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Erroneous Applications of Fractional Calculus: The Catenary as a Prototype

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Abstract: In this work, we study the equation of the catenary curve in the context of the Caputo derivative. We solve this equation and compare the solution with real physical models. From the experiments, we find that the best approximation is achieved in the classical case. Therefore, introducing a fractional parameter arbitrarily can be detrimental. However, we observe that, when adding a certain weight to the chain, fractional calculus produces better results than classical calculus for modeling the minimum height.

Keywords: fractional catenary curve; Caputo differential equations; fractional models

MSC: 34A08; 34A34; 34A06; 35R11

1. Introduction and Statement of the Results

The catenary curve, that elegant and resilient shape describing the configuration of a hanging chain, has left an indelible mark on the modern world. In the field of civil engineering, the catenary has played a fundamental role in the design of iconic structures, such as the Sagrada Familia church in Barcelona, Spain. Furthermore, its ability to evenly distribute loads along its curve makes it an ideal solution for bridges, arches, and other structures that must bear heavy weights.

The influence of the catenary extends far beyond civil engineering. In dentistry, its shape describes the natural curvature of healthy dental arches, providing a basis for understanding oral anatomy and designing dental prostheses. In the oil industry, the catenary is used to optimize well drilling, ensuring an efficient and safe trajectory. Even in the medical field, the catenary is employed to measure breast curvature in cancer diagnosis, providing crucial information for the detection and treatment of this disease (see [1,2], or the references mentioned therein).

The fascination with the catenary dates back to the time of Galileo Galilei, who compared the shape of a hanging chain to a parabola. However, it was Joachim Jungius who demonstrated that hanging chains have a distinct shape, laying the groundwork for a deeper study of this curve. In 1691, the catenary was formally defined and named, opening the doors to countless applications (see [3]).

The catenary curve, with its rich history and diverse applications, continues to inspire researchers and professionals in various fields. Its still untapped potential opens up new possibilities for designing stronger, more efficient, and more aesthetic structures, as well as for better a understanding of natural and complex phenomena. The catenary will undoubtedly remain a cornerstone of the modern world and a symbol of human creativity and ingenuity.

In this context, it is natural to consider the differential equation modeling the catenary curve in the fractional context, as we will do next to explore a new facet of the catenary. It should be noted that the equation modeling the catenary curve has already been studied in [4] using the Caputo–Fabrizio derivative. Here, we will use the fractional derivative in

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the sense of Caputo. The Caputo–Fabrizio derivative involves a non-singular kernel, and in modeling, it often yields different results from the Caputo derivative (see [5]).

Nowadays, it unfortunately happens all too often that a classical differential equation is considered, and instead of the integer-order derivative operator, it is replaced with a fractional derivative operator. In the vast majority of these cases, it is tacitly assumed that the fractional dynamics improve the model obtained using classical differential calculus. Under this premise, that is, assuming there is an improvement using fractional calculus, various authors engage in a series of theoretical developments or numerical manipulations that later make no sense (see [6]).

To illustrate this point, we have chosen the catenary curve because, as we mentioned before, it is an important phenomenon and relatively easy to reproduce. Using a hanging chain and a scanner, one can obtain the shape of the catenary curve (see Section 3). Furthermore, the associated differential equation is a second-order nonlinear differential equation (see (3) and (4)). By replacing the integer-order derivative with the fractional-order derivative (see (2) and (8)), a fractional differential equation is obtained (see (10)). This equation is solved, and the key point here is that the fractional parameters do not model the physical curve of the hanging chain. In other words, it is generally incorrect to think that the fractional model improves the classical model. In some cases, such as in the study of control systems, nonlinear and nonlocal models are often modeled by nonlinear fractional differential equations (see [6]). In our case, we see that this is not the case; indeed, the modeling of the catenary is a nonlinear model, and any (local) perturbation of the chain affects the chain globally.

On the other hand, when a certain weight is applied to the center of the chain, relatively minor compared to the total weight of the chain, it is observed that there are fractional indices that better model the deformed curve than the classical case. This is consistent with what is known (that it is a nonlocal phenomenon), and it opens a possible opportunity for fractional calculus to model this phenomenon, at least improving the classical case. In this case, as we mentioned, by modifying the weight of the chain at its center, the overall structure of the curve is altered; changes are not limited to a neighborhood of the center of the catenary. Since modeling the catenary curve with weight classically is a complicated problem (see [7,8]), and we achieve it with little effort when using fractional calculus, we can say that, for modeling this phenomenon, fractional calculus proves to be useful.

In summary, the fractional index should not be introduced into a differential equation without real evidence that it improves the model (see [6]). At best, the fractional analytical model represents a perturbation of the classical model. This observation should not be taken lightly, as there is a wide variety of fractional derivatives, and theoretical aspects are sometimes studied without any basis other than the aesthetic aspect of mathematics (see [9,10]).

The article is organized as follows. In Section 2, we recall the derivation of the classical catenary, which serves to introduce some concepts, and we propose the relevant modifications in the fractional case. In Section 3, we define some concepts of fractional calculus and state some basic properties. In Section 4, we present the fractional differential equation in the sense of Caputo and solve it. In Section 5, we describe the physical experiment and present some images of catenaries. Finally, in Section 6, we present some conclusions.

2. Classical Catenary Curve

In this section, we will briefly review the classical derivation of the catenary curve. Consider a homogeneous rope or string with linear density ρ and a length greater than a, where a is a fixed positive number. Let T represent the tension at the midpoint $a/2$. The downward displacement of the hanging chain at the point x will be denoted as $y(x)$. With this information, we can create a diagram similar to the one shown in Figure 1.

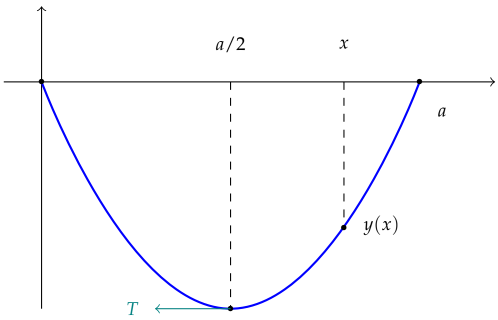


Figure 1. The catenary curve, $y(x)$.

Carrying out an analysis of the forces (see, for example, [11]), we arrive at the following equation,

$$\frac{1}{c}\int_{a/2}^{x}\sqrt{1+(u(t))^2}dt = u(x), \quad x \in (0,a), \tag{1}$$

where

$$u(x) = y'(x), \tag{2}$$

$c = T/(g\rho)$, and g is the gravitational constant. Taking derivatives in (1), we deduce the second-order nonlinear differential equation

$$\frac{1}{c}\sqrt{1+(u(x))^2} = u'(x), \tag{3}$$

with the initial condition

$$u(a/2) = y'(a/2) = 0. \tag{4}$$

The general solution of (3) has the form

$$u(x) = \sinh\left(\frac{1}{c}x + c_0\right), \quad x \in [0,a], \tag{5}$$

where c_0 is a constant. Using the initial conditions (4), we obtain

$$c_0 = -\frac{a}{2c}. \tag{6}$$

Now, from (2) and using $y(0) = 0$, we get the classical catenary curve,

$$y(x) = c\left\{\cosh\left(\frac{x}{c} - \frac{a}{2c}\right) - \cosh\left(\frac{a}{2c}\right)\right\}, \quad x \in [0,a]. \tag{7}$$

The minimum of the curve will be

$$y\left(\frac{a}{2}\right) = c\left\{1 - \cosh\left(\frac{a}{2c}\right)\right\}, \quad x \in [0,a].$$

3. Some Preliminaries on Fractional Calculus

To introduce the fractional model of the catenary curve, it is helpful to revisit some key concepts of fractional calculus. Numerous excellent texts are available on this subject; for our purposes, we adopted the notation and refer to the results presented in [12,13].

Definition 1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. The left Riemann–Liouville integral, $I^{\alpha}_{a+}f$, of f of order $\alpha \in \mathbb{R}$, is defined as*

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad x \in (a,b).$$

Now, let us suppose that $\alpha > 0$ and set $n := [\alpha] + 1$, where $[\cdot]$ is the largest integer less than or equal to α. If $f^{(n)}$ exists and is continuous, then the fractional derivative of Caputo, by the left, ${}^{C}D^{\alpha}_{a+}f$, is defined as

$$\left({}^{C}D^{\alpha}_{a+}f\right)(x) = \left(I^{n-\alpha}_{a+}f^{(n)}\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{1+\alpha-n}}dt, \quad x \in (a,b).$$

In the above definition, $\Gamma(\alpha)$, with $\alpha > 0$, represents the usual gamma function. Since we will not be utilizing any other fractional derivative, we will omit the C in the definition of a Caputo derivative, denoting it simply as $D^{\alpha}f$.

Proposition 1. *Let $\alpha > 0$ and $n := [\alpha] + 1$. If f has continuous derivatives up to order $n-1$, and $f^{(n)}$ is absolutely continuous, then*

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Proof. See Lemma 2.22 in [12]. □

The linearity of the classical integral implies the linearity of the fractional integral. Furthermore, we have the following result.

Proposition 2. *Let $\alpha \geq 0$ and $\beta > 0$; then,*

$$\left(I^{\alpha}_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}.$$

Proof. See formulas (2.1.16) of [12]. □

4. Fractional Catenary Curve

Let $\alpha \in (\mathbb{R}_+ \setminus \mathbb{N})$ and $n := [\alpha] + 1$. A common method to extend the classical model to the fractional context involves introducing the fractional derivative operator into expression (2), thereby replacing the classical derivative with the fractional one; see [4,6],

$$u = D^{\alpha}_{\frac{a}{2}+}y. \tag{8}$$

On the other hand, physical conditions require that the resulting curve be symmetric. Keeping this in mind, we consider the following equation (see Equation (1)):

$$\frac{1}{c}\int_{a/2}^{x}\sqrt{1+(u(t))^2}dt = u(x), \quad \frac{a}{2} < x < a. \tag{9}$$

If we take the derivative, we obtain the fractional differential equation

$$\frac{1}{c}\sqrt{1+\left(D^{\alpha}_{\frac{a}{2}+}y(x)\right)^2} = D^{1+\alpha}_{\frac{a}{2}+}y(x), \tag{10}$$

with boundary conditions

$$y(a) = 0, \; y'\left(\frac{a}{2}\right) = 0, \; \cdots, \; y^{(n)}\left(\frac{a}{2}\right) = 0. \tag{11}$$

Equation (9) is the same as Equation (1); thus, the solution is given by (5),

$$u(x) = \sinh\left(\frac{1}{c}x + c_1\right), \quad \frac{a}{2} < x < a, \tag{12}$$

for some constant, c_1. From Proposition 1, we get

$$y(x) - \sum_{k=0}^{n} \frac{y_+^{(k)}\left(\frac{a}{2}\right)}{k!}\left(x - \frac{a}{2}\right)^k = \left(I^{\alpha}_{\frac{a}{2}+} D^{\alpha}_{\frac{a}{2}+} y\right)(x) = I^{\alpha}_{\frac{a}{2}+} u(x). \tag{13}$$

On the other hand, using (12) and the linearity of fractional integral, we obtain

$$I^{\alpha}_{\frac{a}{2}+} u(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} I^{\alpha}_{\frac{a}{2}+} \left(\frac{1}{c}x + c_1\right)^{2k+1}. \tag{14}$$

Since

$$\left(\frac{1}{c}x + c_1\right)^{2k+1} = \left(\frac{1}{c}\right)^{2k+1} \sum_{j=0}^{2k+1} \binom{2k+1}{j} \left(\frac{a}{2} + c \cdot c_1\right)^{2k+1-j} \left(x - \frac{a}{2}\right)^j,$$

then Propositions 2 and (11) yield

$$\begin{aligned} y(x) &= \sum_{k=0}^{n} \frac{y^{(k)}\left(\frac{a}{2}\right)}{k!}\left(x - \frac{a}{2}\right)^k + I^{\alpha}_{\frac{a}{2}+} u(x) \\ &= y\left(\frac{a}{2}\right) + \sum_{k=0}^{\infty} \left(\frac{1}{c}\right)^{2k+1} \sum_{j=0}^{2k+1} \frac{1}{(2k+1-j)!\,\Gamma(j+1+\alpha)} \left(\frac{a}{2} + c \cdot c_1\right)^{2k+1-j} \left(x - \frac{a}{2}\right)^{j+\alpha}. \end{aligned}$$

By hypothesis, the n-th derivative of y exists at $a/2$. Since $\alpha - n < 0$, then

$$\frac{a}{2} + c \cdot c_1 = 0,$$

otherwise, $y^{(n)}(a/2)$ would not exist. This particular value of c_0 corresponds to the classical case, as depicted in (6). Therefore, (11) suggests

$$y(x) = \sum_{k=0}^{\infty} \left(\frac{1}{c}\right)^{2k+1} \frac{1}{\Gamma(2k+2+\alpha)} \left[\left(x - \frac{a}{2}\right)^{2k+1+\alpha} - \left(\frac{a}{2}\right)^{2k+1+\alpha}\right], \quad \frac{a}{2} \le x \le a.$$

The fractional catenary curve y_α is given by

$$y_\alpha(x) = \begin{cases} y(a-x), & 0 \le x \le \frac{a}{2}, \\ y(x), & \frac{a}{2} \le x \le a. \end{cases} \tag{15}$$

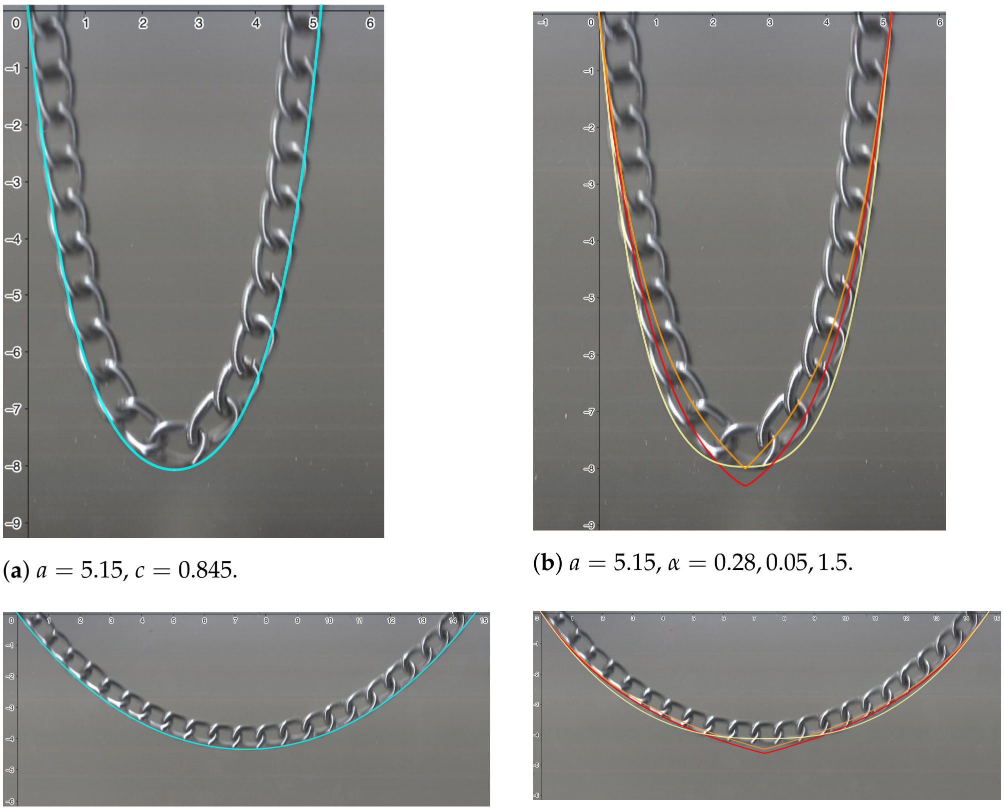
When $\alpha = 1$, employing the Taylor series of the hyperbolic cosine yields the classical catenary (7). In the fractional scenario, the minimum is given by

$$y_\alpha\left(\frac{a}{2}\right) = -c^{\alpha} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2+\alpha)} \left(\frac{a}{2c}\right)^{2k+1+\alpha}. \tag{16}$$

5. Physical Experiments

The experiment consisted of capturing several images of a hanging chain to determine the shape of the curve formed by the chain. A scanner was placed vertically with the chain positioned in front of it. When a photograph of the hanging chain is taken with a camera, the resulting curve depends on the angle of the photograph. However, this method avoids any dependence on the angle in the image.

Note that the function $y(x)$, given in (15), depends on the parameter c, which in turn depends on the linear density of the chain. The criterion we used to determine the value of c is the one that minimizes the error in approximating $y(x)$ to the curve determined via the chain, taking a fixed value of the index α. In Figure 2a,c, the classical catenary curve is shown in blue. As we can observe from Table 1, when α is 1, the errors are smaller compared to the fractional case, with curves in red ($\alpha = 0.28$, $c = 0.9$), yellow ($\alpha = 1.5$, $c = 0.78$), and orange ($\alpha = 0.05$, $c = 0.93$) when $a = 5.1$ and in red ($\alpha = 0.1$, $c = 3.46$), yellow ($\alpha = 1.5$, $c = 11$), and orange ($\alpha = 0.05$, $c = 3.42$) when $a = 14.71$; see Figure 2b,d. Indeed, in the classical case, the error is 0.042 cm when $a = 5.15$ cm and 0.028 cm when $a = 14.71$ cm, considerably lower than in the fractional case; see Table 1. The errors in the figures were measured using Digimizer 2024, an image analysis software package that allows precise manual measurements (see [14]). In this way, in Figure 2b,d, we see that the fractional curves (i.e., $\alpha \neq 1$) deviate from the curve determined according to the chain. In other words, the fractional solution (15) does not model the curve produced via the chain.



(**a**) $a = 5.15$, $c = 0.845$.
(**b**) $a = 5.15$, $\alpha = 0.28, 0.05, 1.5$.
(**c**) $a = 14.71$, $c = 6.85$.
(**d**) $a = 14.71$, $\alpha = 0.1, 0.05, 1.5$.

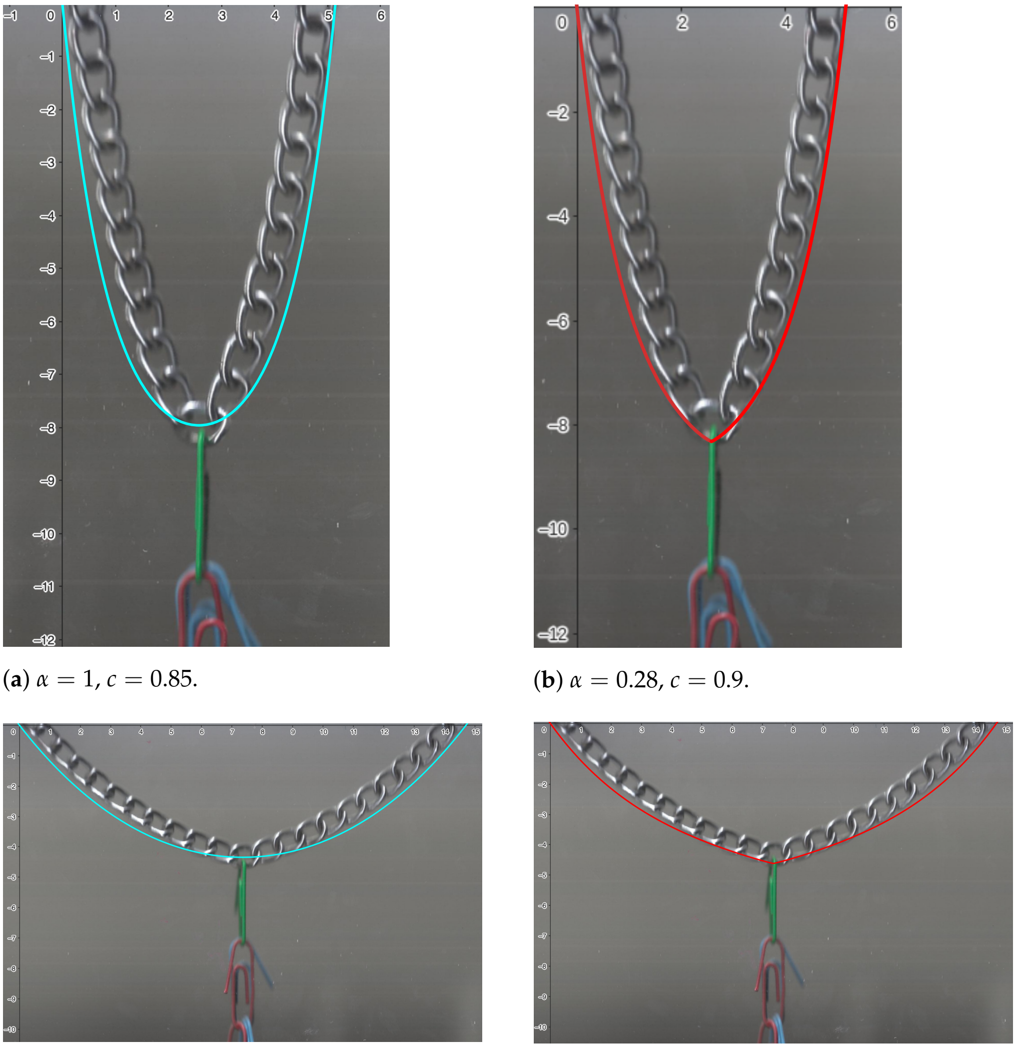
Figure 2. Weightless pendant chain.

Table 1. Different values for fractional catenary parameters; see (15).

a (cm)	α	c	Error (cm)
5.15	0.05	0.925	0.361
	0.28	0.9	0.273
	1	0.845	0.017
	1.5	0.78	0.079
14.71	0.05	3.42	0.264
	0.1	3.46	0.241
	1	6.85	0.009
	1.5	11	0.206

Next, we will modify the system by placing a weight of 4.344 grams at the center of the chain in both cases, i.e., for $a = 5.15$ and $a = 14.71$. In Figure 3a,c, the classical curve, shown in blue, corresponds to $\alpha = 1$. In this case, the errors are 0.258 cm when $a = 5.15$ and 0.307 cm when $a = 14.71$. On the other hand, by selecting the fractional index α that best fits the curve, it turns out that, for $a = 5.15$, the error is 0.19 cm, and when $a = 14.71$, the error is 0.16 cm; see Figure 3b,d. This means that the perturbation is better modeled with a fractional index; see (16).

It may be of particular interest to model the minimum height of the chain with the weight, as this can occur, for example, in a cable car cable. In this case, we see that the actual minimum height is -8.324 when $a = 5.15$. In the classical case, the error is 0.252, and in the fractional case with $\alpha = 0.28$, the error is 0.012. On the other hand, when $a = 14.71$ cm, the minimum height is -4.668. In the classical case, the error is 0.325 cm, and in the fractional case with $\alpha = 0.1$, the error is 0.053 cm.



(**a**) $\alpha = 1$, $c = 0.85$. (**b**) $\alpha = 0.28$, $c = 0.9$.

(**c**) $\alpha = 1$, $c = 6.85$. (**d**) $\alpha = 0.1$, $c = 3.46$.

Figure 3. Weighted pendant chain.

6. Conclusions

It is well known that fractional calculus is useful for modeling certain phenomena, such as temperature controllers [15] or battery charges [16]. However, we can conclude that this is not always the case. Introducing a fractional index arbitrarily into a model based on ordinary differential equations can lead to results that deviate significantly from expectations. Some authors argue that modeling with fractional calculus provides an advantage by adding an additional degree of freedom, the fractional index α. While this can be true, our study shows that the best results may be obtained by not modifying this parameter, i.e., by setting $\alpha = 1$.

In summary, the main contribution of our work lies in advocating for the rational use of fractional calculus. The simplicity and replicability of the model studied here underline its value, demonstrating that this approach is not merely a matter of substituting classical derivatives with fractional ones and performing analytical or numerical manipulations without real value. Our study, conducted with the Caputo derivative, can serve as a starting point for analyzing the real behavior of a chain with other fractional derivatives as well; see, for example, the excellent paper [17].

On the other hand, we have observed that fractional calculus can be useful for modeling a perturbation of the original phenomenon. In our case, the minimum height of the weighted chain is better modeled using fractional calculus. Specifically, the approximation error at the minimum point decreases by approximately twenty percent.

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