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Article

A System of Four Generalized Sylvester Matrix Equations over the Quaternion Algebra

Zhuo-Heng He [1], Jie Tian [1] and Shao-Wen Yu [2,*]

[1] Department of Mathematics and Newtouch Center for Mathematics, Shanghai University, Shanghai 200444, China; zhuohenghe@shu.edu.cn (Z.-H.H.); 18303327484@shu.edu.cn (J.T.)
[2] School of Mathematics, East China University of Science and Technology, Shanghai 200237, China
* Correspondence: yushaowen@ecust.edu.cn

Abstract: In this paper, we make use of the simultaneous decomposition of eight quaternion matrices to study the solvability conditions and general solutions to a system of two-sided coupled Sylvester-type quaternion matrix equations $A_iX_iC_i + B_iX_{i+1}D_i = \Omega_i,\ i = 1,2,3,4$. We design an algorithm to compute the general solution to the system and give a numerical example. Additionally, we consider the application of the system in the encryption and decryption of color images.

Keywords: quaternion matrix equation; matrix decomposition; solvability; general solution

MSC: 15A03; 15A21; 15A23; 15A24

1. Introduction

1.1. Background

Quaternions, the decomposition of quaternion matrices and the matrix equations over quaternions, and so on, play important roles in mathematics and have a wide range of applications in various fields, such as color image processing, robotics, physics, aerospace engineering, control systems, and statistic model and graph theory. There are a great number of papers and monographs that investigate quaternion theory and corresponding applications from different aspects [1–18].

Chen et al. [2] proposed a robust blind watermarking scheme based on quaternion QR decomposition for color image copyright protection. He et al. [19] considered the theory and application of a system of Sylvester-type quaternion matrix equations, the system shown as follows:

$$\begin{cases} X_1A_1 - B_1X_2 = C_1, \\ X_3A_2 - B_2X_2 = C_2, \\ X_3A_3 - B_3X_4 = C_3, \\ X_4A_4 - B_4X_5 = C_4, \\ X_6A_5 - B_5X_5 = C_5. \end{cases}$$

Li et al. [11] came up with a quaternion biconjugate gradient method based on a structure-preserving method for solving non-Hermitian quaternion linear systems arising from color image deblurred problems. Took et al. [17] introduced the quaternion least mean square algorithm for the adaptive filtering of the three- and four-dimensional process.

There are a good deal of papers from various perspectives using various methods to study quaternion matrix equations, including the solvability conditions, general solutions, the properties of the general solution, the extreme rank of solutions, minimum norm least squares solution, ϕ-Hermitian solution, ϕ-skew-Hermitian solution, η-Hermitian solution, η-skew-Hermitian solution, and their applications [5,20–32].

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Kyrchei [23] derived explicit formulas for the determinantal representations of solutions to the systems of the quaternion equations $A_1X = C_1$, $XB_2 = C_2$ and $A_1X = C_1$, $A_2X = C_2$ using the determinantal representations of the Moore–Penrose matrix inverse put forward in [22]. Xu et al. [29] provided some useful necessary and sufficient conditions and a general solution to a constrained system of Sylvester-like matrix equations over the quaternion in terms of ranks and the Moore—Penrose inverse of the coefficient matrices. Xie et al. [33] considered the solvability conditions using ranks and the Moore–Penrose inverse for the system of three Sylvester-type quaternion matrix equations with ten variables $A_iX_i + Y_iB_i + C_iZ_iD_i + F_iZ_{i+1}G_i = E_i (i = 1, 2, 3)$.

To our knowledge, there has been little information on the theory and applications of the following system:

$$A_iX_iC_i + B_iX_{i+1}D_i = \Omega_i,\ i = 1, 2, 3, 4. \tag{1}$$

Motivated by the wide applications of quaternion matrix equations and the needs of their theoretical developments, we, in this paper, consider the solvability conditions, the general solutions and the applications to system (1).

The paper is organized as follows. First, we extend the simultaneous decomposition of seven quaternion matrices, which are shown in [21], to the simultaneous decomposition of eight quaternion matrices. Then, we make use of the simultaneous decomposition to prove that system (1) is consistent if and only if 40 rank equalities or 40 block matrix equalities hold. In the meantime, we also prove that these rank equalities as a whole are equivalent to these block matrix equalities as a whole. Next, we show an algorithm which clearly illustrates the steps taken to obtain the general solution to system (1) and we also give a numerical example. Afterwards, we make use of the system of two-sided coupled Sylvester-type quaternion matrix equations to develop a framework that can be used to encrypt and decrypt four color images simultaneously. Finally, we summarize our work.

1.2. Notation

Let $\mathbb{R}$ and $\mathbb{H}$ stand for the real number field and quaternion algebra, respectively. $\mathbb{H}$ can be viewed as a four-dimensional linear space over $\mathbb{R}$ with the basis: $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are called imaginary units.

Throughout this paper, we denote $\mathbb{H}^{m\times n}$ as all $m \times n$ matrices over the real quaternions. The rank of a quaternion matrix A over $\mathbb{H}$ is defined to be the maximum number of columns of A, which are linearly independent to the right. Quaternion matrix A and PAQ have the same rank if P and Q are invertible quaternion matrices [18]. For convenience, we use $r_{A_{11}A_{12}\ldots A_{1n}|A_{21}A_{22}\ldots A_{2n}|A_{m1}A_{m2}\ldots A_{mn}}$ to represent the rank of a block quaternion matrix, as follows:

$$\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{pmatrix}, \tag{2}$$

and if A_{ij} in the block (2) is a zero matrix, we use "0" to represent it. For instance, $r_{A_i\Omega_iB_i}$, $r_{C_i|\Omega_i|D_i}$ and $r_{A_i\Omega_i|0D_i}$ stand for the ranks of the following block matrices:

$$(A_i, \Omega_i, B_i), \begin{pmatrix} C_i \\ \Omega_i \\ D_i \end{pmatrix}, \begin{pmatrix} A_i & \Omega_i \\ & D_i \end{pmatrix},$$

respectively.

2. The Solvability Conditions of System (1)

In this section, we investigate the solvability conditions of the system of quaternion matrix equations as follows:

$$A_iX_iC_i + B_iX_{i+1}D_i = \Omega_i,\ i = 1,2,3,4.$$

Notice that the sizes of the coefficient matrices A_i, B_i, C_i and D_i have certain rules. They can be arranged into block matrices as

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array}\begin{array}{c} \begin{array}{ccccc} q_1 & q_2 & q_3 & q_4 & q_5 \end{array} \\ \begin{pmatrix} A_1 & B_1 & & & \\ & A_2 & B_2 & & \\ & & A_3 & B_3 & \\ & & & A_4 & B_4 \end{pmatrix} \end{array} \tag{3}$$

and

$$\begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array}\begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} C_1 & & & \\ D_1 & C_2 & & \\ & D_2 & C_3 & \\ & & D_3 & C_4 \\ & & & D_4 \end{pmatrix} \end{array}. \tag{4}$$

Lemma 1 ([19,21])**.** *Considering block matrix (3), there are nonsingular quaternion matrices* $P_i \in \mathbb{H}^{p_i \times p_i}$*,* $Q_i \in \mathbb{H}^{q_i \times q_i}$*, such that*

$$P_iA_iQ_i = S_{a_i},\ P_iB_iQ_{i+1} = S_{b_i},\ \ i = 1, \ldots, 4, \tag{5}$$

where $\begin{array}{c} \\ p_1 \\ p_2 \end{array}\begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{pmatrix} S_{a_1} & S_{b_1} & \\ & S_{a_2} & S_{b_2} \end{pmatrix} \end{array}$*, denoted by* $\widehat{AB}_1$*,* $\begin{array}{c} \\ p_3 \\ p_4 \end{array}\begin{array}{c} \begin{array}{ccc} q_3 & q_4 & q_5 \end{array} \\ \begin{pmatrix} S_{a_3} & S_{b_3} & \\ & S_{a_4} & S_{b_4} \end{pmatrix} \end{array}$*, denoted by* $\widehat{AB}_2$*, and we have*

$$
\widehat{AB}_1 = \begin{pmatrix}
\begin{matrix}
I&0&0&0&0 & 0&0&0&0 \\
0&I&0&0&0 & 0&0&0&0 \\
0&0&I&0&0 & 0&0&0&0 \\
0&0&0&I&0 & 0&0&0&0 \\
0&0&0&0&I & 0&0&0&0 \\
0&0&0&0&0 & I&0&0&0 \\
0&0&0&0&0 & 0&I&0&0 \\
0&0&0&0&0 & 0&0&I&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0
\end{matrix}
&
\begin{matrix}
I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0
\end{matrix}
& \\
&
\begin{matrix}
I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&I & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0
\end{matrix}
&
\begin{matrix}
I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0
\end{matrix}
\end{pmatrix},
$$

$$
\widehat{AB}_2 = \begin{pmatrix}
\begin{matrix}
I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0
\end{matrix}
&
\begin{matrix}
I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0
\end{matrix}
& \\
&
\begin{matrix}
I&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&I&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&I&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&I & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&I&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & I&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&I&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&I&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&I & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0
\end{matrix}
&
\begin{matrix}
I&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&I&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&I&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&I&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&I & 0&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & I&0&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&I&0&0 \\
0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0 & 0&0&I&0 \\
0&0&0&0&0 & 0&0&0&0
\end{matrix}
\end{pmatrix},
$$

where I and 0 stand for the identity matrix and zero matrix with appropriate size, respectively. Similarly, there are the nonsingular quaternion matrices $T_i \in \mathbb{H}^{t_i \times t_i}$, $S_i \in \mathbb{H}^{s_i \times s_i}$, such that

$$T_i C_i S_i = S_{c_i},\ T_{i+1} D_i S_i = S_{d_i},\ \ i = 1, \ldots, 4, \tag{6}$$

where

$$\begin{pmatrix} S_{c_1} & & & \\ S_{d_1} & S_{c_2} & & \\ & S_{d_2} & S_{c_3} & \\ & & S_{d_3} & S_{c_4} \\ & & & S_{d_4} \end{pmatrix}^T = \begin{pmatrix} S_{a_1} & S_{b_1} & & & \\ & S_{a_2} & S_{b_2} & & \\ & & S_{a_3} & S_{b_3} & \\ & & & S_{a_4} & S_{b_4} \end{pmatrix}.$$

He et al. [21] showed the simultaneous decomposition of seven quaternion matrices. Starting from that, and using a similar method as [19], we can reach our conclusion.

According to (5) and (6), we have

$$P_i^{-1} S_{a_i} Q_i^{-1} X_i T_i^{-1} S_{c_i} S_i^{-1} + P_i^{-1} S_{b_i} Q_{i+1}^{-1} X_{i+1} T_{i+1}^{-1} S_{d_i} S_i^{-1} = \Omega_i, \ i = 1,2,3,4.$$

Multiply P_i and S_i to the left and right sides of the equations, respectively. Then, we have

$$S_{a_i} Q_i^{-1} X_i T_i^{-1} S_{c_i} + S_{b_i} Q_{i+1}^{-1} X_{i+1} T_{i+1}^{-1} S_{d_i} = P_i \Omega_i S_i, \ i = 1,2,3,4.$$

Let $\widehat{\Omega_i} = P_i \Omega_i S_i, \ i = 1, \ldots, 4, \ \widehat{X_j} = Q_j^{-1} X_j T_j^{-1}, \ j = 1, \ldots, 5$. Hence, we have

$$S_{a_i} \widehat{X_i} S_{c_i} + S_{b_i} \widehat{X_{i+1}} S_{d_i} = \widehat{\Omega_i}, \ i = 1,2,3,4. \tag{7}$$

According to the equivalence canonical forms of (3) and (4) and system (7), we divide $\widehat{X_j}, j = 1, \ldots, 5$ and $\widehat{\Omega_i}, i = 1,2,3,4$ into the following partitioned matrices:

$$\widehat{X_1} = (X_{ij}^{(1)})_{9\times 9}, \ \widehat{X_2} = (X_{ij}^{(2)})_{21\times 21}, \ \widehat{X_3} = (X_{ij}^{(3)})_{25\times 25}, \ \widehat{X_4} = (X_{ij}^{(4)})_{21\times 21}, \ \widehat{X_5} = (X_{ij}^{(5)})_{9\times 9},$$
$$\widehat{\Omega_1} = (\omega_{ij}^{(1)})_{16\times 16}, \ \widehat{\Omega_2} = (\omega_{ij}^{(2)})_{24\times 24}, \ \widehat{\Omega_3} = (\omega_{ij}^{(3)})_{24\times 24}, \ \widehat{\Omega_4} = (\omega_{ij}^{(4)})_{16\times 16}.$$

Through computation, we have

$$\widehat{\Omega_1} = (\Phi_1^{(1)}, \Phi_2^{(1)}), \tag{8}$$

where

$$\Phi_1^{(1)} = \begin{pmatrix} X_{11}^{(1)} + X_{11}^{(2)} & X_{12}^{(1)} + X_{12}^{(2)} & X_{13}^{(1)} + X_{13}^{(2)} & X_{14}^{(1)} + X_{14}^{(2)} & X_{15}^{(1)} + X_{15}^{(2)} & X_{16}^{(1)} + X_{16}^{(2)} \\ X_{21}^{(1)} + X_{21}^{(2)} & X_{22}^{(1)} + X_{22}^{(2)} & X_{23}^{(1)} + X_{23}^{(2)} & X_{24}^{(1)} + X_{24}^{(2)} & X_{25}^{(1)} + X_{25}^{(2)} & X_{26}^{(1)} + X_{26}^{(2)} \\ X_{31}^{(1)} + X_{31}^{(2)} & X_{32}^{(1)} + X_{32}^{(2)} & X_{33}^{(1)} + X_{33}^{(2)} & X_{34}^{(1)} + X_{34}^{(2)} & X_{35}^{(1)} + X_{35}^{(2)} & X_{36}^{(1)} + X_{36}^{(2)} \\ X_{41}^{(1)} + X_{41}^{(2)} & X_{42}^{(1)} + X_{42}^{(2)} & X_{43}^{(1)} + X_{43}^{(2)} & X_{44}^{(1)} + X_{44}^{(2)} & X_{45}^{(1)} + X_{45}^{(2)} & X_{46}^{(1)} + X_{46}^{(2)} \\ X_{51}^{(1)} + X_{51}^{(2)} & X_{52}^{(1)} + X_{52}^{(2)} & X_{53}^{(1)} + X_{53}^{(2)} & X_{54}^{(1)} + X_{54}^{(2)} & X_{55}^{(1)} + X_{55}^{(2)} & X_{56}^{(1)} + X_{56}^{(2)} \\ X_{61}^{(1)} + X_{61}^{(2)} & X_{62}^{(1)} + X_{62}^{(2)} & X_{63}^{(1)} + X_{63}^{(2)} & X_{64}^{(1)} + X_{64}^{(2)} & X_{65}^{(1)} + X_{65}^{(2)} & X_{66}^{(1)} + X_{66}^{(2)} \\ X_{71}^{(1)} + X_{71}^{(2)} & X_{72}^{(1)} + X_{72}^{(2)} & X_{73}^{(1)} + X_{73}^{(2)} & X_{74}^{(1)} + X_{74}^{(2)} & X_{75}^{(1)} + X_{75}^{(2)} & X_{76}^{(1)} + X_{76}^{(2)} \\ X_{81}^{(1)} & X_{82}^{(1)} & X_{83}^{(1)} & X_{84}^{(1)} & X_{85}^{(1)} & X_{86}^{(1)} \\ X_{81}^{(2)} & X_{82}^{(2)} & X_{83}^{(2)} & X_{84}^{(2)} & X_{85}^{(2)} & X_{86}^{(2)} \\ X_{91}^{(2)} & X_{92}^{(2)} & X_{93}^{(2)} & X_{94}^{(2)} & X_{95}^{(2)} & X_{96}^{(2)} \\ X_{10,1}^{(2)} & X_{10,2}^{(2)} & X_{10,3}^{(2)} & X_{10,4}^{(2)} & X_{10,5}^{(2)} & X_{10,6}^{(2)} \\ X_{11,1}^{(2)} & X_{11,2}^{(2)} & X_{11,3}^{(2)} & X_{11,4}^{(2)} & X_{11,5}^{(2)} & X_{11,6}^{(2)} \\ X_{12,1}^{(2)} & X_{12,2}^{(2)} & X_{12,3}^{(2)} & X_{12,4}^{(2)} & X_{12,5}^{(2)} & X_{12,6}^{(2)} \\ X_{13,1}^{(2)} & X_{13,2}^{(2)} & X_{13,3}^{(2)} & X_{13,4}^{(2)} & X_{13,5}^{(2)} & X_{13,6}^{(2)} \\ X_{14,1}^{(2)} & X_{14,2}^{(2)} & X_{14,3}^{(2)} & X_{14,4}^{(2)} & X_{14,5}^{(2)} & X_{14,6}^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Phi_2^{(1)} = \begin{pmatrix}
x_{17}^{(1)} + x_{17}^{(2)} & x_{18}^{(1)} & x_{18}^{(2)} & x_{19}^{(2)} & x_{1,10}^{(2)} & x_{1,11}^{(2)} & x_{1,12}^{(2)} & x_{1,13}^{(2)} & x_{1,14}^{(2)} & 0 \\
x_{27}^{(1)} + x_{27}^{(2)} & x_{28}^{(1)} & x_{28}^{(2)} & x_{29}^{(2)} & x_{2,10}^{(2)} & x_{2,11}^{(2)} & x_{2,12}^{(2)} & x_{2,13}^{(2)} & x_{2,14}^{(2)} & 0 \\
x_{37}^{(1)} + x_{37}^{(2)} & x_{38}^{(1)} & x_{38}^{(2)} & x_{39}^{(2)} & x_{3,10}^{(2)} & x_{3,11}^{(2)} & x_{3,12}^{(2)} & x_{3,13}^{(2)} & x_{3,14}^{(2)} & 0 \\
x_{47}^{(1)} + x_{47}^{(2)} & x_{48}^{(1)} & x_{48}^{(2)} & x_{49}^{(2)} & x_{4,10}^{(2)} & x_{4,11}^{(2)} & x_{4,12}^{(2)} & x_{4,13}^{(2)} & x_{4,14}^{(2)} & 0 \\
x_{57}^{(1)} + x_{57}^{(2)} & x_{58}^{(1)} & x_{58}^{(2)} & x_{59}^{(2)} & x_{5,10}^{(2)} & x_{5,11}^{(2)} & x_{5,12}^{(2)} & x_{5,13}^{(2)} & x_{5,14}^{(2)} & 0 \\
x_{67}^{(1)} + x_{67}^{(2)} & x_{68}^{(1)} & x_{68}^{(2)} & x_{69}^{(2)} & x_{6,10}^{(2)} & x_{6,11}^{(2)} & x_{6,12}^{(2)} & x_{6,13}^{(2)} & x_{6,14}^{(2)} & 0 \\
x_{77}^{(1)} + x_{77}^{(2)} & x_{78}^{(1)} & x_{78}^{(2)} & x_{79}^{(2)} & x_{7,10}^{(2)} & x_{7,11}^{(2)} & x_{7,12}^{(2)} & x_{7,13}^{(2)} & x_{7,14}^{(2)} & 0 \\
x_{87}^{(1)} & x_{88}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{87}^{(2)} & 0 & x_{88}^{(2)} & x_{89}^{(2)} & x_{8,10}^{(2)} & x_{8,11}^{(2)} & x_{8,12}^{(2)} & x_{8,13}^{(2)} & x_{8,14}^{(2)} & 0 \\
x_{97}^{(2)} & 0 & x_{98}^{(2)} & x_{99}^{(2)} & x_{9,10}^{(2)} & x_{9,11}^{(2)} & x_{9,12}^{(2)} & x_{9,13}^{(2)} & x_{9,14}^{(2)} & 0 \\
x_{10,7}^{(2)} & 0 & x_{10,8}^{(2)} & x_{10,9}^{(2)} & x_{10,10}^{(2)} & x_{10,11}^{(2)} & x_{10,12}^{(2)} & x_{10,13}^{(2)} & x_{10,14}^{(2)} & 0 \\
x_{11,7}^{(2)} & 0 & x_{11,8}^{(2)} & x_{11,9}^{(2)} & x_{11,10}^{(2)} & x_{11,11}^{(2)} & x_{11,12}^{(2)} & x_{11,13}^{(2)} & x_{11,14}^{(2)} & 0 \\
x_{12,7}^{(2)} & 0 & x_{12,8}^{(2)} & x_{12,9}^{(2)} & x_{12,10}^{(2)} & x_{12,11}^{(2)} & x_{12,12}^{(2)} & x_{12,13}^{(2)} & x_{12,14}^{(2)} & 0 \\
x_{13,7}^{(2)} & 0 & x_{13,8}^{(2)} & x_{13,9}^{(2)} & x_{13,10}^{(2)} & x_{13,11}^{(2)} & x_{13,12}^{(2)} & x_{13,13}^{(2)} & x_{13,14}^{(2)} & 0 \\
x_{14,7}^{(2)} & 0 & x_{14,8}^{(2)} & x_{14,9}^{(2)} & x_{14,10}^{(2)} & x_{14,11}^{(2)} & x_{14,12}^{(2)} & x_{14,13}^{(2)} & x_{14,14}^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We have

$$\widehat{\Omega_2} = \begin{pmatrix} \Phi_{11}^{(2)} & \Phi_{12}^{(2)} & \Phi_{13}^{(2)} \\ \Phi_{21}^{(2)} & \Phi_{22}^{(2)} & \Phi_{23}^{(2)} \end{pmatrix}, \tag{9}$$

where

$$\Phi_{11}^{(2)} = \begin{pmatrix}
x_{11}^{(2)} + x_{11}^{(3)} & x_{12}^{(2)} + x_{12}^{(3)} & x_{13}^{(2)} + x_{13}^{(3)} & x_{14}^{(2)} + x_{14}^{(3)} & x_{15}^{(2)} + x_{15}^{(3)} & x_{16}^{(2)} & x_{18}^{(2)} + x_{16}^{(3)} & x_{19}^{(2)} + x_{17}^{(3)} \\
x_{21}^{(2)} + x_{21}^{(3)} & x_{22}^{(2)} + x_{22}^{(3)} & x_{23}^{(2)} + x_{23}^{(3)} & x_{24}^{(2)} + x_{24}^{(3)} & x_{25}^{(2)} + x_{25}^{(3)} & x_{26}^{(2)} & x_{28}^{(2)} + x_{26}^{(3)} & x_{29}^{(2)} + x_{27}^{(3)} \\
x_{31}^{(2)} + x_{31}^{(3)} & x_{32}^{(2)} + x_{32}^{(3)} & x_{33}^{(2)} + x_{33}^{(3)} & x_{34}^{(2)} + x_{34}^{(3)} & x_{35}^{(2)} + x_{35}^{(3)} & x_{36}^{(2)} & x_{38}^{(2)} + x_{36}^{(3)} & x_{39}^{(2)} + x_{37}^{(3)} \\
x_{41}^{(2)} + x_{41}^{(3)} & x_{42}^{(2)} + x_{42}^{(3)} & x_{43}^{(2)} + x_{43}^{(3)} & x_{44}^{(2)} + x_{44}^{(3)} & x_{45}^{(2)} + x_{45}^{(3)} & x_{46}^{(2)} & x_{48}^{(2)} + x_{46}^{(3)} & x_{49}^{(2)} + x_{47}^{(3)} \\
x_{51}^{(2)} + x_{51}^{(3)} & x_{52}^{(2)} + x_{52}^{(3)} & x_{53}^{(2)} + x_{53}^{(3)} & x_{54}^{(2)} + x_{54}^{(3)} & x_{55}^{(2)} + x_{55}^{(3)} & x_{56}^{(2)} & x_{58}^{(2)} + x_{56}^{(3)} & x_{59}^{(2)} + x_{57}^{(3)} \\
x_{61}^{(2)} & x_{62}^{(2)} & x_{63}^{(2)} & x_{64}^{(2)} & x_{65}^{(2)} & x_{66}^{(2)} & x_{68}^{(2)} & x_{69}^{(2)} \\
x_{81}^{(2)} + x_{61}^{(3)} & x_{82}^{(2)} + x_{62}^{(3)} & x_{83}^{(2)} + x_{63}^{(3)} & x_{84}^{(2)} + x_{64}^{(3)} & x_{85}^{(2)} + x_{65}^{(3)} & x_{86}^{(2)} & x_{88}^{(2)} + x_{66}^{(3)} & x_{89}^{(2)} + x_{67}^{(3)} \\
x_{91}^{(2)} + x_{71}^{(3)} & x_{92}^{(2)} + x_{72}^{(3)} & x_{93}^{(2)} + x_{73}^{(3)} & x_{94}^{(2)} + x_{74}^{(3)} & x_{95}^{(2)} + x_{75}^{(3)} & x_{96}^{(2)} & x_{98}^{(2)} + x_{76}^{(3)} & x_{99}^{(2)} + x_{77}^{(3)} \\
x_{10,1}^{(2)} + x_{81}^{(3)} & x_{10,2}^{(2)} + x_{82}^{(3)} & x_{10,3}^{(2)} + x_{83}^{(3)} & x_{10,4}^{(2)} + x_{84}^{(3)} & x_{10,5}^{(2)} + x_{85}^{(3)} & x_{10,6}^{(2)} & x_{10,8}^{(2)} + x_{86}^{(3)} & x_{10,9}^{(2)} + x_{87}^{(3)} \\
x_{11,1}^{(2)} + x_{91}^{(3)} & x_{11,2}^{(2)} + x_{92}^{(3)} & x_{11,3}^{(2)} + x_{93}^{(3)} & x_{11,4}^{(2)} + x_{94}^{(3)} & x_{11,5}^{(2)} + x_{95}^{(3)} & x_{11,6}^{(2)} & x_{11,8}^{(2)} + x_{9,6}^{(3)} & x_{11,9}^{(2)} + x_{97}^{(3)} \\
x_{12,1}^{(2)} + x_{10,1}^{(3)} & x_{12,2}^{(2)} + x_{10,2}^{(3)} & x_{12,3}^{(2)} + x_{10,3}^{(3)} & x_{12,4}^{(2)} + x_{10,4}^{(3)} & x_{12,5}^{(2)} + x_{10,5}^{(3)} & x_{12,6}^{(2)} & x_{12,8}^{(2)} + x_{10,6}^{(3)} & x_{12,9}^{(2)} + x_{10,7}^{(3)} \\
x_{13,1}^{(2)} & x_{13,2}^{(2)} & x_{13,3}^{(2)} & x_{13,4}^{(2)} & x_{13,5}^{(2)} & x_{13,6}^{(2)} & x_{13,8}^{(2)} & x_{13,9}^{(2)}
\end{pmatrix},$$

$$\Phi_{21}^{(2)} = \begin{pmatrix}
x_{15,1}^{(2)} + x_{11,1}^{(3)} & x_{15,2}^{(2)} + x_{11,2}^{(3)} & x_{15,3}^{(2)} + x_{11,3}^{(3)} & x_{15,4}^{(2)} + x_{11,4}^{(3)} & x_{15,5}^{(2)} + x_{11,5}^{(3)} & x_{15,6}^{(2)} & x_{15,8}^{(2)} + x_{11,6}^{(3)} & x_{15,9}^{(2)} + x_{11,7}^{(3)} \\
x_{16,1}^{(2)} + x_{12,1}^{(3)} & x_{16,2}^{(2)} + x_{12,2}^{(3)} & x_{16,3}^{(2)} + x_{12,3}^{(3)} & x_{16,4}^{(2)} + x_{12,4}^{(3)} & x_{16,5}^{(2)} + x_{12,5}^{(3)} & x_{16,6}^{(2)} & x_{16,8}^{(2)} + x_{12,6}^{(3)} & x_{16,9}^{(2)} + x_{12,7}^{(3)} \\
x_{17,1}^{(2)} + x_{13,1}^{(3)} & x_{17,2}^{(2)} + x_{13,2}^{(3)} & x_{17,3}^{(2)} + x_{13,3}^{(3)} & x_{17,4}^{(2)} + x_{13,4}^{(3)} & x_{17,5}^{(2)} + x_{13,5}^{(3)} & x_{17,6}^{(2)} & x_{17,8}^{(2)} + x_{13,6}^{(3)} & x_{17,9}^{(2)} + x_{13,7}^{(3)} \\
x_{18,1}^{(2)} + x_{14,1}^{(3)} & x_{18,2}^{(2)} + x_{14,2}^{(3)} & x_{18,3}^{(2)} + x_{14,3}^{(3)} & x_{18,4}^{(2)} + x_{14,4}^{(3)} & x_{18,5}^{(2)} + x_{14,5}^{(3)} & x_{18,6}^{(2)} & x_{18,8}^{(2)} + x_{14,6}^{(3)} & x_{18,9}^{(2)} + x_{14,7}^{(3)} \\
x_{19,1}^{(2)} + x_{15,1}^{(3)} & x_{19,2}^{(2)} + x_{15,2}^{(3)} & x_{19,3}^{(2)} + x_{15,3}^{(3)} & x_{19,4}^{(2)} + x_{15,4}^{(3)} & x_{19,5}^{(2)} + x_{15,5}^{(3)} & x_{19,6}^{(2)} & x_{19,8}^{(2)} + x_{15,6}^{(3)} & x_{19,9}^{(2)} + x_{15,7}^{(3)} \\
x_{20,1}^{(2)} & x_{20,2}^{(2)} & x_{20,3}^{(2)} & x_{20,4}^{(2)} & x_{20,5}^{(2)} & x_{20,6}^{(2)} & x_{20,8}^{(2)} & x_{20,9}^{(2)} \\
x_{16,1}^{(3)} & x_{16,2}^{(3)} & x_{16,3}^{(3)} & x_{16,4}^{(3)} & x_{16,5}^{(3)} & 0 & x_{16,6}^{(3)} & x_{16,7}^{(3)} \\
x_{17,1}^{(3)} & x_{17,2}^{(3)} & x_{17,3}^{(3)} & x_{17,4}^{(3)} & x_{17,5}^{(3)} & 0 & x_{17,6}^{(3)} & x_{17,7}^{(3)} \\
x_{18,1}^{(3)} & x_{18,2}^{(3)} & x_{18,3}^{(3)} & x_{18,4}^{(3)} & x_{18,5}^{(3)} & 0 & x_{18,6}^{(3)} & x_{18,7}^{(3)} \\
x_{19,1}^{(3)} & x_{19,2}^{(3)} & x_{19,3}^{(3)} & x_{19,4}^{(3)} & x_{19,5}^{(3)} & 0 & x_{19,6}^{(3)} & x_{19,7}^{(3)} \\
x_{20,1}^{(3)} & x_{20,2}^{(3)} & x_{20,3}^{(3)} & x_{20,4}^{(3)} & x_{20,5}^{(3)} & 0 & x_{20,6}^{(3)} & x_{20,7}^{(3)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\Phi_{12}^{(2)} = \begin{pmatrix}
x_{1,10}^{(2)} + x_{18}^{(3)} & x_{1,11}^{(2)} + x_{19}^{(3)} & x_{1,12}^{(2)} + x_{1,10}^{(3)} & x_{1,13}^{(2)} & x_{1,15}^{(2)} + x_{1,11}^{(3)} & x_{1,16}^{(2)} + x_{1,12}^{(3)} & x_{1,17}^{(2)} + x_{1,13}^{(3)} \\
x_{2,10}^{(2)} + x_{28}^{(3)} & x_{2,11}^{(2)} + x_{29}^{(3)} & x_{2,12}^{(2)} + x_{2,10}^{(3)} & x_{2,13}^{(2)} & x_{2,15}^{(2)} + x_{2,11}^{(3)} & x_{2,16}^{(2)} + x_{2,12}^{(3)} & x_{2,17}^{(2)} + x_{2,13}^{(3)} \\
x_{3,10}^{(2)} + x_{38}^{(3)} & x_{3,11}^{(2)} + x_{39}^{(3)} & x_{3,12}^{(2)} + x_{3,10}^{(3)} & x_{3,13}^{(2)} & x_{3,15}^{(2)} + x_{3,11}^{(3)} & x_{3,16}^{(2)} + x_{3,12}^{(3)} & x_{3,17}^{(2)} + x_{3,13}^{(3)} \\
x_{4,10}^{(2)} + x_{48}^{(3)} & x_{4,11}^{(2)} + x_{49}^{(3)} & x_{4,12}^{(2)} + x_{4,10}^{(3)} & x_{4,13}^{(2)} & x_{4,15}^{(2)} + x_{4,11}^{(3)} & x_{4,16}^{(2)} + x_{4,12}^{(3)} & x_{4,17}^{(2)} + x_{4,13}^{(3)} \\
x_{5,10}^{(2)} + x_{58}^{(3)} & x_{5,11}^{(2)} + x_{59}^{(3)} & x_{5,12}^{(2)} + x_{5,10}^{(3)} & x_{5,13}^{(2)} & x_{5,15}^{(2)} + x_{5,11}^{(3)} & x_{5,16}^{(2)} + x_{5,12}^{(3)} & x_{5,17}^{(2)} + x_{5,13}^{(3)} \\
x_{6,10}^{(2)} & x_{6,11}^{(2)} & x_{6,12}^{(2)} & x_{6,13}^{(2)} & x_{6,15}^{(2)} & x_{6,16}^{(2)} & x_{6,17}^{(2)} \\
x_{8,10}^{(2)} + x_{68}^{(3)} & x_{8,11}^{(2)} + x_{69}^{(3)} & x_{8,12}^{(2)} + x_{6,10}^{(3)} & x_{8,13}^{(2)} & x_{8,15}^{(2)} + x_{6,11}^{(3)} & x_{8,16}^{(2)} + x_{6,12}^{(3)} & x_{8,17}^{(2)} + x_{6,13}^{(3)} \\
x_{9,10}^{(2)} + x_{78}^{(3)} & x_{9,11}^{(2)} + x_{79}^{(3)} & x_{9,12}^{(2)} + x_{7,10}^{(3)} & x_{9,13}^{(2)} & x_{9,15}^{(2)} + x_{7,11}^{(3)} & x_{9,16}^{(2)} + x_{7,12}^{(3)} & x_{9,17}^{(2)} + x_{7,13}^{(3)} \\
x_{10,10}^{(2)} + x_{88}^{(3)} & x_{10,11}^{(2)} + x_{89}^{(3)} & x_{10,12}^{(2)} + x_{8,10}^{(3)} & x_{10,13}^{(2)} & x_{10,15}^{(2)} + x_{8,11}^{(3)} & x_{10,16}^{(2)} + x_{8,12}^{(3)} & x_{10,17}^{(2)} + x_{8,13}^{(3)} \\
x_{11,10}^{(2)} + x_{98}^{(3)} & x_{11,11}^{(2)} + x_{99}^{(3)} & x_{11,12}^{(2)} + x_{9,10}^{(3)} & x_{11,13}^{(2)} & x_{11,15}^{(2)} + x_{9,11}^{(3)} & x_{11,16}^{(2)} + x_{9,12}^{(3)} & x_{11,17}^{(2)} + x_{9,13}^{(3)} \\
x_{12,10}^{(2)} + x_{10,8}^{(3)} & x_{12,11}^{(2)} + x_{10,9}^{(3)} & x_{12,12}^{(2)} + x_{10,10}^{(3)} & x_{12,13}^{(2)} & x_{12,15}^{(2)} + x_{10,11}^{(3)} & x_{12,16}^{(2)} + x_{10,12}^{(3)} & x_{12,17}^{(2)} + x_{10,13}^{(3)} \\
x_{13,10}^{(2)} & x_{13,11}^{(2)} & x_{13,12}^{(2)} & x_{13,13}^{(2)} & x_{13,15}^{(2)} & x_{13,16}^{(2)} & x_{13,17}^{(2)}
\end{pmatrix},$$

$$\Phi_{22}^{(2)} = \begin{pmatrix}
x_{15,10}^{(2)}+x_{11,8}^{(3)} & x_{15,11}^{(2)}+x_{11,9}^{(3)} & x_{15,12}^{(2)}+x_{11,10}^{(3)} & x_{15,13}^{(2)} & x_{15,15}^{(2)}+x_{11,11}^{(3)} & x_{15,16}^{(2)}+x_{11,12}^{(3)} & x_{15,17}^{(2)}+x_{11,13}^{(3)} \\
x_{16,10}^{(2)}+x_{12,8}^{(3)} & x_{16,11}^{(2)}+x_{12,9}^{(3)} & x_{16,12}^{(2)}+x_{12,10}^{(3)} & x_{16,13}^{(2)} & x_{16,15}^{(2)}+x_{12,11}^{(3)} & x_{16,16}^{(2)}+x_{12,12}^{(3)} & x_{16,17}^{(2)}+x_{12,13}^{(3)} \\
x_{17,10}^{(2)}+x_{13,8}^{(3)} & x_{17,11}^{(2)}+x_{13,9}^{(3)} & x_{17,12}^{(2)}+x_{13,10}^{(3)} & x_{17,13}^{(2)} & x_{17,15}^{(2)}+x_{13,11}^{(3)} & x_{17,16}^{(2)}+x_{13,12}^{(3)} & x_{17,17}^{(2)}+x_{13,13}^{(3)} \\
x_{18,10}^{(2)}+x_{14,8}^{(3)} & x_{18,11}^{(2)}+x_{14,9}^{(3)} & x_{18,12}^{(2)}+x_{14,10}^{(3)} & x_{18,13}^{(2)} & x_{18,15}^{(2)}+x_{14,11}^{(3)} & x_{18,16}^{(2)}+x_{14,12}^{(3)} & x_{18,17}^{(2)}+x_{14,13}^{(3)} \\
x_{19,10}^{(2)}+x_{15,8}^{(3)} & x_{19,11}^{(2)}+x_{15,9}^{(3)} & x_{19,12}^{(2)}+x_{15,10}^{(3)} & x_{19,13}^{(2)} & x_{19,15}^{(2)}+x_{15,11}^{(3)} & x_{19,16}^{(2)}+x_{15,12}^{(3)} & x_{19,17}^{(2)}+x_{15,13}^{(3)} \\
x_{20,10}^{(2)} & x_{20,11}^{(2)} & x_{20,12}^{(2)} & x_{20,13}^{(2)} & x_{20,15}^{(2)} & x_{20,16}^{(2)} & x_{20,17}^{(2)} \\
x_{16,8}^{(3)} & x_{16,9}^{(3)} & x_{16,10}^{(3)} & 0 & x_{16,11}^{(3)} & x_{16,12}^{(3)} & x_{16,13}^{(3)} \\
x_{17,8}^{(3)} & x_{17,9}^{(3)} & x_{17,10}^{(3)} & 0 & x_{17,11}^{(3)} & x_{17,12}^{(3)} & x_{17,13}^{(3)} \\
x_{18,8}^{(3)} & x_{18,9}^{(3)} & x_{18,10}^{(3)} & 0 & x_{18,11}^{(3)} & x_{18,12}^{(3)} & x_{18,13}^{(3)} \\
x_{19,8}^{(3)} & x_{19,9}^{(3)} & x_{19,10}^{(3)} & 0 & x_{19,11}^{(3)} & x_{19,12}^{(3)} & x_{19,13}^{(3)} \\
x_{20,8}^{(3)} & x_{20,9}^{(3)} & x_{20,10}^{(3)} & 0 & x_{20,11}^{(3)} & x_{20,12}^{(3)} & x_{20,13}^{(3)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\Phi_{13}^{(2)} = \begin{pmatrix}
x_{1,18}^{(2)}+x_{1,14}^{(3)} & x_{1,19}^{(2)}+x_{1,15}^{(3)} & x_{1,20}^{(2)} & x_{1,16}^{(3)} & x_{1,17}^{(3)} & x_{1,18}^{(3)} & x_{1,19}^{(3)} & x_{1,20}^{(3)} & 0 \\
x_{2,18}^{(2)}+x_{2,14}^{(3)} & x_{2,19}^{(2)}+x_{2,15}^{(3)} & x_{2,20}^{(2)} & x_{2,16}^{(3)} & x_{2,17}^{(3)} & x_{2,18}^{(3)} & x_{2,19}^{(3)} & x_{2,20}^{(3)} & 0 \\
x_{3,18}^{(2)}+x_{3,14}^{(3)} & x_{3,19}^{(2)}+x_{3,15}^{(3)} & x_{3,20}^{(2)} & x_{3,16}^{(3)} & x_{3,17}^{(3)} & x_{3,18}^{(3)} & x_{3,19}^{(3)} & x_{3,20}^{(3)} & 0 \\
x_{4,18}^{(2)}+x_{4,14}^{(3)} & x_{4,19}^{(2)}+x_{4,15}^{(3)} & x_{4,20}^{(2)} & x_{4,16}^{(3)} & x_{4,17}^{(3)} & x_{4,18}^{(3)} & x_{4,19}^{(3)} & x_{4,20}^{(3)} & 0 \\
x_{5,18}^{(2)}+x_{5,14}^{(3)} & x_{5,19}^{(2)}+x_{5,15}^{(3)} & x_{5,20}^{(2)} & x_{5,16}^{(3)} & x_{5,17}^{(3)} & x_{5,18}^{(3)} & x_{5,19}^{(3)} & x_{5,20}^{(3)} & 0 \\
x_{6,18}^{(2)} & x_{6,19}^{(2)} & x_{6,20}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{8,18}^{(2)}+x_{6,14}^{(3)} & x_{8,19}^{(2)}+x_{6,15}^{(3)} & x_{8,20}^{(2)} & x_{6,16}^{(3)} & x_{6,17}^{(3)} & x_{6,18}^{(3)} & x_{6,19}^{(3)} & x_{6,20}^{(3)} & 0 \\
x_{9,18}^{(2)}+x_{7,14}^{(3)} & x_{9,19}^{(2)}+x_{7,15}^{(3)} & x_{9,20}^{(2)} & x_{7,16}^{(3)} & x_{7,17}^{(3)} & x_{7,18}^{(3)} & x_{7,19}^{(3)} & x_{7,20}^{(3)} & 0 \\
x_{10,18}^{(2)}+x_{8,14}^{(3)} & x_{10,19}^{(2)}+x_{8,15}^{(3)} & x_{10,20}^{(2)} & x_{8,16}^{(3)} & x_{8,17}^{(3)} & x_{8,18}^{(3)} & x_{8,19}^{(3)} & x_{8,20}^{(3)} & 0 \\
x_{11,18}^{(2)}+x_{9,14}^{(3)} & x_{11,19}^{(2)}+x_{9,15}^{(3)} & x_{11,20}^{(2)} & x_{9,16}^{(3)} & x_{9,17}^{(3)} & x_{9,18}^{(3)} & x_{9,19}^{(3)} & x_{9,20}^{(3)} & 0 \\
x_{12,18}^{(2)}+x_{10,14}^{(3)} & x_{12,19}^{(2)}+x_{10,15}^{(3)} & x_{12,20}^{(2)} & x_{10,16}^{(3)} & x_{10,17}^{(3)} & x_{10,18}^{(3)} & x_{10,19}^{(3)} & x_{10,20}^{(3)} & 0 \\
x_{13,18}^{(2)} & x_{13,19}^{(2)} & x_{13,20}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\Phi_{23}^{(2)} = \begin{pmatrix}
x_{15,18}^{(2)}+x_{11,14}^{(3)} & x_{15,19}^{(2)}+x_{11,15}^{(3)} & x_{15,20}^{(2)} & x_{11,16}^{(3)} & x_{11,17}^{(3)} & x_{11,18}^{(3)} & x_{11,19}^{(3)} & x_{11,20}^{(3)} & 0 \\
x_{16,18}^{(2)}+x_{12,14}^{(3)} & x_{16,19}^{(2)}+x_{12,15}^{(3)} & x_{16,20}^{(2)} & x_{12,16}^{(3)} & x_{12,17}^{(3)} & x_{12,18}^{(3)} & x_{12,19}^{(3)} & x_{12,20}^{(3)} & 0 \\
x_{17,18}^{(2)}+x_{13,14}^{(3)} & x_{17,19}^{(2)}+x_{13,15}^{(3)} & x_{17,20}^{(2)} & x_{13,16}^{(3)} & x_{13,17}^{(3)} & x_{13,18}^{(3)} & x_{13,19}^{(3)} & x_{13,20}^{(3)} & 0 \\
x_{18,18}^{(2)}+x_{14,14}^{(3)} & x_{18,19}^{(2)}+x_{14,15}^{(3)} & x_{18,20}^{(2)} & x_{14,16}^{(3)} & x_{14,17}^{(3)} & x_{14,18}^{(3)} & x_{14,19}^{(3)} & x_{14,20}^{(3)} & 0 \\
x_{19,18}^{(2)}+x_{15,14}^{(3)} & x_{19,19}^{(2)}+x_{15,15}^{(3)} & x_{19,20}^{(2)} & x_{15,16}^{(3)} & x_{15,17}^{(3)} & x_{15,18}^{(3)} & x_{15,19}^{(3)} & x_{15,20}^{(3)} & 0 \\
x_{20,18}^{(2)} & x_{20,19}^{(2)} & x_{20,20}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{16,14}^{(3)} & x_{16,15}^{(3)} & 0 & x_{16,16}^{(3)} & x_{16,17}^{(3)} & x_{16,18}^{(3)} & x_{16,19}^{(3)} & x_{16,20}^{(3)} & 0 \\
x_{17,14}^{(3)} & x_{17,15}^{(3)} & 0 & x_{17,16}^{(3)} & x_{17,17}^{(3)} & x_{17,18}^{(3)} & x_{17,19}^{(3)} & x_{17,20}^{(3)} & 0 \\
x_{18,14}^{(3)} & x_{18,15}^{(3)} & 0 & x_{18,16}^{(3)} & x_{18,17}^{(3)} & x_{18,18}^{(3)} & x_{18,19}^{(3)} & x_{18,20}^{(3)} & 0 \\
x_{19,14}^{(3)} & x_{19,15}^{(3)} & 0 & x_{19,16}^{(3)} & x_{19,17}^{(3)} & x_{19,18}^{(3)} & x_{19,19}^{(3)} & x_{19,20}^{(3)} & 0 \\
x_{20,14}^{(3)} & x_{20,15}^{(3)} & 0 & x_{20,16}^{(3)} & x_{20,17}^{(3)} & x_{20,18}^{(3)} & x_{20,19}^{(3)} & x_{20,20}^{(3)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We have

$$\widehat{\Omega_3} = \begin{pmatrix} \Phi_{11}^{(3)} & \Phi_{12}^{(3)} & \Phi_{13}^{(3)} \\ \Phi_{21}^{(3)} & \Phi_{22}^{(3)} & \Phi_{23}^{(3)} \end{pmatrix}, \tag{10}$$

where

$$\Phi_{11}^{(3)} = \begin{pmatrix}
x_{11}^{(3)}+x_{11}^{(4)} & x_{12}^{(3)}+x_{12}^{(4)} & x_{13}^{(3)}+x_{13}^{(4)} & x_{14}^{(3)} & x_{16}^{(3)}+x_{14}^{(4)} & x_{17}^{(3)}+x_{15}^{(4)} & x_{18}^{(3)}+x_{16}^{(4)} & x_{19}^{(3)} \\
x_{21}^{(3)}+x_{21}^{(4)} & x_{22}^{(3)}+x_{22}^{(4)} & x_{23}^{(3)}+x_{23}^{(4)} & x_{24}^{(3)} & x_{26}^{(3)}+x_{24}^{(4)} & x_{27}^{(3)}+x_{25}^{(4)} & x_{28}^{(3)}+x_{26}^{(4)} & x_{29}^{(3)} \\
x_{31}^{(3)}+x_{31}^{(4)} & x_{32}^{(3)}+x_{32}^{(4)} & x_{33}^{(3)}+x_{33}^{(4)} & x_{34}^{(3)} & x_{36}^{(3)}+x_{34}^{(4)} & x_{37}^{(3)}+x_{35}^{(4)} & x_{38}^{(3)}+x_{36}^{(4)} & x_{39}^{(3)} \\
x_{41}^{(3)} & x_{42}^{(3)} & x_{43}^{(3)} & x_{44}^{(3)} & x_{46}^{(3)} & x_{47}^{(3)} & x_{48}^{(3)} & x_{49}^{(3)} \\
x_{61}^{(3)}+x_{41}^{(4)} & x_{62}^{(3)}+x_{42}^{(4)} & x_{63}^{(3)}+x_{43}^{(4)} & x_{64}^{(3)} & x_{66}^{(3)}+x_{44}^{(4)} & x_{67}^{(3)}+x_{45}^{(4)} & x_{68}^{(3)}+x_{46}^{(4)} & x_{69}^{(3)} \\
x_{71}^{(3)}+x_{51}^{(4)} & x_{72}^{(3)}+x_{52}^{(4)} & x_{73}^{(3)}+x_{53}^{(4)} & x_{74}^{(3)} & x_{76}^{(3)}+x_{54}^{(4)} & x_{77}^{(3)}+x_{55}^{(4)} & x_{78}^{(3)}+x_{56}^{(4)} & x_{79}^{(3)} \\
x_{81}^{(3)}+x_{61}^{(4)} & x_{82}^{(3)}+x_{62}^{(4)} & x_{83}^{(3)}+x_{63}^{(4)} & x_{84}^{(3)} & x_{86}^{(3)}+x_{64}^{(4)} & x_{87}^{(3)}+x_{65}^{(4)} & x_{88}^{(3)}+x_{66}^{(4)} & x_{89}^{(3)} \\
x_{91}^{(3)} & x_{92}^{(3)} & x_{93}^{(3)} & x_{94}^{(3)} & x_{96}^{(3)} & x_{97}^{(3)} & x_{98}^{(3)} & x_{99}^{(3)} \\
x_{11,1}^{(3)}+x_{71}^{(4)} & x_{11,2}^{(3)}+x_{72}^{(4)} & x_{11,3}^{(3)}+x_{73}^{(4)} & x_{11,4}^{(3)} & x_{11,6}^{(3)}+x_{74}^{(4)} & x_{11,7}^{(3)}+x_{75}^{(4)} & x_{11,8}^{(3)}+x_{76}^{(4)} & x_{11,9}^{(3)} \\
x_{12,1}^{(3)}+x_{81}^{(4)} & x_{12,2}^{(3)}+x_{82}^{(4)} & x_{12,3}^{(3)}+x_{83}^{(4)} & x_{12,4}^{(3)} & x_{12,6}^{(3)}+x_{84}^{(4)} & x_{12,7}^{(3)}+x_{85}^{(4)} & x_{12,8}^{(3)}+x_{86}^{(4)} & x_{12,9}^{(3)} \\
x_{13,1}^{(3)}+x_{91}^{(4)} & x_{13,2}^{(3)}+x_{92}^{(4)} & x_{13,3}^{(3)}+x_{93}^{(4)} & x_{13,4}^{(3)} & x_{13,6}^{(3)}+x_{94}^{(4)} & x_{13,7}^{(3)}+x_{95}^{(4)} & x_{13,8}^{(3)}+x_{96}^{(4)} & x_{13,9}^{(3)} \\
x_{14,1}^{(3)} & x_{14,2}^{(3)} & x_{14,3}^{(3)} & x_{14,4}^{(3)} & x_{14,6}^{(3)} & x_{14,7}^{(3)} & x_{14,8}^{(3)} & x_{14,9}^{(3)}
\end{pmatrix},$$

$$\Phi_{21}^{(3)} = \begin{pmatrix}
x^{(3)}_{16,1}+x^{(4)}_{10,1} & x^{(3)}_{16,2}+x^{(4)}_{10,2} & x^{(3)}_{16,3}+x^{(4)}_{10,3} & x^{(3)}_{16,4} & x^{(3)}_{16,6}+x^{(4)}_{10,4} & x^{(3)}_{16,7}+x^{(4)}_{10,5} & x^{(3)}_{16,8}+x^{(4)}_{10,6} & x^{(3)}_{16,9} \\
x^{(3)}_{17,1}+x^{(4)}_{11,1} & x^{(3)}_{17,2}+x^{(4)}_{11,2} & x^{(3)}_{17,3}+x^{(4)}_{11,3} & x^{(3)}_{17,4} & x^{(3)}_{17,6}+x^{(4)}_{11,4} & x^{(3)}_{17,7}+x^{(4)}_{11,5} & x^{(3)}_{17,8}+x^{(4)}_{11,6} & x^{(3)}_{17,9} \\
x^{(3)}_{18,1}+x^{(4)}_{12,1} & x^{(3)}_{18,2}+x^{(4)}_{12,2} & x^{(3)}_{18,3}+x^{(4)}_{12,3} & x^{(3)}_{18,4} & x^{(3)}_{18,6}+x^{(4)}_{12,4} & x^{(3)}_{18,7}+x^{(4)}_{12,5} & x^{(3)}_{18,8}+x^{(4)}_{12,6} & x^{(3)}_{18,9} \\
x^{(3)}_{19,1} & x^{(3)}_{19,2} & x^{(3)}_{19,3} & x^{(3)}_{19,4} & x^{(3)}_{19,6} & x^{(3)}_{19,7} & x^{(3)}_{19,8} & x^{(3)}_{19,9} \\
x^{(3)}_{21,1}+x^{(4)}_{13,1} & x^{(3)}_{21,2}+x^{(4)}_{13,2} & x^{(3)}_{21,3}+x^{(4)}_{13,3} & x^{(3)}_{21,4} & x^{(3)}_{21,6}+x^{(4)}_{13,4} & x^{(3)}_{21,7}+x^{(4)}_{13,5} & x^{(3)}_{21,8}+x^{(4)}_{13,6} & x^{(3)}_{21,9} \\
x^{(3)}_{22,1}+x^{(4)}_{14,1} & x^{(3)}_{22,2}+x^{(4)}_{14,2} & x^{(3)}_{22,3}+x^{(4)}_{14,3} & x^{(3)}_{22,4} & x^{(3)}_{22,6}+x^{(4)}_{14,4} & x^{(3)}_{22,7}+x^{(4)}_{14,5} & x^{(3)}_{22,8}+x^{(4)}_{14,6} & x^{(3)}_{22,9} \\
x^{(3)}_{23,1}+x^{(4)}_{15,1} & x^{(3)}_{23,2}+x^{(4)}_{15,2} & x^{(3)}_{23,3}+x^{(4)}_{15,3} & x^{(3)}_{23,4} & x^{(3)}_{23,6}+x^{(4)}_{15,4} & x^{(3)}_{23,7}+x^{(4)}_{15,5} & x^{(3)}_{23,8}+x^{(4)}_{15,6} & x^{(3)}_{23,9} \\
x^{(3)}_{24,1} & x^{(3)}_{24,2} & x^{(3)}_{24,3} & x^{(3)}_{24,4} & x^{(3)}_{24,6} & x^{(3)}_{24,7} & x^{(3)}_{24,8} & x^{(3)}_{24,9} \\
x^{(4)}_{16,1} & x^{(4)}_{16,2} & x^{(4)}_{16,3} & 0 & x^{(4)}_{16,4} & x^{(4)}_{16,5} & x^{(4)}_{16,6} & 0 \\
x^{(4)}_{17,1} & x^{(4)}_{17,2} & x^{(4)}_{17,3} & 0 & x^{(4)}_{17,4} & x^{(4)}_{17,5} & x^{(4)}_{17,6} & 0 \\
x^{(4)}_{18,1} & x^{(4)}_{18,2} & x^{(4)}_{18,3} & 0 & x^{(4)}_{18,4} & x^{(4)}_{18,5} & x^{(4)}_{18,6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\Phi_{12}^{(3)} = \begin{pmatrix}
x^{(3)}_{1,11}+x^{(4)}_{17} & x^{(3)}_{1,12}+x^{(4)}_{18} & x^{(3)}_{1,13}+x^{(4)}_{19} & x^{(3)}_{1,14} & x^{(3)}_{1,16}+x^{(4)}_{1,10} & x^{(3)}_{1,17}+x^{(4)}_{1,11} & x^{(3)}_{1,18}+x^{(4)}_{1,12} & x^{(3)}_{1,19} \\
x^{(3)}_{2,11}+x^{(4)}_{27} & x^{(3)}_{2,12}+x^{(4)}_{28} & x^{(3)}_{2,13}+x^{(4)}_{29} & x^{(3)}_{2,14} & x^{(3)}_{2,16}+x^{(4)}_{2,10} & x^{(3)}_{2,17}+x^{(4)}_{2,11} & x^{(3)}_{2,18}+x^{(4)}_{2,12} & x^{(3)}_{2,19} \\
x^{(3)}_{3,11}+x^{(4)}_{37} & x^{(3)}_{3,12}+x^{(4)}_{38} & x^{(3)}_{3,13}+x^{(4)}_{39} & x^{(3)}_{3,14} & x^{(3)}_{3,16}+x^{(4)}_{3,10} & x^{(3)}_{3,17}+x^{(4)}_{3,11} & x^{(3)}_{3,18}+x^{(4)}_{3,12} & x^{(3)}_{3,19} \\
x^{(3)}_{4,11} & x^{(3)}_{4,12} & x^{(3)}_{4,13} & x^{(3)}_{4,14} & x^{(3)}_{4,16} & x^{(3)}_{4,17} & x^{(3)}_{4,18} & x^{(3)}_{4,19} \\
x^{(3)}_{6,11}+x^{(4)}_{47} & x^{(3)}_{6,12}+x^{(4)}_{48} & x^{(3)}_{6,13}+x^{(4)}_{49} & x^{(3)}_{6,14} & x^{(3)}_{6,16}+x^{(4)}_{4,10} & x^{(3)}_{6,17}+x^{(4)}_{4,11} & x^{(3)}_{6,18}+x^{(4)}_{4,12} & x^{(3)}_{6,19} \\
x^{(3)}_{7,11}+x^{(4)}_{57} & x^{(3)}_{7,12}+x^{(4)}_{58} & x^{(3)}_{7,13}+x^{(4)}_{59} & x^{(3)}_{7,14} & x^{(3)}_{7,16}+x^{(4)}_{5,10} & x^{(3)}_{7,17}+x^{(4)}_{5,11} & x^{(3)}_{7,18}+x^{(4)}_{5,12} & x^{(3)}_{7,19} \\
x^{(3)}_{8,11}+x^{(4)}_{67} & x^{(3)}_{8,12}+x^{(4)}_{68} & x^{(3)}_{8,13}+x^{(4)}_{69} & x^{(3)}_{8,14} & x^{(3)}_{8,16}+x^{(4)}_{6,10} & x^{(3)}_{8,17}+x^{(4)}_{6,11} & x^{(3)}_{8,18}+x^{(4)}_{6,12} & x^{(3)}_{8,19} \\
x^{(3)}_{9,11} & x^{(3)}_{9,12} & x^{(3)}_{9,13} & x^{(3)}_{9,14} & x^{(3)}_{9,16} & x^{(3)}_{9,17} & x^{(3)}_{9,18} & x^{(3)}_{9,19} \\
x^{(3)}_{11,11}+x^{(4)}_{77} & x^{(3)}_{11,12}+x^{(4)}_{78} & x^{(3)}_{11,13}+x^{(4)}_{79} & x^{(3)}_{11,14} & x^{(3)}_{11,16}+x^{(4)}_{7,10} & x^{(3)}_{11,17}+x^{(4)}_{7,11} & x^{(3)}_{11,18}+x^{(4)}_{7,12} & x^{(3)}_{11,19} \\
x^{(3)}_{12,11}+x^{(4)}_{87} & x^{(3)}_{12,12}+x^{(4)}_{88} & x^{(3)}_{12,13}+x^{(4)}_{89} & x^{(3)}_{12,14} & x^{(3)}_{12,16}+x^{(4)}_{8,10} & x^{(3)}_{12,17}+x^{(4)}_{8,11} & x^{(3)}_{12,18}+x^{(4)}_{8,12} & x^{(3)}_{12,19} \\
x^{(3)}_{13,11}+x^{(4)}_{97} & x^{(3)}_{13,12}+x^{(4)}_{98} & x^{(3)}_{13,13}+x^{(4)}_{99} & x^{(3)}_{13,14} & x^{(3)}_{13,16}+x^{(4)}_{9,10} & x^{(3)}_{13,17}+x^{(4)}_{9,11} & x^{(3)}_{13,18}+x^{(4)}_{9,12} & x^{(3)}_{13,19} \\
x^{(3)}_{14,11} & x^{(3)}_{14,12} & x^{(3)}_{14,13} & x^{(3)}_{14,14} & x^{(3)}_{14,16} & x^{(3)}_{14,17} & x^{(3)}_{14,18} & x^{(3)}_{14,19}
\end{pmatrix},$$

$$\Phi_{22}^{(3)} = \begin{pmatrix}
x^{(3)}_{16,11}+x^{(4)}_{10,7} & x^{(3)}_{16,12}+x^{(4)}_{10,8} & x^{(3)}_{16,13}+x^{(4)}_{10,9} & x^{(3)}_{16,14} & x^{(3)}_{16,16}+x^{(4)}_{10,10} & x^{(3)}_{16,17}+x^{(4)}_{10,11} & x^{(3)}_{16,18}+x^{(4)}_{10,12} & x^{(3)}_{16,19} \\
x^{(3)}_{17,11}+x^{(4)}_{11,7} & x^{(3)}_{17,12}+x^{(4)}_{11,8} & x^{(3)}_{17,13}+x^{(4)}_{11,9} & x^{(3)}_{17,14} & x^{(3)}_{17,16}+x^{(4)}_{11,10} & x^{(3)}_{17,17}+x^{(4)}_{11,11} & x^{(3)}_{17,18}+x^{(4)}_{11,12} & x^{(3)}_{17,19} \\
x^{(3)}_{18,11}+x^{(4)}_{12,7} & x^{(3)}_{18,12}+x^{(4)}_{12,8} & x^{(3)}_{18,13}+x^{(4)}_{12,9} & x^{(3)}_{18,14} & x^{(3)}_{18,16}+x^{(4)}_{12,10} & x^{(3)}_{18,17}+x^{(4)}_{12,11} & x^{(3)}_{18,18}+x^{(4)}_{12,12} & x^{(3)}_{18,19} \\
x^{(3)}_{19,11} & x^{(3)}_{19,12} & x^{(3)}_{19,13} & x^{(3)}_{19,14} & x^{(3)}_{19,16} & x^{(3)}_{19,17} & x^{(3)}_{19,18} & x^{(3)}_{19,19} \\
x^{(3)}_{21,11}+x^{(4)}_{13,7} & x^{(3)}_{21,12}+x^{(4)}_{13,8} & x^{(3)}_{21,13}+x^{(4)}_{13,9} & x^{(3)}_{21,14} & x^{(3)}_{21,16}+x^{(4)}_{13,10} & x^{(3)}_{21,17}+x^{(4)}_{13,11} & x^{(3)}_{21,18}+x^{(4)}_{13,12} & x^{(3)}_{21,19} \\
x^{(3)}_{22,11}+x^{(4)}_{14,7} & x^{(3)}_{22,12}+x^{(4)}_{14,8} & x^{(3)}_{22,13}+x^{(4)}_{14,9} & x^{(3)}_{22,14} & x^{(3)}_{22,16}+x^{(4)}_{14,10} & x^{(3)}_{22,17}+x^{(4)}_{14,11} & x^{(3)}_{22,18}+x^{(4)}_{14,12} & x^{(3)}_{22,19} \\
x^{(3)}_{23,11}+x^{(4)}_{15,7} & x^{(3)}_{23,12}+x^{(4)}_{15,8} & x^{(3)}_{23,13}+x^{(4)}_{15,9} & x^{(3)}_{23,14} & x^{(3)}_{23,16}+x^{(4)}_{15,10} & x^{(3)}_{23,17}+x^{(4)}_{15,11} & x^{(3)}_{23,18}+x^{(4)}_{15,12} & x^{(3)}_{23,19} \\
x^{(3)}_{24,11} & x^{(3)}_{24,12} & x^{(3)}_{24,13} & x^{(3)}_{24,14} & x^{(3)}_{24,16} & x^{(3)}_{24,17} & x^{(3)}_{24,18} & x^{(3)}_{24,19} \\
x^{(4)}_{16,7} & x^{(4)}_{16,8} & x^{(4)}_{16,9} & 0 & x^{(4)}_{16,10} & x^{(4)}_{16,11} & x^{(4)}_{16,12} & 0 \\
x^{(4)}_{17,7} & x^{(4)}_{17,8} & x^{(4)}_{17,9} & 0 & x^{(4)}_{17,10} & x^{(4)}_{17,11} & x^{(4)}_{17,12} & 0 \\
x^{(4)}_{18,7} & x^{(4)}_{18,8} & x^{(4)}_{18,9} & 0 & x^{(4)}_{18,10} & x^{(4)}_{18,11} & x^{(4)}_{18,12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\Phi_{13}^{(3)} = \begin{pmatrix}
x^{(3)}_{1,21}+x^{(4)}_{1,13} & x^{(3)}_{1,22}+x^{(4)}_{1,14} & x^{(3)}_{1,23}+x^{(4)}_{1,15} & x^{(3)}_{1,24} & x^{(4)}_{1,16} & x^{(4)}_{1,17} & x^{(4)}_{1,18} & 0 \\
x^{(3)}_{2,21}+x^{(4)}_{2,13} & x^{(3)}_{2,22}+x^{(4)}_{2,14} & x^{(3)}_{2,23}+x^{(4)}_{2,15} & x^{(3)}_{2,24} & x^{(4)}_{2,16} & x^{(4)}_{2,17} & x^{(4)}_{2,18} & 0 \\
x^{(3)}_{3,21}+x^{(4)}_{3,13} & x^{(3)}_{3,22}+x^{(4)}_{3,14} & x^{(3)}_{3,23}+x^{(4)}_{3,15} & x^{(3)}_{3,24} & x^{(4)}_{3,16} & x^{(4)}_{3,17} & x^{(4)}_{3,18} & 0 \\
x^{(3)}_{4,21} & x^{(3)}_{4,22} & x^{(3)}_{4,23} & x^{(3)}_{4,24} & 0 & 0 & 0 & 0 \\
x^{(3)}_{6,21}+x^{(4)}_{4,13} & x^{(3)}_{6,22}+x^{(4)}_{4,14} & x^{(3)}_{6,23}+x^{(4)}_{4,15} & x^{(3)}_{6,24} & x^{(4)}_{4,16} & x^{(4)}_{4,17} & x^{(4)}_{4,18} & 0 \\
x^{(3)}_{7,21}+x^{(4)}_{5,13} & x^{(3)}_{7,22}+x^{(4)}_{5,14} & x^{(3)}_{7,23}+x^{(4)}_{5,15} & x^{(3)}_{7,24} & x^{(4)}_{5,16} & x^{(4)}_{5,17} & x^{(4)}_{5,18} & 0 \\
x^{(3)}_{8,21}+x^{(4)}_{6,13} & x^{(3)}_{8,22}+x^{(4)}_{6,14} & x^{(3)}_{8,23}+x^{(4)}_{6,15} & x^{(3)}_{8,24} & x^{(4)}_{6,16} & x^{(4)}_{6,17} & x^{(4)}_{6,18} & 0 \\
x^{(3)}_{9,21} & x^{(3)}_{9,22} & x^{(3)}_{9,23} & x^{(3)}_{9,24} & 0 & 0 & 0 & 0 \\
x^{(3)}_{11,21}+x^{(4)}_{7,13} & x^{(3)}_{11,22}+x^{(4)}_{7,14} & x^{(3)}_{11,23}+x^{(4)}_{7,15} & x^{(3)}_{11,24} & x^{(4)}_{7,16} & x^{(4)}_{7,17} & x^{(4)}_{7,18} & 0 \\
x^{(3)}_{12,21}+x^{(4)}_{8,13} & x^{(3)}_{12,22}+x^{(4)}_{8,14} & x^{(3)}_{12,23}+x^{(4)}_{8,15} & x^{(3)}_{12,24} & x^{(4)}_{8,16} & x^{(4)}_{8,17} & x^{(4)}_{8,18} & 0 \\
x^{(3)}_{13,21}+x^{(4)}_{9,13} & x^{(3)}_{13,22}+x^{(4)}_{9,14} & x^{(3)}_{13,23}+x^{(4)}_{9,15} & x^{(3)}_{13,24} & x^{(4)}_{9,16} & x^{(4)}_{9,17} & x^{(4)}_{9,18} & 0 \\
x^{(3)}_{14,21} & x^{(3)}_{14,22} & x^{(3)}_{14,23} & x^{(3)}_{14,24} & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\Phi_{23}^{(3)} = \begin{pmatrix} X_{16,21}^{(3)}+X_{10,13}^{(4)} & X_{16,22}^{(3)}+X_{10,14}^{(4)} & X_{16,23}^{(3)}+X_{10,15}^{(4)} & X_{16,24}^{(3)} & X_{10,16}^{(4)} & X_{10,17}^{(4)} & X_{10,18}^{(4)} & 0 \\ X_{17,21}^{(3)}+X_{11,13}^{(4)} & X_{17,22}^{(3)}+X_{11,14}^{(4)} & X_{17,23}^{(3)}+X_{11,15}^{(4)} & X_{17,24}^{(3)} & X_{11,16}^{(4)} & X_{11,17}^{(4)} & X_{11,18}^{(4)} & 0 \\ X_{18,21}^{(3)}+X_{12,13}^{(4)} & X_{18,22}^{(3)}+X_{12,14}^{(4)} & X_{18,23}^{(3)}+X_{12,15}^{(4)} & X_{18,24}^{(3)} & X_{12,16}^{(4)} & X_{12,17}^{(4)} & X_{12,18}^{(4)} & 0 \\ X_{19,21}^{(3)} & X_{19,22}^{(3)} & X_{19,23}^{(3)} & X_{19,24}^{(3)} & 0 & 0 & 0 & 0 \\ X_{21,21}^{(3)}+X_{13,13}^{(4)} & X_{21,22}^{(3)}+X_{13,14}^{(4)} & X_{21,23}^{(3)}+X_{13,15}^{(4)} & X_{21,24}^{(3)} & X_{13,16}^{(4)} & X_{13,17}^{(4)} & X_{13,18}^{(4)} & 0 \\ X_{22,21}^{(3)}+X_{14,13}^{(4)} & X_{22,22}^{(3)}+X_{14,14}^{(4)} & X_{22,23}^{(3)}+X_{14,15}^{(4)} & X_{22,24}^{(3)} & X_{14,16}^{(4)} & X_{14,17}^{(4)} & X_{14,18}^{(4)} & 0 \\ X_{23,21}^{(3)}+X_{15,13}^{(4)} & X_{23,22}^{(3)}+X_{15,14}^{(4)} & X_{23,23}^{(3)}+X_{15,15}^{(4)} & X_{23,24}^{(3)} & X_{15,16}^{(4)} & X_{15,17}^{(4)} & X_{15,18}^{(4)} & 0 \\ X_{24,21}^{(3)} & X_{24,22}^{(3)} & X_{24,23}^{(3)} & X_{24,24}^{(3)} & 0 & 0 & 0 & 0 \\ X_{16,13}^{(4)} & X_{16,14}^{(4)} & X_{16,15}^{(4)} & 0 & X_{16,16}^{(4)} & X_{16,17}^{(4)} & X_{16,18}^{(4)} & 0 \\ X_{17,13}^{(4)} & X_{17,14}^{(4)} & X_{17,15}^{(4)} & 0 & X_{17,16}^{(4)} & X_{17,17}^{(4)} & X_{17,18}^{(4)} & 0 \\ X_{18,13}^{(4)} & X_{18,14}^{(4)} & X_{18,15}^{(4)} & 0 & X_{18,16}^{(4)} & X_{18,17}^{(4)} & X_{18,18}^{(4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have

$$\widehat{\Omega_4} = (\Phi_1^{(4)}, \Phi_2^{(4)}), \tag{11}$$

where

$$\Phi_1^{(4)} = \begin{pmatrix} X_{11}^{(4)}+X_{11}^{(5)} & X_{12}^{(4)} & X_{14}^{(4)}+X_{12}^{(5)} & X_{15}^{(4)} & X_{17}^{(4)}+X_{13}^{(5)} & X_{18}^{(4)} & X_{1,10}^{(4)}+X_{14}^{(5)} \\ X_{21}^{(4)} & X_{22}^{(4)} & X_{24}^{(4)} & X_{25}^{(4)} & X_{27}^{(4)} & X_{28}^{(4)} & X_{2,10}^{(4)} \\ X_{41}^{(4)}+X_{21}^{(5)} & X_{42}^{(4)} & X_{44}^{(4)}+X_{22}^{(5)} & X_{45}^{(4)} & X_{47}^{(4)}+X_{23}^{(5)} & X_{48}^{(4)} & X_{4,10}^{(4)}+X_{24}^{(5)} \\ X_{51}^{(4)} & X_{52}^{(4)} & X_{54}^{(4)} & X_{55}^{(4)} & X_{57}^{(4)} & X_{58}^{(4)} & X_{5,10}^{(4)} \\ X_{71}^{(4)}+X_{31}^{(5)} & X_{72}^{(4)} & X_{74}^{(4)}+X_{32}^{(5)} & X_{75}^{(4)} & X_{77}^{(4)}+X_{33}^{(5)} & X_{78}^{(4)} & X_{7,10}^{(4)}+X_{34}^{(5)} \\ X_{81}^{(4)} & X_{82}^{(4)} & X_{84}^{(4)} & X_{85}^{(4)} & X_{87}^{(4)} & X_{88}^{(4)} & X_{8,10}^{(4)} \\ X_{10,1}^{(4)}+X_{41}^{(5)} & X_{10,2}^{(4)} & X_{10,4}^{(4)}+X_{42}^{(5)} & X_{10,5}^{(4)} & X_{10,7}^{(4)}+X_{43}^{(5)} & X_{10,8}^{(4)} & X_{10,10}^{(4)}+X_{44}^{(5)} \\ X_{11,1}^{(4)} & X_{11,2}^{(4)} & X_{11,4}^{(4)} & X_{11,5}^{(4)} & X_{11,7}^{(4)} & X_{11,8}^{(4)} & X_{11,10}^{(4)} \\ X_{13,1}^{(4)}+X_{51}^{(5)} & X_{13,2}^{(4)} & X_{13,4}^{(4)}+X_{52}^{(5)} & X_{13,5}^{(4)} & X_{13,7}^{(4)}+X_{53}^{(5)} & X_{13,8}^{(4)} & X_{13,10}^{(4)}+X_{54}^{(5)} \\ X_{14,1}^{(4)} & X_{14,2}^{(4)} & X_{14,4}^{(4)} & X_{14,5}^{(4)} & X_{14,7}^{(4)} & X_{14,8}^{(4)} & X_{14,10}^{(4)} \\ X_{16,1}^{(4)}+X_{61}^{(5)} & X_{16,2}^{(4)} & X_{16,4}^{(4)}+X_{62}^{(5)} & X_{16,5}^{(4)} & X_{16,7}^{(4)}+X_{63}^{(5)} & X_{16,8}^{(4)} & X_{16,10}^{(4)}+X_{64}^{(5)} \\ X_{17,1}^{(4)} & X_{17,2}^{(4)} & X_{17,4}^{(4)} & X_{17,5}^{(4)} & X_{17,7}^{(4)} & X_{17,8}^{(4)} & X_{17,10}^{(4)} \\ X_{19,1}^{(4)}+X_{71}^{(5)} & X_{19,2}^{(4)} & X_{19,4}^{(4)}+X_{72}^{(5)} & X_{19,5}^{(4)} & X_{19,7}^{(4)}+X_{73}^{(5)} & X_{19,8}^{(4)} & X_{19,10}^{(4)}+X_{74}^{(5)} \\ X_{20,1}^{(4)} & X_{20,2}^{(4)} & X_{20,4}^{(4)} & X_{20,5}^{(4)} & X_{20,7}^{(4)} & X_{20,8}^{(4)} & X_{20,10}^{(4)} \\ X_{81}^{(5)} & 0 & X_{82}^{(5)} & 0 & X_{83}^{(5)} & 0 & X_{84}^{(5)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Phi_2^{(4)} = \begin{pmatrix} X_{1,11}^{(4)} & X_{1,13}^{(4)}+X_{15}^{(5)} & X_{1,14}^{(4)} & X_{1,16}^{(4)}+X_{16}^{(5)} & X_{1,17}^{(4)} & X_{1,19}^{(4)}+X_{17}^{(5)} & X_{1,20}^{(4)} & X_{18}^{(5)} & 0 \\ X_{2,11}^{(4)} & X_{2,13}^{(4)} & X_{2,14}^{(4)} & X_{2,16}^{(4)} & X_{2,17}^{(4)} & X_{2,19}^{(4)} & X_{2,20}^{(4)} & 0 & 0 \\ X_{4,11}^{(4)} & X_{4,13}^{(4)}+X_{25}^{(5)} & X_{4,14}^{(4)} & X_{4,16}^{(4)}+X_{26}^{(5)} & X_{4,17}^{(4)} & X_{4,19}^{(4)}+X_{27}^{(5)} & X_{4,20}^{(4)} & X_{28}^{(5)} & 0 \\ X_{5,11}^{(4)} & X_{5,13}^{(4)} & X_{5,14}^{(4)} & X_{5,16}^{(4)} & X_{5,17}^{(4)} & X_{5,19}^{(4)} & X_{5,20}^{(4)} & 0 & 0 \\ X_{7,11}^{(4)} & X_{7,13}^{(4)}+X_{35}^{(5)} & X_{7,14}^{(4)} & X_{7,16}^{(4)}+X_{36}^{(5)} & X_{7,17}^{(4)} & X_{7,19}^{(4)}+X_{37}^{(5)} & X_{7,20}^{(4)} & X_{38}^{(5)} & 0 \\ X_{8,11}^{(4)} & X_{8,13}^{(4)} & X_{8,14}^{(4)} & X_{8,16}^{(4)} & X_{8,17}^{(4)} & X_{8,19}^{(4)} & X_{8,20}^{(4)} & 0 & 0 \\ X_{10,11}^{(4)} & X_{10,13}^{(4)}+X_{45}^{(5)} & X_{10,14}^{(4)} & X_{10,16}^{(4)}+X_{46}^{(5)} & X_{10,17}^{(4)} & X_{10,19}^{(4)}+X_{47}^{(5)} & X_{10,20}^{(4)} & X_{48}^{(5)} & 0 \\ X_{11,11}^{(4)} & X_{11,13}^{(4)} & X_{11,14}^{(4)} & X_{11,16}^{(4)} & X_{11,17}^{(4)} & X_{11,19}^{(4)} & X_{11,20}^{(4)} & 0 & 0 \\ X_{13,11}^{(4)} & X_{13,13}^{(4)}+X_{55}^{(5)} & X_{13,14}^{(4)} & X_{13,16}^{(4)}+X_{56}^{(5)} & X_{13,17}^{(4)} & X_{13,19}^{(4)}+X_{57}^{(5)} & X_{13,20}^{(4)} & X_{58}^{(5)} & 0 \\ X_{14,11}^{(4)} & X_{14,13}^{(4)} & X_{14,14}^{(4)} & X_{14,16}^{(4)} & X_{14,17}^{(4)} & X_{14,19}^{(4)} & X_{14,20}^{(4)} & 0 & 0 \\ X_{16,11}^{(4)} & X_{16,13}^{(4)}+X_{65}^{(5)} & X_{16,14}^{(4)} & X_{16,16}^{(4)}+X_{66}^{(5)} & X_{16,17}^{(4)} & X_{16,19}^{(4)}+X_{67}^{(5)} & X_{16,20}^{(4)} & X_{68}^{(5)} & 0 \\ X_{17,11}^{(4)} & X_{17,13}^{(4)} & X_{17,14}^{(4)} & X_{17,16}^{(4)} & X_{17,17}^{(4)} & X_{17,19}^{(4)} & X_{17,20}^{(4)} & 0 & 0 \\ X_{19,11}^{(4)} & X_{19,13}^{(4)}+X_{75}^{(5)} & X_{19,14}^{(4)} & X_{19,16}^{(4)}+X_{76}^{(5)} & X_{19,17}^{(4)} & X_{19,19}^{(4)}+X_{77}^{(5)} & X_{19,20}^{(4)} & X_{78}^{(5)} & 0 \\ X_{20,11}^{(4)} & X_{20,13}^{(4)} & X_{20,14}^{(4)} & X_{20,16}^{(4)} & X_{20,17}^{(4)} & X_{20,19}^{(4)} & X_{20,20}^{(4)} & 0 & 0 \\ 0 & X_{85}^{(5)} & 0 & X_{86}^{(5)} & 0 & X_{87}^{(5)} & 0 & X_{88}^{(5)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 1. *In terms of system (1), the following conditions are equivalent to each other:*

1. *System (1) is consistent.*
2. *The ranks of A_i, B_i, C_i, D_i, Ω_i , $i = 1, \ldots, 4$ satisfy the following 40 rank equalities:*

$$r_{A_i \Omega_i B_i} = r_{A_i B_i}, \tag{12}$$

$$r_{C_i | \Omega_i | D_i} = r_{C_i | D_i}, \tag{13}$$

$$r_{A_i\Omega_i|0D_i} = r_{A_i} + r_{D_i}, \tag{14}$$

$$r_{C_i0|\Omega_iB_i} = r_{C_i} + r_{B_i}, \tag{15}$$

$$r_{A_1\Omega_1B_100|0D_10C_20|00A_2-\Omega_2B_2} = r_{A_1B_10|0A_2B_2} + r_{D_1C_2}, \tag{16}$$

$$r_{C_100|\Omega_1B_10|D_10C_2|0A_2-\Omega_2|00D_2} = r_{B_1|A_2} + r_{C_10|D_1C_2|0D_2}, \tag{17}$$

$$r_{A_1\Omega_1B_10|0D_10C_2|00A_2-\Omega_2|000D_2} = r_{A_1B_1|0A_2} + r_{D_1C_2|0D_2}, \tag{18}$$

$$r_{C_1000|\Omega_1B_100|D_10C_20|0A_2-\Omega_2B_2} = r_{B_10|A_2B_2} + r_{C_10|D_1C_2}, \tag{19}$$

$$r_{A_2\Omega_2B_200|0D_20C_30|00A_3-\Omega_3B_3} = r_{A_2B_20|0A_3B_3} + r_{D_2C_3}, \tag{20}$$

$$r_{C_200|\Omega_2B_20|D_20C_3|0A_3-\Omega_3|00D_3} = r_{B_2|A_3} + r_{C_20|D_2C_3|0D_3}, \tag{21}$$

$$r_{A_2\Omega_2B_20|0D_20C_3|00A_3-\Omega_3|000D_3} = r_{A_2B_2|0A_3} + r_{D_2C_3|0D_3}, \tag{22}$$

$$r_{C_2000|\Omega_2B_200|D_20C_30|0A_3-\Omega_3B_3} = r_{B_20|A_3B_3} + r_{C_20|D_2C_3}, \tag{23}$$

$$r_{A_3\Omega_3B_300|0D_30C_40|00A_4-\Omega_4B_4} = r_{A_3B_30|0A_4B_4} + r_{D_3C_4}, \tag{24}$$

$$r_{C_300|\Omega_3B_30|D_30C_4|0A_4-\Omega_4|00D_4} = r_{B_3|A_4} + r_{C_30|D_3C_4|0D_4}, \tag{25}$$

$$r_{A_3\Omega_3B_30|0D_30C_4|00A_4-\Omega_4|000D_4} = r_{A_3B_3|0A_4} + r_{D_3C_4|0D_4}, \tag{26}$$

$$r_{C_3000|\Omega_3B_300|D_30C_40|0A_4-\Omega_4B_4} = r_{B_30|A_4B_4} + r_{C_30|D_3C_4}, \tag{27}$$

$$r_{A_1\Omega_1B_10000|0D_10C_2000|00A_2-\Omega_2B_200|000D_20C_30|0000A_3\Omega_3B_3} = r_{A_1B_100|0A_2B_20|00A_3B_3} + r_{D_1C_20|0D_2C_3}, \tag{28}$$

$$r_{C_10000|\Omega_1B_1000|D_10C_200|0A_2-\Omega_2B_20|00D_20C_3|000A_3\Omega_3|0000D_3} = r_{B_10|A_2B_2|0A_3} + r_{C_100|D_1C_20|0D_2C_3|00D_3}, \tag{29}$$

$$r_{A_1\Omega_1B_1000|0D_10C_200|00A_2-\Omega_2B_20|000D_20C_3|0000A_3\Omega_3|00000D_3} = r_{A_1B_10|0A_2B_2|00A_3} + r_{D_1C_20|0D_2C_3|00D_3}, \tag{30}$$

$$r_{C_100000|\Omega_1B_10000|D_10C_2000|0A_2-\Omega_2B_200|00D_20C_30|000A_3\Omega_3B_3} = r_{B_100|A_2B_20|0A_3B_3} + r_{C_100|D_1C_20|0D_2C_3}, \tag{31}$$

$$r_{A_2\Omega_2B_20000|0D_20C_3000|00A_3-\Omega_3B_300|000D_30C_40|0000A_4\Omega_4B_4} = r_{A_2B_200|0A_3B_30|00A_4B_4} + r_{D_2C_30|0D_3C_4}, \tag{32}$$

$$r_{C_20000|\Omega_2B_2000|D_20C_300|0A_3-\Omega_3B_30|00D_30C_4|000A_4\Omega_4|0000D_4} = r_{B_20|A_3B_3|0A_4} + r_{C_200|D_2C_30|0D_3C_4|00D_4}, \tag{33}$$

$$r_{A_2\Omega_2B_2000|0D_20C_300|00A_3-\Omega_3B_30|000D_30C_4|0000A_4\Omega_4|00000D_4} = r_{A_2B_20|0A_3B_3|00A_4} + r_{D_2C_30|0D_3C_4|00D_4}, \tag{34}$$

$$r_{C_2 00000 \mid \Omega_2 B_2 0000 \mid D_2 0 C_3 000 \mid 0 A_3 - \Omega_3 B_3 00 \mid 00 D_3 0 C_4 0 \mid 000 A_4 \Omega_4 B_4} = r_{B_2 00 \mid A_3 B_3 0 \mid 0 A_4 B_4} + r_{C_2 00 \mid D_2 C_3 0 \mid 0 D_3 C_4}, \tag{35}$$

$$\begin{aligned} & r_{A_1 \Omega_1 B_1 000000 \mid 0 D_1 0 C_2 00000 \mid 00 A_2 - \Omega_2 B_2 0000 \mid 000 D_2 0 C_3 000 \mid 0000 A_3 \Omega_3 B_3 00 \mid 00000 D_3 0 C_4 0 \mid 000000 A_4 - \Omega_4 B_4} \\ & = r_{A_1 B_1 000 \mid 0 A_2 B_2 00 \mid 00 A_3 B_3 0 \mid 000 A_4 B_4} + r_{D_1 C_2 00 \mid 0 D_2 C_3 0 \mid 00 D_3 C_4}, \end{aligned} \tag{36}$$

$$\begin{aligned} & r_{C_1 000000 \mid \Omega_1 B_1 00000 \mid D_1 0 C_2 0000 \mid 0 A_2 - \Omega_2 B_2 000 \mid 00 D_2 0 C_3 00 \mid 000 A_3 \Omega_3 B_3 0 \mid 0000 D_3 0 C_4 \mid 00000 A_4 - \Omega_4 \mid 000000 D_4} \\ & = r_{B_1 00 \mid A_2 B_2 0 \mid 0 A_3 B_3 \mid 00 A_4} + r_{C_1 000 \mid D_1 C_2 00 \mid 0 D_2 C_3 0 \mid 00 D_3 C_4 \mid 000 D_4}, \end{aligned} \tag{37}$$

$$\begin{aligned} & r_{A_1 \Omega_1 B_1 00000 \mid 0 D_1 0 C_2 0000 \mid 00 A_2 - \Omega_2 B_2 000 \mid 000 D_2 0 C_3 00 \mid 0000 A_3 \Omega_3 B_3 0 \mid 00000 D_3 0 C_4 \mid 000000 A_4 - \Omega_4 \mid 0000000 D_4} \\ & = r_{A_1 B_1 00 \mid 0 A_2 B_2 0 \mid 00 A_3 B_3 \mid 000 A_4} + r_{D_1 C_2 00 \mid 0 D_2 C_3 0 \mid 00 D_3 C_4 \mid 000 D_4}, \end{aligned} \tag{38}$$

$$\begin{aligned} & r_{C_1 0000000 \mid \Omega_1 B_1 000000 \mid D_1 0 C_2 00000 \mid 0 A_2 - \Omega_2 B_2 0000 \mid 00 D_2 0 C_3 000 \mid 000 A_3 \Omega_3 B_3 00 \mid 0000 D_3 0 C_4 0 \mid 00000 A_4 - \Omega_4 B_4} \\ & = r_{B_1 000 \mid A_2 B_2 00 \mid 0 A_3 B_3 0 \mid 00 A_4 B_4} + r_{C_1 000 \mid D_1 C_2 00 \mid 0 D_2 C_3 0 \mid 00 D_3 C_4}. \end{aligned} \tag{39}$$

3. *The block matrices satisfy*

$$\left\{ \begin{array}{l} \begin{pmatrix} \omega^{(1)}_{16,1} & \omega^{(1)}_{16,2} & \dots & \omega^{(1)}_{16,16} \end{pmatrix} = 0, \\ \begin{pmatrix} \omega^{(2)}_{24,1} & \omega^{(2)}_{24,2} & \dots & \omega^{(2)}_{24,24} \end{pmatrix} = 0, \\ \begin{pmatrix} \omega^{(3)}_{24,1} & \omega^{(3)}_{24,2} & \dots & \omega^{(3)}_{24,24} \end{pmatrix} = 0, \\ \begin{pmatrix} \omega^{(4)}_{16,1} & \omega^{(4)}_{16,2} & \dots & \omega^{(4)}_{16,16} \end{pmatrix} = 0, \end{array} \right. \begin{pmatrix} \omega^{(1)}_{1,16} \\ \omega^{(1)}_{2,16} \\ \vdots \\ \omega^{(1)}_{16,16} \end{pmatrix} = 0, \begin{pmatrix} \omega^{(2)}_{1,24} \\ \omega^{(2)}_{2,24} \\ \vdots \\ \omega^{(2)}_{24,24} \end{pmatrix} = 0, \begin{pmatrix} \omega^{(3)}_{1,24} \\ \omega^{(3)}_{2,24} \\ \vdots \\ \omega^{(3)}_{24,24} \end{pmatrix} = 0, \begin{pmatrix} \omega^{(4)}_{1,16} \\ \omega^{(4)}_{2,16} \\ \vdots \\ \omega^{(4)}_{16,16} \end{pmatrix} = 0, \tag{40}$$

$$\left\{ \begin{array}{l} \omega^{(1)}_{98} = 0, \omega^{(1)}_{10,8} = 0, \omega^{(1)}_{11,8} = 0, \omega^{(1)}_{12,8} = 0, \omega^{(1)}_{13,8} = 0, \omega^{(1)}_{14,8} = 0, \omega^{(1)}_{15,8} = 0, \\ \omega^{(2)}_{19,6} = 0, \omega^{(2)}_{20,6} = 0, \omega^{(2)}_{21,6} = 0, \omega^{(2)}_{22,6} = 0, \omega^{(2)}_{23,6} = 0, \omega^{(2)}_{19,12} = 0, \omega^{(2)}_{20,12} = 0, \omega^{(2)}_{21,12} = 0, \\ \omega^{(2)}_{22,12} = 0, \omega^{(2)}_{23,12} = 0, \omega^{(2)}_{19,18} = 0, \omega^{(2)}_{20,18} = 0, \omega^{(2)}_{21,18} = 0, \omega^{(2)}_{22,18} = 0, \omega^{(2)}_{23,18} = 0, \\ \omega^{(3)}_{21,4} = 0, \omega^{(3)}_{22,4} = 0, \omega^{(3)}_{23,4} = 0, \omega^{(3)}_{21,8} = 0, \omega^{(3)}_{22,8} = 0, \omega^{(3)}_{23,8} = 0, \omega^{(3)}_{21,12} = 0, \omega^{(3)}_{22,12} = 0, \\ \omega^{(3)}_{23,12} = 0, \omega^{(3)}_{21,16} = 0, \omega^{(3)}_{22,16} = 0, \omega^{(3)}_{23,16} = 0, \omega^{(3)}_{21,20} = 0, \omega^{(3)}_{22,20} = 0, \omega^{(3)}_{23,20} = 0, \\ \omega^{(4)}_{15,2} = 0, \omega^{(4)}_{15,4} = 0, \omega^{(4)}_{15,6}, \omega^{(4)}_{15,8} = 0, \omega^{(4)}_{15,10} = 0, \omega^{(4)}_{15,12} = 0, \omega^{(4)}_{15,14} = 0, \end{array} \right. \tag{41}$$

$$\left\{ \begin{array}{l} \omega^{(1)}_{89} = 0, \omega^{(1)}_{8,10} = 0, \omega^{(1)}_{8,11} = 0, \omega^{(1)}_{8,12} = 0, \omega^{(1)}_{8,13} = 0, \omega^{(1)}_{8,14} = 0, \omega^{(1)}_{8,15} = 0, \\ \omega^{(2)}_{6,19} = 0, \omega^{(2)}_{6,20} = 0, \omega^{(2)}_{6,21} = 0, \omega^{(2)}_{6,22} = 0, \omega^{(2)}_{6,23} = 0, \omega^{(2)}_{12,19} = 0, \omega^{(2)}_{12,20} = 0, \omega^{(2)}_{12,21} = 0, \\ \omega^{(2)}_{12,22} = 0, \omega^{(2)}_{12,23} = 0, \omega^{(2)}_{18,19} = 0, \omega^{(2)}_{18,20} = 0, \omega^{(2)}_{18,21}, \omega^{(2)}_{18,22} = 0, \omega^{(2)}_{18,23} = 0, \\ \omega^{(3)}_{4,21} = 0, \omega^{(3)}_{4,22} = 0, \omega^{(3)}_{4,23} = 0, \omega^{(3)}_{8,21} = 0, \omega^{(3)}_{8,22} = 0, \omega^{(3)}_{8,23} = 0, \omega^{(3)}_{12,21} = 0, \omega^{(3)}_{12,22} = 0, \\ \omega^{(3)}_{12,23} = 0, \omega^{(3)}_{16,21} = 0, \omega^{(3)}_{16,22} = 0, \omega^{(3)}_{16,23} = 0, \omega^{(3)}_{20,21} = 0, \omega^{(3)}_{20,22} = 0, \omega^{(3)}_{20,23} = 0, \\ \omega^{(4)}_{2,15} = 0, \omega^{(4)}_{4,15} = 0, \omega^{(4)}_{6,15} = 0, \omega^{(4)}_{8,15} = 0, \omega^{(4)}_{10,15} = 0, \omega^{(4)}_{12,15} = 0, \omega^{(4)}_{14,15} = 0, \end{array} \right. \tag{42}$$

$$\left\{ \begin{array}{l} \omega^{(1)}_{14,1} = \omega^{(2)}_{12,1}, \omega^{(1)}_{14,2} = \omega^{(2)}_{12,2}, \omega^{(1)}_{14,3} = \omega^{(2)}_{12,3}, \omega^{(1)}_{14,4} = \omega^{(2)}_{12,4}, \omega^{(1)}_{14,5} = \omega^{(2)}_{12,5}, \omega^{(1)}_{14,6} = \omega^{(2)}_{12,6}, \\ \omega^{(1)}_{14,9} = \omega^{(2)}_{12,7}, \omega^{(1)}_{14,10} = \omega^{(2)}_{12,8}, \omega^{(1)}_{14,11} = \omega^{(2)}_{12,9}, \omega^{(1)}_{14,12} = \omega^{(2)}_{12,10}, \omega^{(1)}_{14,13} = \omega^{(2)}_{12,11}, \omega^{(1)}_{14,14} = \omega^{(2)}_{12,12}, \end{array} \right. \tag{43}$$

$$\left\{ \begin{array}{l} \omega^{(1)}_{1,14} = \omega^{(2)}_{1,12}, \omega^{(1)}_{2,14} = \omega^{(2)}_{2,12}, \omega^{(1)}_{3,14} = \omega^{(2)}_{3,12}, \omega^{(1)}_{4,14} = \omega^{(2)}_{4,12}, \omega^{(1)}_{5,14} = \omega^{(2)}_{5,12}, \omega^{(1)}_{6,14} = \omega^{(2)}_{6,12}, \\ \omega^{(1)}_{9,14} = \omega^{(2)}_{7,12}, \omega^{(1)}_{10,14} = \omega^{(2)}_{8,12}, \omega^{(1)}_{11,14} = \omega^{(2)}_{9,12}, \omega^{(1)}_{12,14} = \omega^{(2)}_{10,12}, \omega^{(1)}_{13,14} = \omega^{(2)}_{11,12}, \omega^{(1)}_{14,14} = \omega^{(2)}_{12,12}, \end{array} \right. \tag{44}$$

$$\left\{ \begin{array}{l} \omega^{(1)}_{9,6} = \omega^{(2)}_{76}, \omega^{(1)}_{10,6} = \omega^{(2)}_{86}, \omega^{(1)}_{11,6} = \omega^{(2)}_{96}, \omega^{(1)}_{12,6} = \omega^{(2)}_{10,6}, \omega^{(1)}_{13,6} = \omega^{(2)}_{11,6}, \omega^{(1)}_{14,6} = \omega^{(2)}_{12,6}, \\ \omega^{(1)}_{9,14} = \omega^{(2)}_{7,12}, \omega^{(1)}_{10,14} = \omega^{(2)}_{8,12}, \omega^{(1)}_{11,14} = \omega^{(2)}_{9,12}, \omega^{(1)}_{12,14} = \omega^{(2)}_{10,12}, \omega^{(1)}_{13,14} = \omega^{(2)}_{11,12}, \omega^{(1)}_{14,14} = \omega^{(2)}_{12,12}, \end{array} \right. \tag{45}$$

$$\left\{ \begin{array}{l} \omega^{(1)}_{6,9} = \omega^{(2)}_{67}, \omega^{(1)}_{6,10} = \omega^{(2)}_{68}, \omega^{(1)}_{6,11} = \omega^{(2)}_{69}, \omega^{(1)}_{6,12} = \omega^{(2)}_{6,10}, \omega^{(1)}_{6,13} = \omega^{(2)}_{6,11}, \omega^{(1)}_{6,14} = \omega^{(2)}_{6,12}, \\ \omega^{(1)}_{14,9} = \omega^{(2)}_{12,7}, \omega^{(1)}_{14,10} = \omega^{(2)}_{12,8}, \omega^{(1)}_{14,11} = \omega^{(2)}_{12,9}, \omega^{(1)}_{14,12} = \omega^{(2)}_{12,10}, \omega^{(1)}_{14,13} = \omega^{(2)}_{12,11}, \omega^{(1)}_{14,14} = \omega^{(2)}_{12,12}, \end{array} \right. \tag{46}$$

$$\begin{cases} \omega^{(2)}_{22,1} = \omega^{(3)}_{16,1}, \omega^{(2)}_{22,2} = \omega^{(3)}_{16,2}, \omega^{(2)}_{22,3} = \omega^{(3)}_{16,3}, \omega^{(2)}_{22,4} = \omega^{(3)}_{16,4}, \omega^{(2)}_{22,7} = \omega^{(3)}_{16,5}, \omega^{(2)}_{22,8} = \omega^{(3)}_{16,6}, \\ \omega^{(2)}_{22,9} = \omega^{(3)}_{16,7}, \omega^{(2)}_{22,10} = \omega^{(3)}_{16,8}, \omega^{(2)}_{22,13} = \omega^{(3)}_{16,9}, \omega^{(2)}_{22,14} = \omega^{(3)}_{16,10}, \omega^{(2)}_{22,15} = \omega^{(3)}_{16,11}, \omega^{(2)}_{22,16} = \omega^{(3)}_{16,12}, \\ \omega^{(2)}_{22,19} = \omega^{(3)}_{16,13}, \omega^{(2)}_{22,20} = \omega^{(3)}_{16,14}, \omega^{(2)}_{22,21} = \omega^{(3)}_{16,15}, \omega^{(2)}_{22,22} = \omega^{(3)}_{16,16}, \end{cases} \tag{47}$$

$$\begin{cases} \omega^{(2)}_{1,22} = \omega^{(3)}_{1,16}, \omega^{(2)}_{2,22} = \omega^{(3)}_{2,16}, \omega^{(2)}_{3,22} = \omega^{(3)}_{3,16}, \omega^{(2)}_{4,22} = \omega^{(3)}_{4,16}, \omega^{(2)}_{7,22} = \omega^{(3)}_{5,16}, \omega^{(2)}_{8,22} = \omega^{(3)}_{6,16}, \\ \omega^{(2)}_{9,22} = \omega^{(3)}_{7,16}, \omega^{(2)}_{10,22} = \omega^{(3)}_{8,16}, \omega^{(2)}_{13,22} = \omega^{(3)}_{9,16}, \omega^{(2)}_{14,22} = \omega^{(3)}_{10,16}, \omega^{(2)}_{15,22} = \omega^{(3)}_{11,16}, \omega^{(2)}_{16,22} = \omega^{(3)}_{12,16}, \\ \omega^{(2)}_{19,22} = \omega^{(3)}_{13,16}, \omega^{(2)}_{20,22} = \omega^{(3)}_{14,16}, \omega^{(2)}_{21,22} = \omega^{(3)}_{15,16}, \omega^{(2)}_{22,22} = \omega^{(3)}_{16,16}, \end{cases} \tag{48}$$

$$\begin{cases} \omega^{(2)}_{19,4} = \omega^{(3)}_{13,4}, \omega^{(2)}_{20,4} = \omega^{(3)}_{14,4}, \omega^{(2)}_{21,4} = \omega^{(3)}_{15,4}, \omega^{(2)}_{22,4} = \omega^{(3)}_{16,4}, \omega^{(2)}_{19,10} = \omega^{(3)}_{13,8}, \omega^{(2)}_{20,10} = \omega^{(3)}_{14,8}, \\ \omega^{(2)}_{21,10} = \omega^{(3)}_{15,8}, \omega^{(2)}_{22,10} = \omega^{(3)}_{16,8}, \omega^{(2)}_{19,16} = \omega^{(3)}_{13,12}, \omega^{(2)}_{20,16} = \omega^{(3)}_{14,12}, \omega^{(2)}_{21,16} = \omega^{(3)}_{15,12}, \omega^{(2)}_{22,16} = \omega^{(3)}_{16,12}, \\ \omega^{(2)}_{19,22} = \omega^{(3)}_{13,16}, \omega^{(2)}_{20,22} = \omega^{(3)}_{14,16}, \omega^{(2)}_{21,22} = \omega^{(3)}_{15,16}, \omega^{(2)}_{22,22} = \omega^{(3)}_{16,16}, \end{cases} \tag{49}$$

$$\begin{cases} \omega^{(2)}_{4,19} = \omega^{(3)}_{4,13}, \omega^{(2)}_{4,20} = \omega^{(3)}_{4,14}, \omega^{(2)}_{4,21} = \omega^{(3)}_{4,15}, \omega^{(2)}_{4,22} = \omega^{(3)}_{4,16}, \omega^{(2)}_{10,19} = \omega^{(3)}_{8,13}, \omega^{(2)}_{10,20} = \omega^{(3)}_{8,14}, \\ \omega^{(2)}_{10,21} = \omega^{(3)}_{8,15}, \omega^{(2)}_{10,22} = \omega^{(3)}_{8,16}, \omega^{(2)}_{16,19} = \omega^{(3)}_{12,13}, \omega^{(2)}_{16,20} = \omega^{(3)}_{12,14}, \omega^{(2)}_{16,21} = \omega^{(3)}_{12,15}, \omega^{(2)}_{16,22} = \omega^{(3)}_{12,16}, \\ \omega^{(2)}_{22,19} = \omega^{(3)}_{16,13}, \omega^{(2)}_{22,20} = \omega^{(3)}_{16,14}, \omega^{(2)}_{22,21} = \omega^{(3)}_{16,15}, \omega^{(2)}_{22,22} = \omega^{(3)}_{16,16}, \end{cases} \tag{50}$$

$$\begin{cases} \omega^{(3)}_{22,1} = \omega^{(4)}_{12,1}, \omega^{(3)}_{22,2} = \omega^{(4)}_{12,2}, \omega^{(3)}_{22,5} = \omega^{(4)}_{12,3}, \omega^{(3)}_{22,6} = \omega^{(4)}_{12,4}, \omega^{(3)}_{22,9} = \omega^{(4)}_{12,5}, \omega^{(3)}_{22,10} = \omega^{(4)}_{12,6}, \\ \omega^{(3)}_{22,13} = \omega^{(4)}_{12,7}, \omega^{(3)}_{22,14} = \omega^{(4)}_{12,8}, \omega^{(3)}_{22,17} = \omega^{(4)}_{12,9}, \omega^{(3)}_{22,18} = \omega^{(4)}_{12,10}, \omega^{(3)}_{22,21} = \omega^{(4)}_{12,11}, \omega^{(3)}_{22,22} = \omega^{(4)}_{12,12}, \end{cases} \tag{51}$$

$$\begin{cases} \omega^{(3)}_{1,22} = \omega^{(4)}_{1,12}, \omega^{(3)}_{2,22} = \omega^{(4)}_{2,12}, \omega^{(3)}_{5,22} = \omega^{(4)}_{3,12}, \omega^{(3)}_{6,22} = \omega^{(4)}_{4,12}, \omega^{(3)}_{9,22} = \omega^{(4)}_{5,12}, \omega^{(3)}_{10,22} = \omega^{(4)}_{6,12}, \\ \omega^{(3)}_{13,22} = \omega^{(4)}_{7,12}, \omega^{(3)}_{14,22} = \omega^{(4)}_{8,12}, \omega^{(3)}_{17,22} = \omega^{(4)}_{9,12}, \omega^{(3)}_{18,22} = \omega^{(4)}_{10,12}, \omega^{(3)}_{21,22} = \omega^{(4)}_{11,12}, \omega^{(3)}_{22,22} = \omega^{(4)}_{12,12}, \end{cases} \tag{52}$$

$$\begin{cases} \omega^{(3)}_{21,2} = \omega^{(4)}_{11,2}, \omega^{(3)}_{21,6} = \omega^{(4)}_{11,4}, \omega^{(3)}_{21,10} = \omega^{(4)}_{11,6}, \omega^{(3)}_{21,14} = \omega^{(4)}_{11,8}, \omega^{(3)}_{21,18} = \omega^{(4)}_{11,10}, \omega^{(3)}_{21,22} = \omega^{(4)}_{11,12}, \\ \omega^{(3)}_{22,2} = \omega^{(4)}_{12,2}, \omega^{(3)}_{22,6} = \omega^{(4)}_{12,4}, \omega^{(3)}_{22,10} = \omega^{(4)}_{12,6}, \omega^{(3)}_{22,14} = \omega^{(4)}_{12,8}, \omega^{(3)}_{22,18} = \omega^{(4)}_{12,10}, \omega^{(3)}_{22,22} = \omega^{(4)}_{12,12}, \end{cases} \tag{53}$$

$$\begin{cases} \omega^{(3)}_{2,21} = \omega^{(4)}_{2,11}, \omega^{(3)}_{6,21} = \omega^{(4)}_{4,11}, \omega^{(3)}_{10,21} = \omega^{(4)}_{6,11}, \omega^{(3)}_{14,21} = \omega^{(4)}_{8,11}, \omega^{(3)}_{18,21} = \omega^{(4)}_{10,11}, \omega^{(3)}_{22,21} = \omega^{(4)}_{12,11}, \\ \omega^{(3)}_{2,22} = \omega^{(4)}_{2,12}, \omega^{(3)}_{6,22} = \omega^{(4)}_{4,12}, \omega^{(3)}_{10,22} = \omega^{(4)}_{6,12}, \omega^{(3)}_{14,22} = \omega^{(4)}_{8,12}, \omega^{(3)}_{18,22} = \omega^{(4)}_{10,12}, \omega^{(3)}_{22,22} = \omega^{(4)}_{12,12}, \end{cases} \tag{54}$$

$$\begin{cases} \omega^{(1)}_{12,1} + \omega^{(3)}_{81} = \omega^{(2)}_{10,1}, \omega^{(1)}_{12,2} + \omega^{(3)}_{82} = \omega^{(2)}_{10,2}, \omega^{(1)}_{12,3} + \omega^{(3)}_{83} = \omega^{(2)}_{10,3}, \omega^{(1)}_{12,4} + \omega^{(3)}_{84} = \omega^{(2)}_{10,4}, \\ \omega^{(1)}_{12,9} + \omega^{(3)}_{85} = \omega^{(2)}_{10,7}, \omega^{(1)}_{12,10} + \omega^{(3)}_{86} = \omega^{(2)}_{10,8}, \omega^{(1)}_{12,11} + \omega^{(3)}_{87} = \omega^{(2)}_{10,9}, \omega^{(1)}_{12,12} + \omega^{(3)}_{88} = \omega^{(2)}_{10,10}, \end{cases} \tag{55}$$

$$\begin{cases} \omega^{(1)}_{1,12} + \omega^{(3)}_{18} = \omega^{(2)}_{1,10}, \omega^{(1)}_{2,12} + \omega^{(3)}_{28} = \omega^{(2)}_{2,10}, \omega^{(1)}_{3,12} + \omega^{(3)}_{38} = \omega^{(2)}_{3,10}, \omega^{(1)}_{4,12} + \omega^{(3)}_{48} = \omega^{(2)}_{4,10}, \\ \omega^{(1)}_{9,12} + \omega^{(3)}_{58} = \omega^{(2)}_{7,10}, \omega^{(1)}_{10,12} + \omega^{(3)}_{68} = \omega^{(2)}_{8,10}, \omega^{(1)}_{11,12} + \omega^{(3)}_{78} = \omega^{(2)}_{9,10}, \omega^{(1)}_{12,12} + \omega^{(3)}_{88} = \omega^{(2)}_{10,10}, \end{cases} \tag{56}$$

$$\begin{cases} \omega^{(1)}_{94} + \omega^{(3)}_{54} = \omega^{(2)}_{74}, \omega^{(1)}_{10,4} + \omega^{(3)}_{64} = \omega^{(2)}_{84}, \omega^{(1)}_{11,4} + \omega^{(3)}_{74} = \omega^{(2)}_{94}, \omega^{(1)}_{12,4} + \omega^{(3)}_{84} = \omega^{(2)}_{10,4}, \\ \omega^{(1)}_{9,12} + \omega^{(3)}_{58} = \omega^{2}_{7,10}, \omega^{(1)}_{10,12} + \omega^{(3)}_{68} = \omega^{2}_{8,10}, \omega^{(1)}_{11,12} + \omega^{(3)}_{78} = \omega^{2}_{9,10}, \omega^{(1)}_{12,12} + \omega^{(3)}_{88} = \omega^{2}_{10,10}, \end{cases} \tag{57}$$

$$\begin{cases} \omega^{(1)}_{4,9} + \omega^{(3)}_{45} = \omega^{(2)}_{47}, \omega^{(1)}_{4,10} + \omega^{(3)}_{46} = \omega^{(2)}_{48}, \omega^{(1)}_{4,11} + \omega^{(3)}_{47} = \omega^{(2)}_{49}, \omega^{(1)}_{4,12} + \omega^{(3)}_{48} = \omega^{(2)}_{4,10}, \\ \omega^{(1)}_{12,9} + \omega^{(3)}_{85} = \omega^{2}_{10,7}, \omega^{(1)}_{12,10} + \omega^{(3)}_{86} = \omega^{2}_{10,8}, \omega^{(1)}_{12,11} + \omega^{(3)}_{87} = \omega^{2}_{10,9}, \omega^{(1)}_{12,12} + \omega^{(3)}_{88} = \omega^{2}_{10,10}, \end{cases} \tag{58}$$

$$\begin{cases} \omega^{(2)}_{20,1}+\omega^{(4)}_{81}=\omega^{(3)}_{14,1},\omega^{(2)}_{20,2}+\omega^{(4)}_{82}=\omega^{(3)}_{14,2},\omega^{(2)}_{20,7}+\omega^{(4)}_{83}=\omega^{(3)}_{14,5},\omega^{(2)}_{20,8}+\omega^{(4)}_{84}=\omega^{(3)}_{14,6}, \\ \omega^{(2)}_{20,13}+\omega^{(4)}_{85}=\omega^{(3)}_{14,9},\omega^{(2)}_{20,14}+\omega^{(4)}_{86}=\omega^{(3)}_{14,10},\omega^{(2)}_{20,19}+\omega^{(4)}_{87}=\omega^{(3)}_{14,13},\omega^{(2)}_{20,20}+\omega^{(4)}_{88}=\omega^{(3)}_{14,14}, \end{cases} \tag{59}$$

$$\begin{cases} \omega^{(2)}_{1,20}+\omega^{(4)}_{18}=\omega^{(3)}_{1,14},\omega^{(2)}_{2,20}+\omega^{(4)}_{28}=\omega^{(3)}_{2,14},\omega^{(2)}_{7,20}+\omega^{(4)}_{38}=\omega^{(3)}_{5,14},\omega^{(2)}_{8,20}+\omega^{(4)}_{48}=\omega^{(3)}_{6,14}, \\ \omega^{(2)}_{13,20}+\omega^{(4)}_{58}=\omega^{(3)}_{9,14},\omega^{(2)}_{14,20}+\omega^{(4)}_{68}=\omega^{(3)}_{10,14},\omega^{(2)}_{19,20}+\omega^{(4)}_{78}=\omega^{(3)}_{13,14},\omega^{(2)}_{20,20}+\omega^{(4)}_{88}=\omega^{(3)}_{14,14}, \end{cases} \tag{60}$$

$$\begin{cases} \omega^{(2)}_{19,2}+\omega^{(4)}_{72}=\omega^{(3)}_{13,2},\omega^{(2)}_{20,2}+\omega^{(4)}_{82}=\omega^{(3)}_{14,2},\omega^{(2)}_{19,8}+\omega^{(4)}_{74}=\omega^{(3)}_{13,6},\omega^{(2)}_{20,8}+\omega^{(4)}_{84}=\omega^{(3)}_{14,6}, \\ \omega^{(2)}_{19,14}+\omega^{(4)}_{76}=\omega^{(3)}_{13,10},\omega^{(2)}_{20,14}+\omega^{(4)}_{86}=\omega^{(3)}_{14,10},\omega^{(2)}_{19,20}+\omega^{(4)}_{78}=\omega^{(3)}_{13,14},\omega^{(2)}_{20,20}+\omega^{(4)}_{88}=\omega^{(3)}_{14,14}, \end{cases} \tag{61}$$

$$\begin{cases} \omega^{(2)}_{2,19}+\omega^{(4)}_{27}=\omega^{(3)}_{2,13},\omega^{(2)}_{2,20}+\omega^{(4)}_{28}=\omega^{(3)}_{2,14},\omega^{(2)}_{8,19}+\omega^{(4)}_{47}=\omega^{(3)}_{6,13},\omega^{(2)}_{8,20}+\omega^{(4)}_{48}=\omega^{(3)}_{6,14}, \\ \omega^{(2)}_{14,19}+\omega^{(4)}_{67}=\omega^{(3)}_{10,13},\omega^{(2)}_{14,20}+\omega^{(4)}_{68}=\omega^{(3)}_{10,14},\omega^{(2)}_{20,19}+\omega^{(4)}_{87}=\omega^{(3)}_{13,14},\omega^{(2)}_{20,20}+\omega^{(4)}_{88}=\omega^{(3)}_{14,14}, \end{cases} \tag{62}$$

$$\begin{cases} \omega^{(1)}_{10,1}+\omega^{(3)}_{61}=\omega^{(2)}_{81}+\omega^{(4)}_{41},\ \omega^{(1)}_{10,2}+\omega^{(3)}_{62}=\omega^{(2)}_{82}+\omega^{(4)}_{42}, \\ \omega^{(1)}_{10,9}+\omega^{(3)}_{65}=\omega^{(2)}_{87}+\omega^{(4)}_{43},\ \omega^{(1)}_{10,10}+\omega^{(3)}_{66}=\omega^{(2)}_{88}+\omega^{(4)}_{44}, \end{cases} \tag{63}$$

$$\begin{cases} \omega^{(1)}_{1,10}+\omega^{(3)}_{16}=\omega^{(2)}_{18}+\omega^{(4)}_{14},\ \omega^{(1)}_{2,10}+\omega^{(3)}_{26}=\omega^{(2)}_{28}+\omega^{(4)}_{24}, \\ \omega^{(1)}_{9,10}+\omega^{(3)}_{56}=\omega^{(2)}_{78}+\omega^{(4)}_{34},\ \omega^{(1)}_{10,10}+\omega^{(3)}_{66}=\omega^{(2)}_{88}+\omega^{(4)}_{44}, \end{cases} \tag{64}$$

$$\begin{cases} \omega^{(1)}_{92}+\omega^{(3)}_{52}=\omega^{(2)}_{72}+\omega^{(4)}_{32},\omega^{(1)}_{10,2}+\omega^{(3)}_{62}=\omega^{(2)}_{82}+\omega^{(4)}_{42}, \\ \omega^{(1)}_{9,10}+\omega^{(3)}_{56}=\omega^{(2)}_{78}+\omega^{(4)}_{34},\omega^{(1)}_{10,10}+\omega^{(3)}_{66}=\omega^{(2)}_{88}+\omega^{(4)}_{44}, \end{cases} \tag{65}$$

$$\begin{cases} \omega^{(1)}_{29}+\omega^{(3)}_{25}=\omega^{(2)}_{27}+\omega^{(4)}_{23},\omega^{(1)}_{2,10}+\omega^{(3)}_{26}=\omega^{(2)}_{28}+\omega^{(4)}_{24}, \\ \omega^{(1)}_{10,9}+\omega^{(3)}_{65}=\omega^{(2)}_{87}+\omega^{(4)}_{43},\omega^{(1)}_{10,10}+\omega^{(3)}_{66}=\omega^{(2)}_{88}+\omega^{(4)}_{44}. \end{cases} \tag{66}$$

Proof.

(1) $\Rightarrow$ (2): Suppose that $(X_1', X_2', X_3', X_4', X_5')$ is a solution to system (1), that is,

$$A_iX_i'C_i + B_iX_{i+1}'D_i = \Omega_i,\ i = 1,2,3,4,$$

we can employ elementary matrix operations to show that the rank equalities (12)–(39) hold.

(2) $\Rightarrow$ (3):

$$(12) \Leftrightarrow r(S_{a_i}\ \widehat{\Omega_i}\ S_{b_i}) = r(S_{a_i}\ S_{b_i}) \Rightarrow \begin{cases} i=1, \begin{pmatrix}\omega^{(1)}_{16,1} & \omega^{(1)}_{16,2} & \dots & \omega^{(1)}_{16,16}\end{pmatrix}=0, \\ i=2, \begin{pmatrix}\omega^{(2)}_{24,1} & \omega^{(2)}_{24,2} & \dots & \omega^{(2)}_{24,24}\end{pmatrix}=0, \\ i=3, \begin{pmatrix}\omega^{(3)}_{24,1} & \omega^{(3)}_{24,2} & \dots & \omega^{(3)}_{24,24}\end{pmatrix}=0, \\ i=4, \begin{pmatrix}\omega^{(4)}_{16,1} & \omega^{(4)}_{16,2} & \dots & \omega^{(4)}_{16,16}\end{pmatrix}=0. \end{cases}$$

Similarly, we have: $(13) \Rightarrow (40)$, $(14) \Rightarrow (41)$, $(15) \Rightarrow (42)$,
$(16) \Rightarrow (43)$ *with* $(40)-(42)$, $(17) \Rightarrow (44)$ *with* $(40)-(42)$,
$(18) \Rightarrow (45)$ *with* $(40)-(42)$, $(19) \Rightarrow (46)$ *with* $(40)-(42)$,
$(20) \Rightarrow (48)$ *with* $(40)-(42)$, $(22) \Rightarrow (49)$ *with* $(40)-(42)$,
$(23) \Rightarrow (50)$ *with* $(40)-(42)$, $(24) \Rightarrow (51)$ *with* $(40)-(42)$,
$(25) \Rightarrow (52)$ *with* $(40)-(42)$, $(26) \Rightarrow (53)$ *with* $(40)-(42)$,
$(27) \Rightarrow (54)$ *with* $(40)-(42)$, $(28) \Rightarrow (55)$ *with* $(40)-(50)$,
$(29) \Rightarrow (56)$ *with* $(40)-(50)$, $(30) \Rightarrow (57)$ *with* $(40)-(50)$,
$(31) \Rightarrow (58)$ *with* $(40)-(50)$,

(32) $\Rightarrow$ (59) *with* (40) $-$ (42) *and* (47) $-$ (54),
(33) $\Rightarrow$ (60) *with* (40) $-$ (42) *and* (47) $-$ (54),
(34) $\Rightarrow$ (61) *with* (40) $-$ (42) *and* (47) $-$ (54),
(35) $\Rightarrow$ (62) *with* (40) $-$ (42) *and* (47) $-$ (54),
(36) $\Rightarrow$ (63) *with* (40) $-$ (62), (37) $\Rightarrow$ (64) *with* (40) $-$ (62),
(38) $\Rightarrow$ (65) *with* (40) $-$ (62), (39) $\Rightarrow$ (66) *with* (40) $-$ (62).

(3) $\Leftrightarrow$ (1): By (8), (9), (10) and (11), system (1) is consistent if and only if (40)–(66) hold. □

By utilizing the simultaneous decomposition, we give out some necessary and sufficient conditions for system (1) to be solvable. However, it is hard to verify the conditions (40)–(66) because the amount of them is huge. It is easy to check conditions (12)–(39). In terms of conditions (40)–(66), we put more emphasis on their mutual verification with (12)–(39). In addition, by making use of the decomposition, we can obtain some useful properties related to the general solution. We refer the readers to [5].

3. The General Solution to System (1)

In this section, we detail the general solution to system (1) by using the partitioned matrix, and Algorithm 1 which clearly illustrate the steps to obtain the general solution to system (1) is set up.

Theorem 2. *If (12)–(39) or (40)–(66) hold, then $X_j = Q_j \widehat{X_j} T_j$ are the general solution to system (1), where $j = 1,2,3,4,5$. $\widehat{X_j}$ are listed as follows:*

$$\widehat{X_1} = (Y_1^{(1)}, Y_2^{(1)}), \tag{67}$$

where

$$Y_1^{(1)} = \begin{pmatrix} \omega_{11}^{(1)} - \omega_{11}^{(2)} + \omega_{11}^{(3)} - \omega_{11}^{(4)} + X_{11}^{(5)} & \omega_{12}^{(1)} - \omega_{12}^{(2)} + \omega_{12}^{(3)} - \omega_{12}^{(4)} & \omega_{13}^{(1)} - \omega_{13}^{(2)} + \omega_{13}^{(3)} - X_{13}^{(4)} \\ \omega_{21}^{(1)} - \omega_{21}^{(2)} + \omega_{21}^{(3)} - \omega_{21}^{(4)} & \omega_{22}^{(1)} - \omega_{22}^{(2)} + \omega_{22}^{(3)} - \omega_{22}^{(4)} & \omega_{23}^{(1)} - \omega_{23}^{(2)} + \omega_{23}^{(3)} - X_{23}^{(4)} \\ \omega_{31}^{(1)} - \omega_{31}^{(2)} + \omega_{31}^{(3)} - X_{31}^{(4)} & \omega_{32}^{(1)} - \omega_{32}^{(2)} + \omega_{32}^{(3)} - X_{32}^{(4)} & \omega_{33}^{(1)} - \omega_{33}^{(2)} + \omega_{33}^{(3)} - X_{33}^{(4)} \\ \omega_{41}^{(1)} - \omega_{41}^{(2)} + \omega_{41}^{(3)} & \omega_{42}^{(1)} - \omega_{42}^{(2)} + \omega_{42}^{(3)} & \omega_{43}^{(1)} - \omega_{43}^{(2)} + \omega_{43}^{(3)} \\ \omega_{51}^{(1)} - \omega_{51}^{(2)} + X_{51}^{(3)} & \omega_{52}^{(1)} - \omega_{52}^{(2)} + X_{52}^{(3)} & \omega_{53}^{(1)} - \omega_{53}^{(2)} + X_{53}^{(3)} \\ \omega_{61}^{(1)} - \omega_{61}^{(2)} & \omega_{62}^{(1)} - \omega_{62}^{(2)} & \omega_{63}^{(1)} - \omega_{63}^{(2)} \\ \omega_{71}^{(1)} - X_{71}^{(2)} & \omega_{72}^{(1)} - X_{72}^{(2)} & \omega_{73}^{(1)} - X_{73}^{(2)} \\ \omega_{81}^{(1)} & \omega_{82}^{(1)} & \omega_{83}^{(1)} \\ X_{91}^{(1)} & X_{92}^{(1)} & X_{93}^{(1)} \end{pmatrix},$$

$$Y_2^{(1)} = \begin{pmatrix} \omega_{14}^{(1)} - \omega_{14}^{(2)} + \omega_{14}^{(3)} & \omega_{15}^{(1)} - \omega_{15}^{(2)} + X_{15}^{(3)} & \omega_{16}^{(1)} - \omega_{16}^{(2)} & \omega_{17}^{(1)} - X_{17}^{(2)} & \omega_{18}^{(1)} & X_{19}^{(1)} \\ \omega_{24}^{(1)} - \omega_{24}^{(2)} + \omega_{24}^{(3)} & \omega_{25}^{(1)} - \omega_{25}^{(2)} + X_{25}^{(3)} & \omega_{26}^{(1)} - \omega_{26}^{(2)} & \omega_{27}^{(1)} - X_{27}^{(2)} & \omega_{28}^{(1)} & X_{29}^{(1)} \\ \omega_{34}^{(1)} - \omega_{34}^{(2)} + \omega_{34}^{(3)} & \omega_{35}^{(1)} - \omega_{35}^{(2)} + X_{35}^{(3)} & \omega_{36}^{(1)} - \omega_{36}^{(2)} & \omega_{37}^{(1)} - X_{37}^{(2)} & \omega_{38}^{(1)} & X_{39}^{(1)} \\ \omega_{44}^{(1)} - \omega_{44}^{(2)} + \omega_{44}^{(3)} & \omega_{45}^{(1)} - \omega_{45}^{(2)} + X_{45}^{(3)} & \omega_{46}^{(1)} - \omega_{46}^{(2)} & \omega_{47}^{(1)} - X_{47}^{(2)} & \omega_{48}^{(1)} & X_{49}^{(1)} \\ \omega_{54}^{(1)} - \omega_{54}^{(2)} + X_{54}^{(3)} & \omega_{55}^{(1)} - \omega_{55}^{(2)} + X_{55}^{(3)} & \omega_{56}^{(1)} - \omega_{56}^{(2)} & \omega_{57}^{(1)} - X_{57}^{(2)} & \omega_{58}^{(1)} & X_{59}^{(1)} \\ \omega_{64}^{(1)} - \omega_{64}^{(2)} & \omega_{65}^{(1)} - \omega_{65}^{(2)} & \omega_{66}^{(1)} - \omega_{66}^{(2)} & \omega_{67}^{(1)} - X_{67}^{(2)} & \omega_{68}^{(1)} & X_{69}^{(1)} \\ \omega_{74}^{(1)} - X_{74}^{(2)} & \omega_{75}^{(1)} - X_{75}^{(2)} & \omega_{76}^{(1)} - X_{76}^{(2)} & \omega_{77}^{(1)} - X_{77}^{(2)} & \omega_{78}^{(1)} & X_{79}^{(1)} \\ \omega_{84}^{(1)} & \omega_{85}^{(1)} & \omega_{86}^{(1)} & \omega_{87}^{(1)} & \omega_{88}^{(1)} & X_{89}^{(1)} \\ X_{94}^{(1)} & X_{95}^{(1)} & X_{96}^{(1)} & X_{97}^{(1)} & X_{98}^{(1)} & X_{99}^{(1)} \end{pmatrix}.$$

$$\widehat{X_2} = \begin{pmatrix} Y_{11}^{(2)} & Y_{12}^{(2)} & Y_{13}^{(2)} \\ Y_{21}^{(2)} & Y_{22}^{(2)} & Y_{23}^{(2)} \end{pmatrix}, \tag{68}$$

where

$$Y_{11}^{(2)} = \begin{pmatrix} \omega_{11}^{(2)} - \omega_{11}^{(3)} + \omega_{11}^{(4)} - X_{11}^{(5)} & \omega_{12}^{(2)} - \omega_{12}^{(3)} + \omega_{12}^{(4)} & \omega_{13}^{(2)} - \omega_{13}^{(3)} + X_{13}^{(4)} & \omega_{14}^{(2)} - \omega_{14}^{(3)} & \omega_{15}^{(2)} - X_{15}^{(3)} & \omega_{16}^{(2)} & X_{17}^{(2)} \\ \omega_{21}^{(2)} - \omega_{21}^{(3)} + \omega_{21}^{(4)} & \omega_{22}^{(2)} - \omega_{22}^{(3)} + \omega_{22}^{(4)} & \omega_{23}^{(2)} - \omega_{23}^{(3)} + X_{23}^{(4)} & \omega_{24}^{(2)} - \omega_{24}^{(3)} & \omega_{25}^{(2)} - X_{25}^{(3)} & \omega_{26}^{(2)} & X_{27}^{(2)} \\ \omega_{31}^{(2)} - \omega_{31}^{(3)} + X_{31}^{(4)} & \omega_{32}^{(2)} - \omega_{32}^{(3)} + X_{32}^{(4)} & \omega_{33}^{(2)} - \omega_{33}^{(3)} + X_{33}^{(4)} & \omega_{34}^{(2)} - \omega_{34}^{(3)} & \omega_{35}^{(2)} - X_{35}^{(3)} & \omega_{36}^{(2)} & X_{37}^{(2)} \\ \omega_{41}^{(2)} - \omega_{41}^{(3)} & \omega_{42}^{(2)} - \omega_{42}^{(3)} & \omega_{43}^{(2)} - \omega_{43}^{(3)} & \omega_{44}^{(2)} - \omega_{44}^{(3)} & \omega_{45}^{(2)} - X_{45}^{(3)} & \omega_{46}^{(2)} & X_{47}^{(2)} \\ \omega_{51}^{(2)} - X_{51}^{(3)} & \omega_{52}^{(2)} - X_{52}^{(3)} & \omega_{53}^{(2)} - X_{53}^{(3)} & \omega_{54}^{(2)} - X_{54}^{(3)} & \omega_{55}^{(2)} - X_{55}^{(3)} & \omega_{56}^{(2)} & X_{57}^{(2)} \\ \omega_{61}^{(2)} & \omega_{62}^{(2)} & \omega_{63}^{(2)} & \omega_{64}^{(2)} & \omega_{65}^{(2)} & \omega_{66}^{(2)} & X_{67}^{(2)} \\ X_{71}^{(2)} & X_{72}^{(2)} & X_{73}^{(2)} & X_{74}^{(2)} & X_{75}^{(2)} & X_{76}^{(2)} & X_{77}^{(2)} \\ \omega_{91}^{(1)} & \omega_{92}^{(1)} & \omega_{93}^{(1)} & \omega_{94}^{(1)} & \omega_{95}^{(1)} & \omega_{96}^{(1)} & \omega_{97}^{(1)} \\ \omega_{10,1}^{(1)} & \omega_{10,2}^{(1)} & \omega_{10,3}^{(1)} & \omega_{10,4}^{(1)} & \omega_{10,5}^{(1)} & \omega_{10,6}^{(1)} & \omega_{10,7}^{(1)} \\ \omega_{11,1}^{(1)} & \omega_{11,2}^{(1)} & \omega_{11,3}^{(1)} & \omega_{11,4}^{(1)} & \omega_{11,5}^{(1)} & \omega_{11,6}^{(1)} & \omega_{11,7}^{(1)} \end{pmatrix},$$

$$Y_{21}^{(2)} =$$
$$\begin{pmatrix} \omega_{12,1}^{(1)} & \omega_{12,2}^{(1)} & \omega_{12,3}^{(1)} & \omega_{12,4}^{(1)} & \omega_{12,5}^{(1)} & \omega_{12,6}^{(1)} & \omega_{12,7}^{(1)} \\ \omega_{13,1}^{(1)} & \omega_{13,2}^{(1)} & \omega_{13,3}^{(1)} & \omega_{13,4}^{(1)} & \omega_{13,5}^{(1)} & \omega_{13,6}^{(1)} & \omega_{13,7}^{(1)} \\ \omega_{14,1}^{(1)} & \omega_{14,2}^{(1)} & \omega_{14,3}^{(1)} & \omega_{14,4}^{(1)} & \omega_{14,5}^{(1)} & \omega_{14,6}^{(1)} & \omega_{14,7}^{(1)} \\ \omega_{15,1}^{(1)} & \omega_{15,2}^{(1)} & \omega_{15,3}^{(1)} & \omega_{15,4}^{(1)} & \omega_{15,5}^{(1)} & \omega_{15,6}^{(1)} & \omega_{15,7}^{(1)} \\ \omega_{13,1}^{(2)} - \omega_{91}^{(3)} + \omega_{51}^{(4)} - X_{31}^{(5)} & \omega_{13,2}^{(2)} - \omega_{92}^{(3)} + \omega_{52}^{(4)} & \omega_{13,3}^{(2)} - \omega_{93}^{(3)} + X_{73}^{(4)} & \omega_{13,4}^{(2)} - \omega_{94}^{(3)} & \omega_{13,5}^{(2)} - X_{11,5}^{(3)} & \omega_{13,6}^{(2)} & X_{15,7}^{(2)} \\ \omega_{14,1}^{(2)} - \omega_{10,1}^{(3)} + \omega_{61}^{(4)} & \omega_{14,2}^{(2)} - \omega_{10,2}^{(3)} + \omega_{62}^{(4)} & \omega_{14,3}^{(2)} - \omega_{10,3}^{(3)} + X_{83}^{(4)} & \omega_{14,4}^{(2)} - \omega_{10,4}^{(3)} & \omega_{14,5}^{(2)} - X_{12,5}^{(3)} & \omega_{14,6}^{(2)} & X_{16,7}^{(2)} \\ \omega_{15,1}^{(2)} - \omega_{11,1}^{(3)} + X_{91}^{(4)} & \omega_{15,2}^{(2)} - \omega_{11,2}^{(3)} + X_{92}^{(4)} & \omega_{15,3}^{(2)} - \omega_{11,3}^{(3)} + X_{93}^{(4)} & \omega_{15,4}^{(2)} - \omega_{11,4}^{(3)} & \omega_{15,5}^{(2)} - X_{13,5}^{(3)} & \omega_{15,6}^{(2)} & X_{17,7}^{(2)} \\ \omega_{16,1}^{(2)} - \omega_{12,1}^{(3)} & \omega_{16,2}^{(2)} - \omega_{12,2}^{(3)} & \omega_{16,3}^{(2)} - \omega_{12,3}^{(3)} & \omega_{16,4}^{(2)} - \omega_{12,4}^{(3)} & \omega_{16,5}^{(2)} - X_{14,5}^{(3)} & \omega_{16,6}^{(2)} & X_{18,7}^{(2)} \\ \omega_{17,1}^{(2)} - X_{15,1}^{(3)} & \omega_{17,2}^{(2)} - X_{15,2}^{(3)} & \omega_{17,3}^{(2)} - X_{15,3}^{(3)} & \omega_{17,4}^{(2)} - X_{15,4}^{(3)} & \omega_{17,5}^{(2)} - X_{15,5}^{(3)} & \omega_{17,6}^{(2)} & X_{19,7}^{(2)} \\ \omega_{18,1}^{(2)} & \omega_{18,2}^{(2)} & \omega_{18,3}^{(2)} & \omega_{18,4}^{(2)} & \omega_{18,5}^{(2)} & \omega_{18,6}^{(2)} & X_{20,7}^{(2)} \\ X_{21,1}^{(2)} & X_{21,2}^{(2)} & X_{21,3}^{(2)} & X_{21,4}^{(2)} & X_{21,5}^{(2)} & X_{21,6}^{(2)} & X_{21,7}^{(2)} \end{pmatrix},$$

$$Y_{12}^{(2)} = \begin{pmatrix} \omega_{19}^{(1)} & \omega_{1,10}^{(1)} & \omega_{1,11}^{(1)} & \omega_{1,12}^{(1)} & \omega_{1,13}^{(1)} & \omega_{1,14}^{(1)} & \omega_{1,15}^{(1)} \\ \omega_{29}^{(1)} & \omega_{2,10}^{(1)} & \omega_{2,11}^{(1)} & \omega_{2,12}^{(1)} & \omega_{2,13}^{(1)} & \omega_{2,14}^{(1)} & \omega_{2,15}^{(1)} \\ \omega_{39}^{(1)} & \omega_{3,10}^{(1)} & \omega_{3,11}^{(1)} & \omega_{3,12}^{(1)} & \omega_{3,13}^{(1)} & \omega_{3,14}^{(1)} & \omega_{3,15}^{(1)} \\ \omega_{49}^{(1)} & \omega_{4,10}^{(1)} & \omega_{4,11}^{(1)} & \omega_{4,12}^{(1)} & \omega_{4,13}^{(1)} & \omega_{4,14}^{(1)} & \omega_{4,15}^{(1)} \\ \omega_{59}^{(1)} & \omega_{5,10}^{(1)} & \omega_{5,11}^{(1)} & \omega_{5,12}^{(1)} & \omega_{5,13}^{(1)} & \omega_{5,14}^{(1)} & \omega_{5,15}^{(1)} \\ \omega_{69}^{(1)} & \omega_{6,10}^{(1)} & \omega_{6,11}^{(1)} & \omega_{6,12}^{(1)} & \omega_{6,13}^{(1)} & \omega_{6,14}^{(1)} & \omega_{6,15}^{(1)} \\ \omega_{79}^{(1)} & \omega_{7,10}^{(1)} & \omega_{7,11}^{(1)} & \omega_{7,12}^{(1)} & \omega_{7,13}^{(1)} & \omega_{7,14}^{(1)} & \omega_{7,15}^{(1)} \\ \omega_{99}^{(1)} & \omega_{9,10}^{(1)} & \omega_{9,11}^{(1)} & \omega_{9,12}^{(1)} & \omega_{9,13}^{(1)} & \omega_{9,14}^{(1)} & \omega_{9,15}^{(1)} \\ \omega_{10,9}^{(1)} & \omega_{10,10}^{(1)} & \omega_{10,11}^{(1)} & \omega_{10,12}^{(1)} & \omega_{10,13}^{(1)} & \omega_{10,14}^{(1)} & \omega_{10,15}^{(1)} \\ \omega_{11,9}^{(1)} & \omega_{11,10}^{(1)} & \omega_{11,11}^{(1)} & \omega_{11,12}^{(1)} & \omega_{11,13}^{(1)} & \omega_{11,14}^{(1)} & \omega_{11,15}^{(1)} \end{pmatrix},$$

$$Y_{22}^{(2)} =$$
$$\begin{pmatrix} \omega_{12,9}^{(1)} & \omega_{12,10}^{(1)} & \omega_{12,11}^{(1)} & \omega_{12,12}^{(1)} & \omega_{12,13}^{(1)} & \omega_{12,14}^{(1)} & \omega_{12,15}^{(1)} \\ \omega_{13,9}^{(1)} & \omega_{13,10}^{(1)} & \omega_{13,11}^{(1)} & \omega_{13,12}^{(1)} & \omega_{13,13}^{(1)} & \omega_{13,14}^{(1)} & \omega_{13,15}^{(1)} \\ \omega_{14,9}^{(1)} & \omega_{14,10}^{(1)} & \omega_{14,11}^{(1)} & \omega_{14,12}^{(1)} & \omega_{14,13}^{(1)} & \omega_{14,14}^{(1)} & \omega_{14,15}^{(1)} \\ \omega_{15,9}^{(1)} & \omega_{15,10}^{(1)} & \omega_{15,11}^{(1)} & \omega_{15,12}^{(1)} & \omega_{15,13}^{(1)} & \omega_{15,14}^{(1)} & \omega_{15,15}^{(1)} \\ \omega_{13,7}^{(2)} - \omega_{95}^{(3)} + \omega_{53}^{(4)} - X_{32}^{(5)} & \omega_{13,8}^{(2)} - \omega_{96}^{(3)} + \omega_{54}^{(4)} & \omega_{13,9}^{(2)} - \omega_{97}^{(3)} + X_{76}^{(4)} & \omega_{13,10}^{(2)} - \omega_{98}^{(3)} & \omega_{13,11}^{(2)} - X_{11,10}^{(3)} & \omega_{13,12}^{(3)} & X_{15,14}^{(2)} \\ \omega_{14,7}^{(2)} - \omega_{10,5}^{(3)} + \omega_{63}^{(4)} & \omega_{14,8}^{(2)} - \omega_{10,6}^{(3)} + \omega_{64}^{(4)} & \omega_{14,9}^{(2)} - \omega_{10,7}^{(3)} + X_{86}^{(4)} & \omega_{14,10}^{(2)} - \omega_{10,8}^{(3)} & \omega_{14,11}^{(2)} - X_{12,10}^{(3)} & \omega_{14,12}^{(3)} & X_{16,14}^{(2)} \\ \omega_{15,7}^{(2)} - \omega_{11,5}^{(3)} + X_{94}^{(4)} & \omega_{15,8}^{(2)} - \omega_{11,6}^{(3)} + X_{95}^{(4)} & \omega_{15,9}^{(2)} - \omega_{11,7}^{(3)} + X_{96}^{(4)} & \omega_{15,10}^{(2)} - \omega_{11,8}^{(3)} & \omega_{15,11}^{(2)} - X_{13,10}^{(3)} & \omega_{15,12}^{(3)} & X_{17,14}^{(2)} \\ \omega_{16,7}^{(2)} - \omega_{12,5}^{(3)} & \omega_{16,8}^{(2)} - \omega_{12,6}^{(3)} & \omega_{16,9}^{(2)} - \omega_{12,7}^{(3)} & \omega_{16,10}^{(2)} - \omega_{12,8}^{(3)} & \omega_{16,11}^{(2)} - X_{14,10}^{(3)} & \omega_{16,12}^{(3)} & X_{18,14}^{(2)} \\ \omega_{17,7}^{(2)} - X_{15,6}^{(3)} & \omega_{17,8}^{(2)} - X_{15,7}^{(3)} & \omega_{17,9}^{(2)} - X_{15,8}^{(3)} & \omega_{17,10}^{(2)} - X_{15,9}^{(3)} & \omega_{17,11}^{(2)} - X_{15,10}^{(3)} & \omega_{17,12}^{(3)} & X_{19,14}^{(2)} \\ \omega_{18,7}^{(2)} & \omega_{18,8}^{(2)} & \omega_{18,9}^{(2)} & \omega_{18,10}^{(2)} & \omega_{18,11}^{(2)} & \omega_{18,12}^{(3)} & X_{20,14}^{(2)} \\ X_{21,8}^{(2)} & X_{21,9}^{(2)} & X_{21,10}^{(2)} & X_{21,11}^{(2)} & X_{21,12}^{(2)} & X_{21,13}^{(2)} & X_{21,14}^{(2)} \end{pmatrix},$$

$$Y_{13}^{(2)} =$$
$$\begin{pmatrix}
\omega_{1,13}^{(2)} - \omega_{19}^{(3)} + \omega_{15}^{(4)} - X_{13}^{(5)} & \omega_{1,14}^{(2)} - \omega_{1,10}^{(3)} + \omega_{16}^{(4)} & \omega_{1,15}^{(2)} - \omega_{1,11}^{(3)} + X_{19}^{(4)} & \omega_{1,16}^{(2)} - \omega_{1,12}^{(3)} & \omega_{1,17}^{(2)} - X_{1,15}^{(3)} & \omega_{1,18}^{(2)} & X_{1,21}^{(2)} \\
\omega_{2,13}^{(2)} - \omega_{29}^{(3)} + \omega_{25}^{(4)} & \omega_{2,14}^{(2)} - \omega_{2,10}^{(3)} + \omega_{26}^{(4)} & \omega_{2,15}^{(2)} - \omega_{2,11}^{(3)} + X_{29}^{(4)} & \omega_{2,16}^{(2)} - \omega_{2,12}^{(3)} & \omega_{2,17}^{(2)} - X_{2,15}^{(3)} & \omega_{2,18}^{(2)} & X_{2,21}^{(2)} \\
\omega_{3,13}^{(2)} - \omega_{39}^{(3)} + X_{37}^{(4)} & \omega_{3,14}^{(2)} - \omega_{3,10}^{(3)} + X_{38}^{(4)} & \omega_{3,15}^{(2)} - \omega_{3,11}^{(3)} + X_{39}^{(4)} & \omega_{3,16}^{(2)} - \omega_{3,12}^{(3)} & \omega_{3,17}^{(2)} - X_{3,15}^{(3)} & \omega_{3,18}^{(2)} & X_{3,21}^{(2)} \\
\omega_{4,13}^{(2)} - \omega_{49}^{(3)} & \omega_{4,14}^{(2)} - \omega_{4,10}^{(3)} & \omega_{4,15}^{(2)} - \omega_{4,11}^{(3)} & \omega_{4,16}^{(2)} - \omega_{4,12}^{(3)} & \omega_{4,17}^{(2)} - X_{4,15}^{(3)} & \omega_{4,18}^{(2)} & X_{4,21}^{(2)} \\
\omega_{5,13}^{(2)} - X_{5,11}^{(3)} & \omega_{5,14}^{(2)} - X_{5,12}^{(3)} & \omega_{5,15}^{(2)} - X_{5,13}^{(3)} & \omega_{5,16}^{(2)} - X_{5,14}^{(3)} & \omega_{5,17}^{(2)} - X_{5,15}^{(3)} & \omega_{5,18}^{(2)} & X_{5,21}^{(2)} \\
\omega_{6,13}^{(2)} & \omega_{6,14}^{(2)} & \omega_{6,15}^{(2)} & \omega_{6,16}^{(2)} & \omega_{6,17}^{(2)} & \omega_{6,18}^{(2)} & X_{6,21}^{(2)} \\
X_{7,15}^{(2)} & X_{7,16}^{(2)} & X_{7,17}^{(2)} & X_{7,18}^{(2)} & X_{7,19}^{(2)} & X_{7,20}^{(2)} & X_{7,21}^{(2)} \\
\omega_{7,13}^{(2)} - \omega_{59}^{(3)} + \omega_{35}^{(4)} - X_{23}^{(5)} & \omega_{7,14}^{(2)} - \omega_{5,10}^{(3)} + \omega_{36}^{(4)} & \omega_{7,15}^{(2)} - \omega_{5,11}^{(3)} + X_{49}^{(4)} & \omega_{7,16}^{(2)} - \omega_{5,12}^{(3)} & \omega_{7,17}^{(2)} - X_{6,15}^{(3)} & \omega_{7,18}^{(2)} & X_{8,21}^{(2)} \\
\omega_{8,13}^{(2)} - \omega_{69}^{(3)} + \omega_{45}^{(4)} & \omega_{8,14}^{(2)} - \omega_{6,10}^{(3)} + \omega_{46}^{(4)} & \omega_{8,15}^{(2)} - \omega_{6,11}^{(3)} + X_{59}^{(4)} & \omega_{8,16}^{(2)} - \omega_{6,12}^{(3)} & \omega_{8,17}^{(2)} - X_{7,15}^{(3)} & \omega_{8,18}^{(2)} & X_{9,21}^{(2)} \\
\omega_{9,13}^{(2)} - \omega_{79}^{(3)} + X_{67}^{(4)} & \omega_{9,14}^{(2)} - \omega_{7,10}^{(3)} + X_{68}^{(4)} & \omega_{9,15}^{(2)} - \omega_{7,11}^{(3)} + X_{69}^{(4)} & \omega_{9,16}^{(2)} - \omega_{7,12}^{(3)} & \omega_{9,17}^{(2)} - X_{8,15}^{(3)} & \omega_{9,18}^{(2)} & X_{10,21}^{(2)}
\end{pmatrix},$$

$$Y_{23}^{(2)} =$$
$$\begin{pmatrix}
\omega_{10,13}^{(2)} - \omega_{89}^{(3)} & \omega_{10,14}^{(2)} - \omega_{8,10}^{(3)} & \omega_{10,15}^{(2)} - \omega_{8,11}^{(3)} & \omega_{10,16}^{(2)} - \omega_{8,12}^{(3)} & \omega_{10,17}^{(2)} - X_{9,15}^{(3)} & \omega_{10,18}^{(2)} & X_{11,21}^{(2)} \\
\omega_{11,13}^{(2)} - X_{10,11}^{(3)} & \omega_{11,14}^{(2)} - X_{10,12}^{(3)} & \omega_{11,15}^{(2)} - X_{10,13}^{(3)} & \omega_{11,16}^{(2)} - X_{10,14}^{(3)} & \omega_{11,17}^{(2)} - X_{10,15}^{(3)} & \omega_{11,18}^{(2)} & X_{12,21}^{(2)} \\
\omega_{12,13}^{(2)} & \omega_{12,14}^{(2)} & \omega_{12,15}^{(2)} & \omega_{12,16}^{(2)} & \omega_{12,17}^{(2)} & \omega_{12,18}^{(2)} & X_{13,21}^{(2)} \\
X_{14,15}^{(2)} & X_{14,16}^{(2)} & X_{14,17}^{(2)} & X_{14,18}^{(2)} & X_{14,19}^{(2)} & X_{14,20}^{(2)} & X_{14,21}^{(2)} \\
\omega_{13,13}^{(2)} - \omega_{99}^{(3)} + \omega_{55}^{(4)} - X_{33}^{(5)} & \omega_{13,14}^{(2)} - \omega_{9,10}^{(3)} + \omega_{56}^{(4)} & \omega_{13,15}^{(2)} - \omega_{9,11}^{(3)} + X_{79}^{(4)} & \omega_{13,16}^{(2)} - \omega_{9,12}^{(3)} & \omega_{13,17}^{(2)} - X_{11,15}^{(3)} & \omega_{13,18}^{(2)} & X_{15,21}^{(2)} \\
\omega_{14,13}^{(2)} - \omega_{10,9}^{(3)} + \omega_{65}^{(4)} & \omega_{14,14}^{(2)} - \omega_{10,10}^{(3)} + \omega_{66}^{(4)} & \omega_{14,15}^{(2)} - \omega_{10,11}^{(3)} + X_{89}^{(4)} & \omega_{14,16}^{(2)} - \omega_{10,12}^{(3)} & \omega_{14,17}^{(2)} - X_{12,15}^{(3)} & \omega_{14,18}^{(2)} & X_{16,21}^{(2)} \\
\omega_{15,13}^{(2)} - \omega_{11,9}^{(3)} + X_{97}^{(4)} & \omega_{15,14}^{(2)} - \omega_{11,10}^{(3)} + X_{98}^{(4)} & \omega_{15,15}^{(2)} - \omega_{11,11}^{(3)} + X_{99}^{(4)} & \omega_{15,16}^{(2)} - \omega_{11,12}^{(3)} & \omega_{15,17}^{(2)} - X_{13,15}^{(3)} & \omega_{15,18}^{(2)} & X_{17,21}^{(2)} \\
\omega_{16,13}^{(2)} - \omega_{12,9}^{(3)} & \omega_{16,14}^{(2)} - \omega_{12,10}^{(3)} & \omega_{16,15}^{(2)} - \omega_{12,11}^{(3)} & \omega_{16,16}^{(2)} - \omega_{12,12}^{(3)} & \omega_{16,17}^{(2)} - X_{14,15}^{(3)} & \omega_{16,18}^{(2)} & X_{18,21}^{(2)} \\
\omega_{17,13}^{(2)} - X_{15,11}^{(3)} & \omega_{17,14}^{(2)} - X_{15,12}^{(3)} & \omega_{17,15}^{(2)} - X_{15,13}^{(3)} & \omega_{17,16}^{(2)} - X_{15,14}^{(3)} & \omega_{17,17}^{(2)} - X_{15,15}^{(3)} & \omega_{17,18}^{(2)} & X_{19,21}^{(2)} \\
\omega_{18,13}^{(2)} & \omega_{18,14}^{(2)} & \omega_{18,15}^{(2)} & \omega_{18,16}^{(2)} & \omega_{18,17}^{(2)} & \omega_{18,18}^{(2)} & X_{20,21}^{(2)} \\
X_{21,15}^{(2)} & X_{21,16}^{(2)} & X_{21,17}^{(2)} & X_{21,18}^{(2)} & X_{21,19}^{(2)} & X_{21,20}^{(2)} & X_{21,21}^{(2)}
\end{pmatrix}.$$

$$\widehat{X_3} = \begin{pmatrix} Y_{11}^{(3)} & Y_{12}^{(3)} & Y_{13}^{(3)} & Y_{14}^{(3)} \\ Y_{21}^{(3)} & Y_{22}^{(3)} & Y_{23}^{(3)} & Y_{24}^{(3)} \end{pmatrix}, \tag{69}$$

where

$$Y_{11}^{(3)} = \begin{pmatrix}
\omega_{11}^{(3)} - \omega_{11}^{(4)} + X_{11}^{(5)} & \omega_{12}^{(3)} - \omega_{12}^{(4)} & \omega_{13}^{(3)} - X_{13}^{(4)} & \omega_{14}^{(3)} & X_{15}^{(3)} & \omega_{17}^{(2)} - \omega_{19}^{(1)} \\
\omega_{21}^{(3)} - \omega_{21}^{(4)} & \omega_{22}^{(3)} - \omega_{22}^{(4)} & \omega_{23}^{(3)} - X_{23}^{(4)} & \omega_{24}^{(3)} & X_{25}^{(3)} & \omega_{27}^{(2)} - \omega_{29}^{(1)} \\
\omega_{31}^{(3)} - X_{31}^{(4)} & \omega_{32}^{(3)} - X_{32}^{(4)} & \omega_{33}^{(3)} - X_{33}^{(4)} & \omega_{34}^{(3)} & X_{35}^{(3)} & \omega_{37}^{(2)} - \omega_{39}^{(1)} \\
\omega_{41}^{(3)} & \omega_{42}^{(3)} & \omega_{43}^{(3)} & \omega_{44}^{(3)} & X_{45}^{(3)} & \omega_{45}^{(3)} \\
X_{51}^{(3)} & X_{52}^{(3)} & X_{53}^{(3)} & X_{54}^{(3)} & X_{55}^{(3)} & \omega_{57}^{(2)} - \omega_{59}^{(1)} \\
\omega_{71}^{(2)} - \omega_{91}^{(1)} & \omega_{72}^{(2)} - \omega_{92}^{(1)} & \omega_{73}^{(2)} - \omega_{93}^{(1)} & \omega_{54}^{(3)} & \omega_{75}^{(2)} - \omega_{95}^{(1)} & \omega_{77}^{(2)} - \omega_{99}^{(1)} \\
\omega_{81}^{(2)} - \omega_{10,1}^{(1)} & \omega_{82}^{(2)} - \omega_{10,2}^{(1)} & \omega_{83}^{(2)} - \omega_{10,3}^{(1)} & \omega_{64}^{(3)} & \omega_{85}^{(2)} - \omega_{10,5}^{(1)} & \omega_{87}^{(2)} - \omega_{10,9}^{(1)} \\
\omega_{91}^{(2)} - \omega_{11,1}^{(1)} & \omega_{92}^{(2)} - \omega_{11,2}^{(1)} & \omega_{93}^{(2)} - \omega_{11,3}^{(1)} & \omega_{74}^{(3)} & \omega_{95}^{(2)} - \omega_{11,5}^{(1)} & \omega_{97}^{(2)} - \omega_{11,9}^{(1)} \\
\omega_{81}^{(3)} & \omega_{82}^{(3)} & \omega_{83}^{(3)} & \omega_{84}^{(3)} & \omega_{10,5}^{(2)} - \omega_{12,5}^{(1)} & \omega_{85}^{(3)} \\
\omega_{11,1}^{(2)} - \omega_{13,1}^{(1)} & \omega_{11,2}^{(2)} - \omega_{13,2}^{(1)} & \omega_{11,3}^{(2)} - \omega_{13,3}^{(1)} & \omega_{11,4}^{(2)} - \omega_{13,4}^{(1)} & \omega_{11,5}^{(2)} - \omega_{13,5}^{(1)} & \omega_{11,7}^{(2)} - \omega_{13,9}^{(1)} \\
\omega_{91}^{(3)} - \omega_{51}^{(4)} + X_{31}^{(5)} & \omega_{92}^{(3)} - \omega_{52}^{(4)} & \omega_{93}^{(3)} - X_{73}^{(4)} & \omega_{94}^{(3)} & X_{11,5}^{(3)} & \omega_{95}^{(3)} - \omega_{53}^{(4)} + X_{32}^{(5)} \\
\omega_{10,1}^{(3)} - \omega_{61}^{(4)} & \omega_{10,2}^{(3)} - \omega_{62}^{(4)} & \omega_{10,3}^{(3)} - X_{83}^{(4)} & \omega_{10,4}^{(3)} & X_{12,5}^{(3)} & \omega_{10,5}^{(3)} - \omega_{63}^{(4)} \\
\omega_{11,1}^{(3)} - X_{91}^{(4)} & \omega_{11,2}^{(3)} - X_{92}^{(4)} & \omega_{11,3}^{(3)} - X_{93}^{(4)} & \omega_{11,4}^{(3)} & X_{13,5}^{(3)} & \omega_{11,5}^{(3)} - X_{94}^{(4)}
\end{pmatrix},$$

$$Y_{21}^{(3)} = \begin{pmatrix}
\omega_{12,1}^{(3)} & \omega_{12,2}^{(3)} & \omega_{12,3}^{(3)} & \omega_{12,4}^{(3)} & X_{14,5}^{(3)} & \omega_{12,5}^{(3)} \\
X_{15,1}^{(3)} & X_{15,2}^{(3)} & X_{15,3}^{(3)} & X_{15,4}^{(3)} & X_{15,5}^{(3)} & X_{15,6}^{(3)} \\
\omega_{19,1}^{(2)} & \omega_{19,2}^{(2)} & \omega_{19,3}^{(2)} & \omega_{19,4}^{(2)} & \omega_{19,5}^{(2)} & \omega_{19,7}^{(2)} \\
\omega_{20,1}^{(2)} & \omega_{20,2}^{(2)} & \omega_{20,3}^{(2)} & \omega_{20,4}^{(2)} & \omega_{20,5}^{(2)} & \omega_{20,7}^{(2)} \\
\omega_{21,1}^{(2)} & \omega_{21,2}^{(2)} & \omega_{21,3}^{(2)} & \omega_{21,4}^{(2)} & \omega_{21,5}^{(2)} & \omega_{21,7}^{(2)} \\
\omega_{22,1}^{(2)} & \omega_{22,2}^{(2)} & \omega_{22,3}^{(2)} & \omega_{22,4}^{(2)} & \omega_{22,5}^{(2)} & \omega_{22,7}^{(2)} \\
\omega_{23,1}^{(2)} & \omega_{23,2}^{(2)} & \omega_{23,3}^{(2)} & \omega_{23,4}^{(2)} & \omega_{23,5}^{(2)} & \omega_{23,7}^{(2)} \\
\omega_{17,1}^{(3)} - \omega_{91}^{(4)} + X_{51}^{(5)} & \omega_{17,2}^{(3)} - \omega_{92}^{(4)} & \omega_{17,3}^{(3)} - X_{13,3}^{(4)} & \omega_{17,4}^{(3)} & X_{21,5}^{(3)} & \omega_{17,5}^{(3)} - \omega_{93}^{(4)} + X_{52}^{(5)} \\
\omega_{18,1}^{(3)} - \omega_{10,1}^{(4)} & \omega_{18,2}^{(3)} - \omega_{10,2}^{(4)} & \omega_{18,3}^{(3)} - X_{14,3}^{(4)} & \omega_{18,4}^{(3)} & X_{22,5}^{(3)} & \omega_{18,5}^{(3)} - \omega_{10,3}^{(4)} \\
\omega_{19,1}^{(3)} - X_{15,1}^{(4)} & \omega_{19,2}^{(3)} - X_{15,2}^{(4)} & \omega_{19,3}^{(3)} - X_{15,3}^{(4)} & \omega_{19,4}^{(3)} & X_{23,5}^{(3)} & \omega_{19,5}^{(3)} - X_{15,4}^{(4)} \\
\omega_{20,1}^{(3)} & \omega_{20,2}^{(3)} & \omega_{20,3}^{(3)} & \omega_{20,4}^{(3)} & X_{24,5}^{(3)} & \omega_{20,5}^{(3)} \\
X_{25,1}^{(3)} & X_{25,2}^{(3)} & X_{25,3}^{(3)} & X_{25,4}^{(3)} & X_{25,5}^{(3)} & X_{25,6}^{(3)}
\end{pmatrix},$$

$$Y_{12}^{(3)} = \begin{pmatrix}
\omega_{18}^{(2)} - \omega_{1,10}^{(1)} & \omega_{19}^{(2)} - \omega_{1,11}^{(1)} & \omega_{18}^{(3)} & \omega_{1,11}^{(2)} - \omega_{1,13}^{(1)} & \omega_{19}^{(3)} - \omega_{15}^{(4)} + X_{13}^{(5)} & \omega_{1,10}^{(3)} - \omega_{16}^{(4)} \\
\omega_{28}^{(2)} - \omega_{2,10}^{(1)} & \omega_{29}^{(2)} - \omega_{2,11}^{(1)} & \omega_{28}^{(3)} & \omega_{2,11}^{(2)} - \omega_{2,13}^{(1)} & \omega_{29}^{(3)} - \omega_{25}^{(4)} & \omega_{2,10}^{(3)} - \omega_{26}^{(4)} \\
\omega_{38}^{(2)} - \omega_{3,10}^{(1)} & \omega_{39}^{(2)} - \omega_{3,11}^{(1)} & \omega_{38}^{(3)} & \omega_{3,11}^{(2)} - \omega_{3,13}^{(1)} & \omega_{39}^{(3)} - X_{37}^{(4)} & \omega_{3,10}^{(3)} - X_{38}^{(4)} \\
\omega_{46}^{(3)} & \omega_{47}^{(3)} & \omega_{48}^{(3)} & \omega_{4,11}^{(2)} - \omega_{4,13}^{(1)} & \omega_{49}^{(3)} & \omega_{4,10}^{(3)} \\
\omega_{58}^{(2)} - \omega_{5,10}^{(1)} & \omega_{59}^{(2)} - \omega_{5,11}^{(1)} & \omega_{5,10}^{(2)} - \omega_{5,12}^{(1)} & \omega_{5,11}^{(2)} - \omega_{5,13}^{(1)} & X_{5,11}^{(3)} & X_{5,12}^{(3)} \\
\omega_{78}^{(2)} - \omega_{9,10}^{(1)} & \omega_{79}^{(2)} - \omega_{9,11}^{(1)} & \omega_{58}^{(3)} & \omega_{7,11}^{(2)} - \omega_{9,13}^{(1)} & \omega_{59}^{(3)} - \omega_{35}^{(4)} + X_{23}^{(5)} & \omega_{5,10}^{(3)} - \omega_{36}^{(4)} \\
\omega_{88}^{(2)} - \omega_{10,10}^{(1)} & \omega_{89}^{(2)} - \omega_{10,11}^{(1)} & \omega_{68}^{(3)} & \omega_{8,11}^{(2)} - \omega_{10,13}^{(1)} & \omega_{69}^{(3)} - \omega_{45}^{(4)} & \omega_{6,10}^{(3)} - \omega_{46}^{(4)} \\
\omega_{98}^{(2)} - \omega_{11,10}^{(1)} & \omega_{99}^{(2)} - \omega_{11,11}^{(1)} & \omega_{78}^{(3)} & \omega_{9,11}^{(2)} - \omega_{11,13}^{(1)} & \omega_{79}^{(3)} - X_{67}^{(4)} & \omega_{7,10}^{(3)} - X_{68}^{(4)} \\
\omega_{86}^{(3)} & \omega_{87}^{(3)} & \omega_{88}^{(3)} & \omega_{10,11}^{(2)} - \omega_{12,13}^{(1)} & \omega_{89}^{(3)} & \omega_{8,10}^{(3)} \\
\omega_{11,8}^{(2)} - \omega_{13,10}^{(1)} & \omega_{11,9}^{(2)} - \omega_{13,11}^{(1)} & \omega_{11,10}^{(2)} - \omega_{13,12}^{(1)} & \omega_{11,11}^{(2)} - \omega_{13,13}^{(1)} & X_{10,11}^{(3)} & X_{10,12}^{(3)} \\
\omega_{96}^{(3)} - \omega_{54}^{(4)} & \omega_{97}^{(3)} - X_{76}^{(4)} & \omega_{98}^{(3)} & X_{11,10}^{(3)} & \omega_{99}^{(3)} - \omega_{55}^{(4)} + X_{33}^{(5)} & \omega_{9,10}^{(3)} - \omega_{56}^{(4)} \\
\omega_{10,6}^{(3)} - \omega_{64}^{(4)} & \omega_{10,7}^{(3)} - X_{86}^{(4)} & \omega_{10,8}^{(3)} & X_{12,10}^{(3)} & \omega_{10,9}^{(3)} - \omega_{65}^{(4)} & \omega_{10,10}^{(3)} - \omega_{66}^{(4)} \\
\omega_{11,6}^{(3)} - X_{95}^{(4)} & \omega_{11,7}^{(3)} - X_{96}^{(4)} & \omega_{11,8}^{(3)} & X_{13,10}^{(3)} & \omega_{11,9}^{(3)} - X_{97}^{(4)} & \omega_{11,10}^{(3)} - X_{98}^{(4)}
\end{pmatrix},$$

$$Y_{22}^{(3)} = \begin{pmatrix}
\omega_{12,6}^{(3)} & \omega_{12,7}^{(3)} & \omega_{12,8}^{(3)} & X_{14,10}^{(3)} & \omega_{12,9}^{(3)} & \omega_{12,10}^{(3)} \\
X_{15,7}^{(3)} & X_{15,8}^{(3)} & X_{15,9}^{(3)} & X_{15,10}^{(3)} & X_{15,11}^{(3)} & X_{15,12}^{(3)} \\
\omega_{19,8}^{(2)} & \omega_{19,9}^{(2)} & \omega_{19,10}^{(2)} & \omega_{19,11}^{(2)} & \omega_{19,13}^{(2)} & \omega_{19,14}^{(2)} \\
\omega_{20,8}^{(2)} & \omega_{20,9}^{(2)} & \omega_{20,10}^{(2)} & \omega_{20,11}^{(2)} & \omega_{20,13}^{(2)} & \omega_{20,14}^{(2)} \\
\omega_{21,8}^{(2)} & \omega_{21,9}^{(2)} & \omega_{21,10}^{(2)} & \omega_{21,11}^{(2)} & \omega_{21,13}^{(2)} & \omega_{21,14}^{(2)} \\
\omega_{22,8}^{(2)} & \omega_{22,9}^{(2)} & \omega_{22,10}^{(2)} & \omega_{22,11}^{(2)} & \omega_{22,13}^{(2)} & \omega_{22,14}^{(2)} \\
\omega_{23,8}^{(2)} & \omega_{23,9}^{(2)} & \omega_{23,10}^{(2)} & \omega_{23,11}^{(2)} & \omega_{23,13}^{(2)} & \omega_{23,14}^{(2)} \\
\omega_{17,6}^{(3)} - \omega_{94}^{(4)} & \omega_{17,7}^{(3)} - X_{13,6}^{(4)} & \omega_{17,8}^{(3)} & X_{21,10}^{(3)} & \omega_{17,9}^{(3)} - \omega_{95}^{(4)} + X_{53}^{(5)} & \omega_{17,10}^{(3)} - \omega_{96}^{(4)} \\
\omega_{18,6}^{(3)} - \omega_{10,4}^{(4)} & \omega_{18,7}^{(3)} - X_{14,6}^{(4)} & \omega_{18,8}^{(3)} & X_{22,10}^{(3)} & \omega_{18,9}^{(3)} - \omega_{10,5}^{(4)} & \omega_{18,10}^{(3)} - \omega_{10,6}^{(4)} \\
\omega_{19,6}^{(3)} - X_{15,5}^{(4)} & \omega_{19,7}^{(3)} - X_{15,6}^{(4)} & \omega_{19,8}^{(3)} & X_{23,10}^{(3)} & \omega_{19,9}^{(3)} - X_{15,7}^{(4)} & \omega_{19,10}^{(3)} - X_{15,8}^{(4)} \\
\omega_{20,6}^{(3)} & \omega_{20,7}^{(3)} & \omega_{20,8}^{(3)} & X_{24,10}^{(3)} & \omega_{20,9}^{(3)} & \omega_{20,10}^{(3)} \\
X_{25,7}^{(3)} & X_{25,8}^{(3)} & X_{25,9}^{(3)} & X_{25,10}^{(3)} & X_{25,11}^{(3)} & X_{25,12}^{(3)}
\end{pmatrix},$$

$$Y_{13}^{(3)} = \begin{pmatrix}
\omega_{1,11}^{(3)} - X_{19}^{(4)} & \omega_{1,12}^{(3)} & X_{1,15}^{(3)} & \omega_{1,19}^{(2)} & \omega_{1,20}^{(2)} & \omega_{1,21}^{(2)} \\
\omega_{2,11}^{(3)} - X_{29}^{(4)} & \omega_{2,12}^{(3)} & X_{2,15}^{(3)} & \omega_{2,19}^{(2)} & \omega_{2,20}^{(2)} & \omega_{2,21}^{(2)} \\
\omega_{3,11}^{(3)} - X_{39}^{(4)} & \omega_{3,12}^{(3)} & X_{3,15}^{(3)} & \omega_{3,19}^{(2)} & \omega_{3,20}^{(2)} & \omega_{3,21}^{(2)} \\
\omega_{4,11}^{(3)} & \omega_{4,12}^{(3)} & X_{4,15}^{(3)} & \omega_{4,19}^{(2)} & \omega_{4,20}^{(2)} & \omega_{4,21}^{(2)} \\
X_{5,13}^{(3)} & X_{5,14}^{(3)} & X_{5,15}^{(3)} & \omega_{5,19}^{(2)} & \omega_{5,20}^{(2)} & \omega_{5,21}^{(2)} \\
\omega_{5,11}^{(3)} - X_{49}^{(4)} & \omega_{5,12}^{(3)} & X_{6,15}^{(3)} & \omega_{7,19}^{(2)} & \omega_{7,20}^{(2)} & \omega_{7,21}^{(2)} \\
\omega_{6,11}^{(3)} - X_{59}^{(4)} & \omega_{6,12}^{(3)} & X_{7,15}^{(3)} & \omega_{8,19}^{(2)} & \omega_{8,20}^{(2)} & \omega_{8,21}^{(2)} \\
\omega_{7,11}^{(3)} - X_{69}^{(4)} & \omega_{7,12}^{(3)} & X_{8,15}^{(3)} & \omega_{9,19}^{(2)} & \omega_{9,20}^{(2)} & \omega_{9,21}^{(2)} \\
\omega_{8,11}^{(3)} & \omega_{8,12}^{(3)} & X_{9,15}^{(3)} & \omega_{10,19}^{(2)} & \omega_{10,20}^{(2)} & \omega_{10,21}^{(2)} \\
X_{10,13}^{(3)} & X_{10,14}^{(3)} & X_{10,15}^{(3)} & \omega_{11,19}^{(2)} & \omega_{11,20}^{(2)} & \omega_{11,21}^{(2)} \\
\omega_{9,11}^{(3)} - X_{79}^{(4)} & \omega_{9,12}^{(3)} & X_{11,15}^{(3)} & \omega_{13,19}^{(2)} & \omega_{13,20}^{(2)} & \omega_{13,21}^{(2)} \\
\omega_{10,11}^{(3)} - X_{89}^{(4)} & \omega_{10,12}^{(3)} & X_{12,15}^{(3)} & \omega_{14,19}^{(2)} & \omega_{14,20}^{(2)} & \omega_{14,21}^{(2)} \\
\omega_{11,11}^{(3)} - X_{99}^{(4)} & \omega_{11,12}^{(3)} & X_{13,15}^{(3)} & \omega_{15,19}^{(2)} & \omega_{15,20}^{(2)} & \omega_{15,21}^{(2)}
\end{pmatrix},$$

$$Y_{23}^{(3)} = \begin{pmatrix}
\omega_{12,11}^{(3)} & \omega_{12,12}^{(3)} & X_{14,15}^{(3)} & \omega_{16,19}^{(2)} & \omega_{16,20}^{(2)} & \omega_{16,21}^{(2)} \\
X_{15,13}^{(3)} & X_{15,14}^{(3)} & X_{15,15}^{(3)} & \omega_{17,19}^{(2)} & \omega_{17,20}^{(2)} & \omega_{17,21}^{(2)} \\
\omega_{19,15}^{(2)} & \omega_{19,16}^{(2)} & \omega_{19,17}^{(2)} & \omega_{19,19}^{(2)} & \omega_{19,20}^{(2)} & \omega_{19,21}^{(2)} \\
\omega_{20,15}^{(2)} & \omega_{20,16}^{(2)} & \omega_{20,17}^{(2)} & \omega_{20,19}^{(2)} & \omega_{20,20}^{(2)} & \omega_{20,21}^{(2)} \\
\omega_{21,15}^{(2)} & \omega_{21,16}^{(2)} & \omega_{21,17}^{(2)} & \omega_{21,19}^{(2)} & \omega_{21,20}^{(2)} & \omega_{21,21}^{(2)} \\
\omega_{22,15}^{(2)} & \omega_{22,16}^{(2)} & \omega_{22,17}^{(2)} & \omega_{22,19}^{(2)} & \omega_{22,20}^{(2)} & \omega_{22,21}^{(2)} \\
\omega_{23,15}^{(2)} & \omega_{23,16}^{(2)} & \omega_{23,17}^{(2)} & \omega_{23,19}^{(2)} & \omega_{23,20}^{(2)} & \omega_{23,21}^{(2)} \\
\omega_{17,11}^{(3)} - X_{13,9}^{(4)} & \omega_{17,12}^{(3)} & X_{21,15}^{(3)} & \omega_{17,13}^{(3)} - \omega_{97}^{(4)} + X_{54}^{(5)} & \omega_{17,14}^{(3)} - \omega_{98}^{(4)} & \omega_{17,15}^{(3)} - X_{13,12}^{(4)} \\
\omega_{18,11}^{(3)} - X_{14,9}^{(4)} & \omega_{18,12}^{(3)} & X_{22,15}^{(3)} & \omega_{18,13}^{(3)} - \omega_{10,7}^{(4)} & \omega_{18,14}^{(3)} - \omega_{10,8}^{(4)} & \omega_{18,15}^{(3)} - X_{14,12}^{(4)} \\
\omega_{19,11}^{(3)} - X_{15,9}^{(4)} & \omega_{19,12}^{(3)} & X_{23,15}^{(3)} & \omega_{19,13}^{(3)} - X_{15,10}^{(4)} & \omega_{19,14}^{(3)} - X_{15,11}^{(4)} & \omega_{19,15}^{(3)} - X_{15,12}^{(4)} \\
\omega_{20,11}^{(3)} & \omega_{20,12}^{(3)} & X_{24,15}^{(3)} & \omega_{20,13}^{(3)} & \omega_{20,14}^{(3)} & \omega_{20,15}^{(3)} \\
X_{25,13}^{(3)} & X_{25,14}^{(3)} & X_{25,15}^{(3)} & X_{25,16}^{(3)} & X_{25,17}^{(3)} & X_{25,18}^{(3)}
\end{pmatrix},$$

$$Y_{14}^{(3)} = \begin{pmatrix}
\omega_{1,22}^{(2)} & \omega_{1,23}^{(2)} & \omega_{1,17}^{(3)} - \omega_{19}^{(4)} + X_{15}^{(5)} & \omega_{1,18}^{(3)} - \omega_{1,10}^{(4)} & \omega_{1,19}^{(3)} - X_{1,15}^{(4)} & \omega_{1,20}^{(3)} & X_{1,25}^{(3)} \\
\omega_{2,22}^{(2)} & \omega_{2,23}^{(2)} & \omega_{2,17}^{(3)} - \omega_{29}^{(4)} & \omega_{2,18}^{(3)} - \omega_{2,10}^{(4)} & \omega_{2,19}^{(3)} - X_{2,15}^{(4)} & \omega_{2,20}^{(3)} & X_{2,25}^{(3)} \\
\omega_{3,22}^{(2)} & \omega_{3,23}^{(2)} & \omega_{3,17}^{(3)} - X_{3,13}^{(4)} & \omega_{3,18}^{(3)} - X_{3,14}^{(4)} & \omega_{3,19}^{(3)} - X_{3,15}^{(4)} & \omega_{3,20}^{(3)} & X_{3,25}^{(3)} \\
\omega_{4,22}^{(2)} & \omega_{4,23}^{(2)} & \omega_{4,17}^{(3)} & \omega_{4,18}^{(3)} & \omega_{4,19}^{(3)} & \omega_{4,20}^{(3)} & X_{4,25}^{(3)} \\
\omega_{5,22}^{(2)} & \omega_{5,23}^{(2)} & X_{5,21}^{(3)} & X_{5,22}^{(3)} & X_{5,23}^{(3)} & X_{5,24}^{(3)} & X_{5,25}^{(3)} \\
\omega_{7,22}^{(2)} & \omega_{7,23}^{(2)} & \omega_{5,17}^{(3)} - \omega_{39}^{(4)} + X_{25}^{(5)} & \omega_{5,18}^{(3)} - \omega_{3,10}^{(4)} & \omega_{5,19}^{(3)} - X_{4,15}^{(4)} & \omega_{5,20}^{(3)} & X_{6,25}^{(3)} \\
\omega_{8,22}^{(2)} & \omega_{8,23}^{(2)} & \omega_{6,17}^{(3)} - \omega_{49}^{(4)} & \omega_{6,18}^{(3)} - \omega_{4,10}^{(4)} & \omega_{6,19}^{(3)} - X_{5,15}^{(4)} & \omega_{6,20}^{(3)} & X_{7,25}^{(3)} \\
\omega_{9,22}^{(2)} & \omega_{9,23}^{(2)} & \omega_{7,17}^{(3)} - X_{6,13}^{(4)} & \omega_{7,18}^{(3)} - X_{6,14}^{(4)} & \omega_{7,19}^{(3)} - X_{6,15}^{(4)} & \omega_{7,20}^{(3)} & X_{8,25}^{(3)} \\
\omega_{10,22}^{(2)} & \omega_{10,23}^{(2)} & \omega_{8,17}^{(3)} & \omega_{8,18}^{(3)} & \omega_{8,19}^{(3)} & \omega_{8,20}^{(3)} & X_{9,25}^{(3)} \\
\omega_{11,22}^{(2)} & \omega_{11,23}^{(2)} & X_{10,21}^{(3)} & X_{10,22}^{(3)} & X_{10,23}^{(3)} & X_{10,24}^{(3)} & X_{10,25}^{(3)} \\
\omega_{13,22}^{(2)} & \omega_{13,23}^{(2)} & \omega_{9,17}^{(3)} - \omega_{59}^{(4)} + X_{35}^{(5)} & \omega_{9,18}^{(3)} - \omega_{5,10}^{(4)} & \omega_{9,19}^{(3)} - X_{7,15}^{(4)} & \omega_{9,20}^{(3)} & X_{11,25}^{(3)} \\
\omega_{14,22}^{(2)} & \omega_{14,23}^{(2)} & \omega_{10,17}^{(3)} - \omega_{69}^{(4)} & \omega_{10,18}^{(3)} - \omega_{6,10}^{(4)} & \omega_{10,19}^{(3)} - X_{8,15}^{(4)} & \omega_{10,20}^{(3)} & X_{12,25}^{(3)} \\
\omega_{15,22}^{(2)} & \omega_{15,23}^{(2)} & \omega_{11,17}^{(3)} - X_{9,13}^{(4)} & \omega_{11,18}^{(3)} - X_{9,14}^{(4)} & \omega_{11,19}^{(3)} - X_{9,15}^{(4)} & \omega_{11,20}^{(3)} & X_{13,25}^{(3)}
\end{pmatrix},$$

$$Y_{24}^{(3)} = \begin{pmatrix}
\omega_{16,22}^{(2)} & \omega_{16,23}^{(2)} & \omega_{12,17}^{(3)} & \omega_{12,18}^{(3)} & \omega_{12,19}^{(3)} & \omega_{12,20}^{(3)} & X_{14,25}^{(3)} \\
\omega_{17,22}^{(2)} & \omega_{17,23}^{(2)} & X_{15,21}^{(3)} & X_{15,22}^{(3)} & X_{15,23}^{(3)} & X_{15,24}^{(3)} & X_{15,25}^{(3)} \\
\omega_{19,22}^{(2)} & \omega_{19,23}^{(2)} & \omega_{13,17}^{(3)} - \omega_{79}^{(4)} + X_{45}^{(5)} & \omega_{13,18}^{(3)} - \omega_{7,10}^{(4)} & \omega_{13,19}^{(3)} - X_{10,15}^{(4)} & \omega_{13,20}^{(3)} & X_{16,25}^{(3)} \\
\omega_{20,22}^{(2)} & \omega_{20,23}^{(2)} & \omega_{14,17}^{(3)} - \omega_{89}^{(4)} & \omega_{14,18}^{(3)} - \omega_{8,10}^{(4)} & \omega_{14,19}^{(3)} - X_{11,15}^{(4)} & \omega_{14,20}^{(3)} & X_{17,25}^{(3)} \\
\omega_{21,22}^{(2)} & \omega_{21,23}^{(2)} & \omega_{15,17}^{(3)} - X_{12,13}^{(4)} & \omega_{15,18}^{(3)} - X_{12,14}^{(4)} & \omega_{15,19}^{(3)} - X_{12,15}^{(4)} & \omega_{15,20}^{(3)} & X_{18,25}^{(3)} \\
\omega_{22,22}^{(2)} & \omega_{22,23}^{(2)} & \omega_{16,17}^{(3)} & \omega_{16,18}^{(3)} & \omega_{16,19}^{(3)} & \omega_{16,20}^{(3)} & X_{19,25}^{(3)} \\
\omega_{23,22}^{(2)} & \omega_{23,23}^{(2)} & X_{20,21}^{(3)} & X_{20,22}^{(3)} & X_{20,23}^{(3)} & X_{20,24}^{(3)} & X_{20,25}^{(3)} \\
\omega_{17,16}^{(3)} & X_{21,20}^{(3)} & \omega_{17,17}^{(3)} - \omega_{99}^{(4)} + X_{55}^{(5)} & \omega_{17,18}^{(3)} - \omega_{9,10}^{(4)} & \omega_{17,19}^{(3)} - X_{13,15}^{(4)} & \omega_{17,20}^{(3)} & X_{21,25}^{(3)} \\
\omega_{18,16}^{(3)} & X_{22,20}^{(3)} & \omega_{18,17}^{(3)} - \omega_{10,9}^{(4)} & \omega_{18,18}^{(3)} - \omega_{10,10}^{(4)} & \omega_{18,19}^{(3)} - X_{14,15}^{(4)} & \omega_{18,20}^{(3)} & X_{22,25}^{(3)} \\
\omega_{19,16}^{(3)} & X_{23,20}^{(3)} & \omega_{19,17}^{(3)} - X_{15,13}^{(4)} & \omega_{19,18}^{(3)} - X_{15,14}^{(4)} & \omega_{19,19}^{(3)} - X_{15,15}^{(4)} & \omega_{19,20}^{(3)} & X_{23,25}^{(3)} \\
\omega_{20,16}^{(3)} & X_{24,20}^{(3)} & \omega_{20,17}^{(3)} & \omega_{20,18}^{(3)} & \omega_{20,19}^{(3)} & \omega_{20,20}^{(3)} & X_{24,25}^{(3)} \\
X_{25,19}^{(3)} & X_{25,20}^{(3)} & X_{25,21}^{(3)} & X_{25,22}^{(3)} & X_{25,23}^{(3)} & X_{25,24}^{(3)} & X_{25,25}^{(3)}
\end{pmatrix}.$$

$$\widehat{X_4} = \begin{pmatrix} Y_{11}^{(4)} & Y_{12}^{(4)} & Y_{13}^{(4)} \\ Y_{21}^{(4)} & Y_{22}^{(4)} & Y_{23}^{(4)} \end{pmatrix}, \tag{70}$$

where

$$Y_{11}^{(4)} =$$
$$\begin{pmatrix}
\omega_{11}^{(4)} - X_{11}^{(5)} & \omega_{12}^{(4)} & X_{13}^{(4)} & \omega_{15}^{(3)} - \omega_{17}^{(2)} + \omega_{19}^{(1)} & \omega_{14}^{(4)} & \omega_{17}^{(3)} - \omega_{19}^{(2)} + \omega_{1,11}^{(1)} \\
\omega_{21}^{(4)} & \omega_{22}^{(4)} & X_{23}^{(4)} & \omega_{23}^{(4)} & \omega_{24}^{(4)} & \omega_{27}^{(3)} - \omega_{29}^{(2)} + \omega_{2,11}^{(1)} \\
X_{31}^{(4)} & X_{32}^{(4)} & X_{33}^{(4)} & \omega_{35}^{(3)} - \omega_{37}^{(2)} + \omega_{39}^{(1)} & \omega_{36}^{(3)} - \omega_{38}^{(2)} + \omega_{3,10}^{(1)} & \omega_{37}^{(3)} - \omega_{39}^{(2)} + \omega_{3,11}^{(1)} \\
\omega_{51}^{(3)} - \omega_{71}^{(2)} + \omega_{91}^{(1)} & \omega_{32}^{(4)} & \omega_{53}^{(3)} - \omega_{73}^{(2)} + \omega_{93}^{(1)} & \omega_{55}^{(3)} - \omega_{77}^{(2)} + \omega_{99}^{(1)} & \omega_{34}^{(4)} & \omega_{57}^{(3)} - \omega_{79}^{(2)} + \omega_{9,11}^{(1)} \\
\omega_{41}^{(4)} & \omega_{42}^{(4)} & \omega_{63}^{(3)} - \omega_{83}^{(2)} + \omega_{10,3}^{(1)} & \omega_{43}^{(4)} & \omega_{44}^{(4)} & \omega_{67}^{(3)} - \omega_{89}^{(2)} + \omega_{10,11}^{(1)} \\
\omega_{71}^{(3)} - \omega_{91}^{(2)} + \omega_{11,1}^{(1)} & \omega_{72}^{(3)} - \omega_{92}^{(2)} + \omega_{11,2}^{(1)} & \omega_{73}^{(3)} - \omega_{93}^{(2)} + \omega_{11,3}^{(1)} & \omega_{75}^{(3)} - \omega_{97}^{(2)} + \omega_{11,9}^{(1)} & \omega_{76}^{(3)} - \omega_{98}^{(2)} + \omega_{11,10}^{(1)} & \omega_{77}^{(3)} - \omega_{99}^{(2)} + \omega_{11,11}^{(1)} \\
\omega_{51}^{(4)} - X_{31}^{(5)} & \omega_{52}^{(4)} & X_{73}^{(4)} & \omega_{53}^{(4)} - X_{32}^{(5)} & \omega_{54}^{(4)} & X_{76}^{(4)} \\
\omega_{61}^{(4)} & \omega_{62}^{(4)} & X_{83}^{(4)} & \omega_{63}^{(4)} & \omega_{64}^{(4)} & X_{86}^{(4)} \\
X_{91}^{(4)} & X_{92}^{(4)} & X_{93}^{(4)} & X_{94}^{(4)} & X_{95}^{(4)} & X_{96}^{(4)} \\
\omega_{13,1}^{(3)} - \omega_{19,1}^{(2)} & \omega_{72}^{(4)} & \omega_{13,3}^{(3)} - \omega_{19,3}^{(2)} & \omega_{13,5}^{(3)} - \omega_{19,7}^{(2)} & \omega_{74}^{(4)} & \omega_{13,7}^{(3)} - \omega_{19,9}^{(2)} \\
\omega_{81}^{(4)} & \omega_{82}^{(4)} & \omega_{14,3}^{(3)} - \omega_{20,3}^{(2)} & \omega_{83}^{(4)} & \omega_{84}^{(4)} & \omega_{14,7}^{(3)} - \omega_{20,9}^{(2)}
\end{pmatrix},$$

$$Y_{21}^{(4)} = \begin{pmatrix}
\omega_{15,1}^{(3)} - \omega_{21,1}^{(2)} & \omega_{15,2}^{(3)} - \omega_{21,2}^{(2)} & \omega_{15,3}^{(3)} - \omega_{21,3}^{(2)} & \omega_{15,5}^{(3)} - \omega_{21,7}^{(2)} & \omega_{15,6}^{(3)} - \omega_{21,8}^{(2)} & \omega_{15,7}^{(3)} - \omega_{21,9}^{(2)} \\
\omega_{91}^{(4)} - X_{51}^{(5)} & \omega_{92}^{(4)} & X_{13,3}^{(4)} & \omega_{93}^{(4)} - X_{52}^{(5)} & \omega_{94}^{(4)} & X_{13,6}^{(4)} \\
\omega_{10,1}^{(4)} & \omega_{10,2}^{(4)} & X_{14,3}^{(4)} & \omega_{10,3}^{(4)} & \omega_{10,4}^{(4)} & X_{14,6}^{(4)} \\
X_{15,1}^{(4)} & X_{15,2}^{(4)} & X_{15,3}^{(4)} & X_{15,4}^{(4)} & X_{15,5}^{(4)} & X_{15,6}^{(4)} \\
\omega_{21,1}^{(3)} & \omega_{11,2}^{(4)} & \omega_{21,3}^{(3)} & \omega_{21,5}^{(3)} & \omega_{11,4}^{(4)} & \omega_{21,7}^{(3)} \\
\omega_{12,1}^{(4)} & \omega_{12,2}^{(4)} & \omega_{22,3}^{(3)} & \omega_{12,3}^{(4)} & \omega_{12,4}^{(4)} & \omega_{22,7}^{(3)} \\
\omega_{23,1}^{(3)} & \omega_{23,2}^{(3)} & \omega_{23,3}^{(3)} & \omega_{23,5}^{(3)} & \omega_{23,6}^{(3)} & \omega_{23,7}^{(3)} \\
\omega_{13,1}^{(4)} - X_{71}^{(5)} & \omega_{13,2}^{(4)} & X_{19,3}^{(4)} & \omega_{13,3}^{(4)} - X_{72}^{(5)} & \omega_{13,4}^{(4)} & X_{19,6}^{(4)} \\
\omega_{14,1}^{(4)} & \omega_{14,2}^{(4)} & X_{20,3}^{(4)} & \omega_{14,3}^{(4)} & \omega_{14,4}^{(4)} & X_{20,6}^{(4)} \\
X_{21,1}^{(4)} & X_{21,2}^{(4)} & X_{21,3}^{(4)} & X_{21,4}^{(4)} & X_{21,5}^{(4)} & X_{21,6}^{(4)}
\end{pmatrix},$$

$$Y_{12}^{(4)} = \begin{pmatrix} \omega_{15}^{(4)} - X_{13}^{(5)} & \omega_{16}^{(4)} & X_{19}^{(4)} & \omega_{1,13}^{(3)} - \omega_{1,19}^{(2)} & \omega_{18}^{(4)} & \omega_{1,15}^{(3)} - \omega_{1,21}^{(2)} & \omega_{19}^{(4)} - X_{15}^{(5)} \\ \omega_{25}^{(4)} & \omega_{26}^{(4)} & X_{29}^{(4)} & \omega_{27}^{(4)} & \omega_{28}^{(4)} & \omega_{2,15}^{(3)} - \omega_{2,21}^{(2)} & \omega_{29}^{(4)} \\ X_{37}^{(4)} & X_{38}^{(4)} & X_{39}^{(4)} & \omega_{3,13}^{(3)} - \omega_{3,19}^{(2)} & \omega_{3,14}^{(3)} - \omega_{3,20}^{(2)} & \omega_{3,15}^{(3)} - \omega_{3,21}^{(2)} & X_{3,13}^{(4)} \\ \omega_{35}^{(4)} - X_{23}^{(5)} & \omega_{36}^{(4)} & X_{49}^{(4)} & \omega_{5,13}^{(3)} - \omega_{7,19}^{(2)} & \omega_{38}^{(4)} & \omega_{5,15}^{(3)} - \omega_{7,21}^{(2)} & \omega_{39}^{(4)} - X_{25}^{(5)} \\ \omega_{45}^{(4)} & \omega_{46}^{(4)} & X_{59}^{(4)} & \omega_{47}^{(4)} & \omega_{48}^{(4)} & \omega_{6,15}^{(3)} - \omega_{8,21}^{(2)} & \omega_{49}^{(4)} \\ X_{67}^{(4)} & X_{68}^{(4)} & X_{69}^{(4)} & \omega_{7,13}^{(3)} - \omega_{9,19}^{(2)} & \omega_{7,14}^{(3)} - \omega_{9,20}^{(2)} & \omega_{7,15}^{(3)} - \omega_{9,21}^{(2)} & X_{6,13}^{(4)} \\ \omega_{55}^{(4)} - X_{33}^{(5)} & \omega_{56}^{(4)} & X_{79}^{(4)} & \omega_{9,13}^{(3)} - \omega_{13,19}^{(2)} & \omega_{58}^{(4)} & \omega_{9,15}^{(3)} - \omega_{13,21}^{(2)} & \omega_{59}^{(4)} - X_{35}^{(5)} \\ \omega_{65}^{(4)} & \omega_{66}^{(4)} & X_{89}^{(4)} & \omega_{67}^{(4)} & \omega_{68}^{(4)} & \omega_{10,15}^{(3)} - \omega_{14,21}^{(2)} & \omega_{69}^{(4)} \\ X_{97}^{(4)} & X_{98}^{(4)} & X_{99}^{(4)} & \omega_{11,13}^{(3)} - \omega_{15,19}^{(2)} & \omega_{11,14}^{(3)} - \omega_{15,20}^{(2)} & \omega_{11,15}^{(3)} - \omega_{15,21}^{(2)} & X_{9,13}^{(4)} \\ \omega_{13,9}^{(3)} - \omega_{19,13}^{(2)} & \omega_{76}^{(4)} & \omega_{13,11}^{(3)} - \omega_{19,15}^{(2)} & \omega_{13,13}^{(3)} - \omega_{19,19}^{(2)} & \omega_{78}^{(4)} & \omega_{13,15}^{(3)} - \omega_{19,21}^{(2)} & \omega_{79}^{(4)} - X_{45}^{(5)} \\ \omega_{85}^{(4)} & \omega_{86}^{(4)} & \omega_{14,11}^{(3)} - \omega_{20,15}^{(2)} & \omega_{87}^{(4)} & \omega_{88}^{(4)} & \omega_{14,15}^{(3)} - \omega_{20,21}^{(2)} & \omega_{89}^{(4)} \end{pmatrix},$$

$$Y_{22}^{(4)} = \begin{pmatrix} \omega_{15,9}^{(3)} - \omega_{21,13}^{(2)} & \omega_{15,10}^{(3)} - \omega_{21,14}^{(2)} & \omega_{15,11}^{(3)} - \omega_{21,15}^{(2)} & \omega_{15,13}^{(3)} - \omega_{21,19}^{(2)} & \omega_{15,14}^{(3)} - \omega_{21,20}^{(2)} & \omega_{15,15}^{(3)} - \omega_{21,21}^{(2)} & X_{12,13}^{(4)} \\ \omega_{95}^{(4)} - X_{53}^{(5)} & \omega_{9,6}^{(4)} & X_{13,9}^{(4)} & \omega_{97}^{(4)} - X_{54}^{(5)} & \omega_{98}^{(4)} & X_{13,12}^{(4)} & \omega_{99}^{(4)} - X_{55}^{(5)} \\ \omega_{10,5}^{(4)} & \omega_{10,6}^{(4)} & X_{14,9}^{(4)} & \omega_{10,7}^{(4)} & \omega_{10,8}^{(4)} & X_{14,12}^{(4)} & \omega_{10,9}^{(4)} \\ X_{15,7}^{(4)} & X_{15,8}^{(4)} & X_{15,9}^{(4)} & X_{15,10}^{(4)} & X_{15,11}^{(4)} & X_{15,12}^{(4)} & X_{15,13}^{(4)} \\ \omega_{21,9}^{(3)} & \omega_{11,6}^{(4)} & \omega_{21,11}^{(3)} & \omega_{21,13}^{(3)} & \omega_{11,8}^{(4)} & \omega_{21,15}^{(3)} & \omega_{21,17}^{(3)} \\ \omega_{12,5}^{(4)} & \omega_{12,6}^{(4)} & \omega_{22,11}^{(3)} & \omega_{12,7}^{(4)} & \omega_{12,8}^{(4)} & \omega_{22,15}^{(3)} & \omega_{12,9}^{(4)} \\ \omega_{23,9}^{(3)} & \omega_{23,10}^{(3)} & \omega_{23,11}^{(3)} & \omega_{23,13}^{(3)} & \omega_{23,14}^{(3)} & \omega_{23,15}^{(3)} & \omega_{23,17}^{(3)} \\ \omega_{13,5}^{(4)} - X_{73}^{(5)} & \omega_{13,6}^{(4)} & X_{19,9}^{(4)} & \omega_{13,7}^{(4)} - X_{74}^{(5)} & \omega_{13,8}^{(4)} & X_{19,12}^{(4)} & \omega_{13,9}^{(4)} - X_{75}^{(5)} \\ \omega_{14,5}^{(4)} & \omega_{14,6}^{(4)} & X_{20,9}^{(4)} & \omega_{14,7}^{(4)} & \omega_{14,8}^{(4)} & X_{20,12}^{(4)} & \omega_{14,9}^{(4)} \\ X_{21,7}^{(4)} & X_{21,8}^{(4)} & X_{21,9}^{(4)} & X_{21,10}^{(4)} & X_{21,11}^{(4)} & X_{21,12}^{(4)} & X_{21,13}^{(4)} \end{pmatrix},$$

$$Y_{13}^{(4)} = \begin{pmatrix} \omega_{1,10}^{(4)} & X_{1,15}^{(4)} & \omega_{1,21}^{(3)} & \omega_{1,12}^{(4)} & \omega_{1,23}^{(3)} & \omega_{1,13}^{(4)} - X_{17}^{(5)} & \omega_{1,14}^{(4)} & X_{1,21}^{(4)} \\ \omega_{2,10}^{(4)} & X_{2,15}^{(4)} & \omega_{2,11}^{(4)} & \omega_{2,12}^{(4)} & \omega_{2,23}^{(3)} & \omega_{2,13}^{(4)} & \omega_{2,14}^{(4)} & X_{2,21}^{(4)} \\ X_{3,14}^{(4)} & X_{3,15}^{(4)} & \omega_{3,21}^{(3)} & \omega_{3,22}^{(3)} & \omega_{3,23}^{(3)} & X_{3,19}^{(4)} & X_{3,20}^{(4)} & X_{3,21}^{(4)} \\ \omega_{3,10}^{(4)} & X_{4,15}^{(4)} & \omega_{5,21}^{(3)} & \omega_{3,12}^{(4)} & \omega_{5,23}^{(3)} & \omega_{3,13}^{(4)} - X_{27}^{(5)} & \omega_{3,14}^{(4)} & X_{4,21}^{(4)} \\ \omega_{4,10}^{(4)} & X_{5,15}^{(4)} & \omega_{4,11}^{(4)} & \omega_{4,12}^{(4)} & \omega_{6,23}^{(3)} & \omega_{4,13}^{(4)} & \omega_{4,14}^{(4)} & X_{5,21}^{(4)} \\ X_{6,14}^{(4)} & X_{6,15}^{(4)} & \omega_{7,21}^{(3)} & \omega_{7,22}^{(3)} & \omega_{7,23}^{(3)} & X_{6,19}^{(4)} & X_{6,20}^{(4)} & X_{6,21}^{(4)} \\ \omega_{5,10}^{(4)} & X_{7,15}^{(4)} & \omega_{9,21}^{(3)} & \omega_{5,12}^{(4)} & \omega_{9,23}^{(3)} & \omega_{5,13}^{(4)} - X_{37}^{(5)} & \omega_{5,14}^{(4)} & X_{7,21}^{(4)} \\ \omega_{6,10}^{(4)} & X_{8,15}^{(4)} & \omega_{6,11}^{(4)} & \omega_{6,12}^{(4)} & \omega_{10,23}^{(3)} & \omega_{6,13}^{(4)} & \omega_{6,14}^{(4)} & X_{8,21}^{(4)} \\ X_{9,14}^{(4)} & X_{9,15}^{(4)} & \omega_{11,21}^{(3)} & \omega_{11,22}^{(3)} & \omega_{11,23}^{(3)} & X_{9,19}^{(4)} & X_{9,20}^{(4)} & X_{9,21}^{(4)} \\ \omega_{7,10}^{(4)} & X_{10,15}^{(4)} & \omega_{13,21}^{(3)} & \omega_{7,12}^{(4)} & \omega_{13,23}^{(3)} & \omega_{7,13}^{(4)} - X_{47}^{(5)} & \omega_{7,14}^{(4)} & X_{10,21}^{(4)} \\ \omega_{8,10}^{(4)} & X_{11,15}^{(4)} & \omega_{8,11}^{(4)} & \omega_{8,12}^{(4)} & \omega_{14,23}^{(3)} & \omega_{8,13}^{(4)} & \omega_{8,14}^{(4)} & X_{11,21}^{(4)} \end{pmatrix},$$

$$Y_{23}^{(4)} = \begin{pmatrix} X_{12,14}^{(4)} & X_{12,15}^{(4)} & \omega_{15,21}^{(3)} & \omega_{15,22}^{(3)} & \omega_{15,23}^{(3)} & X_{12,19}^{(4)} & X_{12,20}^{(4)} & X_{12,21}^{(4)} \\ \omega_{9,10}^{(4)} & X_{13,15}^{(4)} & \omega_{17,21}^{(3)} & \omega_{9,12}^{(4)} & \omega_{17,23}^{(3)} & \omega_{9,13}^{(4)} - X_{57}^{(5)} & \omega_{9,14}^{(4)} & X_{13,21}^{(4)} \\ \omega_{10,10}^{(4)} & X_{14,15}^{(4)} & \omega_{10,11}^{(4)} & \omega_{10,12}^{(4)} & \omega_{18,23}^{(3)} & \omega_{10,13}^{(4)} & \omega_{10,14}^{(4)} & X_{14,21}^{(4)} \\ X_{15,14}^{(4)} & X_{15,15}^{(4)} & \omega_{19,21}^{(3)} & \omega_{19,22}^{(3)} & \omega_{19,23}^{(3)} & X_{15,19}^{(4)} & X_{15,20}^{(4)} & X_{15,21}^{(4)} \\ \omega_{11,10}^{(4)} & \omega_{21,19}^{(3)} & \omega_{21,21}^{(3)} & \omega_{11,12}^{(4)} & \omega_{21,23}^{(3)} & \omega_{11,13}^{(4)} - X_{67}^{(5)} & \omega_{11,14}^{(4)} & X_{16,21}^{(4)} \\ \omega_{12,10}^{(4)} & \omega_{22,19}^{(3)} & \omega_{12,11}^{(4)} & \omega_{12,12}^{(4)} & \omega_{22,23}^{(3)} & \omega_{12,13}^{(4)} & \omega_{12,14}^{(4)} & X_{17,21}^{(4)} \\ \omega_{23,18}^{(3)} & \omega_{23,19}^{(3)} & \omega_{23,21}^{(3)} & \omega_{23,22}^{(3)} & \omega_{23,23}^{(3)} & X_{18,19}^{(4)} & X_{18,20}^{(4)} & X_{18,21}^{(4)} \\ \omega_{13,10}^{(4)} & X_{19,15}^{(4)} & \omega_{13,11}^{(4)} - X_{76}^{(5)} & \omega_{13,12}^{(4)} & X_{19,18}^{(4)} & \omega_{13,13}^{(4)} - X_{77}^{(5)} & \omega_{13,14}^{(4)} & X_{19,21}^{(4)} \\ \omega_{14,10}^{(4)} & X_{20,15}^{(4)} & \omega_{14,11}^{(4)} & \omega_{14,12}^{(4)} & X_{20,18}^{(4)} & \omega_{14,13}^{(4)} & \omega_{14,14}^{(4)} & X_{20,21}^{(4)} \\ X_{21,14}^{(4)} & X_{21,15}^{(4)} & X_{21,16}^{(4)} & X_{21,17}^{(4)} & X_{21,18}^{(4)} & X_{21,19}^{(4)} & X_{21,20}^{(4)} & X_{21,21}^{(4)} \end{pmatrix}.$$

$$\widehat{X_5} = (Y_1^{(5)}, Y_2^{(5)}), \tag{71}$$

where

$$Y_1^{(5)} = \begin{pmatrix} X_{11}^{(5)} & \omega_{13}^{(4)} - \omega_{15}^{(3)} + \omega_{17}^{(2)} - \omega_{19}^{(1)} & X_{13}^{(5)} & \omega_{17}^{(4)} - \omega_{1,13}^{(3)} + \omega_{1,19}^{(2)} \\ \omega_{31}^{(4)} - \omega_{51}^{(3)} + \omega_{71}^{(2)} - \omega_{91}^{(1)} & \omega_{33}^{(4)} - \omega_{55}^{(3)} + \omega_{77}^{(2)} - \omega_{99}^{(1)} & X_{23}^{(5)} & \omega_{37}^{(4)} - \omega_{5,13}^{(3)} + \omega_{7,19}^{(2)} \\ X_{31}^{(5)} & X_{32}^{(5)} & X_{33}^{(5)} & \omega_{57}^{(4)} - \omega_{9,13}^{(3)} + \omega_{13,19}^{(2)} \\ \omega_{71}^{(4)} - \omega_{13,1}^{(3)} + \omega_{19,1}^{(2)} & \omega_{73}^{(4)} - \omega_{13,5}^{(3)} + \omega_{19,7}^{(2)} & \omega_{75}^{(4)} - \omega_{13,9}^{(3)} + \omega_{19,13}^{(2)} & \omega_{77}^{(4)} - \omega_{13,13}^{(3)} + \omega_{19,19}^{(2)} \\ X_{51}^{(5)} & X_{52}^{(5)} & X_{53}^{(5)} & X_{54}^{(5)} \\ \omega_{11,1}^{(4)} - \omega_{21,1}^{(3)} & \omega_{11,3}^{(4)} - \omega_{21,5}^{(3)} & \omega_{11,5}^{(4)} - \omega_{21,9}^{(3)} & \omega_{11,7}^{(4)} - \omega_{21,13}^{(3)} \\ X_{71}^{(5)} & X_{72}^{(5)} & X_{73}^{(5)} & X_{74}^{(5)} \\ \omega_{15,1}^{(4)} & \omega_{15,3}^{(4)} & \omega_{15,5}^{(4)} & \omega_{15,7}^{(4)} \\ X_{91}^{(5)} & X_{92}^{(5)} & X_{93}^{(5)} & X_{94}^{(5)} \end{pmatrix},$$

$$Y_2^{(5)} = \begin{pmatrix} X_{15}^{(5)} & \omega_{1,11}^{(4)} - \omega_{1,21}^{(3)} & X_{17}^{(5)} & \omega_{1,15}^{(4)} & X_{19}^{(5)} \\ X_{25}^{(5)} & \omega_{3,11}^{(4)} - \omega_{5,21}^{(3)} & X_{27}^{(5)} & \omega_{3,15}^{(4)} & X_{29}^{(5)} \\ X_{35}^{(5)} & \omega_{5,11}^{(4)} - \omega_{9,21}^{(3)} & X_{37}^{(5)} & \omega_{5,15}^{(4)} & X_{39}^{(5)} \\ X_{45}^{(5)} & \omega_{7,11}^{(4)} - \omega_{13,21}^{(3)} & X_{47}^{(5)} & \omega_{7,15}^{(4)} & X_{49}^{(5)} \\ X_{55}^{(5)} & \omega_{9,11}^{(4)} - \omega_{17,21}^{(3)} & X_{57}^{(5)} & \omega_{9,15}^{(4)} & X_{59}^{(5)} \\ \omega_{11,9}^{(4)} - \omega_{21,17}^{(3)} & \omega_{11,11}^{(4)} - \omega_{21,21}^{(3)} & X_{67}^{(5)} & \omega_{11,15}^{(4)} & X_{69}^{(5)} \\ X_{75}^{(5)} & X_{76}^{(5)} & X_{77}^{(5)} & \omega_{13,15}^{(4)} & X_{79}^{(5)} \\ \omega_{15,9}^{(4)} & \omega_{15,11}^{(4)} & \omega_{15,13}^{(4)} & \omega_{15,15}^{(4)} & X_{89}^{(5)} \\ X_{95}^{(5)} & X_{96}^{(5)} & X_{97}^{(5)} & X_{98}^{(5)} & X_{99}^{(5)} \end{pmatrix}.$$

Algorithm 1 An algorithm to Find the General Solution to System (1)

1. Input $A_i \in \mathbb{H}^{p_i \times q_i}$, $B_i \in \mathbb{H}^{p_i \times q_{i+1}}$, $C_i \in \mathbb{H}^{t_i \times s_i}$, $D_i \in \mathbb{H}^{t_{i+1} \times s_i}$, $\Omega_i \in \mathbb{H}^{p_i \times s_i}$, $i = 1,2,3,4$.
2. Compute the decompositions of (3) and (4) and derive the invertible quaternion matrices P_i, S_i, Q_j, T_j, $i = 1,2,3,4$ and $j = 1,2,3,4,5$.
3. Compute $\widehat{X_j}$ and $\widehat{\Omega_i}$ by $\widehat{X_j} = Q_j^{-1} X_j T_j^{-1}$ and $\widehat{\Omega_i} = P_i \Omega_i S_i$, $j = 1,2,3,4,5$ and $i = 1,2,3,4$.
4. Partition $\widehat{X_j}$, $j = 1,2,3,4,5$ and $\widehat{\Omega_i}$, $i = 1,2,3,4$.
5. Check whether (12)–(39) or (40)–(66) hold or not. If one of them holds, then proceed to the following steps.
6. Compute $\widehat{X_j}$ by (67)–(71), $j = 1,2,3,4,5$.
7. Compute X_j by $X_j = Q_j \widehat{X_j} T_j$, $j = 1,2,3,4,5$.

4. A Numerical Example of System (1)

In this section, we show a numerical example of system (1).

Example 1. *Let*

$$A_1 = \begin{pmatrix} 1 & -\mathbf{i} & \mathbf{k} \\ 2 & -2\mathbf{i} & 2\mathbf{k} \\ \mathbf{j} & -\mathbf{k} & -\mathbf{i} \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & -\mathbf{j} & 2+\mathbf{k} \\ \mathbf{j} & 0 & 3+\mathbf{i}-\mathbf{k} \\ 2-\mathbf{k} & 3-\mathbf{i}+\mathbf{k} & 3 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & \mathbf{j} & 2+2\mathbf{j} \\ -\mathbf{j} & 0 & 3+\mathbf{i} \\ 2-\mathbf{j} & 3-\mathbf{i} & 3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -1 & \mathbf{j} & \mathbf{i}-\mathbf{k} \\ \mathbf{k} & 2-\mathbf{k} & -\mathbf{i} \\ \mathbf{j} & \mathbf{j}+\mathbf{k} & -2+\mathbf{i} \end{pmatrix}, B_1 = \begin{pmatrix} 2 & -\mathbf{i} & \mathbf{j} \\ 3 & \mathbf{k} & 1 \\ -2\mathbf{i} & 0 & \mathbf{k} \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & \mathbf{i} & \mathbf{j} \\ \mathbf{j} & 1 & 2\mathbf{k} \\ -\mathbf{i} & 3 & -1 \end{pmatrix}\ B_3 = \begin{pmatrix} \mathbf{j} & \mathbf{k} & \mathbf{i} \\ \mathbf{k} & \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 1 & 2 & -1 \\ \mathbf{i} & 2\mathbf{i} & -\mathbf{i} \\ 1+\mathbf{i} & 2+2\mathbf{i} & -1-\mathbf{i} \end{pmatrix},\ C_1 = \begin{pmatrix} 1 & 2 & \mathbf{j} \\ \mathbf{i} & 2\mathbf{i} & -\mathbf{k} \\ \mathbf{k} & 2\mathbf{k} & \mathbf{i} \end{pmatrix},\ C_2 = \begin{pmatrix} 1 & \mathbf{j} & 2-\mathbf{k} \\ -\mathbf{j} & 0 & 3+\mathbf{k} \\ 2+\mathbf{k} & \mathbf{i}-\mathbf{k} & -3 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 1 & \mathbf{j} & -1 \\ \mathbf{j} & 0 & \mathbf{j} \\ 2+2\mathbf{j} & 2\mathbf{j} & -2+2\mathbf{j} \end{pmatrix},\ C_4 = \begin{pmatrix} -1 & \mathbf{k} & \mathbf{j} \\ \mathbf{j} & 2-\mathbf{k} & \mathbf{j}+\mathbf{k} \\ -\mathbf{i}-\mathbf{k} & \mathbf{i} & -2-\mathbf{i} \end{pmatrix}, D_1 = \begin{pmatrix} 2 & 2\mathbf{i} & -2\mathbf{j} \\ -\mathbf{i} & -1 & \mathbf{k} \\ \mathbf{j} & \mathbf{k} & 1 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 0 & \mathbf{j} & \mathbf{i} \\ -\mathbf{i} & \mathbf{j}-\mathbf{k} & -1-\mathbf{i}+\mathbf{k} \\ 1+\mathbf{i}-\mathbf{k} & 3\mathbf{k} & \mathbf{k} \end{pmatrix}, D_3 = \begin{pmatrix} -\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{j} & 2+\mathbf{k} & 4-\mathbf{j} \\ \mathbf{k} & \mathbf{i}+2\mathbf{j} & \mathbf{j} \end{pmatrix},\ D_4 = \begin{pmatrix} 1 & -\mathbf{i} & \mathbf{k} \\ 2 & 3-2\mathbf{i} & 2\mathbf{k} \\ -1 & \mathbf{i} & -\mathbf{k} \end{pmatrix},$$

$$\Omega_1 = \begin{pmatrix} -\mathbf{i}+7\mathbf{j}-\mathbf{k} & -12-3\mathbf{k} & 5-1\mathbf{i}+3\mathbf{k} \\ -7+6\mathbf{i}+14\mathbf{j}+4\mathbf{k} & -24-9\mathbf{i}+6\mathbf{j}-2\mathbf{k} & 10+4\mathbf{i}+7\mathbf{j}-2\mathbf{k} \\ -5\mathbf{i}+16\mathbf{j}-6\mathbf{k} & -8-2\mathbf{i}+13\mathbf{j} & 12-8\mathbf{i}-2\mathbf{j}+5\mathbf{k} \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} 9+3\mathbf{i}+13\mathbf{j}+7\mathbf{k} & -7-\mathbf{i}+7\mathbf{j} & -18+13\mathbf{i}-9\mathbf{j}-15\mathbf{k} \\ 12+8\mathbf{i}+13\mathbf{j}+3\mathbf{k} & -2+3\mathbf{i}+8\mathbf{j}+3\mathbf{k} & -35+11\mathbf{i}-11\mathbf{j}-4\mathbf{k} \\ 10-4\mathbf{i}-2\mathbf{j}-17\mathbf{k} & -14+6\mathbf{i}+18\mathbf{j}+\mathbf{k} & 3+21\mathbf{i}+10\mathbf{j}+14\mathbf{k} \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 5+3\mathbf{i}+14\mathbf{j}+7\mathbf{k} & -20+4\mathbf{j}-7\mathbf{k} & -17+11\mathbf{i}-10\mathbf{j}-9\mathbf{k} \\ 12+10\mathbf{i}-3\mathbf{j}-2\mathbf{k} & -5-10\mathbf{i}+6\mathbf{j}-3\mathbf{k} & -12-6\mathbf{i}-7\mathbf{j}-2\mathbf{k} \\ 27-10\mathbf{i}+24\mathbf{j}-6\mathbf{k} & -5+16\mathbf{j}-9\mathbf{k} & -23+18\mathbf{i} \end{pmatrix},$$

$$\Omega_4 = \begin{pmatrix} 10+\mathbf{i}-\mathbf{j}+2\mathbf{k} & 18+2\mathbf{i}-9\mathbf{j}-10\mathbf{k} & 5-3\mathbf{j}+13\mathbf{k} \\ 1+3\mathbf{i}-\mathbf{j}-4\mathbf{k} & -6+9\mathbf{i}-4\mathbf{j}+9\mathbf{k} & 5-7\mathbf{i}-7\mathbf{j}-4\mathbf{k} \\ 7+8\mathbf{i}-\mathbf{j}-\mathbf{k} & 12+15\mathbf{i}-\mathbf{j}-15\mathbf{k} & 5-14\mathbf{j}+11\mathbf{k} \end{pmatrix}.$$

Upon examination, (12)–(39) hold. Then, system (1) has a general solution. Note that

$$X_1 = \begin{pmatrix} 1+\mathbf{j} & -2-\mathbf{i}+\mathbf{j} & \mathbf{i}+2\mathbf{k} \\ -2+\mathbf{i} & -1+2\mathbf{k} & \mathbf{i}-\mathbf{j} \\ \mathbf{j}+2\mathbf{k} & -\mathbf{i}-\mathbf{j} & 2 \end{pmatrix},\ X_2 = \begin{pmatrix} \mathbf{j}+\mathbf{k} & \mathbf{i} & 2-\mathbf{j} \\ \mathbf{i} & 1-\mathbf{k} & -3-\mathbf{k} \\ 2\mathbf{i}-\mathbf{j} & -2-\mathbf{k} & 1+\mathbf{j} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & 2+\mathbf{i} & 3+\mathbf{k} \\ 2-\mathbf{i} & 1 & -\mathbf{i}+\mathbf{j} \\ 3+\mathbf{k} & \mathbf{i}-\mathbf{j} & \mathbf{j}-\mathbf{k} \end{pmatrix},\ X_4 = \begin{pmatrix} 2 & -\mathbf{i}+\mathbf{j} & 1+\mathbf{i} \\ \mathbf{i}+\mathbf{j} & -2-\mathbf{j} & \mathbf{i}-\mathbf{k} \\ 1+\mathbf{j} & \mathbf{k} & 0 \end{pmatrix},\ X_5 = \begin{pmatrix} -2-\mathbf{j} & 2+\mathbf{i} & \mathbf{j}-\mathbf{k} \\ 2-\mathbf{i} & 1-\mathbf{k} & 1-\mathbf{i} \\ \mathbf{k} & 1-\mathbf{k} & \mathbf{k} \end{pmatrix},$$

is a solution that satisfies system (1).

This experiment is conducted using MATLAB R2023B (MathWorks, Natick, MA, USA) running on a computer with the Windows 10 operating system.

5. Application of System (1) in Color Image Encryption and Decryption

In section, we make use of system (1) to develop a model which can be used to simultaneously encrypt four color images; this idea is similar to the idea put forward in [19].

The model simultaneously encrypting four color images is shown in Figure 1.

Let X_1, X_2, X_3 and X_4 stand for the four encrypted color images, and X_5 stand for a key used for encryption and decryption. It should be noted that X_5 can be a color image and can also be a general quaternion matrix with a proper size). The cipher consists of the invertible quaternion matrices A_i, B_i, C_i and D_i, where $i = 1, 2, 3, 4$.

Then, we explain the encryption process. First, we randomly select an invertible A_i, B_i, C_i, $D_i (i = 1, 2, 3, 4)$ with a proper size, and select a key X_5. Then, we perform numerical calculations on $X_i (i = 1, 2, 3, 4, 5)$ and A_i, B_i, C_i, $D_i (i = 1, 2, 3, 4)$ to obtain $\Omega_i (i = 1, 2, 3, 4)$ according to system (1). In this way, we can obtain the encrypted quaternion numerical matrices $\Omega_i (i = 1, 2, 3, 4)$. Based on this encryption process and the general solution to system (1), shown in Section 3. It is very difficult to correctly find the original color images when the keys are not disclosed. Hence, the encryption model is effective and secure, since there are an infinite number of choices of the free terms in the general solution.

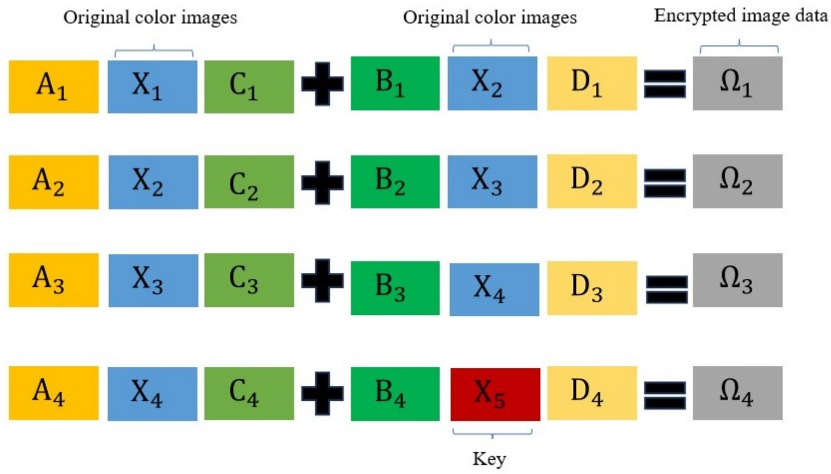

Figure 1. Simultaneous encryption of four color images based on system (1).

In terms of the decryption process, we have utilized a picture to illustrate it. The model simultaneously decrypting four color images is shown in Figure 2.

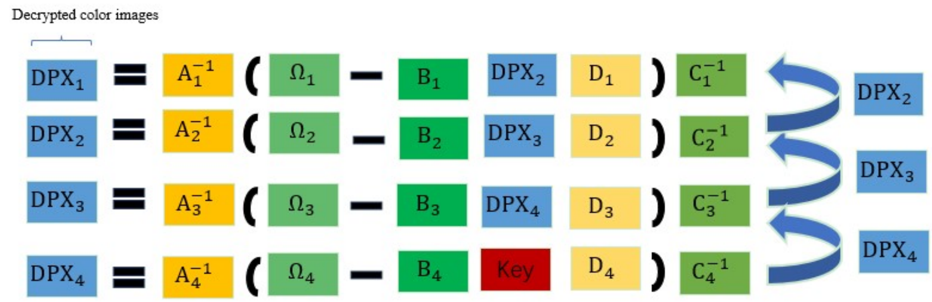

Figure 2. Simultaneous decryption of four color images based on system (1).

In the decryption process, $DPX_i(i = 1, 2, 3, 4)$ represent the decrypted color images. Once the "Key" is given, we can reconstruct the original color images, starting with DPX_4 and going through to DPX_1 based on the process depicted in Figure 2.

Next, we give a numerical example. First, we select four color images to be encrypted and a key from the set of the sample pictures of MATLAB R2023B. The four images are "Indiancorn", "Llama", "Sevilla" and "Strawberries", with the key "Yellowlily". These original color images are shown in Figure 3.

Figure 3. The required encrypted original four color images and key.

Then we carry out the encryption process. Figure 4 shows that the encryption process makes the original image unrecognizable.

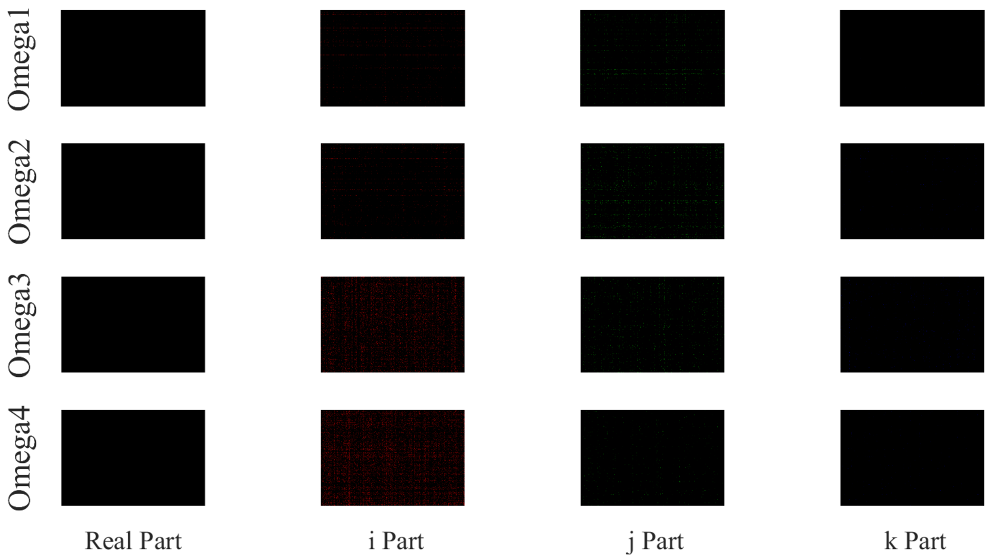

Figure 4. Images display of encrypted data.

Next, we carry out the decryption process. The decrypted color images are shown in Figure 5.

It is easy to see from Figure 5 that the decrypted images are consistent with the original image in Figure 3. We use the Structural Similarity Index (SSIM) as an indicator to measure the decryption effect. Upon computation, the SSIM between the original images and the decrypted images is 1. This shows that the decryption process is effective and the decrypted images are of excellent quality.

Finally, it should be pointed out that encryption process based on the system of two-sided coupled generalized Sylvester quaternion matrix equations makes the decryption process more difficult without a key, and makes the decrypted color images have a stronger similarity with the original images.

Figure 5. Decrypted color image after algorithm restoration.

This experiment is conducted using MATLAB R2023B (MathWorks, Natick, MA, USA) running on a computer with the Windows 10 operating system.

He et al. [19] made use of a system of Sylvester-type quaternion matrix equations

$$\begin{cases} X_1A_1 - B_1X_2 = C_1, \\ X_3A_2 - B_2X_2 = C_2, \\ X_3A_3 - B_3X_4 = C_3, \\ X_4A_4 - B_4X_5 = C_4, \\ X_6A_5 - B_5X_5 = C_5, \end{cases} \tag{72}$$

to develop a frame to encrypt five color images simultaneously, where A_i, B_i, and C_i are given quaternion matrices. The advantages of the encrypted frame developed by system (1) is more complex and safe than system (72). Beyond that, the decrypted images of the frame developed by system (1) is much more similar to the original images than the those of system (72), indicated by comparing the PSNR. The PSNR of the result of our frame is over 200, and is much larger than the result of the frame referred in [19]. The disadvantage of our frame is that it deals with four images simultaneously, which is less than the frame referred in [19]. If one wants to deal with more images, the simultaneous decomposition for more quaternion matrices should be considered.

6. Conclusions

In this paper, we study the solvability conditions and general solutions to a system of two-sided coupled Sylvester-type quaternion matrix equations

$$A_iX_iC_i + B_iX_{i+1}D_i = \Omega_i, \ i = 1, 2, 3, 4,$$

in terms of the partition of quaternion matrices. Meanwhile, we show the equivalent relationship between the rank conditions of the coefficient matrices and the partitioned matrix conditions of the quaternion matrices. We also develop an algorithm to compute the general solution to the system. In addition, we give a numerical example. We also make use of system (1) to build a model that can be used to simultaneously encrypt and decrypt four color images.

We have shown that the simultaneous decomposition of multiple quaternion matrices play an important role in data storage and transmission. Our future work will include extending the simultaneous decomposition of eight quaternion matrices to the simultaneous decomposition of nine quaternion matrices or more, using the simultaneous decomposition of eight quaternion matrices to study the variations of system (1) to adapt to some specific physical systems.

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References

1. Abiad, A.; Brimkov, B.; Hayat, S.; Khramova, A.P.; Koolen, J.H. Extending a conjecture of Graham and Lovász on the distance characteristic polynomial. *Linear Algebra Appl.* **2024**, *693*, 63–82. [CrossRef]
2. Chen, Y.; Jia, Z.G.; Peng, Y.; Peng, Y.X.; Zhang, D. A new structure-preserving quaternion QR decomposition method for color image blind watermarking. *Signal Process.* **2021**, *185*, 108088. [CrossRef]
3. Dmytryshyn, A.; Kågström, B. Coupled Sylvester-type matrix equations and block diagonalization. *SIAM J Matrix Anal Appl.* **2015**, *36*, 580–593. [CrossRef]
4. He, X.; Feng, L.; Stevanović, D. The maximum spectral radius of graphs with a large core. *Electron. J. Linear Algebra* **2023**, *39*, 78–89. [CrossRef]
5. He, Z.H.; Qin, W.L.; Wang, X.X. Some applications of a decomposition for five quaternion matrices in control system and color image processing. *Comput. Appl. Math.* **2021**, *40*, 205. [CrossRef]
6. Jia, Z.G. *The Eigenvalue Problem of Quaternion Matrix: Structure-Preserving Algorithms and Applications*; Science Press: Beijing, China, 2019.
7. Jia, Z.; Ng, M.K.; Song, G.J. Robust quaternion matrix completion with applications to image inpainting. *Numer. Linear Algebra Appl.* **2019**, *26*, e2245. [CrossRef]
8. Bihan, N.L.; Sangwine, S.J. Quaternion principal component analysis of color images. *ICIP* **2003**, *1*, I-809.
9. Bihan, N.L.; Mars, J. Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing. *Signal Process.* **2004** *84*, 1177–1199. [CrossRef]
10. Li, J.; Yu, C.; Gupta, B.B.; Ren, X. Color image watermarking scheme based on quaternion Hadamard transform and Schur decomposition. *Multimed. Tools Appl.* **2018**, *77*, 4545–4561. [CrossRef]
11. Li, T.; Wang, Q.W. Structure Preserving Quaternion Biconjugate Gradient Method. *SIAM J. Matrix Anal. Appl.* **2024**, *45*, 306–326. [CrossRef]
12. Miao, J.; Kou, K.I.; Cheng, D.; Liu, W. Quaternion higher-order singular value decomposition and its applications in color image processing. *Inform. Fusion* **2023**, *92*, 139–153. [CrossRef]
13. Morais, J.P.; Georgiev, S.; Sprößig, W. *Real Quaternionic Calculus Handbook*; Springer: Berlin/Heidelberg, Germany, 2014.
14. Pan, S.; Pati, S.; Kirkland, S. Dominant eigenvalue and universal winners of digraphs. *Linear Algebra Appl.* **2024**, *695*, 79–106. [CrossRef]
15. Rather, B.A.; Ganie, H.A.; Das, K.C.; Shang, Y. The general extended adjacency eigenvalues of chain graphs. *Mathematics* **2024**, *12*, 192. [CrossRef]
16. Rodman, L. *Topics in Quaternion Linear Algebra*; Princeton University Press: Princeton, NJ, USA, 2014.
17. Took, C.C.; Mandic, D.P. The quaternion LMS algorithm for adaptive filtering of hypercomplex processes. *IEEE Trans. Signal Process.* **2008**, *57*, 1316–1327. [CrossRef]
18. Zhang, F.Z. Quaternions and matrices of quaternions. *Linear Algebra Appl.* **1997**, *251*, 21–57. [CrossRef]
19. He, Z.H.; Qin, W.L.; Tian, J.; Wang, X.X.; Zhang, Y. A new Sylvester-type quaternion matrix equation model for color image data transmission. *Comput. Appl. Math.* **2024**, *43*, 227. [CrossRef]
20. Chen, Y.; Wang, Q.W.; Xie, L.M. Dual Quaternion Matrix Equation $AXB = C$ with Applications. *Symmetry* **2024**, *16*, 287. [CrossRef]
21. He, Z.H.; Zhang, X.N.; Zhao, Y.F.; Yu, S.W. The Solvability of a System of Quaternion Matrix Equations Involving ϕ-Skew-Hermicity. *Symmetry* **2022**, *14*, 1273. [CrossRef]
22. Kyrchei, I. Determinantal representations of the Moore–Penrose inverse over the quaternion skew field and corresponding Cramer's rules. *Linear Multilinear Algebra* **2011**, *59*, 413–431. [CrossRef]
23. Kyrchei, I. Determinantal representations of solutions to systems of quaternion matrix equations. *Adv. Appl. Clifford Algebr.* **2018**, *28*, 1–16. [CrossRef]
24. Si, K.W.; Wang, Q.W. The General Solution to a Classical Matrix Equation $AXB = C$ over the Dual Split Quaternion Algebra. *Symmetry* **2024**, *16*, 491. [CrossRef]
25. Wimmer, H.K. Consistency of a pair of generalized Sylvester equations. *IEEE Trans. Automat. Control* **1994**, *39*, 1014–1016. [CrossRef]
26. Wang, Q.W.; Li, C.K. Ranks and the least-norm of the general solution to a system of quaternion matrix equations. *Linear Algebra Appl.* **2009**, *430*, 1626–1640. [CrossRef]
27. Wang, Q.W.; der Woude, J.W.V.; Chang, H.X. A system of real quaternion matrix equations with applications. *Linear Algebra Appl.* **2009**, *431*, 2291–2303. [CrossRef]
28. Wang, Q.W.; Chang, H.X.; Lin, C.Y. P-(skew) symmetric common solutions to a pair of quaternion matrix equations. *Appl. Math. Comput.* **2008**, *195*, 721–732. [CrossRef]
29. Xu, Y.F.; Wang, Q.W.; Liu, L.S.; Mehany, M.S. A constrained system of matrix equations. *Comput. Appl. Math.* **2022**, *41*, 166. [CrossRef]
30. Yuan, S.F.; Wang, Q.W.; Zhang, X. Least-squares problem for the quaternion matrix equation AXB + CYD = E over different constrained matrices. *Int. J. Comput. Math.* **2013**, *90*, 565–576. [CrossRef]
31. Zhang, C.Q.; Wang, Q.W.; Dmytryshyn, A.; He, Z.H. Investigation of some Sylvester-type quaternion matrix equations with multiple unknowns. *Comput. Appl. Math.* **2024**, *43*, 181. [CrossRef]

32. Zhang, Y.; Wang, Q.W.; Xie, L.M. The Hermitian solution to a new system of commutative quaternion matrix equations. *Symmetry* **2024**, *16*, 361. [CrossRef]
33. Xie, M.Y.; Wang, Q.W.; He, Z.H.; Saad, M.M. A system of Sylvester-type quaternion matrix equations with ten variables. *Acta Math. Sin.* **2022**, *38*, 1399–1420. [CrossRef]