mathematics

MDPI

Article

Innovative Methods of Constructing Strict and Strong Fuzzy Negations, Fuzzy Implications and New Classes of Copulas

Panagiotis Georgiou Mangenakis and Basil Papadopoulos *

Department of Civil Engineering, Section of Mathematics and Informatics, Democritus University of Thrace, 67100 Kimeria, Greece; pmangena@civil.duth.gr
* Correspondence: papadob@civil.duth.gr; Tel.: +30-2541079747

Abstract: This paper presents new classes of strong fuzzy negations, fuzzy implications and Copulas. It begins by presenting two theorems with function classes involving the construction of strong fuzzy negations. These classes are based on a well-known equilibrium point theorem. After that, a construction of fuzzy implication is presented, which is not based on any negation. Finally, moving on to the area concerning copulas, we present proof about the third property of copulas. To conclude, we will present two original constructions of copulas. All the above constructions are motivated by a specific formula. For some specific conditions of the variables x, y and other conditions for the function f(x), the formula presented produces strict and strong fuzzy negations, fuzzy implications and copulas.

Keywords: fuzzy negation; fuzzy logic; fuzzy sets; strong fuzzy negation; rational function; formula $f(f^{-1}(y) * x)$; fuzzy implication; copula

MSC: 03B52

check for updates

Citation: Mangenakis, P.G.; Papadopoulos, B. Innovative Methods of Constructing Strict and Strong Fuzzy Negations, Fuzzy Implications and New Classes of Copulas. *Mathematics* **2024**, *12*, 2254. https://doi.org/10.3390/math12142254

Academic Editor: Michael Voskoglou

Received: 6 June 2024
Revised: 12 July 2024
Accepted: 15 July 2024
Published: 19 July 2024

1. Introduction

It is well known that one of the most rapidly growing branches of modern applied mathematics is fuzzy logic and its objects. More and more applications of fuzzy implication and fuzzy negations are widespread; negations either through these implications or autonomously. Fuzzy implication is the generalization of classical (Boolean) implication in the interval [0, 1]. It plays perhaps the most important role in the field of fuzzy logic, decision theory and fuzzy control. The article presents new methods for constructing fuzzy negations [1–4]. Furthermore, the creation of new fuzzy implications [5–14], and through them, new fuzzy negations is necessary. Using the knowledge and information gained through the study of relevant writings [15–19], the article proceeds to study other areas of fuzzy logic. Other interesting objects of fuzzy logic are copulas [20–25]. Since new classes of negations and implications can be defined, the generated negations will be used to construct, additionally, two new classes of copulas.

Methodologically, this article's analysis begins in Section 2 by listing all the theorems and remarks that will be useful in the proof of the constructions below. Those definitions are listed in the order they are used. Definitions 1–5 relate to the construction of the fuzzy negations, Definitions 6 and 7 relate to the construction of fuzzy implications and Definitions 8–12 are helpful in the construction of copulas.

Therefore, for practical reasons, the first object to deal with is the construction of strong negations [1–4]. Some of the areas that strong negations apply are as follows:

1. Artificial Intelligence (AI): Particularly in designing systems that handle uncertain or imprecise information.
2. Control Systems: For instance, in developing controllers for complex systems like washing machines, air conditioners and automotive systems.

3. Decision Making: Assisting in multi-criteria decision-making processes where inputs are not clear-cut.
4. Pattern Recognition: Helping in classifying patterns that are not crisply defined.
5. Robotics: Enabling robots to handle ambiguous or uncertain environments.
6. Data Mining: For analyzing and interpreting data that are noisy or incomplete.
7. Natural Language Processing (NLP): Managing the inherent ambiguity and imprecision in human language.
8. Medical Diagnosis: Supporting systems that need to deal with uncertain or imprecise medical data.

All the fuzzy negations that will be presented will have the form of a multi-branch function and will be based on Definition 5 [1]. This definition constructs negations with the help of the equilibrium point. Two classes of strong fuzzy negations will be constructed. And every class of negations is followed by one example. The second example presents a class of negations with a special property: it makes it very easy to calculate the equilibrium point. This calculation will be obtained by solving a simple secondary equation.

What follows is the construction of a fuzzy implication [8–13,20,21,24] with an alternative way, without the use of fuzzy negations.

This will be achieved with the use of the **formula** $\mathbf{f}\left(\mathbf{f}^{-1}(\mathbf{y}) * \mathbf{x}\right)$. For every x, y into the interval [0, 1] and the use of a decreasing function f(x), the formula $f\left(f^{-1}(y) * x\right)$ helps to construct fuzzy implications. In addition, the formula constructs one branch of a strong fuzzy negation (Remark 1) and, autonomously, a strict fuzzy negation (Remark 2). The construction of the implication will be achieved with the use of Definitions 6 and 7. Let us mention some of the areas where fuzzy implications are important:

1. Artificial Intelligence (AI) and Machine Learning (Expert System, Knowledge Representation) 2. Control Systems (Fuzzy Control) 3. Decision Support System (Multi-Criteria Decision Making (MCDM), Risk Assessment) 4. Pattern Recognition and Image Processing (Classification, Image Segmentation) 5. Robotics (Autonomous Navigation, Sensor Fusion) 6. Natural Language Processing (NLP) (Semantic Analysis, Text Mining) 7. Medical Diagnosis and Healthcare (Diagnostic Systems, Treatment Planning) 8. Economics and Finance (Forecasting, Credit Scoring).

Finally, with the use of the fuzzy negations, newly constructed classes of copulas are built. Making some adjustments to the functions used in the construction of the negations, three-dimensional copulas [20–24] will be generated. Again, the formula $f\left(f^{-1}(y) * x\right)$ will participate in the construction of a class of copulas, with some minor modifications. It is well known that copulas find huge applications in economic problems, portfolio management and risk analysis, specifically in the banking, insurance and investing fields.

In conclusion, this article aims to answer the following questions:

(1) Is there an easy way to calculate the equilibrium point in a two-branch strong negation?
(2) Are there real functions that can provide at the same time the construction of strong negations, implications and copulas?
(3) Is there a point of convergence in the construction of fuzzy negations, fuzzy implications and copulas?
(4) Can the formula $f\left(f^{-1}(y) * x\right)$ provide more alternative options?
(5) Can this article provide knowledge for future use in robotics kai AI technology?

The paper is organized as follows: Section 2 is a reminder of the basic concepts and definitions used in the paper. Section 3 analyzes the newly constructed methods of strong fuzzy negations, fuzzy implications and copulas. One example for every theorem given is presented. Section 4 is about the discussion of the results, and Section 5 is the conclusion.

2. Materials and Methods

Some theorems of fuzzy logic and some definitions will be listed here. This will be conducted so that the theorems concerning the upcoming constructions will be explained and proved. To help the readers get familiar with the theory, some of the concepts and results employed in the rest of the paper shall be recalled below.

In this section all the theorems and propositions necessary to be able to present and fully prove the constructions we have mentioned above will be given.

Theorems from the whole range of the literature concerning the structural definitions of fuzzy negations, fuzzy implications and copulas will be given. In particular, Definitions 1–5 are concerned exclusively with the construction of negations, Definitions 6 and 7 are concerned with the construction of fuzzy implications, and Definitions 8–12 are concerned with the constructions to be presented in the area of copulas.

Note also that between Definitions 5 and 6 there is a table with reference to the most important and best-known classes of fuzzy negations.

At this point, a special bibliographical reference could be made to the articles on the construction of copulas, their properties, Archimedeans and fuzzy copulas [21,24,26].

Definition 1. (see [1–4,8–14] Definition 1.4.1). *The function* $N : [0,1] \to [0,1]$ *is a fuzzy negation if the following properties are applied:*

$$N(0) = 1, N(1) = 0 \tag{1}$$

$$N: \text{ is decreasing} \tag{2}$$

Definition 2. (see [1–4,8–14] Definition 1.4.2 (i)). *A fuzzy negation N is called strict if the following properties are applied:*

$$N \text{ is strictly decreasing} \tag{3}$$

$$N \text{ is continuous} \tag{4}$$

Definition 3. (see [1–4,8–14] Definition 1.4.2 (ii)). *A fuzzy negation N is called strong if*

$$N(N(x)) = x \tag{5}$$

Definition 4. (see [1–4,8–14] Definition 1.4.2 (ii)). *The solution of the equation* $N(x) = x$ *is called the equilibrium point of N. If the function N is continuous, the equilibrium point is unique.*

Definition 5 ([1]). *Strong branching fuzzy negations can be produced, while in every branch is a decreasing function. If* N_1 *is a fuzzy negation, which is not necessary, a strong negation and* $N_1(\varepsilon) = \varepsilon$ *where* ε *is the equilibrium point of* N_1*. So, if* N_1 *is any continuous fuzzy negation in the interval [0, 1], then the following form [12] product is strong fuzzy negations* N_2 *and, in our case, rational fuzzy negations (Figure 1).*

$$N_2(x) = \begin{cases} N_1(x) & , x \in [0, \varepsilon] \\ N_1^{-1}(x) , & x \in (\varepsilon, 1] \end{cases} \tag{6}$$

The above formula will be generalized by using two functions (f, g), one decreasing and one increasing.

Below is Table 1 listing the most well-known classes of fuzzy negations.

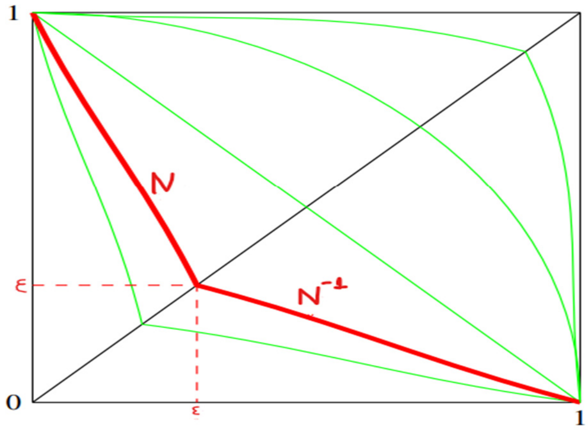

Figure 1. A random example of the form of the negation $N_2(x)$.

Table 1. Some examples of known negation classes.

Name	Fuzzy Negations
Yager class	$N^w(x) = (1 - x^w)^{\frac{1}{w}}$, $w > 0$
Threshold class	$N^t(x) = \begin{cases} 1, \ \alpha\nu \ x < t \\ 1 \ \acute{\eta} \ 0, \ \ \alpha\nu \ x = t, \ \ t \in (0,1) \\ 0, \ \ \alpha\nu \ x > t \end{cases}$
Standard negation	$N(x) = 1 - x$
The least fuzzy negation	$N_{D1}(x) = \begin{cases} 1, \ if \ x = 0 \\ 0, \ if \ x \in (0,1] \end{cases}$
The greatest fuzzy negation	$N_{D2}(x) = \begin{cases} 0, \ if \ x = 1 \\ 1, \ if \ x \in [0,1) \end{cases}$
Sugeno Class	$N_\delta(x) = \frac{1-x}{1+\delta x}, \ \ \delta > -1$

Fuzzy implications have probably become the most important operations in fuzzy logic, approximate reasoning and fuzzy control. These operators not only model fuzzy conditionals but also make inferences in any fuzzy rule-based system. These operators are defined as follows:

Definition 6 (see [8–14] Definition 1.1.1). *A function* $I : [0,1]^2 \to [0,1]$ *is called a fuzzy implication if it satisfies, for all* $x, x_1, x_2, \ y, \ y_1, y_2 \ \in [0,1]$, *the following conditions:*

$$x_1 \leq x_2 \Leftrightarrow I(x_1, y) \geq I(x_2, y), \ i.e., \ I(\cdot, y) \ is \ decreasing. \tag{7}$$

$$y_1 \leq y_2 \Leftrightarrow I(x, \ y_1) \leq \ I(x, \ y_2), \ \ i.e., \ I(x, \cdot) \ is \ increasing. \tag{8}$$

$$I(0,0) = 1 \tag{9}$$

$$I(1,1) = 1 \tag{10}$$

$$I(1,0) = 0 \tag{11}$$

Definition 7 (see [8–14] Definition 1.4.15 (ii)). *If I is a fuzzy implication, then the function* $N_I : [0,1] \to [0,1]$ with the form $N_I(x) = I(x,0)$ *is called natural negation of I.*

Definition 8 ([26]). *Let I be a nonempty interval of R. A function f from I to R is convex if, and only if,* $\frac{\partial^2 f}{\partial x^2} \geq 0$.

Definition 9 **([20–24,26]).** *A function* $C : [0,1]^2 \to [0,1]$ *is called a copula if it satisfies the following properties:*

$$C(0,t) = C(t,0) = 0 \quad \text{for each } 0 \le t \le 1 \tag{12}$$

$$C(1,t) = C(t,1) = t \quad \text{for each } 0 \le t \le 1 \tag{13}$$

The C-volume of a rectangle must be not negative, e.g.,

$$V_H = C(x_1,y_1) - C(x_1,y_2) - C(x_2,y_1) + C(x_2,y_2) \ge 0 \tag{14}$$

for each $x_1 \le x_2$ *and* $y_1 \le y_2$ *where* $0 \le x_1, x_2, y_1, y_2 \le 1$.

Definition 10 **([20–24,26]).** *If the function C is a copula, then the function in form* $C^*(x,y) = x + y - 1 + C(1-x, 1-y)$ *for each* $0 \le x, y \le 1$ *is also a copula, and it is called survival copula.*

Definition 11 **([20–24,26]).** *If f is a decreasing function where* $f(1) = 0$, *then we define the pseudo-inverse of function f*

$$\text{Given by } f^{[-1]} = \begin{cases} f^{-1}(x) & ,if \;\; 0 \le x \le f(0) \\ \\ 0 & ,if \;\; f(0) \le x \le \infty \end{cases} \tag{15}$$

Definition 12 **([20–24,26]).** *Let* $f : [0,1] \to [0,\infty]$ *be a continuous, strictly decreasing and convex function such that* $f(1) = 0$, *and let* $f^{[-1]}$ *be the pseudo-inverse. Let* $C : [0,1] \to [0,1]$, *defined by*

$$C(x,y) = f^{[-1]}(f(x) + f(y)) \tag{16}$$

Then, C is an Archimedean Copula.

3. Results

In this section, this article will present all the constructions resulting from the use of the definitions in the previous section. All the proofs will be presented in detail, with mathematical relations and explanations. In total, five theorems and an interesting proof on the third property of copulas (increasing with respect to the variables x, y) will be presented.

The first theorem proves that a class of multi-branching functions will be a strong fuzzy negation. An example of this follows. The second theorem presents another class of possible fuzzy negations. This second class with some adjustments presents the formula $\mathrm{f}\left(\mathrm{f}^{-1}(\mathrm{y}) * \mathrm{x}\right)$. The following example gives a class of strong fuzzy negations that presents great ease in finding the equilibrium point.

The third proof concerns the presentation of a fuzzy implication using the same formula $\mathrm{f}\left(\mathrm{f}^{-1}(\mathrm{y}) * \mathrm{x}\right)$, avoiding the use of some fuzzy negation.

This article then moves on to the spectrum of copulas.

First, a proof of some classes of copulas will be presented.

This will be followed by the proofs of two propositions on the definition of copulas and, finally, another proof of a class of copulas containing the formula $\mathrm{f}\left(\mathrm{f}^{-1}(\mathrm{y}) * \mathrm{x}\right)$.

3.1. New Forms of Strong Fuzzy Negations

Strong branching fuzzy negations can be produced [1] while in every branch there is a decreasing function. Let N_1 be a fuzzy negation, not necessarily a strong negation, and $\mathrm{N}_1(\varepsilon) = \varepsilon$ where ε is the equilibrium point of N_1. So, if N_1 is any continuous fuzzy negation in the interval [0, 1], then the following form [12] produces strong fuzzy negations N_2 and, in our case, rational fuzzy negations.

$$N_2(x) = \begin{cases} \boldsymbol{N_1(x)} & , \boldsymbol{x \in [0, \varepsilon]} \\ \boldsymbol{N_1^{-1}(x)} & , \boldsymbol{x \in (\varepsilon, 1]} \end{cases} \tag{17}$$

In the Figure 2 below, consider $\boldsymbol{N_1(x) = f(x)}$ and $\boldsymbol{N_1^{-1}(x) = f^{-1}(x)}$

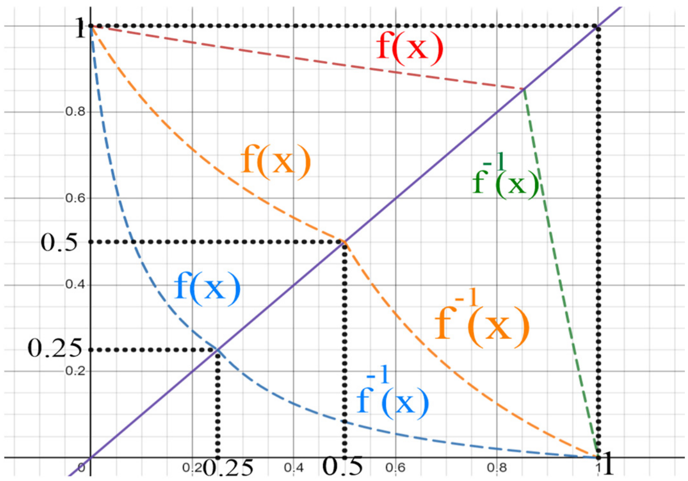

Figure 2. The graph of the negation $N_2(x)$ for three random values of the ε when ε = 0.25, ε = 0.5 and ε = 1.

The above formula will be generalized using two functions (f, g), one decreasing and one increasing.

Generating Classes of Strong Fuzzy Negations

Theorem 1. *Let ε the equilibrium point of N_{PM1}, $f : [0, 1] \to [0, +\infty)$ continuous decreasing function and $g : [0, 1] \to [0, +\infty)$ continuous increasing function with the conditions: f^{-1}, g^{-1} well defined, $f(0) = 1$ and $g(0) = 0$ and $k > 0$ positive real number. Then, the following form is a class of strong fuzzy negations:*

$$\boldsymbol{N_{PM1}}(x) = \begin{cases} \boldsymbol{f(g(x) * k)}, & \boldsymbol{0 \le x \le \varepsilon} \\ \boldsymbol{g^{-1}\left(\frac{f^{-1}(x)}{k}\right)}, & \boldsymbol{\varepsilon < x \le 1} \end{cases} \tag{18}$$

Proof of Theorem 1. The proof that the class of strong negations above is a continuous function will be given first. It is obvious that what is examined is the continuity in the equilibrium point. Assuming that all the values x of the equilibrium points are the solution of the equation $N_{PM1}(x) = x$, then it implies:

$f(g(x) * k) = \mathrm{x} \Leftrightarrow f^{-1}(f(g(x) * k)) = f^{-1}(x) \Leftrightarrow g(x) * k = f^{-1}(x)$, for $k > 0$ implies that $g(x) = \frac{f^{-1}(x)}{k}$ and, finally, $x = g^{-1}\left(\frac{f^{-1}(x)}{k}\right)$. That proves that the two multi-branched functions intersect on the line y = x at the equilibrium point. That proves that the negation $N_{PM1}(x)$ is a continuous function in the interval [0, 1].

Boundary conditions

- For $\mathrm{x} \le \varepsilon$ implies that $N_{PM1}(x) = f(g(x) * k)$

$$N_{PM1}(0) = f(g(0) * k) = f(0) = 1$$

- For x > ε implies that $N_{PM1}(1) = g^{-1}\left(\frac{f^{-1}(1)}{k}\right) = g^{-1}(0) = 0$
  Monotony condition
- For $x \le \varepsilon$ implies that $N_{PM1}(x) = f(g(x) * k)$
  For every $x_1, x_2 \in [0, \varepsilon]$ where

$$x_1 \le x_2 \overset{g\nearrow}{\Leftrightarrow} g(x_1) \le g(x_2)$$

$$\text{thus } k > 0, g(x_1) * k \le g(x_2) * k \overset{f\searrow}{\Leftrightarrow} f(g(x_1) * k) \ge f(g(x_1) * k)$$
$$\Leftrightarrow N_{PM1}(x_1) \ge N_{PM1}(x_2)$$

  So that proves that N_{PM1} is decreasing when $x \le \varepsilon$.
- For $x > \varepsilon$ implies that $N_{PM1}(x) = g^{-1}\left(\frac{f^{-1}(x)}{k}\right)$
  For every $x_1, x_2 \in (\varepsilon, 1]$ where

$$\text{for } x_1 \le x_2 \Leftrightarrow f^{-1}(x_1) \ge f^{-1}(x_2) \text{ and for } \frac{1}{k} > 0 \text{ arises :}$$

$$: \frac{f^{-1}(x_1)}{k} \ge \frac{f^{-1}(x_2)}{k} \text{ and finally}$$
$$\overset{g^{-1}\nearrow}{\Leftrightarrow} g^{-1}\left(\frac{f^{-1}(x_1)}{k}\right) \ge g^{-1}\left(\frac{f^{-1}(x_2)}{k}\right).$$

So $N_{PM1}(x_1) \ge N_{PM1}(x_2)$. That concludes that N_{PM1} is decreasing when x > ε.
Synthesis condition
The most important condition for a negation to be strong is as follows:

$$N_{PM1}(N_{PM1}(x)) = x$$

Because of the way the negation class is constructed, the set of values of one branch is mapped to the definition domain of the other branch. Thus, when synthesizing the negation with itself, the type of one branch inside the other is placed and vice versa.

That equals the following:

$$f\left(g\left(g^{-1}\left(\frac{f^{-1}(x)}{k}\right)\right) * k\right) = f\left(\frac{f^{-1}(x)}{k} * k\right) = f\left(f^{-1}(x)\right) = x.$$

And vice versa the following:

$$g^{-1}\left(\frac{f^{-1}(f(g(x) * k))}{k}\right) = g^{-1}\left(\frac{g(x) * k}{k}\right) = g^{-1}(g(x)) = x.$$

The class of negations is continuous as an operation of continuous functions, and it holds that for any $x \in [0, 1]$ then $N_{PM1}(N_{PM1}(x)) = x$. Therefore, this class of fuzzy negations is a strong one. □

Example 1. *One example of fuzzy negations is presented, generated by Theorem 1.*

Let the decreasing function f(x) be the function f(x) = $\frac{1}{x+1}$. It is easy to check that f(x) is positive, decreasing, f (0) = 1 and continuous. Let g(x) = $\sqrt{x}$, g(x) ≥ 0 and increasing, g(0) = 0 and k > 0. This means that $\boldsymbol{f(g(x)*k)}=\frac{1}{k*\sqrt{x}+1}$. Let us now construct the fuzzy negation proved before, which will be the following:

$$N^k{}_{PM1}(x) = \begin{cases} \frac{1}{k*\sqrt{x}+1}\ , & \boldsymbol{0 \leq x \leq \varepsilon} \\ \\ (\frac{1-x}{kx})^2, & \boldsymbol{\varepsilon < x \leq 1} \end{cases} \tag{19}$$

where ε is the equilibrium point. Quite easily can someone find out that $N^k{}_{PM1}(x)$ completes all the conditions needed.

Boundary conditions

$$N^k{}_{PM1}(0) = \frac{1}{k*\sqrt{0}+1} = 1/1 = 1 \text{ and } \ N^k{}_{PM1}(1) = \left(\frac{1-1}{k*1}\right)^2 = 0$$

Monotony conditions

- For $x \leq \varepsilon$, every $x_1 \leq x_2 \Leftrightarrow$

$$\sqrt{x_1} \leq \sqrt{x_2} \ \Leftrightarrow\ k*\sqrt{x_1} \leq k*\sqrt{x_2} \quad \Leftrightarrow k*\ \sqrt{x_1}+1 \leq k*\sqrt{x_2}+\ 1 \Leftrightarrow$$

$$\frac{1}{k*\sqrt{x_1}+1} \geq \frac{1}{k*\sqrt{x_2}+1} \Leftrightarrow\ N^k{}_{PM1}(x_1) \geq\ N^k{}_{PM1}(x_2) \text{ so, it is decreasing.}$$

- For $\varepsilon < x$ every $\mathrm{x}_1 \leq \mathrm{x}_2 \Leftrightarrow$

$1-x_1\ \geq\ 1-x_2$ (1) and again $x_1 \leq x_2 \Leftrightarrow k*x_1 \leq k*x_2 \Leftrightarrow \frac{1}{k*x_1} \geq \frac{1}{k*x_2}$ (2) I multiply (1) and (2):

$$\frac{1-x_1}{k*x_1} \geq \frac{1-x_2}{k*x_2} \Leftrightarrow (\frac{1-x_1}{kx_1})^2 \geq (\frac{1-x_2}{kx_2})^2 \ \Leftrightarrow\ N^k{}_{PM1}(x_1) \geq\ N^k{}_{PM1}(x_2)$$

it is decreasing.

Synthesis condition

The most important condition for a negation to be strong is the following:

$$N_{PM1}(N_{PM1}(x)) = x$$

Again, because of the way the negation class is constructed, the set of values of one branch is mapped to the definition domain of the other branch. Thus, when synthesizing the negation with itself, the type of one branch is placed inside the other and vice versa.

- For $\leq \varepsilon$:

$$N_{PM1}(N_{PM1}(x)) = \frac{1}{k*\sqrt{(\frac{1-x}{kx})^2}+1} = \frac{1}{k*\ \frac{1-x}{kx}+1} = \frac{1}{\frac{1-x}{x}+1} = \frac{1}{\frac{1}{x}} = x.$$

And vice versa:

- For $\varepsilon < x$

$$N_{PM1}(N_{PM1}(x)) = (\frac{1-\frac{1}{k*\sqrt{x}+1}}{k\frac{1}{k*\sqrt{x}+1}})^2 = (\frac{\frac{k*\sqrt{x}}{k*\sqrt{x}+1}}{k\frac{1}{k*\sqrt{x}+1}})^2 = (\sqrt{x})^2 = x$$

Theorem 2. *Let ε the equilibrium point of N_{PM2}, $f:[0,1] \to [0,+\infty)$ continuous decreasing function and $k > 0$ with the following conditions:*

f^{-1} *well defined,* $f(0) = 1$*. Then, the following form is a class of strong fuzzy negations:*

$$\boldsymbol{N_{PM2}(x)} = \begin{cases} \boldsymbol{f(k*x)}, & \boldsymbol{0 \le x \le \varepsilon} \\ \boldsymbol{\frac{f^{-1}(x)}{k}}, & \boldsymbol{\varepsilon < x \le 1} \end{cases} \tag{20}$$

Proof of Theorem 2. First of all, the proof that the class of strong negations above is a continuous function must be given. It is obvious that the continuity in the equilibrium point must be examined. So, let someone assume that all the values x of the equilibrium points are the solution of the equation $N_{PM2}(x) = x$. That implies the following:

$f(k*x) = x \Leftrightarrow f^{-1}(f(k*x)) = f^{-1}(x) \Leftrightarrow k*x = f^{-1}(x)$, for g(x) > 0 then $x = \frac{f^{-1}(x)}{k}$. That proves that the two bifurcated functions intersect on the line y = x at the equilibrium point. That proves that the negation $N_{PM2}(x)$ is a continuous function in the interval [0, 1].

Boundary conditions

- For $x \le \varepsilon$ implies that

$$N_{PM2}(x) = f(k*x) \Leftrightarrow N_{PM2}(0) = f(k*0) = f(0) = 1$$

- For $x > \varepsilon$ implies that $N_{PM2}(1) = \left(\frac{f^{-1}(1)}{k}\right) = \frac{0}{k} = 0$

Monotony condition

- For $x \le \varepsilon$ implies that $N_{PM2}(x) = f(k*x)$
For every $x_1, x_2 \in [0, \varepsilon]$ where

$$x_1 \le x_2$$

$$\text{and get } k*x_1 \le k*x_2 \overset{f\searrow}{\Leftrightarrow} f(k*x_1) \ge f(k*x_1) \Leftrightarrow N_{PM2}(x_1) \ge N_{PM2}(x_2)$$

So, it is concluded that $\mathrm{N_{PM2}}$ is decreasing when $x \le \varepsilon$.

- For $x > \varepsilon$ then $N_{PM2}(x) = \frac{f^{-1}(x)}{K}$
For every $x_1, x_2 \in (\varepsilon, 1]$ where

$$\text{for } x_1 \le x_2 \overset{f^{-1}\searrow}{\Leftrightarrow} f^{-1}(x_1) \ge f^{-1}(x_2)$$

$$\text{so, we have}: \frac{f^{-1}(x_1)}{k} \ge \frac{f^{-1}(x_2)}{k} \text{ and finally,}$$

$$N_{PM2}(x_1) \ge N_{PM2}(x_2)$$

So, we conclude that N_{PM2} is decreasing when $x > \varepsilon$.

Synthesis condition

The most important condition for a negation to be strong is the following:

$$N_{PM2}(N_{PM2}(x)) = x$$

Again, because of the way the negation class we are studying is constructed, the set of values of one branch is mapped to the definition domain of the other branch. Thus, when we synthesize the negation with itself, we place the type of one branch inside the other and vice versa.

So, we have: $f\left(\frac{f^{-1}(x)}{k} * k\right) = f(f^{-1}(x)) = x$. And vice versa, it is the following:

$$\frac{f^{-1}(f(x*k)\ )}{k} = \frac{x*k}{k} = x.$$

The class of negations is continuous as an operation of continuous functions, and it holds that for any $x \in [0, 1]$ then $N_{PM2}(N_{PM2}(x)) = x$. Therefore, this class of fuzzy negations is a strong one. □

Remark 1. *Let k = $f^{-1}(y) > 0$, for every $0 \leq y < 1$ then $0 < f^{-1}(y) \leq 1$, so the negation takes the following form:*

$$\mathbf{N_{PM2}(x)} = \begin{cases} \mathbf{f\left(f^{-1}(y)*x\right)}, & \mathbf{0 \leq x \leq \varepsilon} \\ \frac{\mathbf{f^{-1}(x)}}{\mathbf{f^{-1}(y)}}, & \mathbf{\varepsilon < x \leq 1} \end{cases} \tag{21}$$

This is a strong fuzzy negation.

Remark 2. *For y = 0 and f(1) = 0 the function $N_{PM2}(x) = f\left(f^{-1}(y) * x\right)$ is a strict fuzzy negation.*

Remark 3. *We will examine later the form $f\left(f^{-1}(y) * x\right)$, which we will prove is a fuzzy implication. The same form of the g(x) function, $g^{-1}(g(y) * x)$ will take part in the construction of a copula.*

Example 2. *One example of fuzzy negations is presented, generated by Theorem 2.*

Let the decreasing function f(x) be the function f(x) = $\frac{1}{x+1}$. It is easy to check that f(x) is positive, decreasing, f (0) = 1 and continuous. Let k > 0. This means that f(k*x) = $\frac{1}{k*x+1}$. Let us now construct the fuzzy negation proved before, which will be the following:

$$\boldsymbol{N^k{}_{PM2}(x)} = \begin{cases} \frac{\mathbf{1}}{\boldsymbol{kx+1}}, & \boldsymbol{0 \leq x \leq \varepsilon} \\ \frac{\boldsymbol{1-x}}{\boldsymbol{kx}}, & \boldsymbol{\varepsilon < x \leq 1} \end{cases} \tag{22}$$

where ε is the equilibrium point. This means that finding the point x=ε is the target. To achieve this someone has to solve the equation $\frac{1}{kx+1}$ = x ⇔ $kx^2 + x - 1$, which is a second-degree equation. We use the type

$$x_{1,2} = \frac{-1 \pm \sqrt{4k+1}}{2k}$$

where the one solution is rejected x_2 = $\frac{-1-\sqrt{4k+1}}{2k}$ because it is negative. That means ε =$\frac{-1+\sqrt{4k+1}}{2k}$. So, the formula takes the form of: $if\ \varepsilon = \frac{-1\pm\sqrt{4k+1}}{2k}$ then

$$\boldsymbol{N^k{}_{PM2}(x)} = \begin{cases} \frac{\mathbf{1}}{\boldsymbol{kx+1}}, & \boldsymbol{0 \leq x \leq \frac{-1+\sqrt{4k+1}}{2k}} \\ \frac{\boldsymbol{1-x}}{\boldsymbol{kx}}, & \boldsymbol{\frac{-1+\sqrt{4k+1}}{2k} < x \leq 1} \end{cases} \tag{23}$$

In this way, a strong negation has been constructed in which someone can quite easily calculate the equilibrium point. That way, negations can exist that satisfy many types of implications or other problems while the class of negations created has a great range of values, easily calculated. For example (Figure 3):

- For k = 12, we calculate that ε = 0.25 and appears at the graph $N^{12}{}_{PM}(x)$.
- For k = 2, we calculate that ε = 0.5 and appears at the graph $N^{2}{}_{PM}(x)$.
- For k = 0.3125, we calculate that ε = 0.8 and appears at the graph $N^{0.3125}{}_{PM}$(x)

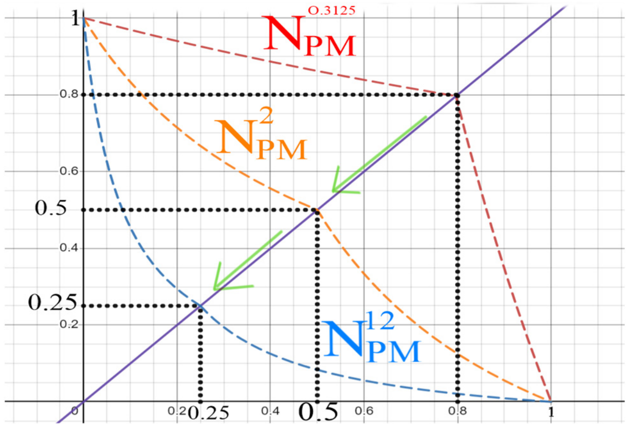

Figure 3. The graph of three specific examples of the negation ${N^k}_{PM2}(x)$.

3.2. Constructing Non-Symmetric Fuzzy Implications without the Use of Fuzzy Negations

Already mentioned above are the conditions that must be met for a function to be a fuzzy implication. In this section, there will be presented a construction of a fuzzy implication, non-symmetric and without the use of fuzzy negation. To perform this, it is necessary to use one of the two functions we have already used so far. The function f(x), which is strictly decreasing, is continuous, and f(0) = 1.

Theorem 3. *Let the function s f, f^{-1} continuous, well defined, then $I:[0,1]^2 \to [0,1]$ as* ***I(x,y) = $f\left(f^{-1}(y)*x\right)$****, and f(x) decreasing and f(0) = 1. Then, I(x,y) is a fuzzy implication.*

Definition 13 ([20–24,26]). *A function $I:[0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies, for all $x, x_1, x_2,\ y\ ,\ y_1, y_2\ \in [0,1]$, the following conditions:*

$$(a)\ \ x_1 \le x_2 \Leftrightarrow I\left(x_{1\,,}\, y\right) \ge I\left(x_{2\,,}\, y\right), \text{ i.e., } I(\cdot, y) \text{ is decreasing.}$$

$$(b)\ \ y_1 \le y_2 \Leftrightarrow I(x,\ y_1) \le\ I(x,\ y_2),\ \text{ i.e., } I(x, \cdot) \text{ is increasing.}$$

$$(c)\ \ I(0,0) = 1$$

$$(d)\ \ I(1,1) = 1$$

$$(e)\ \ I(1,0) = 0$$

Proof of Theorem 3.

(a) For every $x_1 \le x_2 \Leftrightarrow f^{-1}(y)\, x_1\ \ \le f^{-1}(y)x_2$ for $f^{-1}(y)\ \ge\ 0 \overset{f\searrow}{\Leftrightarrow}\ f(f^{-1}(y)x_1)\ \ge f(f^{-1}(y)x_2)$ so, $I(\ x_1{,}y) \ge I(x_2, y)$

(b) For every $y_1\ \le\ y_2 \overset{f^{-1}\searrow}{\Leftrightarrow} f^{-1}(y_1)\ \ge\ f^{-1}(y_2) \Leftrightarrow f^{-1}(y_1)x\ \ge\ f^{-1}(y_2)x$ for $x \ge 0$ $\overset{f\searrow}{\Leftrightarrow}\ f(f^{-1}(y_1)x) \le f(f^{-1}(y_2)x)$ so, $I(\ x,\ y_1) \le I(x, y_2)$

(c) $I(0{,}0) = f\left(f^{-1}(0)*0\right) = f(0) = 1$

(d) $I(1{,}1) = f\left(f^{-1}(1)*1\right) = f\left(f^{-1}(1)\right) = 1$

(e) $I(1{,}0) = f\left(f^{-1}(0)*1\right) = f\left(f^{-1}(0)\right) = 0.$ □

This verifies all five properties of fuzzy implications. So, our function ***I(x,y)*** $= \boldsymbol{f\left(f^{-1}(y)*x\right)}$ is a non-symmetric fuzzy implication.

Example 3. *Now, one example will be presented of the theorem above.*

Let the function f(x) = $\sqrt{1-(\frac{x}{2})}$, which is decreasing, f(0) = 1 and f$^{-1}(x) = 2*(1-x^2)$ be well defined in the interval [0, 1]. Let us prove that *I(x,y)* = $\sqrt{1-(1-y^2)*x}$ is a fuzzy implication.

(1) For every $x_1 \le x_2 \Leftrightarrow 2*(1-y^2)*x_1 \le 2*(1-y^2)*\ x_2 \overset{f\searrow}{\Leftrightarrow} \sqrt{1-(1-y^2)*\ x_1} \ge \sqrt{1-(1-y^2)*\ x_2}$ *so* $I(\ x_1,y) \ge I(x_2, \mathrm{y})$

(2) For every $y_1 \le y_2 \overset{f^{-1}\searrow}{\Leftrightarrow} 2*(1-{y_1}^2)*x \ge 2*(1-{y_2}^2)*x$ *for* $x \ge 0 \overset{f\searrow}{\Leftrightarrow} \sqrt{1-(1-{y_1}^2)*x} \le \sqrt{1-(1-{y_2}^2)*x}$ *so* $I(x,\ y_1) \le I(x,y_2)$

(3) *I*(0,0) = $\sqrt{1-(1-0^2)*0}$ = $\sqrt{1}$ = 1

(4) *I*(1,1) = $\sqrt{1-(1-1^2)*1}$ = $\sqrt{1}$ = 1

(5) *I*(1,0) = $\sqrt{1-(1-0^2)*1}$ = $\sqrt{1-1} = 0$. That means that *I(x,y)* = $\sqrt{1-(1-y^2)*x}$ is a fuzzy implication.

3.3. Generating Copulas Using the Same Functions

In this section, the construction of copulas will be given, functions that are known from their applications in economics and risk analysis, as well as in fuzzy logic in general. So far, our constructions were based on two specific functions called f(x) and g(x) and given some properties. Now, exactly the same functions will be used to construct the copulas. In some of these cases, some additional properties will be given.

Theorem 4. *Let the function* $g:[0,1] \to [0,+\infty)$ *be continuous, strictly increasing and convex, g(0) = 0, g(1) = 1, with* g^{-1} *continuous. The function* $C_1:[0,1]^2 \to [0,1]$, *when* C_1*(x,y) =* $g\left(g^{-1}(x)*g^{-1}(y)\right)$ *is a copula with the symmetric and incentive effect.*

Proof of Theorem 4. Let us remember that there are three conditions that make C(x, y) a copula.

(1) $C(0,t) = C(t,0) = 0$ *for each* $0 \le t \le 1$

(2) $C(1,t) = C(t,1) = t$ *for each* $0 \le t \le 1$

(3) The C-volume of a rectangle must be not negative, e.g.,

$$\mathrm{V_H} = \mathrm{C}(\mathrm{x}_1,\mathrm{y}_1) - \mathrm{C}(\mathrm{x}_1,\mathrm{y}_2) - \mathrm{C}(\mathrm{x}_2,\mathrm{y}_1) + \mathrm{C}(\mathrm{x}_2,\mathrm{y}_2) \ge 0$$

for each $\mathrm{x}_1 \le \mathrm{x}_2$ and $\mathrm{y}_1 \le \mathrm{y}_2$ where $0 \le \mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2 \le 1$.

For the proof of the first condition after replacing the following:

(1) $C_1(t,0) = g\left(g^{-1}(t)*g^{-1}(0)\right) = g\left(g^{-1}(t)*0\right) = g(0) = 0$
$C_1(0,t) = g\left(g^{-1}(0)*g^{-1}(t)\right) = g\left(0*g^{-1}(t)\right) = g(0) = 0$

(2) $C_1(t,1) = g\left(g^{-1}(t)*g^{-1}(1)\right) = g\left(g^{-1}(t)*1\right) = g\left(g^{-1}(t)\right) = t$
$C_1(1,t) = g\left(g^{-1}(1)*g^{-1}(t)\right) = g\left(1*g^{-1}(t)\right) = g\left(g^{-1}(t)\right) = t$

(3) There are two options for proving the third property. If the function g(x) is productive, then it is relatively easy to prove the third property, provided that the derivative $\frac{\partial^2 C(x,y)}{\partial xy} \ge 0$ is positive. But, if the function g(x) is not productive, then the proof becomes much more complex and difficult. Both cases will be listed. □

Proposition 1. *Knowing that for a function to be 2-increasing, must satisfy the inequality* $C(x_1,y_1) + C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) \ge 0$. *This inequality is equivalent to* $\frac{\partial^2 C(x,y)}{\partial xy} \ge 0$ *when C is a differentiable function.*

Proof of Proposition 1. When someone applies the Mean Value Theorem for the function $\mathrm{C}(\mathrm{x},\mathrm{y}_1)$ in the interval $[x_1,x_2]$

$$\exists\ \xi_1\ \in (x_1,x_2): \frac{\partial C(\xi_1\ ,y_1)}{\partial x} = \frac{C(x_2,y_1) - C(x_1,y_1)}{x_2 - x_1}$$

Applying the Mean Value Theorem for the function $C(x, y_2)$ in the interval $[x_1, x_2]$

$$\exists\, \xi_1 \in (x_1, x_2) : \frac{\partial C(\xi_1, y_2)}{\partial x} = \frac{C(x_2, y_2) - C(x_1, y_2)}{x_2 - x_1}$$

Let us suppose that $C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \geq 0$ then

$$\frac{\partial C(\xi_1, y_2)}{\partial x} - \frac{\partial C(\xi_1, y_1)}{\partial x} \geq 0 \Leftrightarrow \frac{\partial^2 C(x, y)}{\partial xy} \geq 0.$$

□

(a) Considering $g(x)$ is convex, we have that $(g)''(x) \geq 0$. So, we have the following:

$$\frac{\partial C(x,y)}{\partial x} = (g)\prime\left(g^{-1}(x) * g^{(-1)}(y)\right) * \left[g^{-1}\right]\prime(x) * g^{-1}(y).$$

And then,

$$\frac{\partial^2 C(x,y)}{\partial xy} =$$
$$(g)''\left(g^{-1}(x) * g^{-1}(y)\right) * g^{-1}(x) * \left[g^{-1}\right]\prime(y) * [g^{-1}]\prime(x) * g^{-1}(y) +$$
$$(g)\prime\left(g^{-1}(x) * g^{-1}(y)\right) \left[g^{-1}\right]\prime(x) * \left[g^{-1}\right]\prime(y) =$$
$$[g^{-1}]\prime(x) * [g^{-1}]\prime(y) * [\, g^{-1}(x) * g^{-1}(y) * (g)''\left(g^{-1}(x) * g^{-1}(y)\right) +$$
$$(g)\prime\left(g^{-1}(x) * g^{-1}(y)\right)] \geq 0$$

Indeed, because $g(x) \geq 0$, $g'(x) \geq 0$, $\left(g^{-1}\right)\prime \geq 0$, $(g)'' \geq 0$, $\left(g^{-1}\right) \geq 0$.

(b) If function g(x) is not productive, the proof of the third property becomes very difficult and interesting and comes with the help of the classical definition of convexity.

Definition 14. *A function $f : A \to \mathbb{R}$ is convex if, for all (x, y) in the domain of f, and for all t in [0, 1] when the inequality*

$$(f(t * x + (1 - t) * y) \leq t * f(x) + (1 - t) * f(y) \tag{24}$$

holds.

Knowing that the copula we constructed is a three-dimensional function, the definition is the following:

Definition 15. *A function $f : A^2 \to \mathbb{R}$ is convex if, for all points (x_1, y_1), (x_2, y_2) in the domain of f (x), and for all t ∈ [0, 1] when the inequality*

$$f(t * x_1 + (1 - t) * x_2,\ t * y_1 + (1 - t) * y_2) \leq t * f(x_1, y_1) + (1 - t) * f(x_2, y_2) \text{ holds} \tag{25}$$

Proposition 2. *Knowing that for a function to be 2-increasing, must satisfy the inequality $C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \geq 0$. If g(x) is convex and strictly increasing yet non-productive, then the function $C_1(x, y) = g\left(g^{-1}(x) * g^{-1}(y)\right)$ is 2-increasing.*

Proof of Proposition 2. First of all, let us give a proof that if $g(x)$ is convex, then C(x, y) is convex. Let

$C_1(x, y) = g\left(g^{-1}(x) * g^{-1}(y)\right)$, $0 \leq x \leq 1$, $0 \leq y \leq 1$, which also means that $0 \leq g^{-1}(x) \leq 1$, $0 \leq g^{-1}(y) \leq 1$.

Also, g(x) is strictly increasing $\overset{g\nearrow}{\Leftrightarrow}$, so let $g^{-1}(x) = u$ and $g^{-1}(y) = w$. That means that $0 \leq u * w \leq 1$.

So, let $u * w = v$. That makes $C_1(x, y) = g\left(g^{-1}(x) * g^{-1}(y)\right) = g(u * w) = g(v)$, which proves that if $g(x)$ is convex, then $C_1(x, y)$ is convex.

Due to the monotony conditions, it is proved that

For every pair of $y_1, y_2 \in A^2$, when

$$\begin{gathered} y_1 \leq y_2 \overset{g^{-1}\nearrow}{\Leftrightarrow} g^{-1}(y_1) \leq g^{-1}(y_2) \Leftrightarrow g^{-1}(x_1) g^{-1}(y_1) \leq g^{-1}(x_1)\, g^{-1}(y_2) \overset{g^{-1}\nearrow}{\Leftrightarrow} \\ g\left(g^{-1}(x_1) * g^{-1}(y_1)\right) \leq g\left(g^{-1}(x_1) * g^{-1}(y_2)\right) \Leftrightarrow C_1(x_1, y_1) \leq C_1(x_1, y_2) \end{gathered} \tag{26}$$

For every pair of $x_1, x_2 \in A^2$, when

$$\begin{gathered} x_1 \leq x_2 \overset{g^{-1}\nearrow}{\Leftrightarrow} g^{-1}(x_1) \leq g^{-1}(x_2) \Leftrightarrow g^{-1}(x_1) g^{-1}(y_1) \leq g^{-1}(x_2)\, g^{-1}(y_1) \overset{g\nearrow}{\Leftrightarrow} \\ g\left(g^{-1}(x_1) * g^{-1}(y_1)\right) \leq g\left(g^{-1}(x_2) * g^{-1}(y_1)\right) \Leftrightarrow C_1(x_1, y_1) \leq C_1(x_2, y_1) \end{gathered} \tag{27}$$

For every pair of $y_1, y_2 \in A^2$, when

$$\begin{gathered} y_1 \leq y_2 \overset{g^{-1}\nearrow}{\Leftrightarrow} g^{-1}(y_1) \leq g^{-1}(y_2) \Leftrightarrow g^{-1}(x_2) g^{-1}(y_1) g^{-1}(x_2)\, g^{-1}(y_2) \overset{g^{-1}\nearrow}{\Leftrightarrow} \\ g\left(g^{-1}(x_2) * g^{-1}(y_1)\right) \leq g\left(g^{-1}(x_2) * g^{-1}(y_2)\right) \Leftrightarrow C_1(x_2, y_1) \leq C_1(x_2, y_2) \end{gathered} \tag{28}$$

For every pair of $x_1, x_2 \in A^2$, when

$$\begin{gathered} x_1 \leq x_2 \overset{g^{-1}\nearrow}{\Leftrightarrow} g^{-1}(x_1) \leq g^{-1}(x_2) \Leftrightarrow g^{-1}(x_1) g^{-1}(y_2) \leq g^{-1}(x_2)\, g^{-1}(y_2) \overset{g\nearrow}{\Leftrightarrow} \\ g\left(g^{-1}(x_1) * g^{-1}(y_2)\right) \leq g\left(g^{-1}(x_2) * g^{-1}(y_2)\right) \Leftrightarrow C_1(x_1, y_2) \leq C_1(x_2, y_2) \end{gathered} \tag{29}$$

Using the four inequalities above, someone can build the inequalities below:

$$C_1(x_1, y_1) \leq C_1(x_1, y_2) \leq C_1(x_2, y_2) \tag{30}$$

$$C_1(x_1, y_1) \leq C_1(x_2, y_1) \leq C_1(x_2, y_2) \tag{31}$$

multiplying relation (30) by t, $t \in [0, 1]$ and relation (31) by $(1 - t)$, by $(1 - t) \in [0, 1]$, and so

$$t * C_1(x_1, y_1) \leq t * C_1(x_1, y_2) \leq t * C_1(x_2, y_2) \tag{32}$$

$$(1 - t) * C_1(x_1, y_1) \leq (1 - t) * C_1(x_2, y_1) \leq (1 - t) * C_1(x_2, y_2) \tag{33}$$

Adding by members the inequalities (32) and (33) it implies the following:

$$C_1(x_2, y_2) \geq t * C_1(x_1, y_2) + (1 - t) * C_1(x_2, y_1) \geq C_1(x_1, y_1) \tag{34}$$

Knowing that the function C(x, y) is a continuous function inside the domain of $[x_1, y_1] \times [x_2, y_2]$, someone can make use of the intermediate value theorem, which means that for every $t \in [0, 1]$ there are points $(x_t, y_t) \in [x_1, y_1] \times [x_2, y_2]$ so that

$$C_1(x_t, y_t) = t * C_1(x_1, y_2) + (1 - t) * C_1(x_2, y_1) \tag{35}$$

Remembering the definition of convexity for points (x_1, y_1), (x_2, y_2) in the domain of f(x) and for all $t \in [0, 1]$, then

$$C_1(t * x_1 + (1 - t) * x_2,\ t * y_1 + (1 - t) * y_2) \geq t * C_1(x_1, y_1) + (1 - t) * C_1(x_2, y_2) \tag{36}$$

It is assumed, without the limitation of generality and knowing, that the function C(x, y) is symmetric

$(C_1(x, y) = C_1(y, x))$ that for every $t \in [0, 1]$ and every $x_1 \leq x_t \leq x_2$, $y_1 \leq y_t \leq y_2$ that

$x_t = t * x_1 + (1 - t) * x_2$ and $y_t = t * y_1 + (1 - t) * y_2$. Thus, substituting in relation (36) we obtain the following:

$C_1(x_t, y_t) \leq t * C_1(x_1, y_1) + (1-t) * C_1(x_2, y_2)$. Now the substitute from relation (35) and obtain:

$$t * C_1(x_1, y_2) + (1-t) * C_1(x_2, y_1) \leq t * C_1(x_1, y_1) + (1-t) * C_1(x_2, y_2) \tag{37}$$

Relation (37) stands for every t ∈ [0, 1], so for t = 0.5, we obtain the following:

$$(0.5) * C(x_1, y_2) + (0.5) * C(x_2, y_1) + (0.5) * C(x_1, y_1) + (0.5) * C(x_2, y_2).$$

Multiply by two, and finally

$$C_1(x_1, y_2) + C_1(x_2, y_1) \leq C_1(x_1, y_1) + C_1(x_2, y_2).$$

This is the third property needed to satisfy for C_1(x, y) to be a copula. □

Remark 3. *In addition, it should be noted that the copula constructed above matches both the symmetric and the prefix property.*

And that is because

$$C_1(x,\ y) = g\left(g^{-1}(x) * g^{-1}(y)\right) = g\left(g^{-1}(y) * g^{-1}(x)\right) = C_1(y, x)$$

also

$$C_1(C_1(x,y), w) = g\left(g^{-1}\left(\ g\left(g^{-1}(x) * g^{-1}(y)\right)\right) * g^{-1}(w)\right) = g\left(\ g^{-1}(x) * g^{-1}(y) * g^{-1}(w)\right)$$

And

$$C_1(x,\ C_1\ (y,\ w)) = g\left(g^{-1}(x) * g^{-1}\left(g\left(g^{-1}(y) * g^{-1}(w)\ \right)\right) = g\left(g^{-1}\ (x) * g^{-1}(y) * g^{-1}(w)\right)$$

thus

$$C_1(C_1(x,y)\ , w) = C(x, C_1(y, w)).$$

Example 4. *Let the function g(x) =* $\sqrt{\chi}$ *when* $0 \leq x \leq 1$*, then,* $g^{-1}(x) = x^2$ *when* $0 \leq x \leq 1$*. So, we construct the copula* C_1 *(x, y) =* $\left(\sqrt{x} * \sqrt{y}\right)^2$*. Let us check the three conditions:*

(1) $C_1(t, 0) = \left(\sqrt{t} * \sqrt{0}\right)^2 = 0 = \left(\sqrt{0} * \sqrt{t}\right)^2 = C_1(0, t)$

(2) $C_1(t, 1) = \left(\sqrt{t} * 1\right)^2 = 0 = \left(\sqrt{1} * \sqrt{t}\right)^2 = C_1(1, t)$

(3) *The function g(x) is productive, so it is relatively easy to prove the third property, provided that the derivative*

$$\frac{\partial^2 C_1(x,y)}{\partial xy} \geq 0 \ \Leftrightarrow$$

$$\frac{\partial C(x,y)}{\partial x} = \left(\frac{1}{\sqrt{x}} * \sqrt{y}\right)\left(\sqrt{x} * \sqrt{y}\right) = y \text{ and } \frac{\partial^2 C(x,y)}{\partial xy} = 1\ \geq 0\ .C_1 \text{ is a copula.}$$

In the next theorem, we will try to combine the construction of the fuzzy implication that we have already constructed with the construction of the last copulas.

Theorem 5. *Let the function* $g : [0,1] \to [0, +\infty)$ *continuous, strictly increasing and convex, g(0) = 0, g(1) = 1 and* g^{-1} *continuous. The function* $C : [0,1]^2 \to [0,1]$*, when*

$$C(x,y) = max\left\{g\left(g^{-1}(x) * y\right), g\left(g^{-1}(y) * x\right)\right\} \text{ is a copula.} \tag{38}$$

Proof of the Theorem 5.

(1) $C(t,0) = C(t,0) = max\{g(g^{-1}(t) * 0), g(g^{-1}(0) * t)\} = max\{g(0), g(0 * t)\} = max\{g(0), g(0)\} = max\{0,0\} = 0$
$C(0,t) = max\{g(g^{-1}(0) * t), g(g^{-1}(t) * 0)\} = max\{g(0 * t), g(0)\} = max\{0,0\} = 0$, which proves that $C(t,0) = C(0,t) = 0$

(a) $C(t,1) = C(t,1) = max\{g(g^{-1}(t) * 1), g(g^{-1}(1) * t)\} = max\{g(g^{-1}(t)), g(t)\} = max\{t, g(t)\} = t$.
and that is because $g(x)$ is convex, which means that $g(t) \leq t$.

(b) $C(1,t) = C(1,t) = max\{g(g^{-1}(1) * t), g(g^{-1}(t) * 1)\} = max\{g(t), g(g^{-1}(t))\} = max\{g(t), t\} = t$. So, C(1,t) = C(t,1) = t.

(2) Let $g^{-1}(x) * y = u$ and $g^{-1}(y) * x = w$, which means that C(x, y) is either equal to $g(u)$ or $g(w)$. It has already been proven before that if the function $g(x)$ is convex, the third property of the copulas is settled. So, there is no need to prove again, as the proof is obvious. □

Example 5. *Let the function* $g(x) = \frac{x}{3-2x}$ *for every x in the interval* $0 \leq x \leq 1$ *be continuous, strictly increasing and convex, with* $g^{-1}(x) = \frac{3x}{2x+1}$ *continuous and strictly increasing. We will prove the following:*

The function $C(x,y) = max\{ \frac{\frac{3x}{2x+1}y}{3-2\frac{3x}{2x+1}y}, \frac{\frac{3y}{2y+1}x}{3-2\frac{3y}{2y+1}x}\}$ *is a copula.*

(1) $C(x,0) = max\{\frac{\frac{3*x}{2x+1}0}{3-2\frac{3x}{2x+1}0}, \frac{\frac{3*0}{2*0+1}x}{3-2\frac{3*0}{2*0+1}x}\} = max\{0,0\} = 0$

$C(0,y) = max\{ \frac{\frac{3*0}{2*0+1}y}{3-2\frac{3*0}{2*0+1}y}, \frac{\frac{3y}{2y+1}0}{3-2\frac{3y}{2y+1}0}\} = max\{0,0\} = 0$

(2) $C(x,1) = max\{ \frac{\frac{3x}{2x+1}1}{3-2\frac{3x}{2x+1}1}, \frac{\frac{3*1}{2*1+1}x}{3-2\frac{3*1}{2*1+1}x}\} = max\{x, \frac{x}{3-2x}\} = x$ *because* $x \geq \frac{x}{3-2x}$.

$C(1,y) = max\{ \frac{\frac{3*1}{2*1+1}y}{3-2\frac{3*1}{2*1+1}y}, \frac{\frac{3y}{2y+1}1}{3-2\frac{3y}{2y+1}1}\} = max\{\frac{y}{3-2y}, y\} = y$ *again because* $y \geq \frac{y}{3-2y}$.

(3) As for the third property, we just have to prove that g(x) is convex. We can easily check that $g'(x) = \frac{3}{(3-2x)^2} \geq 0$ *and* $g''(x) = \frac{4}{(3-2x)^3} \geq 0$*, so g(x) is convex and the third property is automatically proved.*

4. Discussion

The primary main goal of this paper is to present fuzzy negations, fuzzy implications and copulas through a common construction process, using very simple functions with certain properties. In fact, by studying the paper in its entirety, one can see that the present constructions could be performed using a single function.

In an attempt to detail the role played in these constructions by the formula $\mathbf{f}(\mathbf{f^{-1}}(\mathbf{y}) * \mathbf{x})$, the following points should be emphasized: f(x) is strictly decreasing, f(0) = 1 and $f^{-1}(y) > 0$. Going ahead with the constructions, some additional properties are given to f(x), such as it is convex, and f(1) = 0. These additional properties do not negate the previous constructions but merely come to complement them. In Section 3 where the constructions are presented, a reference to two functions is made, since, in addition to the decreasing function f(x), there is also in use an increasing function g(x) with almost similar properties. To be precise, one can easily assume that g(x) = f (1−x). So, what is achieved? The achievement is to represent all of the above constructs by means of a single formula $\boldsymbol{f}(\boldsymbol{f^{-1}}(\boldsymbol{y}) * \boldsymbol{x})$ and a single function f(x). Recall also that strong negations were constructed with the help of the use of the equilibrium point. Negations that help to calculate exactly what the equilibrium point will be. In addition to that, a very interesting construction is presented that is proved to be a copula, the $C_1(x, y) = g(g^{-1}(x) * g^{-1}(y))$. This copula formula is very similar to the other formula presented in Section 3 ($C(x,y) = max\{g(g^{-1}(x) * y), g(g^{-1}(y) * x)\}$). A very detailed proof is given concerning the fact that if the function g(x) is convex, C_1(x, y) will always be a copula.

If we try to talk about the consequences of, for example, applying some of the copulas proved in Section 3 to domains of fuzzy logic such as AI and robotics, this will be a significant prospect. Other areas, such as "Control synthesis for discrete-time T-S fuzzy systems based on membership function-dependent H∞ performance" [27] or "Finite-Time Membership Function-Dependent H∞ Control for T-S Fuzzy Systems via a Dynamic Memory Event-Triggered Mechanism" [28] will have results that could be some of the below:

1. Uncertainty Handling: Copulas can be used to model the dependency between different uncertainties in the system. By incorporating copulas into the T-S fuzzy model, one can more accurately capture the interdependencies between different sources of uncertainty, leading to more robust control designs.
2. Performance Enhancement: Copulas can help in designing membership function-dependent H∞ controllers by accurately modeling the joint behavior of the system's uncertainties. This leads to better performance metrics, such as improved disturbance rejection and enhanced stability under varying operating conditions.
3. Event-Triggered Mechanisms: The use of copulas can optimize event-triggering conditions by better predicting the evolution of system states and disturbances. This optimization can lead to more efficient control actions, reducing unnecessary computations and communications while maintaining desired performance levels.

- Modeling Dependencies: Both papers focus on enhancing control synthesis by considering the dependencies between uncertainties. Copulas offer a sophisticated way to model these dependencies, leading to improved controller performance.
- Robustness and Adaptivity: By using copulas, controllers can be designed to be more adaptive to varying conditions and more robust against disturbances, aligning with the goals of H∞ performance and finite-time stability.
- Efficiency in Control: In event-triggered mechanisms, copulas can optimize the conditions for control actions, leading to more efficient system operation without compromising performance.

In essence, copulas provide a powerful tool to enhance the modeling and control of T-S fuzzy systems by accurately capturing the dependencies between uncertainties, thus improving the robustness and efficiency of control strategies discussed in both papers.

5. Conclusions

Fuzzy negations are necessary in many areas and especially in generating new fuzzy implications. In this article, there have been proposed some novel construction methods of strong fuzzy negations. This is achieved using a specific type of formula to construct at the same time strong fuzzy negations, fuzzy implications and Copulas in an attempt to bring those mathematical concepts a bit closer. Two theorems are presented in negations, one in implications and two theorems in copulas. All of the above are accompanied by their own proofs. Furthermore, there is presented one very interesting proof in the third property of the copulas regarding how one non-productive function g(x) constructs a copula only if it is convex.

The above constructions are intended to provide the mathematical community with the following information:

(a) All of the above constructs can be represented by means of very simple functions, common among the concepts used.
(b) A formula is presented that participates in all three mathematical concepts discussed in this article.
(c) A proof in the area of copulas is presented.

All of the above is intended to bring all the mathematical concepts discussed in this article closer together, from a mathematical point of view, and to give ground for future analysts to build on it and further investigate the convergence and application of these concepts.

6. Patents

The formula $f\left(f^{-1}(y) * x\right)$, which can generate strict fuzzy negations, strong fuzzy negations, fuzzy implications and copulas for a strictly decreasing, positive function, convex with

f (0) = 1 (in some cases f(1) = 0 also).

Author Contributions: Methodology, P.G.M.; Supervision, B.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: Special thanks to all three reviewers for their notes and comments on this article.

Conflicts of Interest: The authors declare no conflicts of interest, financial or otherwise.

References

1. Bustince, H.; Campión, M.J.; De Miguel, L.; Induráin, E. Strong negations and restricted equivalence functions revisited: An analytical and topological approach. *Fuzzy Sets Syst.* **2022**, *441*, 110–129. [CrossRef]
2. Bedregal, B.C. On interval fuzzy negations. *Fuzzy Sets Syst.* **2010**, *161*, 2290–2313. [CrossRef]
3. Gupta, V.K.; Massanet, S.; Vemuri, N.R. Novel construction methods of interval-valued fuzzy negations and aggregation functions based on admissible orders. *Fuzzy Sets Syst.* **2023**, *473*, 108722. [CrossRef]
4. Pradera, A.; Beliakov, G.; Bustince, H.; De Baets, B. A review of the relationships between implication, negation and aggregation functions from the point of view of material. *Implic. Inf. Sci.* **2016**, *329*, 357–380. [CrossRef]
5. Baczynski, M.; Jayaram, B. QL-implications: Some properties and intersections. *Fuzzy Sets Syst.* **2010**, *161*, 158–188. [CrossRef]
6. Baczynski, M.; Jayaram, B. (U, N)-implications and their characterizations. *Fuzzy Sets Syst.* **2009**, *160*, 2049–2062. [CrossRef]
7. Durante, F.; Klement, E.P.; Meriar, R.; Sempi, C. Conjunctors and their residual implicators: Characterizations and construction methods. *Mediterr. J. Math.* **2007**, *4*, 343–356. [CrossRef]
8. Massanet, S.; Torrens, J. An overview of construction methods of fuzzy implications. In *Advances in Fuzzy Implication Functions*; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2013; Volume 300, pp. 1–30. [CrossRef]
9. Baczynski, M.; Jayaram, B.; Massanet, S.; Torrens, J. Fuzzy implications: Past, present, and future. In *Springer Handbook of Computational Intelligence*; Springer Handbooks; Springer: Berlin/Heidelberg, Germany, 2015; pp. 183–202. [CrossRef]
10. Baczynski, M.; Jayaram, B. On the characterization of (S, N)-implications. *Fuzzy Sets Syst.* **2007**, *158*, 1713–1727. [CrossRef]
11. Massanet, S.; Vicente, J.; Clapes, R.; Aguilera, D.R. On fuzzy polynomials implications. In Proceedings of the 2015 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (IFSA-EUSFLAT-15), Gijón, Spain, 30 June–3 July 2015. [CrossRef]
12. Rapti, M.; Papadopoulos, B. A Method of Generating Fuzzy Implications from n Increasing Functions and n+1 Negations. *Mathematics* **2020**, *8*, 886. [CrossRef]
13. Daniilidou, A.; Konguetsof, A.; Souliotis, G.; Papadopoulos, B. Generator of Fuzzy Implications. *Algorithms* **2023**, *16*, 569. [CrossRef]
14. Baczynski, M.; Jayaram, B. *Fuzzy Implications*; Springer: Berlin/Heidelberg, Germany, 2008. [CrossRef]
15. Bustince, H.; Pagola, M.; Barrenechea, E. Construction of fuzzy indices from fuzzy DIsubsethood measures: Application to the global comparison of images. *Inf. Sci.* **2007**, *177*, 906–929. [CrossRef]
16. Bogiatzis, A.C.; Papadopoulos, B.K. Local thresholding of degraded or unevenly illuminated documents using fuzzy inclusion and entropy measures. *Evol. Syst.* **2019**, *10*, 593–619. [CrossRef]
17. Bogiatzis, A.C.; Papadopoulos, B. Global Image Thresholding Adaptive Neuro-Fuzzy Inference System. Trained with Fuzzy Inclusion and Entropy Measures. *Symmetry* **2019**, *11*, 286. [CrossRef]
18. Bogiatzis, A.C.; Papadopoulos, B.K. Producing fuzzy inclusion and entropy measures and their application on 277 global image thresholding. *Evol. Syst.* **2018**, *9*, 331–353. [CrossRef]
19. Betsakos, D. *Introduction to Real Analysis*; Afoi Kyriakidi: Thessaloniki, Greece, 2016.
20. Baczyński, M.; Grzegorzewski, P.; Mesiar, R.; Helbin, P.; Niemyska, W. Fuzzy implications based on semicopulas. *Fuzzy Sets Syst.* **2017**, *323*, 138–151. [CrossRef]
21. Mesiar, R.; Kolesarova, A. Copulas and fuzzy implications. *Int. J. Approx. Reason.* **2020**, *117*, 52–59. [CrossRef]
22. Durante, F.; Sempi, C. *Principles of Copula Theory*; Chapman and Hall/CRC: New York, NY, USA, 2015. [CrossRef]
23. Nelsen, R.B. *An Introduction to Copulas*, 2nd ed.; Springer: New York, NY, USA, 2006.

24. Giakoumakis, S.; Papadopoulos, B. Novel transformation of unimodal (a)symmetric possibility distributions into probability distributions. *Fuzzy Sets Syst.* **2024**, *476*, 108790. [CrossRef]
25. Baczynski, M.; Jayaram, B. (S, N)-and R-implications; a state-of-the-art survey. *Fuzzy Sets Syst.* **2008**, *159*, 1836–1859. [CrossRef]
26. Giakoumakis, S.; Papadopoulos, B. An Algorithm for Fuzzy Negations Based-Intuitionistic Fuzzy Copula Aggregation Operators in Multiple Attribute Decision Making. *Algorithms* **2020**, *13*, 154. [CrossRef]
27. Yang, D.; Xie, X. Relaxed H∞ control design of discrete-time Takagi–Sugeno fuzzy systems: A multi-samples approach. *Neurocomputing* **2016**, *171*, 106–112. [CrossRef]
28. Hou, Q.; Dong, J. Finite-Time Membership Function-Dependent H∞ Control for T-S Fuzzy Systems via a Dynamic Memory Event-Triggered Mechanism. *IEEE Trans. Fuzzy Syst.* **2023**, *31*, 4075–4084. [CrossRef]