mathematics

*Article*

# A Discrete Hamilton–Jacobi Theory for Contact Hamiltonian Dynamics

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**Abstract:** In this paper, we propose a discrete Hamilton–Jacobi theory for (discrete) Hamiltonian dynamics defined on a (discrete) contact manifold. To this end, we first provide a novel geometric Hamilton–Jacobi theory for continuous contact Hamiltonian dynamics. Then, rooting on the discrete contact Lagrangian formulation, we obtain the discrete equations for Hamiltonian dynamics by the discrete Legendre transformation. Based on the discrete contact Hamilton equation, we construct a discrete Hamilton–Jacobi equation for contact Hamiltonian dynamics. We show how the discrete Hamilton–Jacobi equation is related to the continuous Hamilton–Jacobi theory presented in this work. Then, we propose geometric foundations of the discrete Hamilton–Jacobi equations on contact manifolds in terms of discrete contact flows. At the end of the paper, we provide a numerical example to test the theory.

**Keywords:** Hamilton–Jacobi theory; discrete dynamics; contact manifolds; discrete Hamilton–Jacobi

**MSC:** 65P10; 37J55; 70H20



**Citation:** Esen, O.; Sardón, C.; Zajac, M. A Discrete Hamilton–Jacobi Theory for Contact Hamiltonian Dynamics. *Mathematics* **2024**, *12*, 2342. https://doi.org/10.3390/math12152342

Academic Editor: Savin Treanta

Received: 3 June 2024
Revised: 5 July 2024
Accepted: 9 July 2024
Published: 26 July 2024



## 1. Introduction

This work lies at the intersection among the geometric Hamilton–Jacobi (abbreviated as HJ) theory, discrete dynamics and contact geometry. An HJ theory for discrete Hamiltonian dynamics on contact manifolds was missing in the literature, which is why our aim in this work is to fill this gap by introducing a discrete HJ equation for discrete contact Hamiltonian dynamics. Additionally, we shall examine the geometric foundations of the discrete HJ equation. Accordingly, we shall propose a geometric discrete HJ theorem in the contact framework.

The Hamilton–Jacobi equation was first given for classical and continuous Hamiltonian dynamics on symplectic manifolds [1,2]. More recently, a geometrization of the HJ equation was established in [3] and later in [4]. Let us stress that by geometrization, we mean the process of expressing the equation in a coordinate-independent form, which entails representing the equation by associating it with specific objects and relationships on the appropriate manifolds.

Long story short, the Hamilton–Jacobi equation (HJ equation) is a formulation of classical mechanics equivalent to other formulations, such as Newton's equations, or Lagrangian or Hamiltonian mechanics. Our particular choice of the HJ theory to solve the dynamics is rooted in its particular use, for the identification of conserved quantities, of a mechanical system [2], as well as symmetries. The HJ theory is mediated by a generating function, which is key to proposing canonical transformations and a connection to quantum mechanics. Another interesting aspect of the HJ formulation is that generating functions provide transformations which allow us to inspect the integrability properties of the dynamics by using action angle coordinates. Let us take the triple $(T^*Q, \omega, H)$, where $T^*Q$ is our $2n$-dimensional phase space, $\omega$ is a non-degenerate $(0,2)$-skew-symmetric tensor (the canonical symplectic form) and $h$ plays the role of a Hamilton function on $M$.

The standard formulation of the Hamilton–Jacobi theory (HJ theory) consists on finding a function $S(t, q^i)$, known as the generating function, such that

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0, \tag{1}$$

where $H = H(q^i, p_i)$ is the Hamiltonian function of the system. If the function $S$ is separable in space and time,

$$S = W(q^1, \ldots, q^n) - Et \tag{2}$$

where $E$ is the total energy of the system; Equation (1) can be rewritten as [2,5]

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E. \tag{3}$$

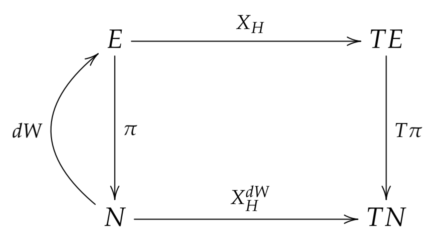
Equations (1) and (3) are the so-called $t$-dependent and time-independent Hamilton–Jacobi equations, respectively.

The Hamilton–Jacobi equation is a useful instrument to solve the dynamics of the system. Indeed, solving

$$\begin{cases} \dot{q}^i = \dfrac{\partial h}{\partial p_i}, \\ \dot{p}_j = -\dfrac{\partial h}{\partial q^j}. \end{cases} \tag{4}$$

is equivalent to solving (3). Indeed, if we find a solution $W$ of (3), then any solution of (4) gives a solution to the Hamilton equations by taking $p_i = \partial W / \partial q^i$.

If we aim at formulating the HJ equation from a geometric point of view, the primordial observation is that if a Hamiltonian vector field $X_H$ can be projected into the configuration manifold by means of a one-form $dW$, then the integral curves of the projected vector field $X_H^{dW}$can be transformed into integral curves of $X_H$ provided that $W$ is a solution of (3) [4,6–8].




Depending on the choice of $E$, we will be constructing different HJ theories on different types of manifolds. For instance, if $E = T^*Q$, we retrieve the classical HJ equation on a symplectic manifold (3). The diagram above can be reformulated on a contact manifold, a cosymplectic manifold, a discrete symplectic manifold, etc. [9–18]. We refer to two recent surveys [19,20] (and references therein) for a contemporary view of the current geometric HJ theory.

In this paper, we will develop an HJ theory for discrete contact mechanics. In the discrete realm, the formulation could be constructed with the aid of a commutative diagram like the one above or by finding a generating function that renders the dynamics trivial. Since the definition of discrete vector fields is not straightforward, not to say very little studied, we will make use of the second approach.

Concerning the discrete formulation of mechanics, in recent years, there has been a growing trend in providing proper discrete analogs of continuous differential equations and designing numerical methods adapted to the type of discretization pursued, as well as the type of differential equations that are dealt with. Numerical methods have shown their utility in solving equations that cannot be managed analytically [21,22].

Discretizing a Hamilton–Jacobi formulation is essential for practical computations, especially in complex systems where analytical solutions are not feasible. It enables numerical simulations, optimization and control; preserves geometric properties; and facilitates the study of both classical and quantum systems. Through various discretization techniques, the Hamilton–Jacobi formulation can be adapted to computational methods, making it a powerful tool in scientific and engineering applications. For example, many modern control systems, such as those in robotics, aerospace and automotive applications, use digital controllers. These controllers operate on discrete-time signals, making the discretization of continuous-time models necessary.

In the branch of geometric mechanics, we deal with a plethora of geometric structures that provide different underlying geometric properties to dynamical systems. This is why when one discretizes a dynamical system, one has to make sure that the discretization is compatible with the geometric structure and that we are applying specific methods that preserve the geometric structure. These specific methods are known as geometric integrators [23,24]. For example, in classical mechanics, there exist numerical methods that preserve the symplectic structure when one works on a phase space [25,26], and other methods are energy-preserving numerical methods [27], momentum-preserving methods [28], etc. Here, we will use very simple numerical integration techniques, as it is the explicit Euler method. The stiffness of our HJ equations has widely varying timescales than can lead to numerical instability if the step size is not chosen appropriately, which is why we carefully chose step sizes in our numerical examples.

Contact geometry is a popular theme in the recent literature, being widely used to describe mechanical dissipative systems, dissipative field theories and generalizations of the Hamilton principle [29–35]. Some of the main uses of contact geometry and its main characteristics can be consulted in [36]. The dissipative character of the formalism provides an important geometric foundation for irreversible dynamics, especially thermodynamics. Here is an incomplete list of works related to contact mechanics and its role in thermodynamics [37–42].

To propose a discrete Hamilton–Jacobi theory on a contact manifold as we shall, one needs to first review the discrete formulation of mechanics on the Lagrangian side [43]. This leads to the discretization of Lagrangian and Hamiltonian systems, as well as the variational principles for dynamical systems and principles of critical action on both the tangent and cotangent bundle [44,45]. Such discretizations have led to discretized versions of Noether's theorem, Legendre transformations, infinitesimal symmetries, etc. The discretization of the Hamiltonian formulation has given rise to optimal control problems by developing a discrete maximum principle that yields discrete necessary conditions for optimality. Furthermore, discrete Hamiltonian theories have been particularly useful in distributed network optimization and derivation of variational integrators [46]. The geometry of the space is also key to performing better discretizations. For this matter, it is important to rely on symmetries and invariants of the geometric space [47]. In this work, we preserve the contact structure under discretization. Some very recent works addressing discrete Lagrangian and Hamiltonian dynamics on contact settings are [48–52]. In these works, one can see the discrete generalized Lagrangian (Herglotz) dynamics on the extended tangent bundle as well as the discrete Hamiltonian dynamics on a contact manifold. Since these works are fundamental for the present study, we shall give a quick review of the theories in the main body of the paper.

We refer to [53] for the discrete HJ equation on a symplectic manifold. In [53], the HJ equation is derived by employing discrete symplectic flows. In the present work, in similar fashion, we extend this discussion to contact geometry. The role of the discrete symplectic flows will be played by discrete contact flows analogously.

More recently, in [11], the geometrization of the discrete HJ equation in [53] has been established in the symplectic category. In this regard, it is possible to consider the present work as a continuation or an extension of the works [11,53] to contact geometry and discrete contact Hamiltonian dynamics. In the present paper, we both derive the discrete HJ equation on contact manifolds and then proceed with its geometrization.

The main body of this work contains three sections. In the upcoming section, Section 2, we shall first review the fundamental principles of contact manifolds as well as Lagrangian and Hamiltonian dynamics on contact manifolds, in order to fix the notation. In Section 2.4, we shall provide an HJ theorem for contact Hamiltonian dynamics. We shall start Section 3 by presenting discrete Lagrangian and Hamiltonian contact dynamics. In Section 3.3, we shall present our main result, which is a Hamilton–Jacobi equation for discrete contact Hamiltonian dynamics. We shall prove this result in terms of discrete contact flows. In Section 3.4, a geometrization of the HJ is provided. In Section 4, we conclude this work by proposing a numerical example applied to the well-known parachute equation in contact dynamics. To avoid mathematical conflict and without loss of generalization, we assume all objects to be smooth and globally defined unless stated otherwise. Manifolds are connected and differentiable.

## 2. Fundamentals of Continuous Contact Dynamics

In this section, we first recall briefly the main definitions and results of the theory of Lagrangian and Hamiltonian dynamics on contact manifolds following [29,31,32]. Later, we introduce a geometric Hamilton–Jacobi theory for Hamiltonian dynamics on contact manifolds. This novel theorem will be the continuous version of the discrete HJ theorem that will be presented in the upcoming section.

### *2.1. Contact Manifolds*

We call a contact manifold a pair $(M, \eta)$, where $M$ is an odd-dimensional manifold, say, $(2n+1)$-dimensional, with a contact form $\eta$, i.e., a one-form on $M$ such that $\eta \wedge \mathrm{d}\eta^n \neq 0$ is a volume form. This type of manifold has a distinguished vector field, the Reeb vector field $\mathcal{R}$, which is the unique vector field that satisfies the following two identities:

$$i_{\mathcal{R}}\mathrm{d}\eta = 0, \qquad \eta(\mathcal{R}) = 1. \tag{5}$$

There exists a Darboux coordinate system $(\mathbf{q}, \mathbf{p}, s) = (q^i, p_i, s)$ on $M$ (with $i$ ranging from 1 to $n$) such that the contact one-form reads

$$\eta = \mathrm{d}s - \mathbf{p} \cdot \mathrm{d}\mathbf{q}. \tag{6}$$

In these coordinates, we have $\mathcal{R} = \nabla_s = \partial/\partial s$. This local observation provides an example of a contact manifold as the extended cotangent bundle

$$\left(T^*Q \times \mathbb{R}, \eta = \mathrm{d}s - \theta_Q\right) \tag{7}$$

where $\theta_Q$ is the pullback of the tautological one-form of $T^*Q$.

**Musical morphisms.** For a contact manifold $(M, \eta)$, there is a musical isomorphism

$$\flat : TM \longrightarrow T^*M, \qquad v \mapsto \iota_v \mathrm{d}\eta + \eta(v)\eta. \tag{8}$$

This mapping takes the Reeb field $\mathcal{R}$ to the contact one-form $\eta$. We denote the inverse of this mapping by $\sharp$. Referring to this, we define a bivector field $\Lambda$ on $M$ as

$$\Lambda(\alpha, \beta) = -\mathrm{d}\eta(\sharp\alpha, \sharp\beta). \tag{9}$$

Then, referring to $\Lambda$, we introduce the following musical mapping:

$$\sharp_\Lambda : T^*M \longrightarrow TM, \qquad \alpha \mapsto \Lambda(\alpha, \bullet) = \sharp\alpha - \alpha(\mathcal{R})\mathcal{R}. \tag{10}$$

Evidently, the mapping $\sharp_\Lambda$ fails to be an isomorphism. Notice that the kernel is spanned by the contact one-form $\eta$.

**Legendrian submanifolds.** Let $(M, \eta)$ be a contact manifold. Recall the associated bivector field $\Lambda$ defined in (9). Consider a linear subbundle $\Xi$ of the tangent bundle $TM$ (that is, a distribution on $M$). We define the contact complement of $\Xi$ as

$$\Xi^\perp := \sharp_\Lambda(\Xi^o), \tag{11}$$

where the sharp map on the right-hand side is the one in (10) and $\Xi^o$ is the annihilator of $\Xi$. We say that $N$ is Legendrian if $TN^\perp = TN$.

Consider the contact manifold $T^*Q \times \mathbb{R}$ in (7), and let $W$ be a real valued function on the base manifold $Q$. Its first prolongation is

$$\mathrm{J}^1\mathrm{W} : \mathrm{Q} \longrightarrow \mathrm{T}^*\mathrm{Q} \times \mathbb{R}, \qquad \mathbf{q} \mapsto \big(\mathbf{q}, \mathrm{W}_{\mathbf{q}}, \mathrm{W}(\mathbf{q})\big), \tag{12}$$

where $W_{\mathbf{q}} = \mathrm{d}W(\mathbf{q})$. The image space of the first prolongation $\mathrm{J}^1\mathrm{W}$ is a Legendrian submanifold of $T^*Q \times \mathbb{R}$. The converse of this assertion is also true, that is, if the image space of a section of $T^*Q \times \mathbb{R} \to Q$ is a Legendrian submanifold, then it is the first prolongation of a function $W$.

**Contact diffeomorphisms.** Let $(M, \eta)$ and $(\widehat{M}, \widehat{\eta})$ be two contact manifolds. A diffeomorphism $\varphi$ from $M$ to $\widehat{M}$ is said to be a contact diffeomorphism (or contactomorphism) if it preserves the contact structures, that is, $T\varphi(\ker \eta) = \ker \widehat{\eta}$. In terms of contact forms, a contact diffeomorphism $\varphi$ is the one satisfying

$$\varphi^*\widehat{\eta} = \mu\eta. \tag{13}$$

where $\mu$ is a conformal factor. If we additionally impose that the conformal factor $\mu$ in definition (13) has to be equal to one, we arrive at the conservation of the contact form. We call such a mapping a strict contact diffeomorphism (or quantomorphism).

Consider two contact manifolds $(M, \eta)$ and $(\widehat{M}, \widehat{\eta})$. A contact product is the product manifold $\widehat{M} \times M \times \mathbb{R}$ with a contact one-form $\tau\widehat{\eta} \ominus \eta$, where $\tau$ is a global coordinate on $\mathbb{R}$ [54]. The operator $\ominus$ is presented in (15). It is possible to validate that the graph of a contact diffeomorphism $\varphi$ is a Legendrian submanifold of the contact product $\widehat{M} \times M \times \mathbb{R}$ [54].

**Generating functions for Legendrian submanifolds.** In particular, consider two same-dimensional extended cotangent bundles $T^*Q \times \mathbb{R}$ and $T^*\widehat{Q} \times \widehat{\mathbb{R}}$ equipped with Darboux' coordinates $(\mathbf{q}, \mathbf{p}, s)$ and $(\widehat{\mathbf{q}}, \widehat{\mathbf{p}}, \widehat{s})$, respectively. Then, we determine the contact product $\big(T^*\widehat{Q} \times \widehat{\mathbb{R}}\big) \times \big(T^*Q \times \mathbb{R}\big) \times \mathbb{R}$ of these two contact manifolds and consider it a fiber bundle over the product manifold $\widehat{Q} \times Q \times \mathbb{R}$ given by

$$\big(T^*\widehat{Q} \times \widehat{\mathbb{R}}\big) \times \big(T^*Q \times \mathbb{R}\big) \times \mathbb{R} \longrightarrow \widehat{Q} \times Q \times \mathbb{R}, \qquad (\widehat{\mathbf{q}}, \widehat{\mathbf{p}}, \widehat{s}; \mathbf{q}, \mathbf{p}, s; \tau) \mapsto (\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}), \tag{14}$$

where $\tau$ is the coordinate on the extended $\mathbb{R}$. In the contact product manifold, the product contact one-form is defined to be

$$\tau\widehat{\eta} - \eta = \tau(\mathrm{d}\widehat{s} - \widehat{\mathbf{p}} \cdot \mathrm{d}\widehat{\mathbf{q}}) - (\mathrm{d}s - \mathbf{p} \cdot \mathrm{d}\mathbf{q}). \tag{15}$$

By determining the isomorphism

$$\big(T^*\widehat{Q} \times \widehat{\mathbb{R}}\big) \times \big(T^*Q \times \mathbb{R}\big) \times \mathbb{R} \simeq T^*\big(\widehat{Q} \times Q \times \mathbb{R}\big) \times \mathbb{R}, \tag{16}$$

we provide Darboux coordinates for the contact product manifold as

$$(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}; \widehat{\boldsymbol{\pi}}, \boldsymbol{\pi}, \widehat{\pi}; z) = (\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}; \tau\widehat{\mathbf{p}}, -\mathbf{p}, -\tau; -s). \tag{17}$$

where $(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}; \widehat{\boldsymbol{\pi}}, \boldsymbol{\pi}, \widehat{\pi})$ are the bundle coordinates of the cotangent bundle $T^*(\widehat{Q} \times Q \times \mathbb{R})$ induced by the coordinates $(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s})$ on $\widehat{Q} \times Q \times \mathbb{R}$. This enables us to recast the contact one-form in (15) in the canonical form given in (6), so that the one-form in (15) turns out to be

$$\tau\widehat{\eta} - \eta = \mathrm{d}z - \widehat{\boldsymbol{\pi}} \cdot \mathrm{d}\widehat{\mathbf{q}} - \boldsymbol{\pi} \cdot \mathrm{d}\mathbf{q} - \widehat{\pi}\mathrm{d}\widehat{s}. \tag{18}$$

On the base manifold $\widehat{Q} \times Q \times \mathbb{R}$, we define a smooth function in the form

$$W(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}) = \widehat{s} + S(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}). \tag{19}$$

for a function $S$, which is the generating function of the discrete contact transformation, and the relation between the new contact variables and the old ones is given in (21) and mediated by $S$. The theory states that the image space of the first jet $\mathrm{J}^1\mathrm{W}$ is a Legendrian submanifold of the contact product manifold in (14). The contact diffeomorphism determined by the Legendrian submanifold $im(\mathrm{J}^1\mathrm{W})$ is computed to be

$$T^*Q \times \mathbb{R} \longrightarrow T^*\widehat{Q} \times \widehat{\mathbb{R}}, \qquad (\mathbf{q}, \mathbf{p}, s) \mapsto (\widehat{\mathbf{q}}, \widehat{\mathbf{p}}, \widehat{s}) \tag{20}$$

where one has the following identifications:

$$\mathbf{p} = D_2S(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}), \qquad s = \widehat{s} + S(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s}), \qquad \widehat{\mathbf{p}} = -\frac{D_1S(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s})}{1 + D_3S(\widehat{\mathbf{q}}, \mathbf{q}, \widehat{s})}, \tag{21}$$

where $D_iS$ represents the partial derivative of $S$ with respect to its $i$-th entry.

### *2.2. Contact Lagrangian Dynamics*

Now, we review the Lagrangian picture of contact systems. Although the Hamilton–Jacobi theory we will develop is based on a discrete contact Hamiltonian, we would like to start with the revision of contact Lagrangian mechanics from Herglotz's action giving rise to Herglotz's equations. From the Lagrangian picture, we will turn by a Legendre transformation to the Hamiltonian contact equations, which are also Herglotz's equations, in the case where the Lagrangian is regular. Let us see this. In [55], one can find the case of singular Lagrangians. Let $Q$ be an $n$-dimensional configuration manifold, and consider the extended tangent bundle $TQ \times \mathbb{R}$. If $\mathbf{q} = (q^i)$ is a local coordinate system, then the induced coordinates on the $(2n+1)$-dimensional manifold $TQ \times \mathbb{R}$ are $(\mathbf{q}, \dot{\mathbf{q}}, s)$.

**Herglotz's action and Herglotz's equation.** Consider two points $\mathbf{q}_0$ and $\mathbf{q}_T$ on $Q$, a Lagrangian $L$ on the extended tangent bundle $TQ \times \mathbb{R}$ and the following initial value problem:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = L\Big(\mathbf{q}, \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}, s\Big), \qquad s(0) = s_0, \tag{22}$$

where $\mathbf{q} = \mathbf{q}(t)$ is the curve containing $\mathbf{q}_0$ and $\mathbf{q}_T$ as initial and final points, i.e., $\mathbf{q}(0) = \mathbf{q}_0$ and $\mathbf{q}(T) = \mathbf{q}_T$, with $0 \leq t \leq T$. Evidently, the Cauchy problem (22) will be different for different curves $\mathbf{q}(t)$. So, a solution $s = s(t)$ to the problem depends on the curve $\mathbf{q}(t)$ substituted in the Lagrangian function.

Let $\mathcal{A}^\infty(Q)$ be the space of all smooth curves on $Q$ joining $\mathbf{q}_0$ and $\mathbf{q}_T$. This space depends on the end points, but we omit this fact in the notation. $\mathcal{A}^\infty(Q)$ is an infinite-dimensional manifold. According to the Cauchy problem in (22), for every initial value $s(0) = s_0$, we determine Herglotz's action as a map from the product space $\mathcal{A}^\infty(Q) \times \mathbb{R}$ to the real numbers as follows:

$$\mathcal{A}^\infty(Q) \times \mathbb{R} \to \mathbb{R}, \qquad (\mathbf{q}, s_0) \mapsto s(T) - s(0) = \int_0^T \frac{\mathrm{d}s}{\mathrm{d}t} dt = \int_0^T L\Big(\mathbf{q}, \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}, s\Big) dt. \tag{23}$$

The extreme values of the action are the curves $\mathbf{q} = \mathbf{q}(t)$ solving Herglotz's (generalized Euler–Lagrange) equations [34,35,56]

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{\mathbf{q}}} - L_{\mathbf{q}} = L_s L_{\dot{\mathbf{q}}}. \tag{24}$$

### 2.3. Contact Hamiltonian Dynamics

Consider a contact manifold $(M, \eta)$. For a Hamiltonian function $H$, the contact Hamiltonian vector field is defined in terms of the contact one-form $\eta$ as

$$\iota_{X_H} \eta = -H, \qquad \iota_{X_H} \mathrm{d}\eta = \mathrm{d}H - \mathcal{R}(H)\eta, \tag{25}$$

where $\mathcal{R}$ is the Reeb vector field. A direct computation determines that a Hamiltonian flow does not preserve the contact one-form

$$\mathcal{L}_{X_H} \eta = \mathrm{d}\iota_{X_H} \eta + \iota_{X_H} \mathrm{d}\eta = -\mathcal{R}(H)\eta. \tag{26}$$

Notice that according to (26), the flow of a contact Hamiltonian system preserves the contact structure, which is defined to be the kernel of the contact one-form. $X_H$ does not preserve the Hamiltonian function nor the volume form $\mathrm{d}\eta^n \wedge \eta$. Instead, we obtain

$$\mathcal{L}_{X_H} H = -\mathcal{R}(H)H, \qquad \mathcal{L}_{X_H} (\mathrm{d}\eta^n \wedge \eta) = -(n+1)\mathcal{R}(H)\mathrm{d}\eta^n \wedge \eta. \tag{27}$$

As shown previously, all contact manifolds locally resemble the extended cotangent bundle $T^*Q \times \mathbb{R}$. In this local view $(\mathbf{q}, \mathbf{p}, s)$, the Hamiltonian vector field turns out to be

$$X_H = H_{\mathbf{p}} \cdot \nabla_{\mathbf{q}} - \big(H_{\mathbf{q}} + \mathbf{p} H_s\big) \cdot \nabla_{\mathbf{p}} + \big(\mathbf{p} \cdot H_{\mathbf{p}} - H\big)\nabla_s, \tag{28}$$

where $\nabla_{\mathbf{q}}$, $\nabla_{\mathbf{p}}$ and $\nabla_{\mathbf{s}}$ are the partial derivatives with respect to the coordinates in the corresponding sub-indices. Thus, an integral curve $(\mathbf{q}(t), \mathbf{p}(t), s(t))$ of $X_H$ satisfies the contact Hamilton equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = H_{\mathbf{p}}, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\big(H_{\mathbf{q}} + \mathbf{p} H_s\big), \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = \mathbf{p} \cdot H_{\mathbf{p}} - H. \tag{29}$$

**The Legendre transformation.** The fiber derivative $\mathbb{F}L$ of a Lagrangian function $L$ determines a mapping from the extended tangent bundle to the extended cotangent bundle as

$$\mathbb{F}L : TQ \times \mathbb{R} \longrightarrow T^*Q \times \mathbb{R}, \qquad (\mathbf{q}, \dot{\mathbf{q}}, s) \mapsto (\mathbf{q}, L_{\dot{\mathbf{q}}}, s). \tag{30}$$

See that for a regular Lagrangian, $\mathbb{F}L$ is a contactomorphism from the contact manifold $TQ \times \mathbb{R}$ equipped with the contact one-form $\eta_L = (\mathbb{F}L)^*\eta$ to the contact manifold $T^*Q \times \mathbb{R}$ equipped with $\eta$ in (7). By regular Lagrangian, we mean a Lagrangian such that $\mathbb{F}L$ is a local diffeomorphism. In this situation, we can define the corresponding Hamiltonian function as

$$H : T^*Q \times \mathbb{R} \to \mathbb{R}, \qquad H = \dot{\mathbf{q}} \cdot L_{\dot{\mathbf{q}}} - L. \tag{31}$$

A direct computation proves that the contact Hamilton equations are the Herglotz's equations; they coincide for regular Lagrangian functions. For the Legendre transformation of non-regular cases, we cite [30,55,57].

### 2.4. A Continuous Geometric Hamilton–Jacobi Theory on Contact Manifolds

We start with the fibration $Q \mapsto \mathbb{R}$, whose first jet bundle is the extended cotangent bundle $T^*Q \times \mathbb{R}$. In this case, the fibration is given by the target map

$$\rho : T^*Q \times \mathbb{R} \longrightarrow Q, \qquad (\mathbf{z}, s) \mapsto \pi_Q(\mathbf{z}), \tag{32}$$

where $\pi_Q$ is the cotangent bundle projection. The first prolongation of a smooth function $F$ on $Q$ is a section of the projection $\rho$. We write the first prolongation as

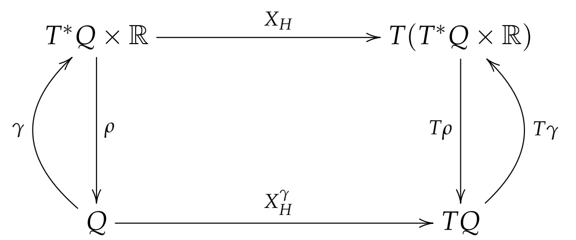
$$\gamma = \mathrm{J}^1\mathrm{F} : \mathrm{Q} \longrightarrow \mathrm{T}^*\mathrm{Q} \times \mathbb{R}, \qquad \mathbf{q} \mapsto (\mathbf{q}, \mathrm{F}_{\mathbf{q}}, \mathrm{F}(\mathbf{q})). \tag{33}$$

Notice that $T^*Q \times \mathbb{R}$ is a contact manifold and the image space of the first prolongation $\gamma$ is a Legendrian submanifold of this space.

Consider a section $\gamma$ in form (33) and a contact Hamiltonian vector field $X_H$ given as in (28). We define a vector field

$$X_H^\gamma := T\rho \circ X_H \circ \gamma, \tag{34}$$

on the base manifold $Q$ according to the commutativity of the following diagram:



(35)

where $T\rho$ is the tangent mapping of the fibration $\rho$ in (32). Next, we recall a geometric Hamilton–Jacobi theorem for contact Hamiltonian dynamics (see [58]).

**Theorem 1.** *For the first prolongation $\gamma$ of a function $F$ on $Q$, the following two conditions are equivalent:*

1. *The vector fields $X_H$ and $X_H^\gamma$ are $\gamma$-related, that is,*

$$T\gamma \circ X_H^\gamma = X_H \circ \gamma \tag{36}$$

*where $T\gamma : TQ \mapsto T(T^*Q \times \mathbb{R})$ is the tangent mapping of the section $\gamma$.*

2. *The equation*

$$H \circ \gamma = 0 \tag{37}$$

*is fulfilled.*

**Proof.** We prove this theorem in local coordinates. The restriction of the Hamiltonian vector field $X_H$ to the image of $\gamma$ is computed to be

$$X_H \circ \gamma(\mathbf{q}) = \Big(\mathbf{q}, F_{\mathbf{q}}, F(\mathbf{q}); H_{\mathbf{p}}\Big|_{im(\gamma)}, -H_{\mathbf{q}}\Big|_{im(\gamma)} - H_s F_{\mathbf{q}}\Big|_{im(\gamma)}, F_{\mathbf{q}} \cdot H_{\mathbf{q}}\Big|_{im(\gamma)} - H\Big|_{im(\gamma)}\Big). \tag{38}$$

By using $T\rho$, we map this vector field down to the tangent bundle of $Q$. This is the projected vector field in (34), that is,

$$X_H^\gamma(\mathbf{q}) = H_{\mathbf{p}}\Big|_{im(\gamma)} \cdot \nabla_{\mathbf{q}}. \tag{39}$$

On the other hand, the tangent mapping of the section $\gamma$ is computed to be

$$T\gamma(\mathbf{q}; \dot{\mathbf{q}}) = (\mathbf{q}, F_{\mathbf{q}}(\mathbf{q}), F(\mathbf{q}); \dot{\mathbf{q}}, F_{\mathbf{qq}}(\mathbf{q})\dot{\mathbf{q}}, F_{\mathbf{q}}(\mathbf{q}) \cdot \dot{\mathbf{q}}), \tag{40}$$

where the notation $F_{\mathbf{qq}}(\mathbf{q})\dot{\mathbf{q}}$ stands for the multiplication of the Hessian matrix $F_{\mathbf{qq}}$ with the column vector $\dot{\mathbf{q}}$. Accordingly, the left-hand side of Equation (36) is computed to be

$$T\gamma \circ X_H^\gamma(\mathbf{q}) = (\mathbf{q}, F_{\mathbf{q}}(\mathbf{q}), F(\mathbf{q}); H_{\mathbf{p}}\Big|_{im(\gamma)}, F_{\mathbf{qq}} H_{\mathbf{p}}\Big|_{im(\gamma)}, F_{\mathbf{q}} \cdot H_{\mathbf{q}}\Big|_{im(\gamma)}). \tag{41}$$

To satisfy (36), the expressions in (38) and (41) must be the same. The first three entries of these local realizations are the same. The first of the fiber variables (which coincides with the fourth entries) are also the same. See that the fifth and the sixth entries of (38) and (41) are not equal. The fifth entries are equal if and only if

$$F_{\mathbf{qq}}H_{\mathbf{p}}\Big|_{im(\gamma)} + H_{\mathbf{q}}\Big|_{im(\gamma)} + H_s F_{\mathbf{q}}\Big|_{im(\gamma)} = 0. \tag{42}$$

See that identity (42) can be compactly written as $d(H \circ \gamma) = 0$. This gives that $H$ turns out to be a constant if it is restricted to the image space of the first prolongation $\gamma$. On the other hand, the sixth entries are the same if and only if $H$ is not only constant, but it also vanishes on the image space of the first prolongation $\gamma$. This is the second condition (37). The inverse of the assertion is proved by reversing the arguments. This completes the proof. □

There exist alternative versions of the geometric HJ theory for contact manifolds where the base space is considered to be the extended configuration space $Q \times \mathbb{R}$. To check these versions, we refer to [11,58,59], which include intrinsic proof.

## 3. Discrete Contact Dynamics

In this section, we first recall discrete Lagrangian and Hamiltonian dynamics on the contact framework. The definitions and the approach we adopt are the ones displayed in [48–51]. We also wish to cite [52]. Then, we introduce the HJ theory for discrete Hamiltonian dynamics. This is the contact version of the approach performed in [53] for the case of symplectic manifolds. In accordance with this, we state a geometric Hamilton–Jacobi theory. This is a contact generalization of the one presented in [11]. One could start with a definition of a discrete contact manifold, but there is no such concept defined so far. Intuitively, let us say that for us, a discrete contact manifold will be the set of discrete positions, momenta and discrete external parameter. Let us introduce such notation in the following paragraphs.

### *3.1. Discrete Contact Lagrangian Dynamics*

In the framework of discrete dynamics, the role of differentiable curves in a configuration manifold $Q$ is played by the finite sequences of points in $Q$. Therefore, we are interested in the space of sequences consisting of $N+1$ points, denoted by $\mathcal{S}^{N+1}(Q)$. It is interesting to note that $\mathcal{S}^{N+1}(Q)$ is a manifold isomorphic to the number of copies $N+1$ of $Q$, denoted by $Q^{N+1}$. We denote such a sequence by a set $[\mathbf{q}] = \{\mathbf{q}_0, \ldots, \mathbf{q}_N\}$, which is a point in $Q^{N+1}$. Here, $\mathbf{q}_0$ is the initial point of the sequence, whereas $\mathbf{q}_N$ is the end point. We use the sub-index $\mathbf{q}_\kappa$ to denote the location of the point in the sequence from $\kappa = 0$ to $\kappa = N$. We call $[\mathbf{q}]$ a discrete curve.

**Herglotz's action and Herglotz's equation.** A discrete Lagrangian in this framework is a real valued function $L_d$ defined on the product space $Q \times Q \times \mathbb{R}$. Comparing with the continuous case, we see that in this geometry, the role of the tangent bundle $TQ$ is played by the pair $Q \times Q$. More technically, the discrete Lagrangian $L_d$ is an approximation of the exact contact discrete Lagrangian $L_d^{ex}$, which is defined to be

$$L_d^{ex}(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa) = \int_{t_\kappa}^{t_{\kappa+1}} L\left(\mathbf{q}, \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}, s_\kappa\right) dt \tag{43}$$

where $\mathbf{q}$ is a solution of the continuous Herglotz Equation (24) with boundary conditions $\mathbf{q}(t_\kappa) = \mathbf{q}_\kappa$ and $\mathbf{q}(t_{\kappa+1}) = \mathbf{q}_{\kappa+1}$. To each discrete curve $[\mathbf{q}]$ and for a fixed initial value point $s_0$, we introduce the discrete Cauchy problem as

$$s_\kappa - s_{\kappa-1} = L_d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, s_{\kappa-1}), \qquad s_0 = s(0), \tag{44}$$

where $\kappa$ runs from 1 to $N$. We determine the discrete Herglotz's action as a map from the product space $\mathcal{S}^{N+1}(Q) \times \mathbb{R}$ to the real numbers, that is,

$$\mathcal{S}^{N+1}(Q) \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad ([\mathbf{q}], s_0) \mapsto s_N - s_0 = \sum_{\kappa=0}^{N-1} L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa). \tag{45}$$

The extreme values of the discrete action in (45) are curves solving the discrete Herglotz's (generalized discrete Euler–Lagrange) equations

$$\begin{aligned} &D_1 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa) + \big(1 + D_3 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa)\big) D_2 L_d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, s_{\kappa-1}) = 0 \\ &s_\kappa - s_{\kappa-1} = L_d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, s_{\kappa-1}), \qquad s_0 = s(0), \end{aligned} \tag{46}$$

provided that $1 + D_3 L_d \neq 0$. Here, $D_i L_d$ refers to the partial derivative of the discrete Lagrangian with respect to its $i$-th entry.

**The discrete Legendre transformation.** Assume a discrete Lagrangian function $L_d$ satisfying the condition $1 + D_3 L_d \neq 0$. We define the following Legendre transformations from the extended discrete space $Q \times Q \times \mathbb{R}$ to the extended cotangent bundle $T^*Q \times \mathbb{R}$ as

$$\begin{aligned} &\mathbb{F}L_d^+ : Q \times Q \times \mathbb{R} \longrightarrow T^*Q \times \mathbb{R}, \\ &\qquad (\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa) \mapsto \big(\mathbf{q}_{\kappa+1}, D_2 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa), s_\kappa + L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa)\big) \\ &\mathbb{F}L_d^- : Q \times Q \times \mathbb{R} \longrightarrow T^*Q \times \mathbb{R}, \\ &\qquad (\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa) \mapsto \left(\mathbf{q}_\kappa, -\frac{D_1 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa)}{1 + D_3 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa)}, s_\kappa\right). \end{aligned} \tag{47}$$

So, we have two alternative definitions of the momenta given as

$$\mathbf{p}^+_{\kappa-1,\kappa,\kappa} = D_2 L_d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, s_\kappa), \qquad \mathbf{p}^-_{\kappa,\kappa+1,\kappa} = -\frac{D_1 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa)}{1 + D_3 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa)}. \tag{48}$$

In light of the discrete Herglotz's Equation (46), one can directly establish the momentum matching equation

$$\mathbb{F}L_d^+(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa) = \mathbb{F}L_d^-(\mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, s_{\kappa+1}). \tag{49}$$

So, in this case, one can define the momenta

$$\mathbf{p}_\kappa := \mathbf{p}^+_{\kappa-1,\kappa,\kappa} = \mathbf{p}^-_{\kappa,\kappa+1,\kappa+1}. \tag{50}$$

A discrete Lagrangian $L_d$ is called regular if the Hessian matrix obtained by taking the partial derivatives of $L_d$ is non-degenerate, that is,

$$\det[D_2 D_2 L_d(\mathbf{q}_{\kappa-1}, \mathbf{q}_\kappa, s_\kappa)] \neq 0. \tag{51}$$

Equivalently, a discrete Lagrangian is a regular Lagrangian if the Legendre transformations in (47) become invertible.

**Diffeomorphisms.** Consider a regular discrete Lagrangian function $L_d$; referring to the discrete Legendre mappings in (47), one can define a local diffeomorphism on the extended discrete space as

$$\begin{aligned} &\Phi : Q \times Q \times \mathbb{R} \longrightarrow Q \times Q \times \mathbb{R}, \\ &\qquad (\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa) \mapsto (\mathbf{q}_{\kappa+1}, \mathbf{q}_{\kappa+2}, s_{\kappa+1}) := (\mathbb{F}L_d^-)^{-1} \circ \mathbb{F}L_d^+(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa). \end{aligned} \tag{52}$$

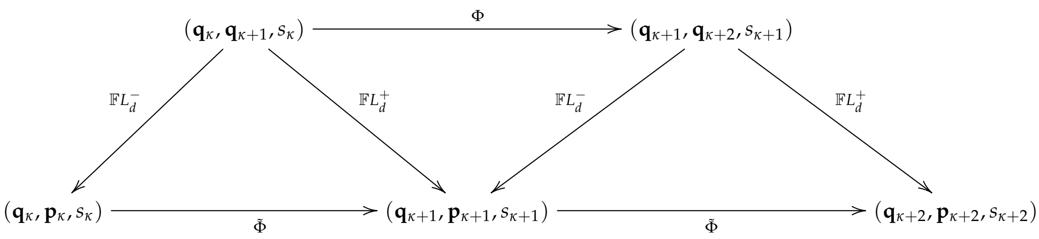
Similarly, one can define a diffeomorphism on the extended cotangent bundle as follows:

$$\tilde{\Phi} := \mathbb{F}L_d^+ \circ (\mathbb{F}L_d^-)^{-1}. \tag{53}$$

A direct comparison shows that the diffeomorphism $\Phi$ given in (52) and the diffeomorphism $\tilde{\Phi}$ given in (53) are related by the Legendre transformations

$$\tilde{\Phi} = \mathbb{F}L_d^{+} \circ \Phi \circ (\mathbb{F}L_d^{+})^{-1}, \qquad \tilde{\Phi} = \mathbb{F}L_d^{-} \circ \Phi \circ (\mathbb{F}L_d^{-})^{-1}. \tag{54}$$

In order to be more specific about the diffeomorphism, we plot the following commutative diagram.



(55)

See that in this diagram, in the bottom row, we have the flow on the extended cotangent bundle $T^*Q \times \mathbb{R}$, whereas the top row corresponds to the flow on the extended discrete space $Q \times Q \times \mathbb{R}$. Here, the Legendre transformations establish the equivalency between the flows.

### *3.2. Discrete Contact Hamiltonian Dynamics*

To derive Hamiltonian dynamics, we use the fact that a discrete contact Lagrangian is essentially a generating function of type one [1] and that we can apply the defined Legendre transformations to the discrete Lagrangian to find a discrete Hamiltonian [1,2]. We start with the definition of momentum $\mathbf{p}_{\kappa+1}$ and the local inversion operation $\Psi$ in order to obtain $\mathbf{q}_{\kappa+1}$ in terms of $\mathbf{p}_{\kappa+1}$.

$$\mathbf{p}_{\kappa+1} = D_2 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa), \qquad \mathbf{q}_{\kappa+1} = \Psi(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa), \tag{56}$$

provided that the regularity condition holds, i.e., the operator $D_2 D_2 L_d$ does not vanish.

**Right discrete contact Hamilton equations.** Referring to these local identifications, we introduce the right discrete Hamiltonian function on the extended cotangent bundle as

$$H_d^{+}(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) = \mathbf{p}_{\kappa+1} \cdot \mathbf{q}_{\kappa+1} - L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa). \tag{57}$$

In the above and subsequent expressions, we identify each point in $T^*Q \times \mathbb{R}$ with its coordinates. By taking the partial derivatives of the Hamiltonian function $H_d^{+}$ with respect to its arguments (applying the chain rule referring to (56)), we arrive at the following expressions:

$$\begin{aligned} D_1 H_d^{+}(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) &= -D_1 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa) \\ D_2 H_d^{+}(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) &= \mathbf{q}_{\kappa+1} \\ D_3 H_d^{+}(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) &= -D_3 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_\kappa). \end{aligned} \tag{58}$$

Notice that these equations hold under the identifications in (56). The second equation in the list determines $\mathbf{q}_{\kappa+1}$. We substitute the first and third identities in (58) into the discrete Euler–Lagrange Equation (46), and by using (56), we obtain

$$\mathbf{p}_\kappa = \frac{D_1 H_d^{+}(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa)}{1 - D_3 H_d^{+}(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa)}. \tag{59}$$

Considering the discrete Cauchy problem in (44) as part of the set, we arrive at the following system of equations, which we call right discrete contact Hamilton equations:

$$
\begin{aligned}
\mathbf{q}_{\kappa+1} &= D_2 H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa), \\
\mathbf{p}_\kappa &= \frac{D_1 H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa)}{(1 - D_3 H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa))}, \\
s_{\kappa+1} &= s_\kappa + \mathbf{p}_{\kappa+1} \cdot D_2 H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) - H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa).
\end{aligned}
\tag{60}
$$

**Left discrete contact Hamilton equations.** The left discrete Hamiltonian function on the extended cotangent bundle is

$$
H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) = -\mathbf{p}_\kappa \cdot \mathbf{q}_\kappa - L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}). \tag{61}
$$

By taking the partial derivatives of the Hamiltonian function $H_d^-$, we arrive at the following expressions:

$$
\begin{aligned}
D_1 H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) &= -\mathbf{q}_\kappa \\
D_2 H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) &= \mathbf{p}_{\kappa+1} \\
D_3 H_d^+(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) &= -D_3 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}).
\end{aligned}
\tag{62}
$$

The second equation in the list determines $\mathbf{p}_{\kappa+1}$.

$$
\mathbf{p}_{\kappa+1} = D_2 H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) = -D_2 L_d(\mathbf{q}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}). \tag{63}
$$

Considering the discrete Cauchy problem in (44) as part of the set, we arrive at the following system of equations, which we call the left discrete contact Hamilton equations:

$$
\begin{aligned}
\mathbf{q}_\kappa &= -D_1 H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) \\
\mathbf{p}_{\kappa+1} &= D_2 H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) \\
s_\kappa &= s_{\kappa+1} - \mathbf{p}_\kappa \cdot D_1 H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1}) + H_d^-(\mathbf{p}_\kappa, \mathbf{q}_{\kappa+1}, s_{\kappa+1})
\end{aligned}
\tag{64}
$$

From now on, we shall focus on the right discrete dynamics, since everything can be reenacted in terms of left discrete approach straightforwardly.

### *3.3. Discrete Contact Hamilton–Jacobi Theory*

How does one derive a discrete contact Hamilton–Jacobi equation? We will follow the classical steps that one can find in Goldstein's book [2]. The process for obtaining this equation is rooted in the derivation of a generating function of a coordinate transformation that trivializes the dynamics, as in the classical continuous theory [2]. For this, we need a generating function to introduce a contact diffeomorphism from the contact manifold $T^*Q \times \mathbb{R}$ to the contact manifold $T^*\widehat{Q} \times \widehat{\mathbb{R}}$, that is,

$$
T^*Q \times \mathbb{R} \longrightarrow T^*\widehat{Q} \times \widehat{\mathbb{R}}, \qquad (\mathbf{q}_\kappa, \mathbf{p}_\kappa, s_\kappa) \to (\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_\kappa, \widehat{s}_\kappa). \tag{65}
$$

Recall from (21) that in Section 2.1, we have obtained a generating function defined on the product space $\widehat{Q} \times Q \times \widehat{\mathbb{R}}$, realizing the transformation. The main result for the generating function is described in the next theorem, following the lines of [53] in the symplectic case.

**Theorem 2.** *Consider the right discrete contact Hamilton Equation (60) and a discrete phase space $(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, s_\kappa)$. Consider also a transformation (65) that satisfies the following:*

1. *The old and new coordinates are related by a generating function $S^\kappa : \widehat{Q} \times Q \times \widehat{\mathbb{R}} \to \mathbb{R}$ of the type*

$$
\mathbf{p}_\kappa = D_2 S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa), \quad s_\kappa = \widehat{s}_\kappa + S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa), \quad \widehat{\mathbf{p}}_\kappa = -\frac{D_1 S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa)}{1 + D_3 S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa)}. \tag{66}
$$

2. *The dynamics in the new coordinates* $(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_\kappa, \widehat{s}_\kappa)$ *is rendered trivial, i.e.,*

$$(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_\kappa, \widehat{s}_\kappa) = (\widehat{\mathbf{q}}_{\kappa+1}, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_{\kappa+1}). \tag{67}$$

*Then, the set of functions* $\{S^\kappa\}$ *satisfies the right discrete contact Hamilton–Jacobi equation*

$$\begin{aligned} S^{\kappa+1}(\widehat{\mathbf{q}}_0, \mathbf{q}_{\kappa+1}, \widehat{s}_0) - S^\kappa(\widehat{\mathbf{q}}_0, \mathbf{q}_\kappa, \widehat{s}_0) - D_2 S^{\kappa+1}(\widehat{\mathbf{q}}_0, \mathbf{q}_{\kappa+1}, \widehat{s}_0) \cdot \mathbf{q}_{\kappa+1} \\ + H_d^+(\mathbf{q}_\kappa, D_2 S^{\kappa+1}(\widehat{\mathbf{q}}_0, \mathbf{q}_{\kappa+1}, \widehat{s}_0), \widehat{s}_0 + S^\kappa(\widehat{\mathbf{q}}_0, \mathbf{q}_\kappa, \widehat{s}_0)) = 0. \end{aligned} \tag{68}$$

If we prove this theorem, step by step, we will describing the derivation of the discrete HJ equation.

**Proof.** We establish the proof in four steps.

*Step 1.* Consider the right discrete contact Hamilton equations (60) in the new variables $(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_\kappa, \widehat{s}_\kappa)$, that is,

$$\begin{aligned} \widehat{\mathbf{q}}_{\kappa+1} &= D_2 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa), \\ \widehat{\mathbf{p}}_\kappa &= D_1 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa) / \left(1 - D_3 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa)\right), \\ \widehat{s}_{\kappa+1} &= \widehat{s}_\kappa + \widehat{\mathbf{p}}_{\kappa+1} \cdot D_2 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa) - \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa). \end{aligned} \tag{69}$$

These equations can be recast in the form of a total derivative of the right discrete contact Hamiltonian in the following way:

$$\begin{aligned} \mathrm{d}\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa) &= D_1 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa) \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa + D_2 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa) \cdot \mathrm{d}\widehat{\mathbf{p}}_{\kappa+1} \\ &\quad + D_3 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa) \mathrm{d}\widehat{s}_\kappa \\ &= \widehat{\mathbf{p}}_\kappa \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa + \widehat{\mathbf{q}}_{\kappa+1} \cdot \mathrm{d}\widehat{\mathbf{p}}_{\kappa+1} + D_3 \widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa, \widehat{\mathbf{p}}_{\kappa+1}, \widehat{s}_\kappa) \widehat{\eta}_\kappa \end{aligned} \tag{70}$$

where we have employed (69) in the total derivative. More explicitly, we have replaced $D_1 \widehat{H}_d^+$ from the second equation in (69). Notice that we have used the following notation for the discrete contact form:

$$\widehat{\eta}_\kappa = \mathrm{d}\widehat{s}_\kappa - \widehat{\mathbf{p}}_\kappa \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa. \tag{71}$$

*Step 2.* Start with the generating function $S = S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa)$ satisfying (66). The total derivative of the generating function is

$$\begin{aligned} \mathrm{d}S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa) &= D_1 S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa) \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa + D_2 S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa) \cdot d\mathbf{q}_\kappa + D_3 S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa) \mathrm{d}\widehat{s}_\kappa \\ &= -\widehat{\mathbf{p}}_\kappa \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa + \mathbf{p}_\kappa \cdot d\mathbf{q}_\kappa + D_3 S^\kappa(\widehat{\mathbf{q}}_\kappa, \mathbf{q}_\kappa, \widehat{s}_\kappa) \widehat{\eta}_\kappa \end{aligned} \tag{72}$$

in which we can introduce $D_1 S^k$ and $D_2 S^k$ from (66). We can write the expression for $\mathrm{d}S^{\kappa+1}$ as

$$\mathrm{d}S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1}, \mathbf{q}_{\kappa+1}, \widehat{s}_{\kappa+1}) = -\widehat{\mathbf{p}}_{\kappa+1} \cdot \mathrm{d}\widehat{\mathbf{q}}_{\kappa+1} + \mathbf{p}_{\kappa+1} \cdot d\mathbf{q}_{\kappa+1} + D_3 S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1}, \mathbf{q}_{\kappa+1}, \widehat{s}_{\kappa+1}) \widehat{\eta}_{\kappa+1} \tag{73}$$

where $\widehat{\eta}_{\kappa+1} = \mathrm{d}\widehat{s}_{\kappa+1} - \widehat{\mathbf{p}}_{\kappa+1} \cdot \mathrm{d}\widehat{\mathbf{q}}_{\kappa+1}$.

Coming back to the expression of the total derivative of the Hamiltonian in (70), realize that the second term $\widehat{\mathbf{q}}_{\kappa+1} \cdot \mathrm{d}\widehat{\mathbf{p}}_{\kappa+1}$ can be written as $d(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}) - \widehat{\mathbf{p}}_{\kappa+1} \cdot \mathrm{d}\widehat{\mathbf{q}}_{\kappa+1}$. Then, we substitute expressions (72) and (73) into the total derivative of the Hamiltonian function. With these, we can continue the calculation in (70) as follows:

$$
\begin{aligned}
\mathrm{d}\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa) &= \widehat{\mathbf{p}}_\kappa \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa + \widehat{\mathbf{q}}_{\kappa+1} \cdot \mathrm{d}\widehat{\mathbf{p}}_{\kappa+1} + D_3\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa)\widehat{\eta}_\kappa \\
&= \widehat{\mathbf{p}}_\kappa \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa - \widehat{\mathbf{p}}_{\kappa+1} \cdot \mathrm{d}\widehat{\mathbf{q}}_{\kappa+1} + \mathrm{d}(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}) + D_3\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa)\widehat{\eta}_\kappa \\
&= \mathbf{p}_\kappa \cdot \mathrm{d}\mathbf{q}_\kappa + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa)\widehat{\eta}_\kappa - \mathrm{d}S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) \\
&\quad - \big(\mathbf{p}_{\kappa+1} \cdot \mathrm{d}\mathbf{q}_{\kappa+1} + D_3S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1},\mathbf{q}_{\kappa+1},\widehat{s}_{\kappa+1})\widehat{\eta}_{\kappa+1} - \mathrm{d}S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1},\mathbf{q}_{\kappa+1},\widehat{s}_{\kappa+1})\big) \\
&\quad + \mathrm{d}(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}) + D_3\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa)\widehat{\eta}_\kappa.
\end{aligned} \tag{74}
$$

*Step 3.* We take the exterior derivative of the second identity in (66). See that

$$
\begin{aligned}
ds_\kappa &= \mathrm{d}\widehat{s}_\kappa + \mathrm{d}S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) \\
&= \mathrm{d}\widehat{s}_\kappa + D_1S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa + D_2S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) \cdot \mathrm{d}\mathbf{q}_\kappa + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) \cdot \mathrm{d}\widehat{s}_\kappa \\
&= \mathrm{d}\widehat{s}_\kappa - (1 + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa))\widehat{\mathbf{p}}_\kappa \cdot \mathrm{d}\widehat{\mathbf{q}}_\kappa + \mathbf{p}_\kappa \cdot \mathrm{d}\mathbf{q}_\kappa + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) \cdot \mathrm{d}\widehat{s}_\kappa \\
&= \big(1 + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa)\big)\widehat{\eta}_\kappa + \mathbf{p}_\kappa \cdot \mathrm{d}\mathbf{q}_\kappa.
\end{aligned} \tag{75}
$$

So, we arrive at two expressions. One is the relationship between the contact forms in terms of the generating function

$$
\eta_\kappa = \big(1 + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa)\big)\widehat{\eta}_\kappa. \tag{76}
$$

Notice that we have the followings identities:

$$
\begin{aligned}
&\mathbf{p}_\kappa \cdot \mathrm{d}\mathbf{q}_\kappa + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa)\widehat{\eta}_\kappa = ds_\kappa - \widehat{\eta}_\kappa, \\
&\mathbf{p}_{\kappa+1} \cdot \mathrm{d}\mathbf{q}_{\kappa+1} + D_3S^\kappa(\widehat{\mathbf{q}}_{\kappa+1},\mathbf{q}_{\kappa+1},\widehat{s}_{\kappa+1})\widehat{\eta}_{\kappa+1} = \mathrm{d}s_{\kappa+1} - \widehat{\eta}_{\kappa+1}.
\end{aligned} \tag{77}
$$

We continue the calculation in (74) by substituting these equations. Accordingly, we have that

$$
\begin{aligned}
\mathrm{d}\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa) &= \mathbf{p}_\kappa \cdot \mathrm{d}\mathbf{q}_\kappa + D_3S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa)\widehat{\eta}_\kappa - \mathrm{d}S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) \\
&\quad - \big(\mathbf{p}_{\kappa+1} \cdot \mathrm{d}\mathbf{q}_{\kappa+1} + D_3S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1},\mathbf{q}_{\kappa+1},\widehat{s}_{\kappa+1})\widehat{\eta}_{\kappa+1} - \mathrm{d}S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1},\mathbf{q}_{\kappa+1},\widehat{s}_{\kappa+1}))\big) \\
&\quad + d(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}) + D_3\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa)\widehat{\eta}_\kappa. \\
&= \mathrm{d}s_\kappa - \widehat{\eta}_\kappa - \mathrm{d}S^\kappa(\widehat{\mathbf{q}}_\kappa,\mathbf{q}_\kappa,\widehat{s}_\kappa) - (\mathrm{d}s_{\kappa+1} - \widehat{\eta}_{\kappa+1} - \mathrm{d}S^{\kappa+1} + \mathrm{d}(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}) \\
&\quad + D_3\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa)\widehat{\eta}_\kappa.
\end{aligned} \tag{78}
$$

Since on the image space, i.e., the space with coordinates $(\widehat{\mathbf{q}},\widehat{\mathbf{p}},\widehat{s})$, the dynamics are rendered trivial (67), we take the Hamiltonian function as

$$
\widehat{H}_d^+(\widehat{\mathbf{q}}_\kappa,\widehat{\mathbf{p}}_{\kappa+1},\widehat{s}_\kappa) = \widehat{\mathbf{q}}_\kappa \cdot \widehat{\mathbf{p}}_{\kappa+1}. \tag{79}
$$

This gives that the Hamiltonian function is independent of $\widehat{s}_\kappa$, so $D_3\widehat{H}_d^+$ vanishes identically. We substitute this into (78) and arrive at

$$
\begin{aligned}
\mathrm{d}\widehat{H}_d^+ &= \mathrm{d}s_\kappa - \widehat{\eta}_\kappa - \mathrm{d}S^\kappa - (\mathrm{d}s_{\kappa+1} - \widehat{\eta}_{\kappa+1} - \mathrm{d}S^{\kappa+1}) + \mathrm{d}(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}) + D_3\widehat{H}_d^+\widehat{\eta}_\kappa. \\
\mathrm{d}(\widehat{\mathbf{q}}_\kappa \cdot \widehat{\mathbf{p}}_{\kappa+1}) &= \mathrm{d}s_\kappa - \widehat{\eta}_\kappa - \mathrm{d}S^\kappa - (ds_{\kappa+1} - \widehat{\eta}_{\kappa+1} - \mathrm{d}S^{\kappa+1}) + \mathrm{d}(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}).
\end{aligned} \tag{80}
$$

*Step 4.* Since the dynamics are trivial on the image space, the discrete contact forms coincide as

$$
\widehat{\eta}_\kappa = \widehat{\eta}_{\kappa+1} \tag{81}
$$

and so do the coupling functions

$$
\widehat{\mathbf{q}}_\kappa \cdot \widehat{\mathbf{p}}_{\kappa+1} = \widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1} = \widehat{\mathbf{q}}_0 \cdot \widehat{\mathbf{p}}_0. \tag{82}
$$

Now, we take the exterior derivative of the third equation in (60) and replace the first equation in (60). We also use the first equation in (66) to replace the value of $p_{k+1}$. This gives

$$\begin{aligned} \mathrm{d}s_{\kappa+1} - \mathrm{d}s_\kappa &= \mathrm{d}(\mathbf{p}_{\kappa+1} \cdot \mathbf{q}_{\kappa+1}) - \mathrm{d}H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) \\ &= \mathrm{d}(D_2 S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1}, \mathbf{q}_{\kappa+1}, \widehat{s}_{\kappa+1}) \cdot \mathbf{q}_{\kappa+1}) - \mathrm{d}H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) \end{aligned} \tag{83}$$

We collect all the equations in (81)–(83) and substitute them into (80). This reads as follows:

$$\begin{aligned} \mathrm{d}(\widehat{\mathbf{q}}_\kappa \cdot \widehat{\mathbf{p}}_{\kappa+1}) &= \mathrm{d}s_\kappa - \widehat{\eta}_\kappa - \mathrm{d}S^\kappa - (\mathrm{d}s_{\kappa+1} - \widehat{\eta}_{\kappa+1} - \mathrm{d}S^{\kappa+1}) + \mathrm{d}(\widehat{\mathbf{q}}_{\kappa+1} \cdot \widehat{\mathbf{p}}_{\kappa+1}) \\ \mathrm{d}(\widehat{\mathbf{q}}_0 \cdot \widehat{\mathbf{p}}_0) &= \mathrm{d}s_\kappa - \widehat{\eta}_\kappa - \mathrm{d}S^\kappa - (\mathrm{d}s_{\kappa+1} - \widehat{\eta}_{\kappa+1} - \mathrm{d}S^{\kappa+1}) + \mathrm{d}(\widehat{\mathbf{q}}_0 \cdot \widehat{\mathbf{p}}_0) \\ 0 &= \mathrm{d}s_\kappa - \mathrm{d}s_{\kappa+1} + \mathrm{d}S^{\kappa+1} - \mathrm{d}S^\kappa \\ 0 &= -\mathrm{d}(D_2 S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1}, \mathbf{q}_{\kappa+1}, \widehat{s}_{\kappa+1}) \cdot \mathbf{q}_{\kappa+1}) + \mathrm{d}H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa) + \mathrm{d}S^{\kappa+1} - \mathrm{d}S^\kappa \\ 0 &= \mathrm{d}\big(S^{\kappa+1} - S^\kappa - D_2 S^{\kappa+1}(\widehat{\mathbf{q}}_{\kappa+1}, \mathbf{q}_{\kappa+1}, \widehat{s}_{\kappa+1}) \cdot \mathbf{q}_{\kappa+1} + H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa)\big). \end{aligned} \tag{84}$$

Now, fix the initial point $(\widehat{\mathbf{q}}_0, \mathbf{q}_0, \widehat{s}_0)$. For the generating function (66) presented in the previous subsection, we introduce a new notation as follows:

$$S_d^\kappa(\mathbf{q}_\kappa) := \widehat{s}_0 + S^\kappa(\widehat{\mathbf{q}}_0, \mathbf{q}_\kappa, \widehat{s}_0). \tag{85}$$

This denotation determines a function from $Q$ to the real numbers. In this notation, the discrete contact Hamilton–Jacobi Equation (68) turns out to be

$$S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) - S_d^\kappa(\mathbf{q}_\kappa) - \mathrm{d}S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) \cdot \mathbf{q}_{\kappa+1} + H_d^+(\mathbf{q}_\kappa, \mathrm{d}S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}), S_d^\kappa(\mathbf{q}_\kappa)) = 0. \tag{86}$$

□

One can obtain a similar result in terms of the left discrete Hamiltonian. By using the new notation (85), we present the left discrete contact Hamilton–Jacobi equation

$$S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) - S_d^\kappa(\mathbf{q}_\kappa) + \mathrm{d}S_d^\kappa(\mathbf{q}_\kappa) \cdot \mathbf{q}_\kappa + H_d^-(\mathbf{q}_{\kappa+1}, \mathrm{d}S_d^\kappa(\mathbf{q}_\kappa), S_d^{\kappa+1}(\mathbf{q}_{\kappa+1})) = 0. \tag{87}$$

### *3.4. A Geometric Discrete Hamilton–Jacobi Theory on Contact Manifolds*

Having obtained a discrete contact Hamilton–Jacobi Equation (86) in the previous section, it is easy to see that that the differential of the discrete generating function in (85) is precisely the discrete version of the continuous section $\gamma$ in Theorem 1. Here, we aim at interpreting the discrete contact Hamilton–Jacobi Equation (86) in terms of discrete flows.

**Projection of discrete flow.** Consider a Hamiltonian function $H_d^+ = H_d^+(\mathbf{q}_\kappa, \mathbf{p}_{\kappa+1}, s_\kappa)$. Referring to the commutativity of diagram (55) and in the light of the discrete contact Hamiltonian dynamics in (60), we compute the Hamiltonian flow on the extended cotangent bundle as

$$\tilde{\Phi} : \left(\mathbf{q}_\kappa, \frac{D_1 H_d^+}{1 - D_3 H_d^+}, s_\kappa\right) \longrightarrow \left(D_2 H_d^+, \mathbf{p}_{\kappa+1}, s_\kappa + \mathbf{p}_{\kappa+1} \cdot D_2 H_d^+ - H_d^+\right) \tag{88}$$

Recall the projection $\rho : T^*Q \times \mathbb{R} \mapsto Q \times \mathbb{R}$ given in (32). For a function $S_d = S_d(\mathbf{q}_\kappa)$ on the base space, we define local sections of the projection $\rho$ as

$$\begin{aligned} \gamma^\kappa : Q &\longrightarrow T^*Q \times \mathbb{R}, \qquad \mathbf{q}_\kappa \mapsto (\mathbf{q}_\kappa, \mathrm{d}S_d^\kappa(\mathbf{q}_\kappa), S_d^\kappa(\mathbf{q}_\kappa)), \\ \gamma^{\kappa+1} : Q &\longrightarrow T^*Q \times \mathbb{R}, \qquad \mathbf{q}_{\kappa+1} \mapsto (\mathbf{q}_{\kappa+1}, \mathrm{d}S_d^\kappa(\mathbf{q}_{\kappa+1}), S_d^{\kappa+1}(\mathbf{q}_{\kappa+1})). \end{aligned} \tag{89}$$

According to the commutative diagram

$$\begin{array}{ccc} (\mathbf{q}_\kappa, \mathbf{p}_\kappa, s_\kappa) & \xrightarrow{\tilde{\Phi}} & (\mathbf{q}_{\kappa+1}, \mathbf{p}_{\kappa+1}, s_{\kappa+1}) \\ \gamma^\kappa \uparrow \downarrow \rho & & \rho \downarrow \uparrow \gamma^{\kappa+1} \\ (\mathbf{q}_\kappa, s_\kappa) & \xrightarrow{\tilde{\Phi}^{\gamma^\kappa}} & (\mathbf{q}_{\kappa+1}, s_{\kappa+1}) \end{array} \tag{90}$$

we pull down the flow $\tilde{\Phi}$ to the base manifold $Q$ and obtain the following projected discrete flow:

$$\tilde{\Phi}^\gamma : Q \longrightarrow Q, \qquad \mathbf{q}_\kappa \mapsto \mathbf{q}_{\kappa+1} = D_2 H_d^+\left(\mathbf{q}_\kappa, \mathrm{d}s_d^{\kappa+1}(\mathbf{q}_{\kappa+1}), S_d^\kappa(\mathbf{q}_\kappa)\right). \tag{91}$$

This procedure reduces the number of dependent variables to the coordinates $\mathbf{q}$ on the base manifold $Q$. By means of the section $\gamma$, one can lift a solution of the projected flow to the extended cotangent bundle level. The lifted solution becomes a solution to the Hamiltonian flow if and only if the commutation relation

$$\tilde{\Phi} \circ \gamma^\kappa = \gamma^{\kappa+1} \circ \tilde{\Phi}^\gamma. \tag{92}$$

holds.

**Theorem 3.** *For a section $\gamma$ admitting the local forms in (89), the following conditions are equivalent:*

1. *The flows $\tilde{\Phi}$ and $\tilde{\Phi}^\gamma$ commute, i.e., $\tilde{\Phi} \circ \gamma^\kappa = \gamma^{\kappa+1} \circ \tilde{\Phi}^\gamma$.*
2. *$S$ solves the HJ Equation (86).*

**Proof.** In terms of the local coordinates, the commutation condition (that is, the first of the conditions in the statement of the theorem) gives rise to the following equations:

$$\begin{aligned} S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) &= S_d^\kappa(\mathbf{q}_\kappa) + \mathrm{d}s_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) \cdot \mathbf{q}_{\kappa+1} - H_d^+\left(\mathbf{q}_\kappa, \mathrm{d}s_d^{\kappa+1}(\mathbf{q}_{\kappa+1}), S_d^\kappa(\mathbf{q}_\kappa)\right), \\ \mathrm{d}S_d^\kappa(\mathbf{q}_\kappa) &= D_1 H_d^+\left(\mathbf{q}_\kappa, \mathrm{d}S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}), S_d^\kappa(\mathbf{q}_\kappa)\right) \big/ \left(1 - D_3 H_d^+\left(\mathbf{q}_\kappa, \mathrm{d}S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}), S_d^\kappa(\mathbf{q}_\kappa)\right)\right). \end{aligned} \tag{93}$$

A direct comparison gives that the second condition in (93) is the infinitesimal version of the first one. Indeed, if we take the derivative of the first equation in (93) with respect to $\mathbf{q}_\kappa$, we have that

$$0 = \mathrm{d}S_d^\kappa(\mathbf{q}_\kappa) - D_1 H_d^+\left(\mathbf{q}_\kappa, \mathrm{d}S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}), S_d^\kappa(\mathbf{q}_\kappa)\right) - D_3 H_d^+\left(\mathbf{q}_\kappa, \mathrm{d}S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}), S_d^\kappa(\mathbf{q}_\kappa)\right) \mathrm{d}S_d^\kappa(\mathbf{q}_\kappa) \tag{94}$$

By collecting all the terms involving $\mathrm{d}S_d^\kappa$ on the left-hand side, it is immediate to see that it is precisely the second condition in (93). We remark that the first equation is the discrete Hamilton–Jacobi Equation (86). The inverse of the assertion is straightforwardly proved by reversing the arguments. □

## 4. Application

### 4.1. Free Single Particle Contact Hamiltonian

Given a mechanical contact Lagrangian with an Euclidean metric and a potential function $V : Q \to \mathbb{R}$ of the type

$$L(q, \dot{q}, z) = \frac{1}{2}\dot{q}^2 - V(q) + \gamma z, \quad (q, \dot{q}, z) \in TQ \times \mathbb{R}, \quad \gamma > 0.$$

one usually approximates the exact discrete Lagrangian associated to $L$ by means of a quadrature rule. Note that the restriction of $\gamma$ to positive values is necessary to model a withholding oscillator receiving energy, though we could define the integrator for any value of $\gamma \in \mathbb{R}$. If we use the middle point rule to approximate the positions, i.e., $q \approx \frac{q_{k+1}+q_k}{2}$, one may define the discrete Lagrangian $L_d : Q \times Q \times \mathbb{R} \to \mathbb{R}$ in the following way:

$$L_d(q_k, q_{k+1}, s_k) = \frac{1}{2h}(q_{k+1} - q_k)^2 - hV\left(\frac{q_{k+1} + q_k}{2}\right) + h\gamma s_k.$$

We remark that the value of $h$ should be chosen small. In this case, the right Hamiltonian reads

$$H^+(q_k, p_{k+1}, s_k) = \frac{1}{2}hp_{k+1}^2 + p_{k+1}q_k - h\gamma s_k. \tag{95}$$

If we compute the Hamilton–Jacobi equation with this Hamiltonian, we obtain

$$S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) - S_d^{\kappa}(\mathbf{q}_\kappa) - \mathrm{d}S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) \cdot \mathbf{q}_{\kappa+1} + \frac{1}{2}hp_{k+1}^2 + p_{k+1}q_k - h\gamma s_k = 0. \tag{96}$$

We integrate (96) numerically with a first-order Euler method and fixed step $h = 0.001$. The choice of a not-so-relatively small $h$ is due to instability or numerical oscillations of the first-order Euler method in stiff systems. Since our purpose is to show the qualitative behavior of the first derivative of the generating function in comparison with the momentum, we leave higher-order precision methods for further research. In the following graphics, one can see that $p$ and $dS$ qualitatively behave in the same way for the same set of initial values $(q_0 = 1, p_0 = 1, s_0 = 0.1, \gamma = 0.1)$. The figures were generated with the library *matplolib* (Figures 1 and 2).

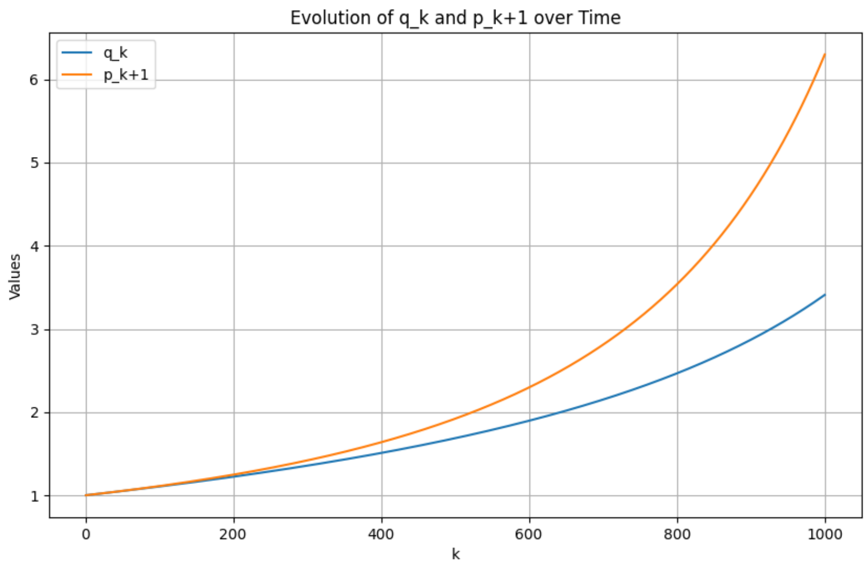



**Figure 1.** $q$ vs. $p$.

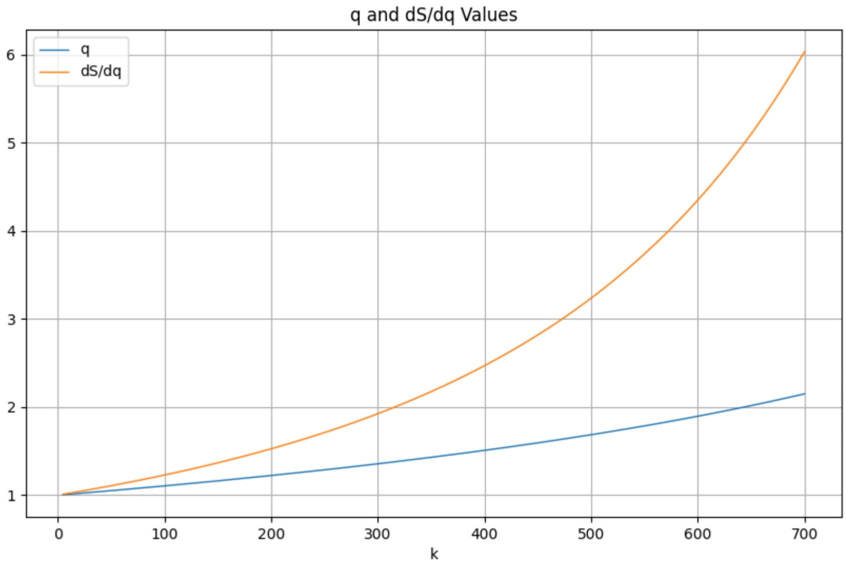



**Figure 2.** $q$ vs. $dS$.

In the case that we choose a proper $V$ for the description of a parachute, we obtain the Hamiltonian

$$H_d^+(q_\kappa, p_{\kappa+1}, s_\kappa) = \frac{1}{2m}(p_{\kappa+1} + 2\lambda s_\kappa)^2 + \frac{mg}{2\lambda}(e^{2\lambda q_\kappa} - 1), \tag{97}$$

where $\lambda, g \in \mathbb{R}$.

This Hamiltonian describes the vertical motion of a particle falling in a fluid under the action of constant gravity. The friction is modeled by the drag equation, and it is proportional to the square of the velocity.

When calculating the discrete right contact Hamiltonian equations, we have the following discrete dynamics:

$$\begin{aligned} q_{\kappa+1} &= \frac{1}{m}(p_{\kappa+1} + 2\lambda s_\kappa), \\ p_\kappa &= \frac{mge^{2\lambda q_\kappa}}{1 - \frac{2\lambda}{m}(p_{\kappa+1} + 2\lambda s_\kappa)}, \\ s_{\kappa+1} &= s_\kappa + \frac{p_{\kappa+1}^2}{2m} - \frac{2\lambda^2 s_\kappa}{m} - \frac{mg}{2\lambda}\left(e^{2\lambda q_\kappa} - 1\right) \end{aligned}$$

This dynamic is represented in the following diagram obtained by plotting the points $(q(\kappa), p(\kappa), s(\kappa))$, which have been integrated recursively for different values of $\kappa$. For $\lambda = -0.01$, $m = 1$, $g = 10$ and initial condition $(q(0), p(0), s(0)) = (100, 1, 1)$, we obtain the dynamics of the parachute as follows.

Here, it is easy to see from Figures 3 and 4 that the position coordinate (red line on the left) descends from the initial position $q(0) = 100$ until the parachute reaches the ground. The momentum (blue line on the left) grows until it crosses the red line. This crossing point represents the instant in which the parachute opens and the momentum decreases until the parachute touches the ground, i.e., the parachute decelerates so that the parachuter can touch the ground safely. Nonetheless, as it happens, the velocity is not zero when the parachuter touches the ground.

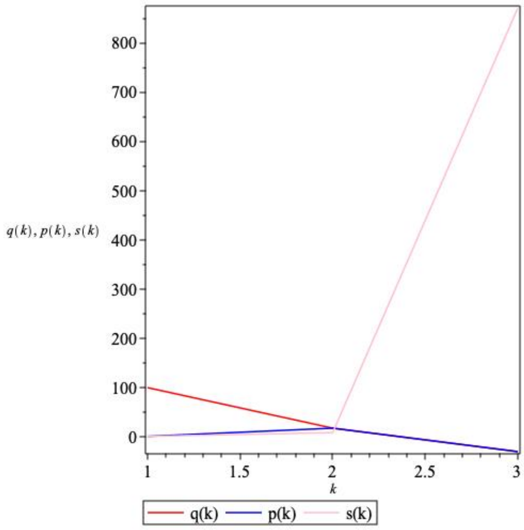



**Figure 3.** Dynamics of the parachute.

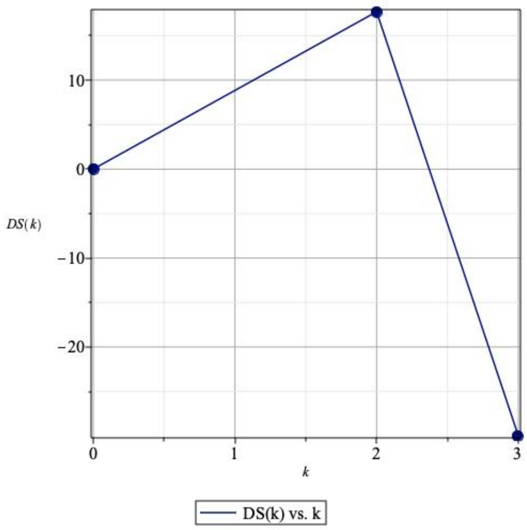



**Figure 4.** Evolution of $DS(k)$.

Now, in order to prove the accuracy of the discrete contact Hamilton–Jacobi equation, we plot the differential of the generating function, i.e., $DS^{\kappa}(q_{\kappa})$, which, according to the first equation in (66), should match the dynamic of the blue line in the diagram depicted above. Let us corroborate this fact by plotting $\gamma = DS^{\kappa}(q_{\kappa})$ for different discrete values of $\kappa$. For $\lambda = -0.01$, $m = 1$, $g = 10$ and initial condition $(q(0), S(0), s(0)) = (100, 1, 1)$, we obtain Figure 2.

It is easy to check that the line representing $DS(k)$ (blue line on the right in Figures 3 and 4) is very similar to the blue line depicted on the left. Indeed, both lines should be equal (see that for $k = 2$, both reach the maximum momentum $p$) since in the vicinity of the discrete Hamilton–Jacobi equation, $p(k) = DS(k)$.

### 4.2. Rayleigh Systems

Physically, the Rayleigh equation represents systems with oscillatory motion influenced by nonlinear damping forces. Consider the discrete left Rayleigh Hamiltonian with a time-dependent mass which accounts for the third parameter in the contact framework [60].

$$H_d^-(q_{j+1}, p_j, s_{j+1}) = -p_j q_j - \frac{h}{2} s_{j+1} p_{j+1}^2 + \frac{h}{2} k q_{j+1}^2. \tag{98}$$

By making use of the left discrete contact HJ equation this time and using the formulation in (87), we obtain the equation

$$S_d^{\kappa+1}(\mathbf{q}_{\kappa+1}) - S_d^{\kappa}(\mathbf{q}_\kappa) + \mathrm{d}S_d^{\kappa}(\mathbf{q}_\kappa) \cdot \mathbf{q}_\kappa - p_j q_j - \frac{h}{2} s_{j+1} p_{j+1}^2 + \frac{h}{2} k q_{j+1}^2. \tag{99}$$

We integrate (99) numerically with a first-order Euler method and fixed step $h = 0.1$. The choice of a not-so-relatively small $h$ is due to instability or numerical oscillations of the first-order Euler method. We represent the momentum versus the discrete-time step with conditions for $j = 0$ and $(q(0) = 0, p(0) = 1, s(0) = 1)$. It is easy to see that the Rayleigh systems end up losing all the momentum stored and the oscillations tend to zero because of dissipation. One can check a more precise simulation in [60], but even a few-step integration like ours already shows the same qualitative behavior that our collaborators previously detected in [60] (Figure 5).

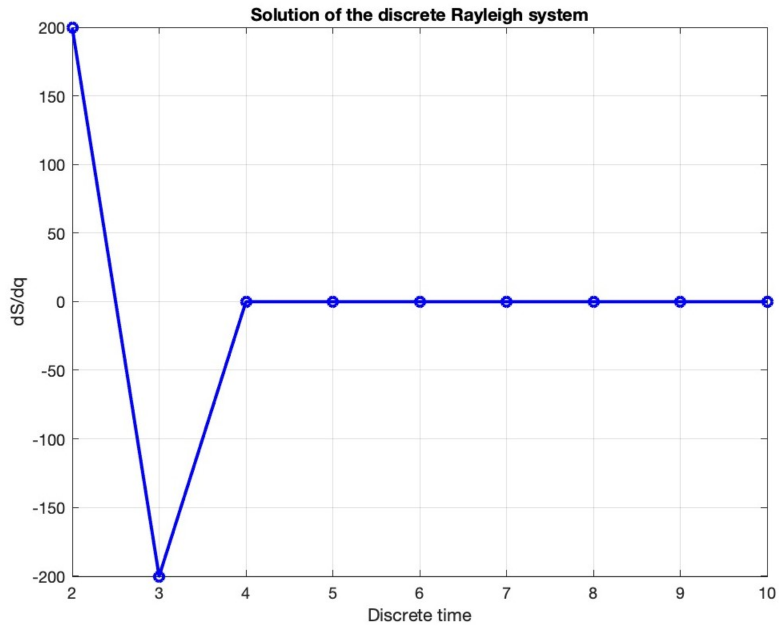



**Figure 5.** Solution of the discrete Rayleigh system.

## 5. Commentaries

In this paper, we have proposed a discrete Hamilton–Jacobi equation (stated in Theorem 2) for contact discrete Hamiltonian dynamics. We have presented its geometric foundations in terms of contact discrete flows in Theorem 3. We have exhibited the relationship between the continuous geometric HJ Theorem 1 for contact Hamiltonian dynamics and Theorem 3 for discrete contact Hamiltonian dynamics. One may wonder if all of these examples could be solved with the aid of other theories, and the answer is yes. Indeed, it would be fairly easy to retrieve the equations of the discrete contact dynamics by using the discrete contact Hamilton equations. Nonetheless, here, we have made the choice of developing a Hamilton–Jacobi theory so that we can also obtain the generating functions, which are crucial in the identification of conserved quantities and symmetries, and they serve as a bridge to the quantum formulation.

We wish to continue in the following directions:

- Contact Hamiltonian dynamics do not preserve the Hamiltonian function. There exists an alternative characterization of Hamiltonian dynamics on contact manifolds that preserves the energy, known as evolution dynamics [50,61]. We wish to examine the discretization of evolution dynamics and their HJ formulation.
- For the extended cotangent bundle $T^*Q \times \mathbb{R}$, the continuous HJ theory for contact Hamiltonian dynamics was presented in [11,58,59]. The authors consider the base

manifold as the extended configuration space $Q \times \mathbb{R}$. In this work, we consider the base manifold to be $Q$. We wish to write a discrete HJ equation on contact manifolds with base manifold $Q \times \mathbb{R}$.

- If a Lagrangian is degenerate, then one cannot arrive at explicit Euler–Lagrange equations. In this case, the Legendre transformation is not immediate. Tulczyjew's triple is a geometric formulation that allows us to also achieve this in singular cases [62]. In a discrete framework, Tulczyjew's triple was constructed in [63]. This determines a proper geometry for implicit discrete Lagrangian and Hamiltonian dynamics [64]. For the continuous case, a geometric HJ theory has been recently given in [18,65] in the symplectic framework and in [59] in the contact framework. We wish to concentrate on generalizing the discrete HJ theories both for symplectic and contact geometry including the implicit case. On the other hand, Tulczyjew's triple for contact geometry has been recently constructed in [57]. In the future, we aim at constructing a discrete contact Tulczyjew's triple.
- One could wonder what are the limitations of the discrete contact Hamilton–Jacobi equation. The equation has been exactly derived, so the limitations are up to the numerical integration of choice. For a simple visualization like the one we have displayed, the behavior is fair, given that the Euler method is the simplest method we can choose and the number of iterations is small. One could greatly diminish the error with a different integration procedure.
- In the limit of very big number of steps in the discretization, one can easily depict that one retrieves the continuous contact HJ equation.
- In the future, we would like to explore more intricate examples and implement more sophisticated numerical methods for the integration, since Euler is unstable. As we have mentioned, in modern control systems, controllers operate on discrete-time signals, making the discretization of continuous-time models necessary.
- The discretization of field theories and the formulation of a corresponding HJ theory to study families of discrete Hamiltonian mappings as in [66] and the effects of diffusion.
- Study of integrability properties of discrete solitons as in [67] by developing a discrete field HJ theory.
- To redefine step by step our HJ theory to solve more real-world scenarios. The generalization of an HJ theory to classical mechanics and field theory from a discrete or quantum point of view will bring several potential difficulties. These can arise from both the theoretical aspects of the method and the practical considerations of implementation. Some challenges will be nonlinear effects, chaos and sensitivity, as in [66–68].
- To find specific geometric integrators that preserve the contact discrete geometry. In this paper, we have used a very simple Euler method to depict the dynamic of simple discrete contact systems. Nonetheless, the Euler method does not preserve the geometric background of the system, and this is why we need to come up with specific integrators for these systems.

**Author Contributions:** Conceptualization, O.E.; Methodology, O.E. and C.S.; Investigation, O.E. and C.S.; Writing—original draft, O.E. and C.S.; Writing—review & editing, C.S. and M.Z. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was partially supported by MINECO MTM 2013-42-870-P and ICMAT Severo Ochoa project SEV-2011-0087.

**Data Availability Statement:** The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

**Acknowledgments:** Marcin Zajac and Cristina Sardón acknowledge the hospitability of Universidad Politécnica de Madrid and the department of Applied Mathematics at ETSII for hosting Marcin Zajac to work alongside Cristina Sardón.

**Conflicts of Interest:** The authors declare no conflict of interest.

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