies the largest number of vertices, form a disjoint clique by Theorem 14, we obtain (2) and (3).

To prove (4), let C_m denote a component that consists of vertices that correspond to the elements of order $d_m = \prod_{j=1}^{m} p_{i_j}$, where $1 \leq m \leq k$ in the group. Then we have $k+1$ components, including the identity. From Theorem 14, we obtain that the identity and one vertex from C_k belong to the dominating set, say S. The two components C_1 and C_{k-1} consist of $k = \binom{k}{1} = \binom{k}{k-1}$ connected cliques. Hence, taking into view the number of cliques in these components and the difference between the divisors, at least one vertex of each of them can be in S. Each one of the remaining $(k-3)$ components consists of $\binom{k}{j}$ connected cliques, which is greater than k for all $2 \leq j \leq k-2$. So again, based on a similar reason, at least two vertices of each component belong to S. Thus, the dominating set S consists of at least $4 + 2(k-3) = 2k - 2$ components. The highest value that S can attain is $\lceil \frac{2^k - 2}{3} \rceil + 2$ since each divisor of n corresponds to a clique in this graph and, in our case, n has 2^k divisors. As previously explained, the identity and one vertex of C_k are included in the dominating set. Therefore, $2^k - 2$ cliques remain, in which, for each three consecutive cliques, one vertex can be in S, which has an order equal to the middle ones. □

Proposition 4. *Let G be a cyclic equitable group of order $n = \prod_{i=1}^{k} p_i$, where $k > 2$ is a positive integer and p_i are distinct primes. Consider the connected graph $\mathcal{E}_1(G)$. Then $\mathcal{E}_1(G)$ has the following properties:*

1. $\delta(\mathcal{E}_1) = 1$.
2. $\Delta(\mathcal{E}_1) = \phi(n) + \phi(\frac{n}{2}) + \phi(\frac{n}{3}) - 1$.
3. $\chi(\mathcal{E}_1) = \omega(\mathcal{E}_1) = \phi(n) + \phi(\frac{n}{2})$.
4. $\gamma(\mathcal{E}_1) \geq k + 1$ *unless* $k = 2, 3$ *or* 4, *in which case,* $\gamma(\mathcal{E}_1) = k$.
5. $\mathrm{diam}(\mathcal{E}_1) \geq 2\gamma(\mathcal{E}_1)$ *unless* $k = 2, 3$ *or* 4, *in which case,* $\mathrm{diam}(\mathcal{E}_1) = 3, 6$ *or* 10, *respectively.*

Proof. Let G be a cyclic equitable group of order $n = \prod_{i=1}^{k} p_i$, where $k > 2$ is a positive integer and p_i are distinct primes, and consider the connected graph $\mathcal{E}_1(G)$. Then the number of vertices that are associated with the elements of order 2 in G is $\phi(2) = 1$, for which the identity is uniquely adjacent to it, and this yields the result of (1). Now, the two following differences $|n - \frac{n}{2}| \leq \min\{n, \frac{n}{2}\}$ and $|\frac{n}{2} - \frac{n}{3}| \leq \min\{\frac{n}{2}, \frac{n}{3}\}$ lead to any vertex associated with an element of order $\frac{n}{2}$ being adjacent to all symmetrical vertices and all vertices that are associated with elements of order n and $\frac{n}{3}$. Moreover, as $|\frac{n}{2} - \frac{n}{5}| > \min\{\frac{n}{2}, \frac{n}{5}\}$ and $|n - \frac{n}{3}| > \min\{n, \frac{n}{3}\}$, taking into consideration the number of elements in each order, this gives (2) and (3), respectively.

The diameter and the domination number of the graph when $k = 2, 3\, or\, 4$ is obtained obviously. On the other hand, let k be greater than four. Then the first four primes are always $2,\ 3,\ 5\ and\ 7$, which means that the divisors of n begin as $1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ \ldots,\ \frac{n}{2},\ n$. Let $v_{(i)}$ denote a single vertex associated with an element of order i in G for all $1 \leq i \leq n$. Then we have $v_{(2)}, v_{(10)}, v_{(42)}, v_{(\frac{n}{2})}$ or $v_{(n)}$ and one vertex from the component C_{k-1}, where C_m is defined as in Proposition 3 are always belonging to S. Now for the residue components (C_3 to C_{k-2}), at least $k-4$ vertices from these components can be included in the dominating set regarding the connectedness of the graph and the difference between the divisors. Thus, $\gamma(\mathcal{E}_1) \geq k+1$.

Moreover, the diameter of the graph is clearly the shortest path from the identity to $v_{(n)}$, which, in this case, usually starts as $v_{(1)} \to v_{(2)} \to v_{(3)} \to v_{(6)} \to v_{(10)} \to \ldots \to v_{(n)}$.

So we obtain that each vertex in S gives at least two edges in this path in addition to the edge between $v_{(3)}$ and $v_{(6)}$. Hence, we conclude the result. □

4. Zagreb Indices of the Equitable Graph

Topological indices are crucial for analyzing the physico–chemical characteristics of chemical compounds. They include degree-based and distance-based molecular structures and hybrid formulations. These indices are leading tools for identifying physical properties, chemical reactivity and biological activities of compounds. For any graph Γ with vertex set V and edge set E, the first and second Zagreb indices are defined as $M_1(\Gamma) = \sum\limits_{u \in V} \left(d(u)\right)^2$ and $M_2(\Gamma) = \sum\limits_{uv \in E} d(u)d(v)$. The forgotten index is similar to the first Zagreb index, which is defined as $F(\Gamma) = \sum\limits_{u \in V} \left(d(u)\right)^3$. For more details, see [15,16]. Through this section, we determine these three indices for the equitable graph Type I from some specific cyclic groups.

Theorem 15. *Let G be a cyclic group of order 2^k; $k > 1$ is a positive integer. Then the first, second and forgotten Zagreb indices of $\mathcal{E}_1(G)$ will be as follows:*

1. $M_1(\mathcal{E}_1(G)) \ = \ 10 + 2^{k-1}\left((3)2^{k-2} - 1\right)^2 + \sum\limits_{i=1}^{k-2} 2^i\left((7)2^{i-1} - 1\right)^2.$
2. $M_2(\mathcal{E}_1(G)) \ = \ 3 \ + \ \sum\limits_{i=1}^{k-1} 2^{2i-1}[d(v_{2^i})d(v_{2^{i+1}})] \ + \ \sum\limits_{i=2}^{k} [\frac{s_i(s_i+1)}{2}]\,[d(v_{2^i})]^2$
   where $s_i \ = \ \phi(2^i) - 1 \ = \ 2^{i-1} - 1.$
3. $F(\mathcal{E}_1(G)) \ = \ 28 + 2^{k-1}\left((3)2^{k-2} - 1\right)^3 + \sum\limits_{i=1}^{k-2} 2^i\left((7)2^{i-1} - 1\right)^3.$

Proof. Let G be a cyclic group of order 2^k; $k > 1$ is a positive integer. Then as $\phi(2^i) \ = \ 2^{i-1}$ and for any vertex v that associates with an element of order 2^i; $2 \leq i \leq k-1$, we have $d(v) \ = \ \phi(2^{i-1}) + \phi(2^i) - 1 + \phi(2^{i+1})$, and if $i = 1\ or\ k$, $d(v) \ = \ 3\ or\ [\phi(2^{k-1}) + \phi(2^k)]$, respectively. Then, computing $M_1(\mathcal{E}_1) \ = \sum\limits_{v \in V(\mathcal{E})} d^2(v)$, we obtain

$$M_1(\mathcal{E}_1(G)) \ = \ 1 \ + \ 9 \ + \ \phi(2^2)[1 + \phi(2^2) - 1 + \phi(2^3)]^2 \ + \ \phi(2^3)[\phi(2^2) + \phi(2^3) - 1 + \phi(2^4)]^2 + \ \ldots \ + \ \phi(2^{k-1})[\phi(2^{k-2}) + \phi(2^{k-1}) - 1 + \phi(2^k)]^2 \ + \ \phi(2^k)[\phi(2^{k-1}) + \phi(2^k) - 1]^2.$$

Hence, substituting the value of $\phi(2^i)$ and using Lemma 2, we obtain the result.

Now for the second Zagreb index, let $v_{(2^i)}$ denote the vertex corresponding to an element of order 2^i; then

$$\begin{aligned}
M_2(\mathcal{E}_1(G)) &= \sum_{uv \in E(\mathcal{E}_1)} d(u)d(v) \\
&= (1)(3) + 2[d(v_{(2)})d(v_{(2^2)})] + (\sum_{j=1}^{\phi(2^3)-1} j)[d(v_{(2^3)})]^2 + (2^2)(2^3)[d(v_{(2^3)})d(v_{(2^4)})] \\
&+ (\sum_{j=1}^{\phi(2^4)-1} j)[d(v_{(2^4)})]^2 + \ldots\ldots + (2^{k-2})(2^{k-1})[d(v_{(2^{k-1})})d(v_{(2^k)})] + \\
&(\sum_{j=1}^{\phi(2^k)-1} j)[d(v_{(2^k)})]^2.
\end{aligned}$$

Setting $s_i = \phi(2^i) - 1$, using the fact that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$, and since all the vertices that correspond to elements of the same order have the same degree, we obtain what is required. For the forgotten index, we have the following:

$$F(\mathcal{E}_1(G)) = 1 + 27 + \phi(2^2)[1 + \phi(2^2) - 1 + \phi(2^3)]^3 + \phi(2^3)[\phi(2^2) + \phi(2^3) - 1 + \phi(2^4)]^3 + \ldots + \phi(2^{k-1})[\phi(2^{k-2}) + \phi(2^{k-1}) - 1 + \phi(2^k)]^3 + \phi(2^k)[\phi(2^{k-1}) + \phi(2^k) - 1]^3.$$

Then, similarly to the first index, we obtain the desired outcome. □

Example 4. *Let G be a cyclic group of order 2^k; $k > 1$. Table 1 shows the value of the topological indices of $\mathcal{E}_1(G)$.*

Table 1. The topological indices of a cyclic group G of order 2^k for some $k > 1$.

	$\lvert G\rvert = 2^2$	$\lvert G\rvert = 2^3$	$\lvert G\rvert = 2^4$	$\lvert G\rvert = 2^5$	$\lvert G\rvert = 2^6$
$M_1(\mathcal{E}_1(G))$	18	182	1726	15,054	125,678
$M_2(\mathcal{E}_1(G))$	19	465	9677	176,325	300,5621
$F(\mathcal{E}_1(G))$	44	960	19,896	361,384	6,151,048

Theorem 16. *Let G be a cyclic group of order p^k; $p > 2$ is a prime number, and $k > 1$ is a positive integer. Then the first, second and forgotten Zagreb indices of $\mathcal{E}_1(G)$ will be as follows:*

1. $M_1(\mathcal{E}_1(G)) = \sum_{i=1}^{k} (p^i - p^{i-1})[p^i - p^{i-1} - 1]^2.$
2. $M_2(\mathcal{E}_1(G)) = \sum_{i=1}^{k} [\frac{s_i(s_i+1)}{2}.(s_i)^2]$
   where $s_i = \phi(p^i) - 1;\ 1 \le i \le k.$
3. $F(\mathcal{E}_1(G)) = \sum_{i=1}^{k} (p^i - p^{i-1})[p^i - p^{i-1} - 1]^3.$

Proof. Let G be a cyclic group of order p^k; $p > 2$ is a prime number, and $k > 1$ is a positive integer. Then the result for the first Zagreb and the forgotten indices follows from the fact that each clique in this graph has $\phi(p^i)$ vertices, where p^i is the order of the group elements that correspond to these vertices for all $1 \le i \le k$, and hence, the degree of any vertex v in such a clique is $\phi(p^i) - 1$.

Now for the second Zagreb index, since each vertex is adjacent only to the vertices that associate with elements of the same order, consider the clique, say Q, of vertices that correspond to elements of order p^i for some $1 \le i \le k$. Let v_1 , v_2 , $\ldots$, v_t, where $t = \phi(p^i)$.

Then $d(u) = d(v),\ for\ all\ u \neq v\ in\ Q$, and by computing $\sum d(u).d(v)$, $uv \in E(\mathcal{E}_1)$, we obtain

$$\begin{aligned}
& d(v_1)d(v_2) + d(v_1)d(v_3) + \ldots + d(v_1)d(v_t) \\
& + d(v_2)d(v_3) + d(v_2)d(v_4) + \ldots + d(v_2)d(v_t) \\
& \vdots \\
& + d(v_{t-2})d(v_{t-1}) + d(v_{t-2})d(v_t) + d(v_{t-1})d(v_t) \\
& = d(v_1)[(\phi(p^i) - 1)(d(v_1))] + d(v_2)[(\phi(p^i) - 2)(d(v_2))] + \ldots + d(v_{t-2})[2(d(v_{t-2}))] \\
& + d(v_{t-1})[1(d(v_{t-1}))] \\
& = \sum_{j=1}^{t-1} j[d(v_j)]^2 = [\frac{t-1(t)}{2}](t-1)^2.
\end{aligned}$$

Therefore, by generalizing this sum to all $1 \leq i \leq k$, we obtain the result. □

Example 5. *Let G be a cyclic group of order p^k; $k > 1$ and $p > 2$. Table 2 shows the value of the topological indices of $\mathcal{E}_1(G)$.*

Table 2. The topological indices of a cyclic group G of order p^k; $p > 2$ for some $k > 1$.

	$\|G\| = 3^2$	$\|G\| = 3^3$	$\|G\| = 5^2$	$\|G\| = 5^3$	$\|G\| = 7^2$	$\|G\| = 7^3$
$M_1(\mathcal{E}_1(G))$	152	157,040	7256	987,356	70,752	1,307,720
$M_2(\mathcal{E}_1(G))$	376	4,064,272	68,644	48,583,594	1,447,716	71,230,240
$F(\mathcal{E}_1(G))$	752	8,128,544	137,288	97,167,188	2,895,432	142,460,480

Theorem 17. *Let G be a cyclic group of order $2^k q$; $q > 2$ a prime number, and $k > 1$. Then the first Zagreb index is given by the following formula:*

- *If $2^t \leq q \leq 2^{t+1}$ for some $1 \leq t < k$, we have*

$$M_1(\mathcal{E}_1(G)) = 10 + \sum_{i=1}^{t-1} 2^i[(7)2^{i-1} - 1]^2 + 2^{t-1}[(7)2^{t-2} + q - 2]^2 + (q-1)[(3)2^{t-1} + 2q - 3]^2 + \sum_{i=t+1}^{k-1} 2^{i-1}[(7)2^{i-2} + (3)2^{i-t-2}(q-1) - 1]^2 + \sum_{i=1}^{k-t-1} 2^{i-1}(q-1)[(7)2^{i-2}(q-1) + (3)2^{i+t-1} - 1]^2 + 2^{k-1}[(3)2^{k-2} + (3)2^{k-t-2}(q-1) - 1]^2 + 2^{k-t-1}(q-1)[2^{k-1} + (7)2^{k-t-2}(q-1) - 1]^2 + \sum_{i=k-t+1}^{k-1} 2^{i-1}(q-1)[(7)2^{i-2}(q-1) - 1]^2 + 2^{k-1}(q-1)[(3)2^{k-2}(q-1) - 1]^2.$$

- *If $q > 2^k$, and $|q - 2^k| \leq \min\{q, 2^k\}$, we have*

$$M_1(\mathcal{E}_1(G)) = 10 + \sum_{i=1}^{k-2} 2^i[(7)2^{i-1} - 1]^2 + 2^{k-1}[(3)2^{k-2} + q - 2]^2 + (q-1)[2^{k-1} + 2q - 3]^2 + \sum_{i=1}^{k-2} 2^i(q-1)[(7)2^{i-1}(q-1) - 1]^2 + 2^{k-1}(q-1)[(3)2^{k-2}(q-1) - 1]^2.$$

- *If $q > 2^k$, and $|q - 2^k| > \min\{q, 2^k\}$, we have*

$$M_1(\mathcal{E}_1(G)) = 10 + \sum_{i=1}^{k-2} 2^i[(7)2^{i-1} - 1]^2 + 2^{k-1}[(3)2^{k-2} - 1]^2 + (q-1)[2q-3]^2 + \sum_{i=1}^{k-2} 2^i(q-1)[(7)2^{i-1}(q-1) - 1]^2 + 2^{k-1}(q-1)[(3)2^{k-2}(q-1) - 1]^2.$$

Proof. Let G be a cyclic group of order $2^k q$; $q > 2$ a prime number, and $k > 1$. Then consider the arrangement of the divisors according to the position of the prime number q.

Assume the first case; then the divisors will be as follows:
$1, 2, 2^2, \ldots, 2^t, q, 2^{t+1}, 2q, 2^{t+2}, 2^2q, \ldots, 2^k, 2^{k-t}, \ldots, 2^kq$.
For the later cases, the divisors will be as mentioned in the proof of Theorem 6. Hence, applying identical procedures as outlined in Theorem 15 and using Lemma 2, we achieve the desired outcome. □

Theorem 18. *Let G be a cyclic group of order 2^kq; $q > 2$ is a prime number, and $k > 1$. Then the second Zagreb index is given by the following formula:*

- *If $2^t \le q \le 2^{t+1}$ for some $1 \ge t < k$, then*

$$\begin{aligned} M_2(\mathcal{E}_1(G)) &= 3 + \sum_{i=1}^{k-1} 2^{2i-1}[d(v_{(2^i)})d(v_{(2^{i+1})})] + \sum_{i=2^2}^{2^kq}[\frac{s_i(s_i+1)}{2}][d(v_{(i)})]^2 \\ &+ \sum_{i=0}^{k}\phi(2^iq)\phi(2^{i+1}q)[d(v_{(2^iq)})d(v_{(2^{i+1}q)})] + \sum_{i=t}^{k}2^{2i-t-2}(q-1)[d(v_{(2^i)})d(v_{(2^{i-t}q)})] \\ &+ \sum_{i=t+1}^{k}2^{2i-t-3}(q-1)[d(v_{(2^{i-t-1}q)})d(v_{(2^i)})]. \end{aligned}$$

- *If $q > 2^k$, and $|q-2^k| \le \min\{q, 2^k\}$, then*

$$\begin{aligned} M_2(\mathcal{E}_1(G)) &= 3 + \sum_{i=1}^{k-1} 2^{2i-1}[d(v_{(2^i)})d(v_{(2^{i+1})})] + \sum_{i=2^2}^{2^kq}[\frac{s_i(s_i+1)}{2}][d(v_{(i)})]^2 + \\ & 2^{k-1}(q-1)[d(v_{(2^k)})d(v_{(q)})] + \sum_{i=0}^{k-1}\phi(2^iq)\phi(2^{i+1}q)[d(v_{(2^iq)})d(v_{(2^{i+1}q)})]. \end{aligned}$$

- *If $q > 2^k$, and $|q-2^k| > \min\{q, 2^k\}$, then*

$$\begin{aligned} M_2(\mathcal{E}_1(G)) &= 3 + \sum_{i=1}^{k-1} 2^{2i-1}[d(v_{(2^i)})d(v_{(2^{i+1})})] + \sum_{i=2^2}^{2^kq}[\frac{s_i(s_i+1)}{2}][d(v_{(i)})]^2 \\ &+ \sum_{i=0}^{k-1}\phi(2^iq)\phi(2^{i+1}q)[d(v_{(2^iq)})d(v_{(2^{i+1}q)})]. \end{aligned}$$

where $s_i = \phi(i) - 1$ and $d(v_{(j)})$ denote the degree of a vertex that is associated with an element of order j.

Proof. Let G be a cyclic group of order 2^kq; $q > 2$ a prime number, and $k > 1$. Then applying the same procedure as in Theorems 15 and 17, we obtain the result. □

Theorem 19. *Let G be a cyclic group of order 2^kq; $q > 2$ is a prime number, and $k > 1$. Then the forgotten index is given by the following formula:*

- *If $2^t \le q \le 2^{t+1}$ for some $1 \ge t < k$, we have*

$$\begin{aligned} F(\mathcal{E}_1(G)) &= 28 + \sum_{i=1}^{t-1} 2^i[(7)2^{i-1}-1]^3 + 2^{t-1}[(7)2^{t-2}+q-2]^3 + (q-1)[(3)2^{t-1}+ \\ & 2q-3]^3 + \sum_{i=t+1}^{k-1} 2^{i-1}[(7)2^{i-2}+(3)2^{i-t-2}(q-1)-1]^3 + \sum_{i=1}^{k-t-1} 2^{i-1}(q-1)[(7)2^{i-2}(q- \\ & 1)+(3)2^{i+t-1}-1]^3 + 2^{k-1}[(3)2^{k-2}+(3)2^{k-t-2}(q-1)-1]^3 + 2^{k-t-1}(q-1)[2^{k-1}+ \\ & (7)2^{k-t-2}(q-1)-1]^3 + \sum_{i=k-t+1}^{k-1} 2^{i-1}(q-1)[(7)2^{i-2}(q-1)-1]^3 + 2^{k-1}(q-1)[(3)2^{k-2}(q- \\ & 1)-1]^3. \end{aligned}$$

- *If $q > 2^k$, and $|q-2^k| \le \min\{q, 2^k\}$, we have*

$$F(\mathcal{E}_1(G)) \;=\; 28 + \sum_{i=1}^{k-2} 2^i[(7)2^{i-1} - 1]^3 + 2^{k-1}[(3)2^{k-2} + q - 2]^3 + (q-1)[2^{k-1} + 2q-$$
$$3]^3 + \sum_{i=1}^{k-2} 2^i(q-1)[(7)2^{i-1}(q-1) - 1]^3 + 2^{k-1}(q-1)[(3)2^{k-2}(q-1) - 1]^3.$$

- *If $q > 2^k$, and $|q - 2^k| > \min\{q, 2^k\}$, we have*

$$F(\mathcal{E}_1(G)) \;=\; 28 + \sum_{i=1}^{k-2} 2^i[(7)2^{i-1} - 1]^3 + 2^{k-1}[(3)2^{k-2} - 1]^3 + (q-1)[2q-3]^3 +$$
$$\sum_{i=1}^{k-2} 2^i(q-1)[(7)2^{i-1}(q-1) - 1]^3 + 2^{k-1}(q-1)[(3)2^{k-2}(q-1) - 1]^3.$$

Proof. Let G be a cyclic group of order $2^k q$; $q > 2$ is a prime number, and $k > 1$. Then the proof is similar to Theorems 15 and 17. □

5. The Adjacency Matrix $A(\mathcal{E}_1(G))$

In graph theory, the adjacency matrix of a simple graph Γ is a symmetric matrix $A(\Gamma) = (a_{ij})$ of size $n \times n$, where n represents the number of vertices in the graph. The matrix is defined such that $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and 0 otherwise.

This section deals with obtaining the adjacency matrix of the equitable graph of Type I that arises from cyclic p groups.

Proposition 5. *Let G be a cyclic group of order 2^k; $k > 2$ (or p^k; $k > 1$, and $p > 2$ is a prime number). Then the adjacency matrix of the equitable graph Type I of G will be as follows:*

$$A(\mathcal{E}_1(G)) \;=\; \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & J^* & \vdots & J & \vdots & \cdots & \cdots & \vdots & J \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & J & \vdots & J^* & \vdots & \cdots & \cdots & \vdots & J \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & J & \vdots & \cdots & \ddots & \cdots & \cdots & \vdots & J \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & J & \vdots & \cdots & \cdots & \cdots & J^* & \vdots & J \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & J & \vdots & \cdots & \cdots & \cdots & J & \vdots & J^* \end{pmatrix}$$

Proof. Let G be a cyclic group of order n and assume that $n = 2^k$; $k > 2$. Then according to the adjacency method, let J be a 3×3 matrix for which each entry equals one, and J^* is similar to J except that it has zeros in the main diagonal. In $A(\mathcal{E}_1(G))$, the first row consists of zeros except for in the $(2^{k-1})th$ position. The middle row, (2^{k-1}), has one only in the positions $(2^{k-1}\ ,\ 0)$, $(2^{k-1}\ ,\ 2^{k-2})$ and $(2^{k-1}\ ,\ 2^{k-2}3)$. Now for each $(4m)th$ row, where $m \geq 1$, if m is odd, then there are zeros in the positions $(4m\ ,\ i)$, where $i \;=\; 0,\ 4m,\ 2^{k-1}$, and all odd numbers. On the other hand, if m is even such that $4m \;\neq 2^{k-1}$, this row has ones in the positions $(4m, 4i)$ for all $i \geq 1$ and $i \neq m$. The corresponding rows and columns are symmetric.

Now suppose that $n \;=\; p^k$, where $p > 2$, and $k > 1$. Then by the definition of the graph, in this case, J and J^* are $(p - 1 \times p - 1)$ matrices, and they are as defined as before.

The first row (column) is zeros, and the remaining rows (columns) in $A(\mathcal{E}_1(G))$ have the following explanations

First, if $p = 3$, the $(3m)th$ rows, where $m \geq 1$ and $3m \neq 3^{k-1}$ *or* $(2)3^{k-1}$, have ones in the positions $(3m\ ,\ 3i)$, where $i \geq 1$, except for the case when $i = 3^{k-2}$ *or* $(2)3^{k-2}$ or $i = m$. For the $(3^{k-1})th$ and $((2)3^{k-1})th$ rows (columns), they have one at a unique position where the row and the column intersect mutually. Now if $p > 3$, then the $(pr)th$ rows, where $r \geq 1$, consist of ones only in the positions $(pr\ ,\ pi)$ for all $i \geq 1$ and $i \neq r$. □

Example 6. *Let $G \cong \mathbb{Z}_8$. Then*

$$A(\mathcal{E}_1(G)) = \left(\begin{array}{c|ccc|c|ccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ \hline
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0
\end{array}\right)$$

Example 7. *Let G be a cyclic group of order 2^k; $k = 2$. Then*

$$A(\mathcal{E}_1(G)) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}$$

Example 8. *Let $G \cong \mathbb{Z}_9$. Then*

$$A(\mathcal{E}_1(G)) = \left(\begin{array}{c|cc|c|cc|c|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0
\end{array}\right)$$

6. Conclusions

In this research, we introduced the equitable graphs Type I on groups. We studied the connectedness of these graphs for some groups and explored some of their theoretical properties. Additionally, the equitable square-free number and the equitable group were established. Furthermore, the connectedness and characteristics of the graph of cyclic equitable groups were investigated. The first, second and forgotten Zagreb indices were determined for the equitable graph Type I of specific groups. Finally, the adjacency matrix for the equitable graph Type I of cyclic p-groups was obtained. The newly introduced graph has significant potential for further investigation into its properties. Promising avenues for future research include analyzing equitable graph Type I, examining its perfectness, computing spectral properties, and elucidating connections with other well-known graph classes associated with finite groups. Addressing these open problems can provide valuable insights into theoretical and practical aspects, advancing our understanding of finite group theory and its interplay with graph theory.

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