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Criteria of a Two-Weight, Weak-Type Inequality in Orlicz Classes for Maximal Functions Defined on Homogeneous Spaces

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Abstract: In this study, some new necessary and sufficient conditions for a two-weight, weak-type maximal inequality of the form $\varphi_1(\lambda)\int_{\{x\in X:Mf(x)>\lambda\}}\varrho(x)d\mu(x)\leq c\int_X\varphi_2(c|f(x)|)\sigma(x)d\mu(x)$ are obtained in Orlicz classes, where Mf is a Hardy–Littlewood maximal function defined on homogeneous spaces and ϱ is a weight function.

Keywords: weight; weak-type inequality; Hardy–Littlewood maximal function; Orlicz classes

MSC: 42B25, 46E30

1. Introduction

Weighted inequalities play an important role in weighted theory research and have been extensively studied. For instance, Hardy and Littlewood [1] proved the weighted norm inequality for fractional integrals for the one-dimensional case. Mamedov and Harman [2] investigated two-weighted Hardy inequalities in the norms of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Sawyer [3] characterized the weak weighted inequalities for the one-sided Hardy–Littlewood maximal function on $\mathbb{R}$. Ghosh and Mohanty [4] obtained the extra-weak and weak-type inequalities for the one-sided maximal function on $\mathbb{R}^2$. Moen [5] studied a class of two-weight inequalities for multilinear fractional integral operators and maximal functions. Ren [6] explored a four-weight weak type maximal inequality for martingales. Other relevant studies could be found in [7–10].

Let f be a locally integrable function on $\mathbb{R}^n$; the Hardy–Littlewood maximal function is defined as

$$Mf(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)|dy,$$

where Mf is a Hardy–Littlewood maximal function and the supremum is taken over all cubes Q containing x in $\mathbb{R}^n$. In 1972, Muckenhoupt [11] proved the following critical conclusion.

Theorem 1 ([11]). *Let $1<p<\infty$ and u,v be a pair of weight functions. The following statements are equivalent:*

(i) There exists a constant $c>0$ such that

$$\int_{\{x\in\mathbb{R}^n:Mf(x)>\lambda\}}u(x)dx\leq\frac{c}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^pv(x)dx, \tag{1}$$

holds for arbitrary $\lambda>0$.

(ii) (Muckenhoupt A_p condition) There exists a constant $c>0$ such that, for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-p'/p}dx\right)^{p/p'}\leq c,$$

where $p'=\frac{p}{p-1}$.

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Since the Muckenhoupt A_p condition is central to weighted theory, many scholars have extended Muckenhoupt's result to various function spaces. Research on the correspondence of inequality (1) with the framework of Orlicz classes has aroused significant academic interest (see [12–19]). In [16], Gogatishvili and Kokilashvili provided a generalization form of (1) and its weighted characteristics; if ϱ is a weight function, then the weighted weak-type inequality

$$\varphi(\lambda)\int_{\{x\in X:Mf(x)>\lambda\}}\varrho(x)d\mu(x)\le c_1\int_X\varphi(c_1|f(x)|)\varrho(x)d\mu(x) \tag{2}$$

holds if and only if the inequality

$$\int_B\widetilde{\varphi}\left(\varepsilon\frac{\varphi(\lambda)}{\lambda}\frac{\int_B\varrho(x)d\mu(x)}{\varrho(x)\mu(B)}\right)\varrho(x)d\mu(x)\le c_2\varphi(\lambda)\int_B\varrho(x)d\mu(x) \tag{3}$$

is true. Then, they introduced the four-weight extension forms of inequalities (2) and (3) [17]. On the basis of [16,17], in 2020, Ding and Ren [18] obtained suitable four-weight extension inequalities, which were new necessary and sufficient conditions of the four-weight, weak-type maximal inequalities in [17]. They are all extended forms of Muckenhoupt's result in Orlicz classes.

In this study, we continue to investigate the extended forms of Muckenhoupt's results in Orlicz classes. Our research is motivated by the question of whether the two-weight, weak-type inequalities shown in (6) have new equivalent characterization inequalities. Combined with the work of Ding and Ren [18] and inspired by Lai's research on two-weight mixed inequalities for the Hardy–Littlewood maximal operator [20], we obtain a pair of two-weight, weak-type inequalities (see (9) and (10) in Theorem 2) in Orlicz classes for maximal functions defined on homogeneous spaces, which are new equivalent characterization inequalities of the two-weight, weak-type maximal inequality (6) in Theorem 2.

The remainder of this article is organized as follows: In Section 2, as preliminaries, we recapitulate some basic notions. The main result and its proof are given in the final section.

2. Preliminaries

In this section, we give a brief summary of facts about the homogeneous spaces, Young functions, and Orlicz spaces that we require; see [9,14,16,21] for more details.

A homogeneous space (X,d,μ) is a metric space with a complete measure μ in which a class of compactly supported continuous functions is densely organized in the space $L^1(X,\mu)$.

$d : X\times X\to\mathbb{R}^1$ is a nonnegative real-valued function and satisfies the following conditions:

(i) $d(x,x)=0$ for all $x\in X$;
(ii) $d(x,y)>0$ for all $x\neq y$ in X;
(iii) There is a constant $a_0>0$ such that $d(x,y)\le a_0 d(y,x)$ for all x,y in X;
(iv) There is a constant $a_1>0$ such that $d(x,y)\le a_1(d(x,z)+d(z,y))$ for all x,y,z in X;
(v) For each neighborhood V of x in X, there is an $r>0$ such that the ball $B(x,r)=\{y\in X : d(x,y)<r\}$ is contained in V;
(vi) The balls $B(x,r)$ are measurable for all $x\in X$ and $r>0$;
(vii) There is a constant $b>0$ such that $\mu B(x,2r)\le b\mu B(x,r)$ for all $x\in X$ and $r>0$.

Let f be a μ-measurable locally integrable function on X, and let B be a ball; we set

$$(f)_B:=\frac{1}{\mu(B)}\int_B f(x)d\mu.$$

The maximal function of f is defined by

$$Mf(x) := \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y)|d\mu(y), \;\; x \in X,$$

where the supremum is taken over all balls B containing x.

A μ-measurable locally integrable function that is positive almost everywhere is called a weight function.

We use the symbol Φ to denote the set of all functions $\varphi : \mathbb{R}^1 \to \mathbb{R}^1$ that are nonnegative, even, and increasing on $(0,\infty)$ such that $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$.

We call ω a Young function if $\omega \in \Phi$ and it is convex on $(0,\infty)$; it may have a jump up to ∞ at some point if $t > 0$, but in that case, it should be left continuous at t. A function φ is called a quasi-convex function if there is a Young function ω and a constant $c > 1$ such that $\omega(t) \le \varphi(t) \le \omega(ct)$ for any $t \ge 0$. A function φ is said to be quasi-increasing if there is a constant $c > 0$ such that

$$\varphi(t_1) \le c\varphi(ct_2)$$

for each t_1 and t_2, $0 < t_1 < t_2$. It was proved in [16] that φ is quasi-convex if and only if $\frac{\varphi(t)}{t}$ is quasi-increasing.

For a quasi-convex function φ, we define its complementary function $\tilde{\varphi}$ as

$$\tilde{\varphi}(t) = \sup_{s\ge 0}(st - \varphi(s)).$$

The subadditivity of the supremum readily implies that $\tilde{\varphi}$ is a Young function, and from the definition of the complementary function $\tilde{\varphi}$, we obtain the following Young inequality:

$$st \le \varphi(s) + \tilde{\varphi}(t).$$

Let φ be a Young function; we define its inverse function $\varphi^{-1} : [0,\infty] \to [0,\infty]$ as

$$\varphi^{-1}(s) = \inf\{t \ge 0 : \varphi(t) \ge s\}.$$

Lemma 1 ([14])**.** *Let φ be a Young function; then, $\frac{\varphi(t)}{t}$ and $\frac{\tilde{\varphi}(t)}{t}$ are continuous and increasing on $(0,\infty)$, and they satisfy*

$$\tilde{\varphi}\left(\frac{\varphi(t)}{t}\right) \le \varphi(t) \le \tilde{\varphi}\left(2\frac{\varphi(t)}{t}\right) \tag{4}$$

and

$$\varphi\left(\frac{\tilde{\varphi}(t)}{t}\right) \le \tilde{\varphi}(t) \le \varphi\left(2\frac{\tilde{\varphi}(t)}{t}\right) \tag{5}$$

for all $t > 0$.

Lemma 2 ([22])**.** *Let $\mathcal{F}$ be a family $\{B(x,r)\}$ of balls with bounded radii. Then, there is a countable subfamily $\{B(x_i,r_i)\}$ consisting of pairwise disjoint balls such that each ball in $\mathcal{F}$ is contained in one of the balls $B(X_i, ar_i)$, where $a = 3a_1^2 + 2a_0a_1$. The constants a_0, a_1 are from the definition of the space (X,d,μ).*

Let (X,μ) be a measured space, let v be a weight function, and let φ be a Young function. The weighted modular is defined by

$$m_v(f,\varphi) = \int_X \varphi(|f(x)|)v(x)d\mu(x);$$

the Orlicz space $L_{\varphi,v}(X,\mu)$ is equipped with the Orlicz norm

$$|||f|||_{\varphi,v} = \sup\{\int_X f(x)g(x)v(x)d\mu(x) : \int_X \widetilde{\varphi}(|g(x)|)v(x)d\mu(x) \leq 1\},$$

and the Luxemburg norm

$$||f||_{\varphi,v} = \inf\{\lambda > 0 : \int_X \varphi\left(\frac{|f(x)|}{\lambda}\right)v(x)d\mu(x) \leq 1\}.$$

The above two norms are equivalent, i.e.,

$$||f||_{\varphi,v} \leq |||f|||_{\varphi,v} \leq 2||f||_{\varphi,v}.$$

In addition, in [23], Luxemburg showed the following:

(i) The closed unit ball in $L_{\varphi,v}$ with respect to the Luxemburg norm coincides with the closed unit ball with respect to the modular, i.e.,

$$\int_X \varphi(|f(x)|)v(x)d\mu(x) \leq 1 \textit{ if and only if } ||f||_{\varphi,v} \leq 1;$$

(ii) The Hölder inequality

$$\int_X f(x)g(x)v(x)d\mu(x) \leq \|f\|_{\varphi,v} \cdot |||g|||_{\widetilde{\varphi},v}$$

holds for all μ-measurable functions f, g.

Throughout this article, we use c_i and c to denote positive constants. They may denote different values at different occurrences.

3. Main Result and Proof

On the basis of [18], in this section, we provide two equivalent characterization inequalities for the two-weight, weak-type maximal inequality (6). Let us first present Lemma 3 and Corollary 1 before stating and proving the main result.

Lemma 3 ([18]). *Let $\varphi \in \Phi$ and ω_i $(i = 1,2,3,4)$ be weight functions. Then, the following statements are equivalent:*

(i) The inequality

$$\int_{\{x\in X:Mf(x)>\lambda\}} \varphi(\lambda\omega_1(x))\omega_2(x)d\mu(x) \leq c_1 \int_X \varphi(c_1|f(x)|\omega_3(x))\omega_4(x)d\mu(x)$$

holds with a constant $c_1 > 0$, independent of f and $\lambda > 0$;

(ii) The function φ is quasi-convex, and the inequality

$$\int_{\{x\in X:Mf(x)>\lambda\}} \widetilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)d\mu(x) \leq c_2 \int_X \widetilde{\varphi}\left(c_2\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)d\mu(x)$$

holds with a constant $c_2 > 0$, independent of f and $\lambda > 0$;

(iii) The function φ is quasi-convex, and there is a constant $c_3 > 0$ such that the inequality

$$\int_B \widetilde{\varphi}\left(\frac{|f|_B}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)d\mu(x) \leq c_3 \int_B \widetilde{\varphi}\left(c_3\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary ball B;

(iv) *There is a constant $c_4 > 0$ such that the inequality*

$$\int_B \varphi(|f|_B\omega_1(x))\omega_2(x)d\mu(x) \le c_4 \int_B \varphi(c_4|f(x)|\omega_3(x))\omega_4(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary ball B;

(v) *The function φ is quasi-convex, and there are positive constants c_5 and ε_1 such that the inequality*

$$\int_B \varphi\left(\frac{\varepsilon_1}{\lambda\mu(B)} \int_B \widetilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)d\mu(x)\cdot\omega_1(x)\right)\omega_2(x)d\mu(x) \le c_5 \int_B \widetilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)d\mu(x)$$

holds for any $\lambda > 0$ and an arbitrary ball B;

(vi) *There are positive constants c_6 and ε_2 such that the inequality*

$$\int_B \widetilde{\varphi}\left(\varepsilon_2\frac{\int_B \varphi(\lambda\omega_1(x))\omega_2(x)d\mu(x)}{\lambda\mu(B)\omega_3(x)\omega_4(x)}\right)\omega_4(x)d\mu(x) \le c_6 \int_B \varphi(\lambda\omega_1(x))\omega_2(x)d\mu(x)$$

holds for any $\lambda > 0$ and an arbitrary ball B.

Remark 1. *The conclusion in Lemma 3 remains valid upon the replacement of quasi-convex functions with Young functions. Thus, let φ be a Young function, and $\omega_1 = \omega_3 = 1$, $\omega_2 = \varrho$, $\omega_4 = \sigma$; then, we reach the following conclusion.*

Corollary 1. *Let $(\varphi, \widetilde{\varphi})$ be a pair of complementary Young functions, and let ϱ and σ be weight functions. Then, the following statements are equivalent:*

(i) *There is a constant $c_1 > 0$ such that the inequality*

$$\varphi(\lambda)\int_{\{x\in X: Mf(x)>\lambda\}} \varrho(x)d\mu(x) \le c_1 \int_X \varphi(c_1|f(x)|)\sigma(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary $\lambda > 0$;

(ii) *There is a constant $c_2 > 0$ such that the inequality*

$$\int_{\{x\in X: Mf(x)>\lambda\}} \widetilde{\varphi}\left(\frac{\lambda}{\sigma(x)}\right)\sigma(x)d\mu(x) \le c_2 \int_X \widetilde{\varphi}\left(c_2\frac{|f(x)|}{\varrho(x)}\right)\varrho(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary $\lambda > 0$;

(iii) *There is a constant $c_3 > 0$ such that the inequality*

$$\int_B \widetilde{\varphi}\left(\frac{|f|_B}{\sigma(x)}\right)\sigma(x)d\mu(x) \le c_3 \int_B \widetilde{\varphi}\left(c_3\frac{|f(x)|}{\varrho(x)}\right)\varrho(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary ball B;

(iv) *There is a constant $c_4 > 0$ such that the inequality*

$$\varphi(|f|_B)\int_B \varrho(x)d\mu(x) \le c_4 \int_B \varphi(c_4|f(x)|)\sigma(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary ball B;

(v) *There are constants $c_5 > 0$ and $\varepsilon_1 > 0$ such that the inequality*

$$\varphi\left(\frac{\varepsilon_1}{\lambda\mu(B)}\int_B \widetilde{\varphi}\left(\frac{\lambda}{\sigma(x)}\right)\sigma(x)d\mu(x)\right)\int_B \varrho(x)d\mu(x) \le c_5 \int_B \widetilde{\varphi}\left(\frac{\lambda}{\sigma(x)}\right)\sigma(x)d\mu(x)$$

holds for arbitrary $\lambda > 0$ and ball B;

(vi) *There are constants $c_6 > 0$ and $\varepsilon_2 > 0$ such that the inequality*

$$\int_B \widetilde{\varphi}\left(\varepsilon_2 \frac{\varphi(\lambda)}{\lambda} \frac{\int_B \varrho(x)d\mu(x)}{\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x) \le c_6\varphi(\lambda)\int_B \varrho(x)d\mu(x)$$

holds for arbitrary $\lambda > 0$ and ball B.

In Theorem 2, we obtain a pair of two-weight, weak-type inequalities in Orlicz classes for maximal functions defined on homogeneous spaces, which are new necessary and sufficient conditions of the two-weight, weak-type maximal inequality (6).

Theorem 2. *Let $(\varphi_1, \widetilde{\varphi}_1)$ and $(\varphi_2, \widetilde{\varphi}_2)$ be two pairs of complementary Young functions, and let ϱ and σ be weight functions. Then, the following statements are equivalent:*

(i) *There is a constant $c_1 > 0$ such that the inequality*

$$\varphi_1(\lambda)\int_{\{x\in X: Mf(x)>\lambda\}} \varrho(x)d\mu(x) \le c_1\int_X \varphi_2(c_1|f(x)|)\sigma(x)d\mu(x) \tag{6}$$

holds for any nonnegative μ-measurable function f and arbitrary $\lambda > 0$;

(ii) *There is a constant $c_2 > 0$ such that the inequality*

$$\int_{\{x\in X: Mf(x)>\lambda\}} \widetilde{\varphi}_2\left(\frac{\lambda}{\sigma(x)}\right)\sigma(x)d\mu(x) \le c_2\int_X \widetilde{\varphi}_1\left(c_2\frac{|f(x)|}{\varrho(x)}\right)\varrho(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary $\lambda > 0$;

(iii) *There is a constant $c_3 > 0$ such that the inequality*

$$\int_B \widetilde{\varphi}_2\left(\frac{|f|_B}{\sigma(x)}\right)\sigma(x)d\mu(x) \le c_3\int_B \widetilde{\varphi}_1\left(c_3\frac{|f(x)|}{\varrho(x)}\right)\varrho(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary ball B;

(iv) *There is a constant $c_4 > 0$ such that the inequality*

$$\varphi_1(|f|_B)\int_B \varrho(x)d\mu(x) \le c_4\int_B \varphi_2(c_4|f(x)|)\sigma(x)d\mu(x)$$

holds for any nonnegative μ-measurable function f and arbitrary ball B;

(v) *There are constants $c_5 > 0$ and $\varepsilon_1 > 0$ such that the inequality*

$$\varphi_1\left(\frac{\varepsilon_1}{\lambda\mu(B)}\int_B \widetilde{\varphi}_2\left(\frac{\lambda}{\sigma(x)}\right)\sigma(x)d\mu(x)\right)\int_B \varrho(x)d\mu(x) \le c_5\int_B \widetilde{\varphi}_2\left(\frac{\lambda}{\sigma(x)}\right)\sigma(x)d\mu(x) \tag{7}$$

holds for arbitrary $\lambda > 0$ and ball B;

(vi) *There are constants $c_6 > 0$ and $\varepsilon_2 > 0$ such that the inequality*

$$\int_B \widetilde{\varphi}_2\left(\varepsilon_2 \frac{\varphi_1(\lambda)}{\lambda} \frac{\int_B \varrho(x)d\mu(x)}{\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x) \le c_6\varphi_1(\lambda)\int_B \varrho(x)d\mu(x) \tag{8}$$

holds for arbitrary $\lambda > 0$ and ball B;

(vii) *There are constants $c_7 > 0$ and $\varepsilon_3 > 0$ such that the inequality*

$$\int_B \widetilde{\varphi}_2\left(\varepsilon_3 \frac{\lambda\int_B \varrho(x)d\mu(x)}{\sigma(x)\mu(B)}\right)\sigma(x)d\mu(x) \le c_7\widetilde{\varphi}_1(\lambda)\int_B \varrho(x)d\mu(x) \tag{9}$$

holds for arbitrary $\lambda > 0$ and ball B;

(viii) There are constants $c_8 > 0$ and $\varepsilon_4 > 0$ such that the inequality

$$\varphi_1\left(\frac{1}{c_8\mu(B)}||\frac{\chi_B}{\varepsilon_4\sigma}||_{\widetilde{\varphi}_2(\varepsilon_4\sigma)}\right) \leq \frac{c_8}{\varepsilon_4\int_B \varrho(x)d\mu(x)} \tag{10}$$

holds for arbitrary ball B.

Proof. In Corollary 1, we replace the Young function φ with a pair of Young functions, φ_1 and φ_2, in the form of (i)–(vi) in Theorem 2. Since the proof of the equivalence relation ((i)–(vi)) in Theorem 2 is similar to that of Theorem 3.6 in [18], it is omitted. So, we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$. Now, we complete the proof by showing that $(v) \Rightarrow (vii) \Rightarrow (vi)$ and $(i) \Leftrightarrow (viii)$.

$(v) \Rightarrow (vii)$. In (7), we replace λ with $\frac{\lambda\varepsilon_1\int_B \varrho(x)d\mu(x)}{2c_5\mu(B)}$; then, we can obtain

$$\varphi_1\left(\frac{\varepsilon_1}{\mu(B)}\cdot\frac{2c_5\mu(B)}{\lambda\varepsilon_1\int_B\varrho(x)d\mu(x)}\int_B\widetilde{\varphi}_2\left(\frac{\lambda\varepsilon_1\int_B\varrho(x)d\mu(x)}{2c_5\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x)\right)\int_B\varrho(x)d\mu(x)$$
$$\leq c_5\int_B\widetilde{\varphi}_2\left(\frac{\lambda\varepsilon_1\int_B\varrho(x)d\mu(x)}{2c_5\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x).$$

So, we have

$$\frac{2\varphi_1\left(\frac{2c_5}{\lambda\int_B\varrho(x)d\mu(x)}\int_B\widetilde{\varphi}_2\left(\frac{\lambda\varepsilon_1\int_B\varrho(x)d\mu(x)}{2c_5\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x)\right)}{\frac{2c_5}{\lambda\int_B\varrho(x)d\mu(x)}\int_B\widetilde{\varphi}_2\left(\frac{\lambda\varepsilon_1\int_B\varrho(x)d\mu(x)}{2c_5\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x)} \leq \lambda. \tag{11}$$

If we set

$$s = \frac{2c_5}{\lambda\int_B\varrho(x)d\mu(x)}\int_B\widetilde{\varphi}_2\left(\frac{\lambda\varepsilon_1\int_B\varrho(x)d\mu(x)}{2c_5\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x), \tag{12}$$

then (11) becomes

$$\frac{2\varphi_1(s)}{s} \leq \lambda.$$

Notice that $\frac{\widetilde{\varphi}_1(t)}{t}$ is increasing; then, we have

$$\frac{\widetilde{\varphi}_1\left(\frac{2\varphi_1(s)}{s}\right)}{\left(\frac{2\varphi_1(s)}{s}\right)} \leq \frac{\widetilde{\varphi}_1(\lambda)}{\lambda}.$$

It follows from the above inequality and (4) that we have

$$\frac{s}{2} = \frac{\varphi_1(s)}{\left(\frac{2\varphi_1(s)}{s}\right)} \leq \frac{\widetilde{\varphi}_1\left(\frac{2\varphi_1(s)}{s}\right)}{\left(\frac{2\varphi_1(s)}{s}\right)} \leq \frac{\widetilde{\varphi}_1(\lambda)}{\lambda},$$

from which we obtain

$$\int_B\widetilde{\varphi}_2\left(\varepsilon_3\frac{\lambda\int_B\varrho(x)d\mu(x)}{\sigma(x)\mu(B)}\right)\sigma(x)d\mu(x) \leq c_7\widetilde{\varphi}_1(\lambda)\int_B\varrho(x)d\mu(x),$$

where $\varepsilon_3 = \frac{\varepsilon_1}{2c_5}, c_7 = \frac{1}{c_5}$.

$(vii) \Rightarrow (vi)$. In (9), we replace λ with $\frac{\varphi_1(\lambda)}{\lambda}$; then, we have

$$\int_B \widetilde{\varphi}_2\left(\varepsilon_3 \frac{\varphi_1(\lambda)}{\lambda} \frac{\int_B \varrho(x)d\mu(x)}{\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x) \leq c_7 \widetilde{\varphi}_1\left(\frac{\varphi_1(\lambda)}{\lambda}\right) \int_B \varrho(x)d\mu(x).$$

Using (4), we obtain

$$\int_B \widetilde{\varphi}_2\left(\varepsilon_2 \frac{\varphi_1(\lambda)}{\lambda} \frac{\int_B \varrho(x)d\mu(x)}{\mu(B)\sigma(x)}\right)\sigma(x)d\mu(x) \leq c_6 \varphi_1(\lambda) \int_B \varrho(x)d\mu(x),$$

where $\varepsilon_2 = \varepsilon_3$, $c_6 = c_7$.

$(i) \Rightarrow (viii)$. For any ball B, we clearly see that $B \subset \{x : M(2f\chi_B)(x) > |f|_B\}$. So, we have

$$\varphi_1(|f|_B) \int_B \varrho(x)d\mu(x) \leq c_1 \int_B \varphi_2(c_1|f(x)|)\sigma(x)d\mu(x).$$

We set $f(x) = \frac{1}{c_1}\widetilde{\varphi}_2(\frac{1}{\delta\sigma(x)})\delta\sigma(x)$, where δ is an arbitrary positive constant. From the above inequality and (5), we have

$$\begin{aligned} &\int_B \varrho(x)d\mu(x)\varphi_1\left(\frac{1}{c_1\mu(B)} \int_B \widetilde{\varphi}_2\left(\frac{1}{\delta\sigma(x)}\right)\delta\sigma(x)d\mu(x)\right) \\ \leq c_1 &\int_B \varphi_2\left(\widetilde{\varphi}_2\left(\frac{1}{\delta\sigma(x)}\right)\delta\sigma(x)\right)\sigma(x)d\mu(x) \\ \leq c_1 &\int_B \widetilde{\varphi}_2\left(\frac{1}{\delta\sigma(x)}\right)\sigma(x)d\mu(x). \end{aligned} \tag{13}$$

Setting

$$\frac{1}{\eta} = c_1\mu(B)\varphi_1^{-1}\left(\frac{c_1}{\int_B \varrho(x)d\mu(x)} \int_B \widetilde{\varphi}_2\left(\frac{1}{\delta\sigma(x)}\right)\sigma(x)d\mu(x)\right), \tag{14}$$

(13) and (14) yield

$$\int_B \widetilde{\varphi}_2\left(\frac{1}{\delta\sigma(x)}\right)\delta\eta\sigma(x)d\mu(x) \leq 1, \tag{15}$$

hence, we have

$$\|\frac{\chi_B}{\delta\eta\sigma}\|_{\widetilde{\varphi}_2(\delta\eta\sigma)} \leq \frac{1}{\eta}, \tag{16}$$

where χ_B is the characteristic function of B.

From (14), we obtain

$$\int_B \widetilde{\varphi}_2\left(\frac{1}{\delta\sigma(x)}\right)\sigma(x)d\mu(x) = \varphi_1\left(\frac{1}{c_1\eta\mu(B)}\right) \cdot \frac{\int_B \varrho(x)d\mu(x)}{c_1}. \tag{17}$$

Then, using (15) and (17), we have

$$\delta\eta\varphi_1\left(\frac{1}{c_1\eta\mu(B)}\right) \cdot \frac{\int_B \varrho(x)d\mu(x)}{c_1} \leq 1. \tag{18}$$

Furthermore, we obtain

$$\frac{1}{\eta} \leq \varphi_1^{-1}\left(\frac{c_1}{\delta\eta \int_B \varrho(x)d\mu(x)}\right) \cdot c_1\mu(B). \tag{19}$$

According to (16) and (19), we have

$$\|\frac{\chi_B}{\delta\eta\sigma}\|_{\widetilde{\varphi}_2(\delta\eta\sigma)} \leq c_1\mu(B)\varphi_1^{-1}\left(\frac{c_1}{\delta\eta\int_B \varrho(x)d\mu(x)}\right),$$

and then, we obtain

$$\varphi_1\left(\frac{1}{c_1\mu(B)}\|\frac{\chi_B}{\delta\eta\sigma}\|_{\widetilde{\varphi}(\delta\eta\sigma)}\right) \leq \frac{c_1}{\delta\eta\int_B \varrho(x)d\mu(x)}. \tag{20}$$

In (20), by setting $\varepsilon_4 = \delta\eta$, we obtain

$$\varphi_1\left(\frac{1}{c_8\mu(B)}||\frac{\chi_B}{\varepsilon_4\sigma}||_{\widetilde{\varphi}_2(\varepsilon_4\sigma)}\right) \leq \frac{c_8}{\varepsilon_4\int_B \varrho(x)d\mu(x)},$$

where $c_8 = c_1$.

$(viii) \Rightarrow (i)$. For each natural number n, we set

$$M^n f(x) = \sup \frac{1}{\mu(B)}\int_B |f(x)|d\mu,$$

where the supremum is taken over all balls B in X, which contains x and $r_B \leq n$.

For any point $x \in \{x : M^n f(x) > \lambda\}$, there is a ball B_x $(x \in B_x, 0 < r_B \leq n)$ such that

$$\frac{1}{\mu(B_x)}\int_{B_x} |f(y)|d\mu > \lambda.$$

According to Lemma 2, from the family $\mathcal{F}=\{B_x\}$, we can choose a sequence of pairwise disjoint balls $\{B_i = B(x_i, r_i)\}$ such that each ball in $\mathcal{F}$ is contained in one of the balls $B(x_j, ar_j)$. Then,

$$\{x : M^n f(x) > \lambda\} \subset \bigcup_i B_i, \ \sum_i \chi_{B_i} \leq c',$$

where χ_{B_i} is the characteristic function of B_i.

So, we have

$$\begin{aligned}\varphi_1(\lambda)\int_{\{x:M^n f(x)>\lambda\}} \varrho(x)d\mu(x) &\leq \sum_{i=1}\int_{B_i} \varrho(x)d\mu(x)\varphi_1(\lambda) \\ &\leq \sum_{i=1}\int_{B_i} \varrho(x)d\mu(x)\varphi_1\left(\frac{1}{c_9\mu(B_i)}\int_{B_i} c_9|f(x)|d\mu(x)\right).\end{aligned}$$

According to the Hölder inequality, we have

$$\int_{B_i} c_9|f(x)|d\mu(x) \leq 2\|c_9 f_i(x)\|_{\varphi_2(\varepsilon_i\sigma)} \cdot \|\frac{\chi_{B_i}}{\varepsilon_i\sigma}\|_{\widetilde{\varphi}_2(\varepsilon_i\sigma)},$$

where $f_i(x) = f(x)\chi_{B_i}$.

We choose ε_i such that

$$\int_X \varphi_2(c_9|f_i(x)|)\varepsilon_i\sigma(x)d\mu(x) = 1, \tag{21}$$

and then, we have $\|c_9 f_i(x)\|_{\varphi_2(\varepsilon_i\sigma)} \leq 1$.

Consequently, using (10) and (21), we can obtain

$$\begin{aligned}\varphi_1(\lambda)\int_{\{x:M^n f(x)>\lambda\}}\varrho(x)d\mu(x) &\le \sum_{i=1}\int_{B_i}\varrho(x)d\mu(x)\varphi_1\left(\frac{2}{c_9}\frac{1}{\mu(B_i)}\|\frac{\chi_{B_i}}{\varepsilon_i v}\|_{\widetilde{\varphi}_2(\varepsilon_i\sigma)}\right)\\ &\le \sum_{i=1}\frac{c_9}{2}\int_X \varphi_2(c_9|f(x)\chi_{B_i}|)\sigma(x)d\mu(x)\\ &\le \frac{c'c_9}{2}\int_X \varphi_2(c_9|f(x)|)\sigma(x)d\mu(x).\end{aligned}$$

Now, let $n\to\infty$; we then obtain

$$\varphi_1(\lambda)\int_{\{x:Mf(x)>\lambda\}}\varrho(x)d\mu(x)\le c_1\int_X \varphi_2(c_1|f(x)|)\sigma(x)d\mu(x),$$

where $c_1=\max\{\frac{c'c_9}{2},c_9\}$.

In summary, we find that $(i)\Leftrightarrow(ii)\Leftrightarrow(iii)\Leftrightarrow(iv)\Leftrightarrow(v)\Leftrightarrow(vi)\Leftrightarrow(vii)\Leftrightarrow(viii)$. The proof is complete. □

Finally, we present two new equivalent characterization inequalities (i.e., (9) and (10)) for the two-weight, weak-type maximal inequality (6) in Orlicz classes for maximal functions defined on homogeneous spaces. Our future work will focus on the excavation of the corresponding four-weight extension forms of inequalities (9) and (10) and the demonstration of corresponding four-weight equivalent characterization inequalities.

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