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Pairs of Positive Solutions for a Carrier $p(x)$-Laplacian Type Equation

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Abstract: The existence of multiple pairs of smooth positive solutions for a Carrier problem, driven by a $p(x)$-Laplacian operator, is studied. The approach adopted combines sub-super solutions, truncation, and variational techniques. In particular, after an explicit computation of a sub-solution, obtained combining a monotonicity type hypothesis on the reaction term and the Giacomoni–Takáč's version of the celebrated Díaz–Saá's inequality, we derive a multiplicity of solution by investigating an associated one-dimensional fixed point problem. The nonlocal term involved may be a sign-changing function and permit us to obtain the existence of multiple pairs of positive solutions, one for each "positive bump" of the nonlocal term. A new result, also for a constant exponent, is established and an illustrative example is proposed.

Keywords: $p(x)$-Laplacian; variable exponent; multiple positive solutions; variational methods; sub-super solutions methods; fixed-point methods; truncation techniques

MSC: 35J62; 35J92; 35B09; 35Q74

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1. Introduction

Since the work of G. R. Kirchhoff [1] and G. F. Carrier [2], owing also to the fine and seminal investigations developed by J. L. Lions [3], nonlocal problems have been widely studied due to their high applicability in biology, engineering, and physics; see, for instance, [4–10]. For example, the nonlocal elliptic problem can be used to describe population diffusion where the velocity of the dispersion depends on the whole population; see [11–13]. Meanwhile, [2] is devoted to the study of deflection of beams. Additionally, see also [14,15] and the references therein for a more general overview on the findings on these topics.

We emphasize that the $p(x)$-Laplacian operator is a gainful generalization of the classical p-Laplacian, since it allows us to consider more general and tricky physical aspects. The most famous results have been obtained in [16] (Section 14.4) and [17,18], which are devoted to studying models on electrorheological fluids, which are viscous fluids with the characteristic of changing, when activated by an electric field, their mechanical properties, changing from liquid to gel, reversibly, in a few milliseconds. See [18,19] for applications to actuators, clutches, shock absorbers, and rehabilitation equipment. Another fascinating application of this operator is the process of image restoration, as shown in [20–22] and the references therein. Other useful applications can be found for biological aspects in [23] and for chemical reactions and fluid dynamics in [24,25].

Here, we consider the following Carrier $p(x)$-Laplacian type equation with Dirichlet boundary value conditions:

$$\begin{cases} -a\left(\int_\Omega u^q dx\right)\Delta_{p(x)}u = \beta(x)f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \ \partial\Omega, \end{cases} \qquad (P_{p(x),\beta,f})$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with C^2-boundary, $q \geq 1$ and

$$\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$$

denotes the $p(x)$-Laplacian operator, with p being a function belonging to $C^1(\bar{\Omega})$ such that

(p_1) $1 < p_- := \min_{x\in\Omega} p(x) \leq p(x) \leq p_+ := \max_{x\in\Omega} p(x) < N$;

(p_2) *there exists a vector* $l \in \mathbb{R}^N \setminus \{0\}$ *such that for all* $x \in \Omega$ *the function* $p(x + tl)$ *is monotone for* $t \in \Sigma = \{t : x + tl \in \Omega\}$.

The first eigenvalue and eigenfunction of $\left(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)\right)$ are indicated with λ_1 and φ_1, respectively, and let K be a positive integer; in addition, we assume that

(i_0) $a : [0, +\infty) \to \mathbb{R}$ *is a continuous function and there exist positive numbers* $0 =: \alpha_0 \leq \alpha_1 < \alpha_2 \leq \alpha_3 < \alpha_4 \leq \ldots \leq \alpha_{2k-1} < \alpha_{2k}$ *such that* $a > 0$ *in* $(\alpha_{2k-1}, \alpha_{2k})$ *and* $a(\alpha_{2k-1}) = a(\alpha_{2k}) = 0$ *for all* $k \in \{1, \ldots, K\}$.

(i_1) $\beta \in L^\infty(\Omega)$, *with* $\underline{\beta} := \operatorname{essinf} \beta > 0$, $f : \mathbb{R} \to \mathbb{R}$ *is a function and there exists* $t_* > 0$ *such that* $f(t) > 0$ *in* $(0, t_*)$, $f(t_*) = 0$, $f \in C([0, t_*])$, $r \in [1, p_-]$, *and the map*

$$(0, t^*) \ni t \overset{\psi_r}{\mapsto} \frac{f(t)}{t^{r-1}}$$

is strictly decreasing.

(i_2) $\alpha_{2K} < \frac{t_*^q}{\|\varphi_1\|_\infty^q} \int_\Omega \varphi_1^q dx.$

(i_3) $\max_{\alpha\in[\alpha_{2k-1},\alpha_{2k}]} a(\alpha) < \frac{\underline{\beta}\gamma\|\varphi_1\|_\infty^{r-1}}{\lambda_1 \max\left\{1, \|\varphi_1\|_\infty^{p_+-1}\right\}}$ *for all* $k \in \{1, \ldots, K\}$, *being* $\gamma := \lim_{t\to 0^+} \frac{f(t)}{t^{r-1}}$.
Put

$$\alpha_* = \left[\frac{2}{\sqrt{\pi}}\left(1 - \frac{1}{p_+}\right)\left(1 + \frac{1}{N}\right)\Gamma\left(1 + \frac{N}{2}\right)^{\frac{1}{N}}\right]^q |\Omega|^{1+\frac{q}{N}},$$

with Γ being the Euler gamma function.

(i_4) *One has that, for all* $k \in \{1, \ldots, K\}$ *such that* $\alpha_* < \alpha_{2k-1}$, *there exists* $\alpha_k \in (\alpha_{2k-1}, \alpha_{2k})$ *satisfying one of the following conditions:*

(i_{4_1}) $a(\alpha_k)p_- > \frac{2}{N\sqrt{\pi}}\left(|\Omega|\Gamma(1 + \frac{N}{2})\right)^{\frac{1}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)$;

(i_{4_2}) $a(\alpha_k)\alpha_k^{\frac{p_--1}{q}} > \left(\frac{(N+1)(p_+-1)}{p_+}\right)^{p_--1} \left(\frac{2^{p_-}(\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}}}{p_-(N\sqrt{\pi})^{p_-}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)\right) |\Omega|^{\frac{p_-}{N}+\frac{p_--1}{q}}$,

if $a(\alpha_k)p_- \leq \frac{2}{N\sqrt{\pi}}\left(|\Omega|\Gamma(1 + \frac{N}{2})\right)^{\frac{1}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)$.

Remark 1. *Regarding our hypotheses, the following is worth noting: the first eigenpair* (λ_1, φ_1) *of* $\left(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)\right)$ *satisfies* $\varphi_1 \in \operatorname{int}(C_+)$ *(p. 56 and p. 76, [15]), being*

$$\operatorname{int}(C_+) := \left\{u \in C_0^1(\bar{\Omega}) : u > 0, \forall x \in \Omega, \text{ and }, \frac{\partial u}{\partial \nu} < 0, \forall x \in \partial\Omega\right\},$$

and hypotheses (p_2) *guarantees us that* $\lambda_1 > 0$ *(Theorem 3.3, [26]); see also, [27].*

In (i_4), you can find the constant

$$C_0 = \frac{\left(\Gamma(1+\frac{N}{2})\right)^{\frac{1}{N}}}{N\sqrt{\pi}}, \tag{1}$$

which is the best constant in the continuous embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, which is related to the classical isoperimetric inequality in $\mathbb{R}^N$; see, for instance, [28] (p. 46) or [29] (p. 355).

The nonlocal term a admits a finite number of "positive bumps" and it can be a continuously changing sign function as in (Theorem 2.1, [30]).

This paper follows some aspects introduced in [31–33] for the Laplacian case and, more recently, in [30], where a p-Laplacian operator with $1 < p < +\infty$ is considered. However, with respect to these previous works, here, to overcome the difficulties arising from the lack of homogeneity of the $p(x)$-Laplacian operator, we arrange a similar approach combining the sub-super solutions, truncation, and variational techniques, inspired by the classic result of Brezis and Nirenberg [34]; see also the more recent papers [35–41].

For completeness, we should mention that variational and non-variational techniques have been successfully used to study variable growth problems; see, for instance, [26,42–48] and the related references. In particular for sub-super solutions approaches, we refer the reader to [49].

Our paper is organized as follows: Section 2 is dedicated to introducing the mathematical background; Section 3 contains the proof of the main results, where first we obtain an explicit computation of a sub-solution for problem ($P_{p(x),\beta,f}$) and then the multiplicity of solutions as fixed points of a suitable continuous map.

2. Mathematical Background

2.1. Setting of Function Spaces

We introduce the functional space in which we set our problem; see [50,51] for further details. The variable exponent Lebesgue space, $L^{p(x)}(\Omega)$, is defined as

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \rho_{p(x)}(u) < +\infty \right\},$$

where

$$\rho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)} dx$$

is called the modular of $L^{p(x)}(\Omega)$. We endow these spaces with the Luxemburg norm, i.e.,

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Moreover, we consider

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}.$$

We should point out that on $W_0^{1,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,p(x)}(\Omega)}}$, due to the Poincaré's inequality on variable exponent spaces (Proposition 2.5(iii) [51]), we can use the equivalent norm

$$\|u\| = \||\nabla u|\|_{L^{p(x)}(\Omega)}$$

(see also [26,42,52]). Now, we state a useful lemma.

Lemma 1. *(Theorem 1.2, [45]) Let $u \in W_0^{1,p(x)}(\Omega)$ and*

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx.$$

Then, the following hold:

1. $\|u\| < 1 (= 1; > 1) \Leftrightarrow \rho_{p(x)}(\nabla u) < 1 (= 1; > 1)$;
2. *if* $\|u\| > 1$, *then* $\frac{1}{p_+}\|u\|^{p_-} \le \Phi(u) \le \frac{1}{p_-}\|u\|^{p_+}$;
3. *if* $\|u\| < 1$, *then* $\frac{1}{p_+}\|u\|^{p_+} \le \Phi(u) \le \frac{1}{p_-}\|u\|^{p_-}$.

In particular, Φ is a coercive functional.

For the reader's convenience, we recall some valuable properties of the $p(x)$-Laplace operator. To have a complete overview on these arguments, see [48,52,53].

Lemma 2. *(Theorem 3.1, [48]) Let $L : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}\right)^*$ such that*

$$\langle Lu, v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega). \tag{2}$$

Then, $L : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}\right)^*$ is a continuous bounded operator, strictly monotone and of (S_+)-type, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $\limsup_{n\to+\infty} \langle -\Delta_{p(x)} u_n, u_n - u \rangle \le 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

2.2. Sub-Super Solutions with Variational Structure

In order to apply our techniques, we will use an auxiliary problem:

$$\begin{cases} -a(\alpha)\Delta_{p(x)} u = \beta(x) f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{k,\alpha,\beta f}$}$$

with $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, $k \in \{1, \ldots, K\}$.

As in [49] (Definition 2.1) or [54] (Definition 11.5), we define a sub- and super-solution for problem ($P_{k,\alpha,\beta f}$) as follows:

Definition 1. *Let $L_+^{p(x)}(\Omega) = \{v \in L^{p(x)}(\Omega) : v \ge 0\}$. We say that $\underline{u} \in W^{1,p(x)}(\Omega)$ is a sub-solution for ($P_{k,\alpha,\beta f}$) if $\underline{u} \le 0$ on $\partial\Omega$ and*

$$a(\alpha) \int_\Omega |\nabla \underline{u}|^{p(x)-2} \nabla \underline{u} \nabla v dx \le \int_\Omega \beta(x) f(\underline{u}) v dx \quad \forall v \in W_0^{1,p(x)}(\Omega) \cap L_+^{p(x)}(\Omega).$$

Analogously, $\bar{u} \in W^{1,p(x)}(\Omega)$ is called a super-solution for ($P_{k,\alpha,\beta f}$) if $\bar{u} \ge 0$ on $\partial\Omega$ and

$$a(\alpha) \int_\Omega |\nabla \bar{u}|^{p(x)-2} \nabla \bar{u} \nabla v dx \ge \int_\Omega \beta(x) f(\bar{u}) v dx \quad \forall v \in W_0^{1,p(x)}(\Omega) \cap L_+^{p(x)}(\Omega).$$

Similarly, $u \in W_0^{1,p(x)}(\Omega)$ is called a (weak) solution for ($P_{k,\alpha,\beta f}$) if $u = 0$ on $\partial\Omega$ and

$$a(\alpha) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx = \int_\Omega \beta(x) f(u) v dx \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

In all three cases, the definition holds true, provided that the right-hand side is well defined. To the truncation of f, we put

$$f_*(t) = \begin{cases} f(\underline{u}) & \text{if} \quad t \leq \underline{u}, \\ f(t) & \text{if} \quad \underline{u} \leq t \leq \bar{u}, \\ f(\bar{u}) & \text{if} \quad t \geq \bar{u}, \end{cases} \tag{3}$$

and we indicate the related problem with $(P_{k,\alpha,\beta f_*})$. Now, by (3), we prove that every weak solution of $(P_{k,\alpha,\beta f_*})$ is a weak solution of $(P_{k,\alpha,\beta f})$ (see, for instance, [34] for $p(x) = 2$, (Proposition 11.8, [54]) for $p(x) \equiv p$, $1 < p < +\infty$ and (Theorem 3.3, [49]) for variable exponent).

Lemma 3. *Let $\underline{u}$ be a sub-solution and let $\bar{u}$ be a super-solution of $(P_{k,\alpha,\beta f})$, such that $\underline{u} \leq \bar{u}$. Then, every solution u of $(P_{k,\alpha,\beta f_*})$ is such that $\underline{u} \leq u \leq \bar{u}$, that is u is a weak solution of $(P_{k,\alpha,\beta f})$.*

Proof. Assume there exists a weak solution u of $(P_{k,\alpha,\beta f_*})$, then it satisfies

$$a(\alpha) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx = \int_\Omega \beta(x) f_*(u) v dx \quad \forall v \in W_0^{1,p(x)}(\Omega). \tag{4}$$

We will indicate with $u^+ = \max\{0, u\}$ and $u^- = \max\{0, -u\}$. It is clear that $u = u^+ - u^-$. Now, choosing, as test functions, $v_1 = (u - \underline{u})^-$ and $v_2 = (u - \bar{u})^+$, which, lying in $W_0^{1,p(x)}$, from (4) (see Remark 1.35, [54]), we get, first

$$a(\alpha) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla (u - \underline{u})^- dx = \int_\Omega \beta(x) f_*(u)(u - \underline{u})^- dx,$$

i.e.,

$$a(\alpha) \int_{\{u<\underline{u}\}} |\nabla u|^{p(x)-2} \nabla u \nabla (u - \underline{u}) dx = \int_{\{u<\underline{u}\}} \beta(x) f(\underline{u})(u - \underline{u}) dx. \tag{5}$$

On the other hand, $\underline{u}$ is a sub-solution for $(P_{k,\alpha,\beta f})$ and by (3), one has

$$a(\alpha) \int_\Omega |\nabla \underline{u}|^{p(x)-2} \nabla \underline{u} \nabla (u - \underline{u})^- dx \leq \int_\Omega \beta(x) f_*(u)(u - \underline{u})^- dx,$$

that is,

$$-a(\alpha) \int_{\{u<\underline{u}\}} |\nabla \underline{u}|^{p(x)-2} \nabla \underline{u} \nabla (u - \underline{u}) dx \leq - \int_{\{u<\underline{u}\}} \beta(x) f(\underline{u})(u - \underline{u}) dx. \tag{6}$$

Adding (5) and (6), we get

$$a(\alpha) \int_{\{u<\underline{u}\}} \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla \underline{u}|^{p(x)-2} \nabla \underline{u} \right) \nabla (u - \underline{u}) dx \leq 0. \tag{7}$$

Next, for $v = v_2$, we have

$$a(\alpha) \int_{\{u>\bar{u}\}} |\nabla u|^{p(x)-2} \nabla u \nabla (u - \bar{u}) dx = \int_{\{u>\bar{u}\}} \beta(x) f(\bar{u})(u - \bar{u}) dx. \tag{8}$$

Moreover, since $\bar{u}$ is a super-solution for $(P_{k,\alpha,\beta f})$ and by (3), we have

$$a(\alpha) \int_\Omega |\nabla \bar{u}|^{p(x)-2} \nabla \bar{u} \nabla (u - \bar{u})^+ dx \geq \int_\Omega \beta(x) f_*(u)(u - \bar{u})^+ dx,$$

i.e.,

$$a(\alpha)\int_{\{u>\bar{u}\}}|\nabla\bar{u}|^{p(x)-2}\nabla\bar{u}\nabla(u-\bar{u})dx\geq\int_{\{u>\bar{u}\}}\beta(x)f(\bar{u})(u-\bar{u})dx.$$

Then, subtracting from the previous inequality (8), we get

$$a(\alpha)\int_{\{u>\bar{u}\}}\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla\bar{u}|^{p(x)-2}\nabla\bar{u}\right)\nabla(u-\bar{u})dx\leq 0. \tag{9}$$

Since $a(\alpha)>0$, the strictly monotonicity of $-\Delta_{p(x)}$ (Lemma 2) ensures that the measure of the sub-levels $\{u<\underline{u}\}$ and $\{u>\bar{u}\}$ are zero; then, $\underline{u}\leq u\leq\bar{u}$. Finally, we highlight that, by (3), every weak solution of $(P_{k,\alpha,\beta f_*})$ is a weak solution of $(P_{k,\alpha,\beta f})$. □

3. Main Results

Our main result is the following.

Theorem 1. *Assume that (p_1), (p_2), (i_0), (i_1), (i_2), (i_3), and (i_4) hold. Then, for all $k\in\{1,\dots,K\}$, such that $\alpha_*<\alpha_{2k-1}$, the problem $(P_{p(x),\beta,f})$ admits at least $K-k+1$ pair of solutions $u_{k,1},u_{k,2}\in\operatorname{int}(C_+)$ such that*

$$\alpha_{2k-1}<\int_\Omega u_{k,1}^q dx<\int_\Omega u_{k,2}^q dx<\alpha_{2k}\quad\forall k\in\{1,\dots,K\}.$$

We split the proof of Theorem 1 in some steps. Clearly, by (i_1), since $f(t_*)=0$, $\bar{u}=t_*$ turns to be a super-solution for problem $(P_{k,\alpha,\beta f})$. In the following lemma, we show the existence of a sub-solution for $(P_{k,\alpha,\beta f})$.

Lemma 4. *Assume that (i_0), (i_1), and (i_3) hold. For every $k\in\{1,\dots,K\}$ and $\alpha\in(\alpha_{2k-1},\alpha_{2k})$, let us consider the following function:*

$$z_\alpha=\begin{cases}\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)\frac{\varphi_1}{\|\varphi_1\|_\infty} & \text{if}\quad \|\varphi_1\|_\infty\geq\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right);\\ c\frac{\varphi_1}{\|\varphi_1\|_\infty} & \text{if}\quad \|\varphi_1\|_\infty<\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right),\end{cases} \tag{10}$$

where $k_\alpha=\frac{a(\alpha)\lambda_1\max\left\{1,\|\varphi_1\|_\infty^{p_+-1}\right\}}{\|\varphi_1\|_\infty^{r-1}}$ and $\frac{\|\varphi_1\|_\infty}{m}\leq c\leq\|\varphi_1\|_\infty$, being $m\geq\frac{\|\varphi_1\|_\infty}{t_*}$, and $\psi_r^{-1}:(0,\gamma)\to(0,t^*)$ being the inverse function of ψ_r. Then, z_α is a sub-solution for $(P_{k,\alpha,\beta f})$.

Proof. From (i_1), we have that the map $\psi_r:t\mapsto\frac{f(t)}{t^{r-1}}$ is strictly decreasing, and by (i_3), one has $\frac{k_\alpha}{\underline{\beta}}<\gamma$; then, there exists the inverse function $\psi_r^{-1}:(0,\gamma)\to(0,t_*)$, so z_α is well posed. Furthermore, it is not restrictive to assume that $\gamma<+\infty$, because, if $\gamma=+\infty$ we can argue in a similar way. By definitions of z_α and ψ_r^{-1}, we have $0<z_\alpha<t_*$, and we get

$$\frac{k_\alpha}{\underline{\beta}}=\frac{f\left(\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)\right)}{\left(\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)\right)^{r-1}}\leq\frac{f(z_\alpha)}{z_\alpha^{r-1}}, \tag{11}$$

that is, we obtain

$$k_\alpha z_\alpha^{r-1}\leq\underline{\beta}f(z_\alpha)\leq\beta(x)f(z_\alpha),\quad\text{for a.a. } x\in\Omega. \tag{12}$$

Now, we analyze the two cases: *Case 1:* $\|\varphi_1\|_\infty<\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)$. For every $v\in W_0^{1,p(x)}(\Omega)$, with $v\geq 0$, arguing by duality, we get

$$\begin{aligned}
a(\alpha)\int_\Omega |\nabla z_\alpha|^{p(x)-2}\nabla z_\alpha \nabla v dx &= a(\alpha)\int_\Omega \left|\nabla\left(\frac{c}{\|\varphi_1\|_\infty}\right)\varphi_1\right|^{p(x)-2}\nabla\left(\frac{c}{\|\varphi_1\|_\infty}\varphi_1\right)\nabla v\, dx \\
&= a(\alpha)\int_\Omega \left|\left(\frac{c}{\|\varphi_1\|_\infty}\right)\right|^{p(x)-1}|\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla\, vdx \\
&\leq a(\alpha)\left(\frac{c}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega |\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla\, vdx \\
&= a(\alpha)\lambda_1\left(\frac{c}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega \varphi_1^{p(x)-1} vdx \\
&= a(\alpha)\lambda_1\left(\frac{c}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega \frac{\|\varphi_1\|_\infty^{p(x)-1}}{\|\varphi_1\|_\infty^{p(x)-1}}\varphi_1^{p(x)-1} vdx \\
&\leq \frac{a(\alpha)\lambda_1\max\{1,\|\varphi_1\|_\infty^{p_+-1}\}}{\|\varphi_1\|_\infty^{r-1}}\int_\Omega c^{r-1}\left(\frac{\varphi_1}{\|\varphi_1\|_\infty}\right)^{p(x)-1} vdx \\
&\leq k_\alpha\int_\Omega c^{r-1}\left(\frac{\varphi_1}{\|\varphi_1\|_\infty}\right)^{r-1} vdx \\
&= \int_\Omega k_\alpha z_\alpha^{r-1} vdx \leq \int_\Omega \beta(x) f(z_\alpha) vdx.
\end{aligned}$$

Case 2: $\|\varphi_1\|_\infty \geq \psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)$, i.e., $\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty} \leq 1$. As above, we have

$$\begin{aligned}
& a(\alpha)\int_\Omega |\nabla z_\alpha|^{p(x)-2}\nabla z_\alpha\nabla\, vdx \\
= {} & a(\alpha)\int_\Omega \left|\nabla\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\varphi_1\right)\right|^{p(x)-2}\nabla\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\varphi_1\right)\nabla\, vdx \\
= {} & a(\alpha)\int_\Omega \left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{p(x)-1}|\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla\, vdx \\
\leq {} & a(\alpha)\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega |\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla\, vdx \\
= {} & a(\alpha)\lambda_1\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega \varphi_1^{p(x)-1} vdx \\
= {} & a(\alpha)\lambda_1\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega \frac{\|\varphi_1\|_\infty^{p(x)-1}}{\|\varphi_1\|_\infty^{p(x)-1}}\varphi_1^{p(x)-1} vdx \\
\leq {} & a(\alpha)\lambda_1\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1}\max\{1,\|\varphi_1\|_\infty^{p_+-1}\}\int_\Omega \frac{\varphi_1^{p(x)-1}}{\|\varphi_1\|_\infty^{p(x)-1}} vdx \\
\leq {} & \frac{a(\alpha)\lambda_1\max\{1,\|\varphi_1\|_\infty^{p_+-1}\}}{\|\varphi_1\|_\infty^{r-1}}\int_\Omega\left(\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)\frac{\varphi_1}{\|\varphi_1\|_\infty}\right)^{r-1} vdx \\
= {} & \frac{a(\alpha)\lambda_1\max\{1,\|\varphi_1\|_\infty^{p_+-1}\}}{\|\varphi_1\|_\infty^{r-1}}\int_\Omega z_\alpha^{r-1} vdx \\
= {} & \int_\Omega k_\alpha z_\alpha^{r-1} vdx \leq \int_\Omega \beta(x) f(z_\alpha) vdx.
\end{aligned}$$

Then, in both cases, we have that z_α is a sub-solution for ($P_{k,\alpha,\beta f}$). □

Proposition 1. *Assume that* (i_0), (i_1), *and* (i_3) *hold. Then, there exists a unique solution,* u_α, *of* $(P_{k,\alpha,\beta f_*})$ *such that* $0 < z_\alpha \le u_\alpha \le t_*$ *and* $u_\alpha \in \text{int}(C_+)$.

Proof. Fix $k \in \{1, \dots K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$. Consider the functional $I_{k,\alpha} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ with

$$I_{k,\alpha}(u) = a(\alpha) \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_\Omega \beta(x) F_*(u) dx, \tag{13}$$

for all $u \in W_0^{1,p(x)}(\Omega)$, where $F_*(u) = \int_0^u f_*(s) ds$, being $\underline{u} = z_\alpha$ and $\bar{u} = t_*$ in (3). Since f_* is bounded and continuous, owing to Lemma 1, one has that the energy functional, $I_{k,\alpha}$ is coercive and weakly lower semi-continuous (see Lemma 3 [15] or [16] (Section 13.2). Then, by the Direct Methods' Theorem (Theorem 1.2, [55]), there exists u_α, a global minimum of $I_{k,\alpha}$, so u_α is a critical point for $I_{k,\alpha}$ and standard arguments show that u_α is a weak solution of $(P_{k,\alpha,\beta f_*})$; therefore, by Lemma 3, u_α is a weak solution of $(P_{k,\alpha,\beta f})$, such that $z_\alpha \le u_\alpha \le t_*$. Now, we prove that u_α is the unique solution of $(P_{k,\alpha,\beta f})$ such that $z_\alpha \le u_\alpha \le t_*$. For this scope, we use the version of Díaz–Saá inequality (Lemma 2, [56]) obtained for the $p(x)$-Laplacian in (Theorem 2.4, [57]) by J. Giacomoni and P. Takáč. Suppose there exists a $v_\alpha \ne u_\alpha$ solution of $(P_{k,\alpha,\beta f})$, such that $z_\alpha \le v_\alpha \le t_*$. Then, by (i_1) $\left(\text{decreasing of } t \mapsto \frac{f(t)}{t^{r-1}}\right)$, we have

$$0 \le \int_\Omega \left(-\frac{\Delta_{p(x)} u_\alpha}{u_\alpha^{r-1}} + \frac{\Delta_{p(x)} v_\alpha}{v_\alpha^{r-1}} \right)(u_\alpha^r - v_\alpha^r) dx = \int_\Omega \beta(x) \left(\frac{f(u_\alpha)}{u_\alpha^{r-1}} - \frac{f(v_\alpha)}{v_\alpha^{r-1}} \right)(u_\alpha^r - v_\alpha^r) dx < 0,$$

which is a contradiction. Moreover, since u_α is bounded, (Theorem 1, [58]) implies that u_α belongs to the Hölder's space $C^{1,\nu}(\bar{\Omega})$, for some $0 < \nu < 1$. Finally, by (i_1), $f \in C([0, t_*])$ and $0 \le f(u_\alpha) \le \max_{t \in [0,t_*]} f(t) < +\infty$, then $\Delta_{p(x)} u_\alpha \in L^\infty(\Omega)$ and $-\Delta_{p(x)} u_\alpha \ge 0$, hence the Maximum Principle for $p(x)$-Laplacian, (Proposition 3.1, [49]), guarantees that $u_\alpha \in \text{int}(C_+)$. □

Now, to prove the multiplicity of solutions for $(P_{p(x),\beta,f})$, we associate to $(P_{p(x),\beta,f})$ an auxiliary one-dimension fixed-point problem. In particular, for each $k \in \{1, \dots, K\}$ and for the u_α minimizer of $I_{k,\alpha}$ (unique solution of $(P_{k,\alpha,\beta f_*})$ such that $z_\alpha \le u_\alpha \le t_*$, the previous proposition allows us to define the following map: $\mathcal{P}_k : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$, with

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx$$

for all $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$. It plays a key role in our approach since it is true that

$$\text{if } \alpha \in \text{Fix}(\mathcal{P}_k), \text{ then } u_\alpha \text{ is a solution of problem } (P_{p(x),\beta,f}), \tag{14}$$

where $\text{Fix}(\mathcal{P}_k) = \{\alpha \in (\alpha_{2k-1}, \alpha_{2k}) : \mathcal{P}_k(\alpha) = \alpha\}$. Indeed, fix $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ such that $\mathcal{P}_k(\alpha) = \alpha$, then, we have

$$-a\left(\int_\Omega u_\alpha^q dx\right) \Delta_{p(x)} u_\alpha = -a(\alpha) \Delta_{p(x)} u_\alpha = \beta(x) f(u_\alpha) \quad \text{in} \quad \Omega.$$

Proposition 2. *Assume that* (i_0), (i_1), *and* (i_3) *hold. Then, for every* $k \in \{1, \dots, K\}$, *the map* $\mathcal{P}_k : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$ *is continuous.*

Proof. Let $\{\alpha_n\} \subseteq (\alpha_{2k-1}, \alpha_{2k})$, such that $\alpha_n \to \overline{\alpha} \in (\alpha_{2k-1}, \alpha_{2k})$ and, for every $n \in \mathbb{N}$, $u_n = u_{\alpha_n}$; by Proposition 1, one has $z_{\alpha_n} \le u_n \le t_*$ and u_n is a global minimum of

$$I_{k,a_n}(u_n) = a(\alpha_n) \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \int_\Omega \beta(x) F_*(u_n) dx. \tag{15}$$

Our aim is to show that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Clearly, if $\rho_{p(x)}(\nabla u_n) \leq 1$, exploiting Lemma 1, we have done. On the other hand, if $\rho_{p(x)}(\nabla u_n) > 1$ then, since $I_{k,a_n}(u_n) \leq 0$ ($I_{k,a}(0) = 0$), one has

$$\frac{a(\alpha_n)}{p_+}\|u_n\|^{p_-} - \int_\Omega \beta(x)F_*(u_n)dx \leq a(\alpha_n)\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \int_\Omega \beta(x)F_*(u_n)dx \leq 0,$$

i.e.,

$$\frac{a(\alpha_n)}{p_+}\|u_n\|^{p_-} \leq \|\beta\|_\infty F_*(t_*)|\Omega| < +\infty,$$

Hence, also in this case, $\{u_n\}$ is a bounded sequence. Therefore, up to a subsequence, $u_n \rightharpoonup u_*$ in $W_0^{1,p(x)}(\Omega)$ and, by (Theorem 4.9, [59]),

$$u_n \to u_* \text{ in } L^1(\Omega) \text{ and } u_n(x) \to u_*(x) \text{ a.e. in } \Omega, \tag{16}$$

for some $u_* \in W_0^{1,p(x)}(\Omega)$. Moreover, for every $n \in \mathbb{N}$, one has

$$a(\alpha_n)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla v dx = \int_\Omega \beta(x) f_*(u_n) v dx \quad \forall v \in W_0^{1,p(x)}(\Omega). \tag{17}$$

Since a and f_* are continuous and bounded functions by Lebesgue's Dominate Convergence Theorem, we can test (17) with $v = u_n - u_*$, and passing to the lim sup, we get

$$a(\overline{\alpha})\limsup_{n\to+\infty}\langle -\Delta_{p(x)}u_n, u_n - u_*\rangle \leq 0.$$

By (S_+)-property (Lemma 2) of $-\Delta_{p(x)}$, we get that $u_n \to u_*$ in $W_0^{1,p(x)}(\Omega)$. Taking the limit of (17), we obtain that u_* is a weak solution of $(P_{k,\bar{\alpha},\beta f_*})$. Now, we prove that $u_* \neq 0$. Moreover, $z_{\alpha_n} \leq u_n \leq t_*$ for all $n \in \mathbb{N}$, and $k_{\alpha_n} \to k_{\overline{\alpha}}$ as $n \to +\infty$, i.e., by continuity of ψ_r, $\psi_r^{-1}\left(\frac{k_{\alpha_n}}{\underline{\beta}}\right) \to \psi_r^{-1}\left(\frac{k_{\overline{\alpha}}}{\underline{\beta}}\right)$, we derive

$$\lim_{n\to+\infty} z_{\alpha_n} = z_{\overline{\alpha}} > 0.$$

From this, we get $z_{\bar{\alpha}} \leq u_{\bar{\alpha}} \leq t_*$, then $u_{\overline{\alpha}} \neq 0$ and, by Proposition 1, one has that $u_* = u_{\overline{\alpha}}$; then, from (16) and by Lebesgue's Dominate Convergence Theorem, we pass to the limit in (17) and obtain that

$$\mathcal{P}_k(\alpha_n) \to \mathcal{P}_k(\overline{\alpha}). \tag{18}$$

□

Proposition 3. *Assume that (i_0), (i_1), (i_2), (i_3), and (i_4) hold. For every $k \in \{1,\ldots,K\}$, the map $\mathcal{P}_k$ possesses at least two fixed point, $\alpha_{k,1}$ and $\alpha_{k,2}$, such that $\alpha_{2k-1} < \alpha_{k,1} < \alpha_{k,2} < \alpha_{2k}$.*

Proof. Fix $k \in \{1,\ldots,K\}$. First, we show that

$$\lim_{\alpha\to\alpha_{2k-1}^+}\mathcal{P}_k(\alpha) > \alpha_{2k-1} \quad \text{and} \quad \lim_{\alpha\to\alpha_{2k}^-}\mathcal{P}_k(\alpha) > \alpha_{2k}. \tag{19}$$

By Lemma 4, we have

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \geq \int_\Omega z_\alpha^q dx \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}). \tag{20}$$

We observe that, if $z_\alpha = \frac{\psi_r^{-1}\left(\frac{k_\alpha}{\beta}\right)}{\|\varphi_1\|_\infty}\varphi_1$, then

$$\int_\Omega z_\alpha dx \geq \left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\beta}\right)}{\|\varphi_1\|_\infty}\right)^q \int_\Omega \varphi_1^q dx.$$

On the other hand, if $z_\alpha = \frac{c}{\|\varphi_1\|_\infty}\varphi_1$, then

$$\int_\Omega z_\alpha dx \geq \left(\frac{1}{m}\right)^q \int_\Omega \varphi_1^q dx.$$

Now, by (i_0), (i_2), and Lemma 4, we get

$$\begin{aligned}
\lim_{\alpha\to\alpha_{2k-1}^+} \mathcal{P}_k(\alpha) &\geq \frac{t_*^q}{\|\varphi_1\|_\infty^q}\int_\Omega \varphi_1^q dx > \alpha_{2K} > \alpha_{2k-1},\\
\lim_{\alpha\to\alpha_{2k}^-} \mathcal{P}_k(\alpha) &\geq \frac{t_*^q}{\|\varphi_1\|_\infty^q}\int_\Omega \varphi_1^q dx > \alpha_{2K} \geq \alpha_{2k}.
\end{aligned}$$

In the subsequent, we realize that there exists $\alpha_0 \in (\alpha_{2k-1}, \alpha_{2k})$, such that

$$\mathcal{P}_k(\alpha_0) < \alpha_0. \tag{21}$$

First, we observe that, since u_{α_0} is a solution of $(P_{k,\alpha,\beta f_*})$, and, by (i_0), $a(\alpha_0) > 0$, one has

$$-\Delta_{p(x)} u_{\alpha_0} = \frac{\beta(x) f(u_{\alpha_0})}{a(\alpha_0)} \leq \frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_0)},$$

i.e., setting

$$M(\alpha_0) = M := \frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_0)}, \tag{22}$$

we focus on the following problem:

$$\begin{cases} -\Delta_{p(x)} w = M & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

We introduce these suitable constants as in [49]:

$$C^* = \frac{(N+1)(2C_0)^{(p_-)'}}{(p_+)'(p_-)^{\frac{(p_-)'}{p_-}}} |\Omega|^{\frac{(p_-)'}{N}}, \tag{24}$$

and

$$C_* = \frac{(N+1)(2C_0)^{(p_+)'}}{(p_+)'(p_-)^{\frac{(p_+)'}{p_+}}} |\Omega|^{\frac{(p_+)'}{N}}, \tag{25}$$

with C_0 as in (1). Due to the result in (Lemma 2.1, [49]), we have

$$\|w\|_\infty \leq \begin{cases} C^* M^{\frac{1}{p_- -1}} & \text{if } M \geq \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0};\\ C_* M^{\frac{1}{p_+ -1}} & \text{if } M < \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0}, \end{cases} \tag{26}$$

with C_0, C^*, and C_*, respectively, as in (1), (24), and (25). Now, keeping in mind (Proposition 2.3, [49]), since $u_\alpha \leq w$ on $\partial\Omega$ and $-\Delta_{p(x)} u_\alpha \leq -\Delta_{p(x)} w$, one has $u_\alpha \leq w$ on Ω; we

analize the two cases separately.
Case 1: Let us assume that $M < \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0}$, then

$$\begin{aligned}
\|w\|_\infty &\leq C_* M^{\frac{1}{p_+-1}} \\
&< \frac{(N+1)(2C_0)^{(p_+)'}}{(p_+)'(p_-)^{\frac{(p_+)'}{p_+}}} |\Omega|^{\frac{(p_+)'}{N}} \left(\frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0} \right)^{\frac{1}{p_+-1}} \\
&= \frac{(N+1)(p_+-1)}{p_+} \frac{2^{\frac{p_+}{p_+-1}} 2^{\frac{-1}{p_+-1}} C_0^{\frac{p_+}{p_+-1}} C_0^{\frac{-1}{p_+-1}} |\Omega|^{\frac{p_+}{N(p_+-1)}} |\Omega|^{\frac{-1}{N(p_+-1)}}}{(p_-)^{\frac{1}{p_+-1}} (p_-)^{\frac{-1}{p_+-1}}} \\
&= \frac{(N+1)(p_+-1)}{p_+} 2C_0 |\Omega|^{\frac{1}{N}} \\
&= \frac{(N+1)(p_+-1)}{p_+} 2|\Omega|^{\frac{1}{N}} \frac{\left(\Gamma(1+\frac{N}{2})\right)^{\frac{1}{N}}}{N\sqrt{\pi}}.
\end{aligned}$$

This computation implies that

$$\begin{aligned}
\mathcal{P}_k(\alpha) &= \int_\Omega u_\alpha^q dx \leq \|w\|_\infty^q |\Omega| \\
&< \left(\frac{(N+1)(p_+-1)}{p_+} \right)^q \left[2 \frac{\Gamma(1+\frac{N}{2})^{\frac{1}{N}}}{N\sqrt{\pi}} \right]^q |\Omega|^{\frac{q+N}{N}}.
\end{aligned}$$

Then, we need

$$\alpha_0 > \left(\frac{(N+1)(p_+-1)}{p_+} \right)^q \left[2 \frac{\Gamma(1+\frac{N}{2})^{\frac{1}{N}}}{N\sqrt{\pi}} \right]^q |\Omega|^{\frac{q+N}{N}}. \tag{27}$$

Moreover, by (22), one has

$$M < \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0} \Leftrightarrow \frac{\|\beta\|_\infty \max_{[0,t_*]} f(t)}{a(\alpha_0)} < \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0},$$

by (i_{4_1}); our claim follows with $\alpha_0 = \alpha_k$, i.e.,

$$a(\alpha_k) p_- > \frac{2}{N\sqrt{\pi}} \left(|\Omega| \Gamma(1+\frac{N}{2}) \right)^{\frac{1}{N}} \|\beta\|_\infty \max_{[0,t_*]} f(t). \tag{28}$$

Case 2: Let us assume that $M \geq \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0}$, then

$$\begin{aligned}
\|w\|_\infty &\leq C^* M^{\frac{1}{p_--1}} \\
&= \frac{(N+1)(2C_0)^{(p_-)'}}{(p_+)'(p_-)^{\frac{(p_-)'}{p_-}}} |\Omega|^{\frac{(p_-)'}{N}} \left(\frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_0)} \right)^{\frac{1}{p_--1}}.
\end{aligned}$$

Then, we have,

$$\begin{aligned}\mathcal{P}_k(\alpha) &= \int_\Omega u_\alpha^q dx \leq \|w\|_\infty^q |\Omega| \\ &\leq \left(\frac{(N+1)(p_+-1)}{p_+} \frac{2^{\frac{p_-}{p_--1}} C_0^{\frac{p_-}{p_--1}}}{(p_-)^{\frac{1}{p_--1}}} |\Omega|^{\frac{p_-}{N(p_--1)}} |\Omega|^{\frac{1}{q}} \left(\frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_0)} \right)^{\frac{1}{p_--1}} \right)^q \\ &= \left(\frac{(N+1)(p_+-1)}{p_+} \left[\frac{2^{p_-} C_0^{p_-} |\Omega|^{\frac{qp_-+N(p_--1)}{qN}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_0)p_-} \right]^{\frac{1}{p_--1}} \right)^q \\ &= \left(\frac{(N+1)(p_+-1)}{p_+} \right)^q \left[\frac{2^{p_-} (\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{p_-(N\sqrt{\pi})^{p_-} a(\alpha_0)} \right]^{\frac{q}{p_--1}} |\Omega|^{\frac{qp_-+N(p_--1)}{N(p_--1)}}.\end{aligned}$$

In order to ensure (21), we have

$$a(\alpha_0)\alpha_0^{\frac{p_--1}{q}} > \left(\frac{(N+1)(p_+-1)}{p_+} \right)^{p_--1} \left(\frac{2^{p_-} (\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}}}{p_-(N\sqrt{\pi})^{p_-}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t) \right) |\Omega|^{\frac{qp_-+N(p_--1)}{qN}},$$

which is true by (i_{4_2}) with $\alpha_0 = \alpha_k$. Furthermore, since

$$M \geq \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0} \Leftrightarrow \frac{\|\beta\|_\infty \max_{[0,t_*]} f(t)}{a(\alpha_0)} \geq \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0},$$

i.e.,

$$a(\alpha_0)p_- \leq \frac{2}{N\sqrt{\pi}} \left(|\Omega| \Gamma\left(1+\frac{N}{2}\right) \right)^{\frac{1}{N}} \|\beta\|_\infty \max_{[0,t_*]} f(t), \tag{29}$$

one has

$$\begin{aligned}\alpha_0 &> \left(\frac{(N+1)(p_+-1)}{p_+} \right)^q \left[\frac{2^{p_-} (\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{p_-(N\sqrt{\pi})^{p_-} a(\alpha_0)} \right]^{\frac{q}{p_--1}} |\Omega|^{\frac{qp_-+N(p_--1)}{N(p_--1)}} \\ &\geq \left(\frac{(N+1)(p_+-1)}{p_+} \right)^q \left[\frac{2^{p_-} (\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}}}{p_-(N\sqrt{\pi})^{p_-}} \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0} \right]^{\frac{q}{p_--1}} |\Omega|^{\frac{qp_-+N(p_--1)}{N(p_--1)}} \\ &= \left(\frac{(N+1)(p_+-1)}{p_+} \right)^q \left[2\frac{\Gamma(1+\frac{N}{2})^{\frac{1}{N}}}{N\sqrt{\pi}} \right]^q |\Omega|^{\frac{q+N}{N}},\end{aligned}$$

which is guaranteed by (i_4) with $\alpha_0 = \alpha_k$. Now, from the continuity of $\mathcal{P}_k$ (Proposition 2), the two limits in (19) guarantee that there exist $\alpha_{k,1}, \alpha_{k,2}$, such that $\mathcal{P}_k(\alpha_{k,1}) > \alpha_{2k-1}$ and $\mathcal{P}_k(\alpha_{k,2}) > \alpha_{2k}$. By (21) and the Intermediate Value's Theorem, we conclude that there exists at least one interval in which $\mathcal{P}_k$ possess at least two fixed points. □

Now, we are ready to prove our main result.

Proof of Theorem 1. By Proposition 3, we obtain the existence of at least two fixed points for the map $\mathcal{P}_k$. Therefore, by (14), this means that we obtain at least two positive solutions for problem ($P_{k,\alpha,\beta f}$) for each $k \in \{1,\ldots,K\}$, such that $\alpha_{2k-1} > \alpha_*$. Moreover, for such k, we get

$$\alpha_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_{2k}. \tag{30}$$

□

Remark 2. *Obviously, if one has that $\alpha_* < \alpha_{2k-1}$ for all $k \in \{1,\ldots,K\}$, Theorem 1 furnishes the existence of at least K pairs of positive solutions.*

We obtain a new result also in the cases of $p(x) \equiv p$ and $r = p$. Indeed, if we come back to the constant exponent framework, i.e., when $p_- = p_+$, clearly, one has

$$C^* = C_* = \frac{(N+1)(2C_0)^{p'}}{p' p^{\frac{p'}{p}}} |\Omega|^{\frac{p'}{N}}.$$

Moreover, in this setting, φ_1 denotes the first eigenfunction of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$, normalized in $L^\infty(\Omega)$-norm, i.e., $\|\varphi_1\|_\infty = 1$. Combining the classical results on p-Laplacian spectrum (Chapter 9, [54]) with our investigations, Theorem 1, with $p = p_- = p_+ = r$, yields the following:

Theorem 2. *Assume that (i_0), (i_1), and (i_2) hold. In addition, we suppose that*

(i_3') $\max_{t\in[\alpha_{2k-1},\alpha_{2k}]} a(t) < \frac{\beta\gamma}{\lambda_1}$ *for all $k \in \{1,\ldots,K\}$, being $\gamma := \lim_{t\to 0^+} \frac{f(t)}{t^{p-1}}$ and λ_1 the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$.*
Set

$$\alpha_* = \left[\frac{2}{\sqrt{\pi}}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{N}\right)\Gamma\left(1+\frac{N}{2}\right)^{\frac{1}{N}}\right]^q |\Omega|^{1+\frac{q}{N}},$$

(i_4') *one has that, for all $k \in \{1,\ldots,K\}$ such that $\alpha_* < \alpha_{2k-1}$, there exists $\alpha_k \in (\alpha_{2k-1}, \alpha_{2k})$, satisfying one of the following conditions:*

(i_{4_1}') $a(\alpha_k)p > \frac{2}{N\sqrt{\pi}}\left(|\Omega|\Gamma(1+\frac{N}{2})\right)^{\frac{1}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t);$

(i_{4_2}') $a(\alpha_k)\alpha_k^{\frac{p-1}{q}} > \left(\frac{(N+1)(p-1)}{p}\right)^{p-1}\left(\frac{2^p(\Gamma(1+\frac{N}{2}))^{\frac{p}{N}}}{p(N\sqrt{\pi})^p}\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)\right)|\Omega|^{\frac{p}{N}+\frac{p-1}{q}},$

if $a(\alpha_k)p \le \frac{2}{N\sqrt{\pi}}\left(|\Omega|\Gamma(1+\frac{N}{2})\right)^{\frac{1}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t).$

Then, the problem $(P_{p,\beta,f})$ admits at least, $K-k+1$ pairs of solutions, $u_{k,1}, u_{k,2} \in \mathrm{int}(C_+)$, such that

$$\alpha_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_{2k} \quad \forall k \in \{1,\ldots,K\}.$$

Remark 3. *We stress that Theorem 2 is a new result also in the constant case. Indeed, although (Theorem 2.1, [30]) works under the same assumptions, (i_0), (i_1), (i_2), (i_3'), here we have the new and deeper condition (i_4').*

Finally, we would like to conclude this paper by presenting an example

Example 1. *Let $\Omega \subset \mathbb{R}^3$ be the sphere centered at the origin with unit radius, i.e.,*

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1\right\},$$

$p : \bar{\Omega} \to \mathbb{R}$ defined by $p(x,y,z) = \frac{1}{\sqrt[4]{12}}(x+y+z) + 2$ for all $(x,y,z) \in \Omega$, β any function fulfils (i_1), K a positive integer and $A, b, \vartheta \in \mathbb{R}_+$, such that

(B_1) $b \min\left\{ \sqrt[4]{\|\varphi_1\|_\infty}, \sqrt[4]{\|\varphi_1\|_\infty^7} \right\} > \frac{4K+1}{2}\pi$,

(B_2) $\frac{A}{\|\beta\|_\infty b^\vartheta} > \frac{2}{5}\sqrt[12]{\frac{5488}{1089}}$.

Consider the problem

$$\begin{cases} -A\cos\left(\int_\Omega u dx\right)\Delta_{p(x,y,z)}u = \beta(x)(b-u)^\vartheta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

then, for each $k \in \{1, \dots, K\}$, problem (P) admits at least K pairs, $u_{k,1}, u_{k,2} \in \mathrm{int}(C_+)$, of solutions.

Indeed, under the conditions (B_1) and (B_2), it is easy to verify that all the hypotheses of Theorem 1 are satisfied by choosing $a(\alpha) = A\cos(\alpha)$, for all $\alpha \in [0, +\infty)$, $\alpha_0 \le \alpha_1 = \frac{3}{2}\pi < \alpha_2 = \frac{5}{2}\pi \le \dots \le \alpha_{2k-1} = (2k - \frac{1}{2})\pi < \alpha_{2k} = (2k + \frac{1}{2})\pi$, $\alpha_k = 2k\pi$, for $k = 1, 2, \dots, K$, $p_- = r = \frac{5}{4}$, $p_+ = \frac{11}{4}$, $f(t) = (b-t)^\vartheta$ for all $t \in (0, +\infty)$ and $t_* = b$; taking into account that $|\Omega| = \frac{3}{4}\pi$, $\alpha_* = \frac{7}{11}\pi\sqrt[3]{\frac{9}{2}}$ and $\gamma = +\infty$, in particular, we point out that if $\frac{A}{\|\beta\|_\infty b^\vartheta} > \frac{2}{5}\sqrt[3]{\frac{4}{3}}$ then (i_{4_1}) holds, otherwise, for $\frac{2}{5}\sqrt[12]{\frac{5488}{1089}} < \frac{A}{\|\beta\|_\infty b^\vartheta} \le \frac{2}{5}\sqrt[3]{\frac{4}{3}}$, (i_{4_2}) is satisfied. Finally, we have the following estimates for the L^1-norms of the solutions obtained:

$$(2k - \frac{1}{2})\pi < \int_\Omega u_{k,1} dx < \int_\Omega u_{k,2} dx < (2k + \frac{1}{2})\pi \quad \forall k \in \{1, \dots, K\}.$$

4. Conclusions

In this paper, we propose a useful generalization of the results contained in [30–32], where the constant case $p(x) \equiv p$ has been studied. As a consequence of our results, one could consider performing more mathematical models. For example, in analogy to what has been done in [11–13], one can consider biological diffusion processes, in which the dispersion velocity

$$\Omega \ni x \to v(x) = -a\left(\int_\Omega u^q dx\right)|\nabla u(x)|^{p(x)-2}\nabla u(x)$$

depends on the gradient of concentration, point-wise also through the function $p(x)$.

From a mathematical point of view, compared to the approach developed in the constant case setting, the main novelty introduced here is the use of the sub-super solution method, which allows us to overcome the technical difficulties deriving from the lack of homogeneity of the $p(x)$-Laplacian operator. This different technicality leads to a new result, even when the constant case occurs, namely $p(x) \equiv p$.

Finally, we wish to stress some potential directions for future research involving the following:

- nonlocal operators, such as the fractional p-Laplacian;
- non-homogeneous operators, such as the a-Laplacian, in the framework of the Sobolev spaces, or the Φ-Laplacian, where the presence of Young's functions in the divergence operator could raise new and interesting mathematical questions;
- nonlocal term also in the right-hand side, as in [33], with possible blow-up phenomenons;
- the variable exponent $q(x)$ also in the nonlocal term.

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