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The Global Existence and Boundedness of Solutions to a Chemotaxis–Haptotaxis Model with Nonlinear Diffusion and Signal Production

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Abstract: In this paper, we investigate the following chemotaxis–haptotaxis system (1) with nonlinear diffusion and signal production under homogenous Neumann boundary conditions in a bounded domain with smooth boundary. Under suitable conditions on the data we prove the following: (i) For $0 < \gamma \le \frac{2}{n}$, if $\alpha > \gamma - k + 1$ and $\beta > 1 - k$, problem (1) admits a classical solution (u, v, w) which is globally bounded. (ii) For $\frac{2}{n} < \gamma \le 1$, if $\alpha > \gamma - k + \frac{1}{e} + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n} - k + 1\}$ or $\alpha > \gamma - k + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\alpha+k-1)}{n} - k + 1\}$, problem (1) admits a classical solution (u, v, w) which is globally bounded.

Keywords: boundedness; chemotaxis–haptotaxis; nonlinear diffusion; signal production

MSC: 35K55; 35K65; 35A07; 35B35

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1. Introduction

In the present work, we consider the following chemotaxis–haptotaxis system with nonlinear diffusion and signal production:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(H(u)\nabla v) - \nabla\cdot(I(u)\nabla w) + u(a - \mu u^{k-1} - \lambda w), & x\in\Omega, t>0,\\ v_t = \triangle v - v + g(u), & x\in\Omega,\ t>0,\\ w_t = -vw, & x\in\Omega,\ t>0,\\ D(u)\frac{\partial u}{\partial\nu} - H(u)\frac{\partial v}{\partial\nu} - I(u)\frac{\partial w}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n (n \ge 2)$ is a bounded domain with a smooth boundary, the function $u = u(x,t)$ denotes the cancer cell density, $v = v(x,t)$ represents the concentration of matrix-degrading enzymes, and $w = w(x,t)$ represents the density of an extracellular matrix. We assume that $D, H, I \in C^2([0,\infty))$ fulfils, for all $s \ge 0$,

$$D(s) \ge K_D(s+1)^{m-1}, \quad (2)$$

$$0 \le H(s) \le \chi s(s+1)^{-\alpha} \ \text{ and } H(0) = 0, \quad (3)$$

$$0 \le I(s) \le \xi s(s+1)^{-\beta} \ \text{ and } I(0) = 0, \quad (4)$$

with $K_D, \chi, \xi > 0$ and $\alpha, \beta, m \in \mathbb{R}$. Moreover, we assume $g \in C^1([0,\infty))$ such that

$$0 \le g(s) \le K_g s^{\gamma} \text{ for all } s \ge 0, \quad (5)$$

where $K_g, \gamma > 0$. To this end, we assume that the initial data satisfy

$$\begin{cases} u_0, v_0, w_0 \in C^{2+\delta}(\overline{\Omega}), \delta \in (0,1), \\ u_0, v_0, w_0 \geq 0, \\ \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0, \frac{\partial v_0}{\partial \nu}|_{\partial\Omega} = 0, \frac{\partial w_0}{\partial \nu}|_{\partial\Omega} = 0. \end{cases} \tag{6}$$

The model (1) is reduced to the chemotaxis system if $w \equiv 0$, which has been widely researched by many authors over the past several decades (see [1–15]). In the case of $g(u) = u$, Zheng [5] proved that all solutions are global and uniformly bounded if $0 < 2 - \alpha - m < \max\{k - m, \frac{2}{N}\}$ or $2 - \alpha = k$ and μ is large enough. In the case of $g(u) = u^\gamma$, Tao et al. [6] considered problem (1), showing that if $1 + \gamma - \alpha < k$ or $1 + \gamma - \alpha = k$ and μ is large enough, then the solutions of (1) are globally bounded. When cell growth is neglected and $1 - \alpha - m + \gamma < \frac{2}{N}$, they also proved that system (1) possesses a non-negative classical solution (u, v) which is globally bounded. Later, Ding et al. [7] provided a boundedness result under $1 - \alpha - m + \gamma < \frac{2}{n}$ and proved the asymptotic stability when the damping effects of the logistic source are strong enough. Nowadays, there are more and more mathematical models used to describe complex natural phenomena, and the results are also very impressive (see [16–34]).

The chemotaxis–haptotaxis model was first proposed by Chalain and Lolas [35]:

$$\begin{cases} u_t = \triangle u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + u(a - \mu u^{k-1} - \lambda w), \\ v_t = \triangle v - v + u, \\ w_t = -vw. \end{cases} \tag{7}$$

and described the process of cancer cells invading surrounding healthy tissue. In the absence of reconstruction mechanisms, problem (7), with $\eta = 0$, has been studied by many authors. For instance, Tao Wang [36–38] proved the global solvability and uniform boundedness for $n = 1, 2$. For the case of $n = 3$, the global existence and boundedness was proved for $\frac{\mu}{\chi}$ and is sufficiently large (see [36,39]). Later, Tao and Winkler [40] researched how, under the fully explicit condition $\mu > \frac{\chi^2}{8}$, the solution (u, v, w) exponentially stabilizes to a constant stationary solution $(1, 1, 0)$. When $\mu u(1 - u - w)$ was replaced by $u(a - \mu u^{k-1} - \lambda w)$, Zheng and Ke [41] proved that model (7) possesses a global classical solution which is bounded for $k > 2$ or $k = 2$, with μ being sufficiently large. And they demonstrated that if μ is large enough, the corresponding solution of (7) exponentially decays to $((\frac{a}{\mu})^{\frac{1}{k-1}}, (\frac{a}{\mu})^{\frac{1}{k-1}}, 0)$.

In recent years, many authors have begun to study the chemotaxis–haptotaxis model with nonlinear diffusion, that is

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\ v_t = \triangle v - v + u, & x \in \Omega,\ t > 0, \\ w_t = -vw, & x \in \Omega,\ t > 0, \end{cases} \tag{8}$$

where $D(u) \geq C_D(u + 1)^{m-1}$ for all $u > 0$. Tao and Winkler [42] showed the global existence of solutions to (8) if $m > \frac{2n^2+4n-4}{n(n+4)}$ for $n \leq 8$ or $m > \frac{2n^2+3n+2-\sqrt{8n(n+1)}}{n(n+2)}$ for $n \geq 9$. Further, Li Wang et al. [43,44] proved the boundedness of solutions for $m > 2 - \frac{2}{n}$, and Wang and Zheng et al. [45,46] extended the results to $m > \frac{2n}{n+2}$. Later, Jin [47] obtained similar results for any $m > 0$, under a smallness assumption on $\frac{\chi}{\mu}$.

Next, we consider problem (1) with $g(u) = u, k = 2, a = \lambda = \mu$. Liu et al. [48] proved the global existence and boundedness of solutions for $n = 2$ if $\max\{1 - \alpha, 1 - \beta\} < m + \frac{2}{n} - 1$ or for $n \geq 3$ if $\max\{1 - \alpha, 1 - \beta\} < m + \frac{2}{n} - 1$ with either $m > 2 - \frac{2}{n}$ or $m \leq 1$. Afterwards, Xu et al. [49] proved that if $m > 0, \alpha > 0, \beta \geq 0$ for $n = 3$, problem (1) possesses a globally bounded weak solution. Subsequently, they discussed the large time behavior of solutions and showed that when $0 < m \leq 1$, for appropriately large μ, $(u,v,w) \to (1,1,0)$ as $t \to \infty$. Later, Jia et al. [50] extended the boundedness result of [49], which deals with the global boundedness of solutions with $\alpha > 0, \beta > -\frac{1}{6}$. This paper is devoted to researching the boundedness of the solution of (1) with nonlinear diffusion and signal production in the case of $n \geq 2$.

Now, we present the primary result of this paper.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with a smooth boundary and (u_0, v_0, w_0) that satisfies (6). Suppose that D, H, I and g fulfill (2)–(5). Then,*

(i) For $0 < \gamma \leq \frac{2}{n}$, if $\alpha > \gamma - k + 1$ and $\beta > 1 - k$, problem (1) possesses a classical solution (u,v,w) which is globally bounded.

(ii) For $\frac{2}{n} < \gamma \leq 1$, if $\alpha > \gamma - k + \frac{1}{e} + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n} - k + 1\}$ or $\alpha > \gamma - k + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\alpha+k-1)}{n} - k + 1\}$, problem (1) possesses a classical solution (u,v,w) which is globally bounded.

This paper is organized as follows. In Section 2, we present the local existence of classical solutions to system (1) and recall some preliminaries. In Section 3, we establish the global existence and boundedness of solutions to system (1).

2. Preliminaries

We first state the local existence result of classical solutions to (1) as follows. In fact, by a fixed point argument similar to [13,51], it can be proved.

Lemma 1. *Assume that u_0, v_0, w_0 satisfy (6) and D, H, I and g fulfill (2)–(5). Then, there exists $T_{\max} \in (0, \infty]$ such that the system (1) admits a classical solution $(u,v,w) \in C^{2+\delta, 1+\frac{\delta}{2}}(\Omega \times (0, T_{\max}))$ with*

$$u \geq 0, v \geq 0, w \geq 0 \text{ for all } (x,t) \in \Omega \times (0, T_{\max}) \tag{9}$$

such that either $T_{\max} = \infty$, or

$$\lim_{t \to T_{\max}} \sup(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)}) = \infty.$$

Then, we will give a useful lemma referred to as a variation of maximal Sobolev regularity, as obtained in (Lemma 4 [52]).

Lemma 2. *Let $z_0 \in W^{2,p}(\Omega)$ and $f \in L^p(0,T; L^p(\Omega))$. Then, the following problem*

$$\begin{cases} z_t = \triangle z - z + f, \\ \frac{\partial z}{\partial \nu} = 0, \\ z(x,0) = z_0(x), \end{cases} \tag{10}$$

possesses a unique solution: $z \in L^p_{loc}((0,+\infty); W^{2,p}(\Omega))$ and $z_t \in L^p_{loc}((0,+\infty); L^p(\Omega))$. If $t_0 \in (0,T)$, then

$$\int_{t_0}^T \int_\Omega e^{pt} |\triangle z|^p dxdt \leq C_p \int_{t_0}^T \int_\Omega e^{pt} |f|^p dxdt + C_p \|z(\cdot,t_0)\|^p_{W^{2,p}(\Omega)}, \tag{11}$$

where C_p is a constant independent of t_0.

According to [38], we have the following lemma.

Lemma 3. *Assume (u,v,w) be the solution of model (1). Then,*

$$-\triangle w(x,t) \leq \|w_0\|_{L^\infty(\Omega)} v(x,t) + C \text{ for } (x,t) \in \Omega \times (0, T_{\max}), \tag{12}$$

where

$$C := \|w_0\|_{L^\infty(\Omega)} + 4\|\nabla\sqrt{w_0}\|^2_{L^\infty(\Omega)} + \frac{\|w_0\|_{L^\infty(\Omega)}}{e}. \tag{13}$$

In order to prove Theorem 1, following the ideas in [43], we firstly state the lemma.

Lemma 4. *Assume that D, H, I and g fulfill (2)–(5) with $0 < \gamma \le 1$, then we have*

(i) There exists $K > 0$ such that for all $t \in (0, T_{\max})$

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le K\mu^{-\frac{1}{k-1}}. \tag{14}$$

(ii) For $r \in [1, \frac{n}{(n\gamma-2)_+})$, there exists $K_r > 0$ such that for all $t \in (0, T_{\max})$

$$\|v(\cdot,t)\|_{L^r(\Omega)} \le K_r. \tag{15}$$

where $(n\gamma - 2)_+ := \max\{n\gamma - 2, 0\}$.

(iii) Assume that $p > \max\{\frac{n\gamma}{2}, \gamma\}$ and $\|u(\cdot,t)\|_{L^p(\Omega)} \le K$. Then, there exists $K_p > 0$ such that for all $t \in (0, T_{\max})$

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le K_p. \tag{16}$$

(iv) Assume that $q > n\gamma$ and $\|u(\cdot,t)\|_{L^q(\Omega)} \le K$. Then, there exists a positive constant K_q such that for all $t \in (0, T_{\max})$

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le K_q. \tag{17}$$

3. Proof of Theorem 1

In this section, we deal with the global existence and boundedness of system (1). We firstly devote time to establishing the L^p-boundedness of u . For convenience, we denote $T = T_{\max}$.

Lemma 5. *Assume that D, H, I and g fulfill (2)–(5) with $\beta > 1 - k$. Then,*

(i) Let $\alpha > \gamma - k + \frac{1}{e} + 1$ and $p > \max\{1, \beta, \frac{n\gamma}{2} + 1 - k, \gamma - k + \frac{1}{e} + 1\}$. If $K_0 > 0$ fulfills for all $t \in (0, T)$

$$\|v(\cdot,t)\|_{L^{\frac{p+k-1}{\beta+k-1}}(\Omega)} \le K_0, \tag{18}$$

then,

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le K \ \text{for}\ t \in (0, T), \tag{19}$$

where $K > 0$ depends on K_0, μ.

(ii) Let $\alpha > \gamma - k + 1$ and $p > \max\{1, \alpha, \beta\}$. If there exists $K_0 > 0$ fulfills for all $t \in (0, T)$

$$\|v(\cdot,t)\|_{L^{\frac{p+k-1}{\beta+k-1}}(\Omega)} \le K_0, \tag{20}$$

then,

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le K \ \text{for}\ t \in (0, T), \tag{21}$$

where $K > 0$ depends on K_0, μ.

Proof. Multiplying the first equation in (1) with $p(1+u)^{p-1}$ and integrating by parts yields

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u+1)^p dx \\ \le& -p(p-1)K_D\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 dx + p(p-1)\int_\Omega (u+1)^{p-2}H(u)\nabla u\cdot\nabla v dx \\ &+p(p-1)\int_\Omega (u+1)^{p-2}I(u)\nabla u\cdot\nabla w dx + p\int_\Omega (u+1)^{p-1}u(a - \mu u^{k-1} - \lambda w)dx. \end{aligned} \tag{22}$$

Since $(u+1)^k \le 2^{k-1}(u^k+1)$, we have

$$\begin{aligned} & p\int_\Omega (u+1)^{p-1}u(a-\mu u^{k-1}-\lambda w)dx \\ \le\ & |a|p\int_\Omega (u+1)^p dx - \frac{\mu p}{2^{k-1}}\int_\Omega (u+1)^{k+p-1}dx + \mu p\int_\Omega (u+1)^{p-1}dx \\ \le\ & -\frac{5\mu}{6\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_1, \end{aligned} \tag{23}$$

where $C_1 = (3\cdot 2^{k+1})^{\frac{p}{k-1}}(|a|p)^{\frac{p+k-1}{k-1}}|\Omega|\mu^{-\frac{p}{k-1}} + (3\cdot 2^{k+1})^{\frac{p-1}{k}}|\Omega|\mu p^{\frac{k+p-1}{k}}$. It follows from (22) and (23) that

$$\begin{aligned} & \frac{d}{dt}\int_\Omega (u+1)^p dx \\ & \le p(p-1)\int_\Omega (u+1)^{p-2}H(u)\nabla u\cdot\nabla v dx + p(p-1)\int_\Omega (u+1)^{p-2}I(u)\nabla u\cdot\nabla\, w dx \\ & \quad -\frac{5\mu}{6\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_1. \end{aligned} \tag{24}$$

Define

$$\varphi(u) := \int_0^u (1+\sigma)^{p-2}H(\sigma)d\sigma \quad \text{for } u \ge 0.$$

We infer from (3) that

$$0 \le \varphi(u) \le \chi\int_0^u (1+\sigma)^{p-\alpha-1}d\sigma.$$

This implies for $u \ge 0$

$$\varphi(u) \le \begin{cases} \frac{2\chi}{|p-\alpha|}, & \text{for } \ p<\alpha, \\ \chi\ln(1+u), & \text{for } \ p=\alpha, \\ \frac{\chi}{p-\alpha}(1+u)^{p-\alpha}, & \text{for } \ p>\alpha. \end{cases} \tag{25}$$

Integrating by parts the first term of (24), we obtain that

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla\, v dx \\ =\ & p(p-1)\int_\Omega \nabla\varphi(u)\cdot\nabla v\, dx \\ \le\ & p(p-1)\int_\Omega \varphi(u)|\triangle v| dx. \end{aligned} \tag{26}$$

Case (i). Combining (25) with (26) yields, for $\gamma - k + \frac{1}{e} + 1 < p < \alpha$ and $n \ge 2$,

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla v dx \le \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v| dx \\ & \le \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v|^{\frac{n}{2}}dx + C_2, \end{aligned} \tag{27}$$

where $C_2 = \frac{2\chi p(p-1)|\Omega|}{|p-\alpha|}$. For $p>\alpha$, we obtain

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla\, v dx \\ \le\ & \frac{\chi p(p-1)}{p-\alpha}\int_\Omega (1+u)^{p-\alpha}|\triangle v| dx, \\ \le\ & \frac{\mu}{3\cdot 2^k}\int_\Omega (1+u)^{p+k-1} + C_3\int_\Omega |\triangle v|^{\frac{p+k-1}{k+\alpha-1}}, \end{aligned} \tag{28}$$

where $C_3 = (3\cdot 2^k)^{\frac{p-\alpha}{\alpha+k-1}}(\frac{\chi p(p-1)}{p-\alpha})^{\frac{p+k-1}{\alpha+k-1}}\mu^{-\frac{p-\alpha}{\alpha+k-1}}$. For $p=\alpha$, we obtain

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla\, v dx \\ \le\ & p(p-1)\chi\int_\Omega \ln(1+u)|\triangle v| dx, \\ \le\ & p(p-1)\chi\int_\Omega (1+u)^{\frac{1}{e}}|\triangle v| dx, \\ \le\ & \frac{\mu}{3\cdot 2^k}\int_\Omega (1+u)^{p+k-1} + C_4\int_\Omega |\triangle v|^{\frac{e(p+k-1)}{e(p+k-1)-1}}, \end{aligned} \tag{29}$$

where $C_4 = (3\cdot 2^k)^{\frac{1}{e(p+k-1)-1}}(\chi(p-1))^{\frac{e(p+k-1)}{e(p+k-1)-1}}\mu^{-\frac{1}{e(p+k-1)-1}}$.

Denote $\psi(u) = \int_0^u (1+\sigma)^{p-2} I(\sigma) d\sigma$ for all $u \geq 0$. We infer from (4) and $p > \beta$ that

$$0 \leq \psi(u) \leq \frac{\xi}{p-\beta}(1+u)^{p-\beta}, \tag{30}$$

for $u \geq 0$. This, together with Lemma 3 and $\beta > 1-k$, means that

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2} I(u)\nabla u \cdot \nabla w dx \\ = & -p(p-1)\int_\Omega \psi(u)\cdot \triangle w dx \\ \leq & p(p-1)\|w_0\|_{L^\infty(\Omega)}\int_\Omega v\psi(u)dx + p(p-1)C\int_\Omega \psi(u)dx \\ \leq & \frac{\xi p(p-1)}{p-\beta}\|w_0\|_{L^\infty(\Omega)}\int_\Omega (1+u)^{p-\beta} v dx + \frac{\xi C p(p-1)}{p-\beta}\int_\Omega (1+u)^{p-\beta} dx \\ \leq & \frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_5\int_\Omega v^{\frac{p+k-1}{\beta+k-1}}dx + C_6, \end{aligned} \tag{31}$$

where

$$C_5 = (3\cdot 2^k)^{\frac{p-\beta}{\beta+k-1}}\left(\frac{\xi p(p-1)}{p-\beta}\|w_0\|_{L^\infty(\Omega)}\right)^{\frac{p+k-1}{\beta+k-1}}\mu^{-\frac{p-\beta}{\beta+k-1}},$$

$$C_6 = (3\cdot 2^k)^{\frac{p-\beta}{\beta+k-1}}\left(\frac{\xi C p(p-1)}{p-\beta}\right)^{\frac{p+k-1}{\beta+k-1}}|\Omega|\mu^{-\frac{p-\beta}{\beta+k-1}}.$$

For $\gamma - k + \frac{1}{e} + 1 < p < \alpha$, we infer from (18), (24), (27) and (31) that

$$\frac{d}{dt}\int_\Omega (1+u)^p \leq -\frac{\mu}{2^k}\int_\Omega (1+u)^{p+k-1}dx + \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v|^{\frac{n}{2}}dx + C_7, \tag{32}$$

where $C_7 = C_5 K_0^{\frac{p+k-1}{\beta+k-1}} + C_1 + C_2 + C_6$. Since

$$\frac{n}{2}\int_\Omega (1+u)^p dx \leq \frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_8, \tag{33}$$

where $C_8 = (\frac{n}{2})^{\frac{p+k-1}{k-1}}(3\cdot 2^{k-1})^{\frac{p}{k-1}}|\Omega|\mu^{-\frac{p}{k-1}}$. Combining (32) with (33), we obtain

$$\begin{aligned} & \frac{d}{dt}\int_\Omega (1+u)^p dx + \frac{n}{2}\int_\Omega (1+u)^p dx \\ \leq & -\frac{\mu}{3\cdot 2^k}\int_\Omega (1+u)^{p+k-1}dx + \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v|^{\frac{n}{2}}dx + C_9, \end{aligned} \tag{34}$$

where $C_9 = C_7 + C_8$; this, together with the variation-of-constants formula, shows that

$$\begin{aligned} & \int_\Omega (1+u)^p dx \\ \leq & -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}(1+u)^{p+k-1}dxds + \frac{2\chi p(p-1)}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}|\triangle v|^{\frac{n}{2}}dxds \\ & + e^{-\frac{n}{2}(t-t_0)}\int_\Omega (1+u(\cdot,t_0))^p dx + C_9\int_{t_0}^t e^{-\frac{n}{2}(t-s)}ds \\ \leq & -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}(1+u)^{p+k-1}dxds + \frac{2\chi p(p-1)}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}|\triangle v|^{\frac{n}{2}}dxds + M_0 + C_9, \end{aligned} \tag{35}$$

where $M_0 = \int_\Omega (1+u(\cdot,t_0))^p dx$ is a positive constant. Since $p > \frac{n\gamma}{2} + 1 - k$, we have from Lemma 2 and (5) that

$$\begin{aligned} & \frac{2\chi p(p-1)}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}|\triangle v|^{\frac{n}{2}}dxds \\ \leq & \frac{2\chi p(p-1)C_n}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}u^{\frac{n\gamma}{2}}dxds + \frac{2\chi p(p-1)C_n}{|p-\alpha|}\|v(\cdot,t_0)\|^{\frac{n}{2}}_{W^{2,\frac{n}{2}}(\Omega)} \\ \leq & \frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}(u+1)^{p+k-1}dxds + C_{10}, \end{aligned} \tag{36}$$

where C_n, C_{10} is a positive constant related to n and independent of t_0. From the combination of (35) and (36), we conclude that

$$\int_\Omega (u+1)^p dx \leq C_{11}. \tag{37}$$

where $C_{11} = M_0 + C_9 + C_{10}$.

For $p > \alpha$, we infer from (18), (24), (28) and (31) that

$$\frac{d}{dt}\int_\Omega (1+u)^p dx \le -\frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_3\int_\Omega |\triangle v|^{\frac{p+k-1}{\alpha+k-1}}dx + C_{12}. \tag{38}$$

where $C_{12} = C_5 K_0^{\frac{p+k-1}{\beta+k-1}} + C_1 + C_6$. Define

$$m := \frac{p+k-1}{\alpha+k-1},$$

we have from (38) that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (1+u)^p dx + m\int_\Omega (1+u)^p dx\\ &\le -\frac{\mu}{3\cdot 2^{k}}\int_\Omega (1+u)^{p+k-1}dx + C_3\int_\Omega |\triangle v|^{m}dx + C_{13},\end{aligned} \tag{39}$$

where $C_{13} = (3\cdot 2^k)^{\frac{p}{k-1}} m^{\frac{p+k-1}{k-1}}|\Omega|\mu^{-\frac{p}{k-1}} + C_{12}$. Recalling Lemma 2, it can be obtained from (39) that

$$\begin{aligned}&\int_\Omega (1+u)^p dx\\ &\le -\frac{\mu}{3\cdot 2^{k}}\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{p+k-1}dxds + C_3\int_{t_0}^t\int_\Omega e^{-m(t-s)}|\triangle v|^m dxds\\ &\quad + e^{-m(t-t_0)}\int_\Omega (1+u(\cdot,t_0))^p dx + C_{13}\int_{t_0}^t e^{-m(t-s)}ds\\ &\le -\frac{\mu}{3\cdot 2^{k}}\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{p+k-1}dxds + C_3C_m\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{m\gamma}dxds\\ &\quad + C_3C_m\|v(\cdot,t_0)\|^m_{W^{2,m}(\Omega)} + C_{14},\end{aligned} \tag{40}$$

where $C_{14} = M_0 + C_{13}$ and C_m is a positive constant related to m and independent of t_0. Since $\alpha > r - k + \frac{1}{e} + 1$, $m\gamma < p + k - 1$; we have from Young's inequality that

$$\begin{aligned}&C_3C_m\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{m\gamma}dxds\\ &\le \frac{\mu}{3\cdot 2^{k}}\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{p+k-1}dxds + C_{15},\end{aligned} \tag{41}$$

where $C_{15} = \frac{1}{m}(3\cdot 2^k)^{\frac{m\gamma}{p+k-1-m\gamma}}(C_3C_m)^{\frac{p+k-1}{p+k-1-m\gamma}}|\Omega|\mu^{-\frac{m\gamma}{p+k-1-m\gamma}}$.

Inserting (41) into (40), we have

$$\int_\Omega (u+1)^p dx \le C_{16}, \tag{42}$$

where $C_{16} = C_3C_m\|v(\cdot,t_0)\|^m_{W^{2,m}(\Omega)} + C_{14} + C_{15}$.

For $p = \alpha$, we infer from (18), (24), (29) and (31) that

$$\frac{d}{dt}\int_\Omega (1+u)^p dx \le -\frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_4\int_\Omega |\triangle v|^{\frac{e(p+k-1)}{e(p+k-1)-1}}dx + C_{17}, \tag{43}$$

where $C_{17} = C_5 K_0^{\frac{p+k-1}{\beta+k-1}} + C_1 + C_6$. Define

$$\widetilde{m} := \frac{e(p+k-1)}{e(p+k-1)-1},$$

Similar to (40), we have

$$\begin{aligned}&\int_\Omega (1+u)^p dx\\ &\le -\frac{\mu}{3\cdot 2^{k}}\int_{t_0}^t\int_\Omega e^{-\widetilde{m}(t-s)}(1+u)^{p+k-1}dxds + C_4C_{\widetilde{m}}\int_{t_0}^t\int_\Omega e^{-\widetilde{m}(t-s)}(1+u)^{\widetilde{m}\gamma}dxds\\ &\quad + C_4C_{\widetilde{m}}\|v(\cdot,t_0)\|^{\widetilde{m}}_{W^{2,\widetilde{m}}(\Omega)} + C_{18},\end{aligned} \tag{44}$$

where $C_{\widetilde{m}}, C_{18}$ is a positive constant related to $\widetilde{m}$ and independent of t_0.

Since $\alpha = p > r - k + \frac{1}{e} + 1$, $\widetilde{m}\gamma < p + k - 1$; we have from Young's inequality that

$$C_4 C_{\widetilde{m}} \int_{t_0}^{t} \int_{\Omega} e^{-\widetilde{m}(t-s)} (1+u)^{\widetilde{m}\gamma} dxds \leq \frac{\mu}{3 \cdot 2^k} \int_{t_0}^{t} \int_{\Omega} e^{-\widetilde{m}(t-s)} (1+u)^{p+k-1} dxds + C_{19}, \quad (45)$$

where $C_{19} = \frac{1}{\widetilde{m}} (3 \cdot 2^k)^{\frac{\widetilde{m}\gamma}{p+k-1-\widetilde{m}\gamma}} (C_4 C_{\widetilde{m}})^{\frac{p+k-1}{p+k-1-\widetilde{m}\gamma}} |\Omega| \mu^{-\frac{\widetilde{m}\gamma}{p+k-1-\widetilde{m}\gamma}}$. Inserting (45) into (44), we have

$$\int_{\Omega} (u+1)^p dx \leq C_{20}, \quad (46)$$

where $C_{20} = C_4 C_{\widetilde{m}} \|v(\cdot, t_0)\|_{W^{2,\widetilde{m}}(\Omega)}^{\widetilde{m}} + C_{18} + C_{19}$.

Case (ii). For $p > \alpha$ and $\alpha > \gamma - k + 1$, define

$$m := \frac{p+k-1}{\alpha+k-1},$$

we have from (40) that

$$\begin{aligned} & \int_{\Omega} (1+u)^p dx \\ \leq & -\frac{\mu}{3 \cdot 2^k} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-s)} (1+u)^{p+k-1} dxds + C_3 C_m \int_{t_0}^{t} \int_{\Omega} e^{-m(t-s)} (1+u)^{m\gamma} dxds \\ & + C_3 C_m \|v(\cdot, t_0)\|_{W^{2,m}(\Omega)}^{m} + C_{21}. \end{aligned} \quad (47)$$

Since $\alpha > r - k + 1$, then $m\gamma < p + k - 1$. We infer from Young's inequality and (47) that

$$\int_{\Omega} (u+1)^p dx \leq C_{22}, \quad (48)$$

where C_{22} is a positive constant. This completes the proof of Lemma 5. □

Lemma 6. *Assume that D, H, I and g fulfill (2)–(5). Then,*

(i) For $0 < \gamma \leq \frac{2}{n}$, if $\alpha > \gamma - k + 1$ and $\beta > 1 - k$, there exists a constant $C > 0$ such that $\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C$.

(ii) For $\frac{2}{n} < \gamma \leq 1$, if $\alpha > \gamma - k + \frac{1}{e} + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n} - k + 1\}$ or $\alpha > \gamma - k + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\alpha+k-1)}{n} - k + 1\}$, there exists a constant $C > 0$ such that $\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C$.

Proof. Case (i). Since $0 < \gamma \leq \frac{2}{n}$, we have

$$n\gamma - 2 \leq 0. \quad (49)$$

Lemma 4(ii) yields that

$$\|v(\cdot, t)\|_{L^s(\Omega)} \leq C_{23} \quad \text{for all } t \in (0, T), \quad (50)$$

for any $s \geq 1$. Taking $p_1 > \max\{\frac{n\gamma}{2}, 1, \alpha, \beta\}$, this implies $\frac{p_1+k-1}{\beta+k-1} > 1$, and so we obtain

$$\|v(\cdot, t)\|_{L^{\frac{p_1+k-1}{\beta+k-1}}(\Omega)} \leq C_{24} \quad \text{for all } t \in (0, T), \quad (51)$$

which, along with Lemma 5(ii), we have for all $t \in (0, T)$

$$\|u(\cdot, t)\|_{L^{p_1}(\Omega)} \leq C_{25}. \quad (52)$$

Since $p_1 > \frac{n\gamma}{2}$, we infer from Lemma 4(iii) that

$$\|v(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C_{26} \quad \text{for all } t \in (0, T). \quad (53)$$

By Lemma 5(ii), again, and letting $p_2 > \max\{n\gamma, 1, \alpha, \beta\}$, one can find

$$\|u(\cdot,t)\|_{L^{p_2}(\Omega)} \le C_{27} \quad \text{for all } t \in (0,T). \tag{54}$$

From this, together with Lemma 4(iv), we obtain

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{28} \quad \text{for all } t \in (0,T). \tag{55}$$

This completes the proof of Case (i).

Case (ii). For $\frac{2}{n} < \gamma \le 1$. Since $\beta > \frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1$ and $\beta > \frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n} - k + 1$, we have

$$\begin{aligned} \tfrac{n\gamma}{2} &< \tfrac{n}{nr-2}(\beta+k-1)+1-k, \\ \gamma - k + \tfrac{1}{e} + 1 &< \tfrac{n}{n\gamma-2}(\beta+k-1)+1-k, \\ \beta &< \tfrac{n}{nr-2}(\beta+k-1)+1-k. \end{aligned} \tag{56}$$

Taking $\max\{\frac{n\gamma}{2}, \gamma - k + \frac{1}{e} + 1, \beta\} < p_3 < \frac{n}{nr-2}(\beta+k-1)+1-k$, then

$$\tfrac{p_3+k-1}{\beta+k-1} \in (1, \tfrac{n}{n\gamma-2}). \tag{57}$$

By Lemma 4(ii), we obtain

$$\|v(\cdot,t)\|_{L^{\frac{p_3+k-1}{\beta+k-1}}(\Omega)} \le C_{29} \quad \text{for all } t \in (0,T), \tag{58}$$

from which, along with Lemma 5(i), we have

$$\|u(\cdot,t)\|_{L^{p_3}(\Omega)} \le C_{30} \quad \text{for all } t \in (0,T). \tag{59}$$

Since $p_3 > \frac{n\gamma}{2}$, applying Lemma 4(iii), we obtain

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{31} \quad \text{for all } t \in (0,T). \tag{60}$$

By Lemma 5(i), again, and letting $p_4 > \max\{n\gamma, \gamma - k + \frac{1}{e} + 1, \beta\}$, one can find

$$\|u(\cdot,t)\|_{L^{p_4}(\Omega)} \le C_{32} \quad \text{for all } t \in (0,T). \tag{61}$$

From this, together with Lemma 4(iv), we obtain for all $t \in (0,T)$

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{33}. \tag{62}$$

Similarly, we infer from $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\alpha+k-1)}{n} - k + 1\}$ that there exists a positive constant p_5 such that

$$\max\{\tfrac{n\gamma}{2}, \alpha, \beta\} < p_5 < \tfrac{n}{nr-2}(\beta+k-1)+1-k, \tag{63}$$

thus $\frac{p_5+k-1}{\beta+k-1} \in (1, \frac{n}{n\gamma-2})$. Combining Lemmas 4(iii) and 5(ii), we have $\|u(\cdot,t)\|_{L^{p_5}(\Omega)} \le C_{34}$ and $\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{35}$. Using Lemmas 5(ii) and 4(iv), we deduce that $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{36}$. This completes the proof of Case (ii). □

Proof of Theorem 1. From Lemma 6 and the well-known Moser iteration (Lemma 3.6, [49]), we obtain the boundedness of $\|u\|_{L^\infty(\Omega)}$. The proof of Theorem 1 is complete by Lemma 1. □

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