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Hirota Bilinear Approach to Multi-Component Nonlocal Nonlinear Schrödinger Equations

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Abstract: Nonlocal nonlinear Schrödinger equations are among the important models of nonlocal integrable systems. This paper aims to present a general formula for arbitrary-order breather solutions to multi-component nonlocal nonlinear Schrödinger equations by using the Hirota bilinear method. In particular, abundant wave solutions of two- and three-component nonlocal nonlinear Schrödinger equations, including periodic and mixed-wave solutions, are obtained by taking appropriate values for the involved parameters in the general solution formula. Moreover, diverse wave structures of the resulting breather and periodic wave solutions with different parameters are discussed in detail.

Keywords: multi-component nonlocal nonlinear Schrödinger equations; Hirota bilinear method; breather solutions

MSC: 35Q99; 35C99

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1. Introduction

Solving the nonlinear Schrödinger (NLS) equation is a frequent subject in classical integrable systems [1,2]. Various wave solutions of the NLS equation can be obtained by using the inverse scattering transformation [3,4], the Darboux transformation [5], and the Hirota bilinear method [6]. In 2013, Ablowitz [7] and his collaborators derived a nonlocal nonlinear Schrödinger (NNLS) equation with PT symmetry, which provides a new research direction for the integrable systems, namely nonlocal integrable systems. It has been shown in [8] that a nonlocal system can be derived from the nonlocal reduction of the general AKNS system, and its soliton solution can also be constructed by the inverse scattering method, which means that the classical methods for the integrable systems can be used to study the nonlocal systems. Further studies have also proved that different nonlocal reductions from the general AKNS system lead to many new nonlocal systems with PT symmetry, such as the nonlocal Klein–Gordon equation, nonlocal modified KdV equation, and so on [9–12].

The NNLS equation introduced in [7] is

$$i\mathcal{P}_t(x,t) + \frac{1}{2}\mathcal{P}_{xx}(x,t) + \kappa\mathcal{P}^2(x,t)\mathcal{P}^*(-x,t) = 0, \kappa = \pm 1 \tag{1}$$

where $\mathcal{P}$ is a complex valued function of the real variables x and t. $*$ refers to the complex conjugation. The case with $\kappa = +1$ is called focusing, and the case with $\kappa = -1$ is called defocusing. In Equation (1), for a fixed time, the nonlinear term $U(x,t) = \mathcal{P}(x,t)\mathcal{P}^*(-x,t)$ satisfies the PT symmetry condition $U(x,t) = U^*(-x,t)$ [11]. Following Ablowitz's work, many achievements in integrable nonlocal systems have emerged in nonlinear mathematical physics. In [13], Equation (1) was investigated by the Darboux transformation, and, a chain of nonsingular localized wave solutions was obtained. In [14], three types of rogue waves for Equation (1) corresponding to the focusing case were derived by using the Darboux transformation. In [15], the integrability of a two-component nonlocal system with PT symmetry was studied, and the inverse scattering transformation gave its soliton solutions. In [16] Wen-Xiu Ma introduced a more generalized PT-symmetric NNLS system, namely, MNNLS equations, investigated their integrability, and presented soliton solutions employing the Riemann–Hilbert approach. Many other interesting applications and theoretical developments about PT symmetry and nonlocal integrable systems have appeared in [17–23].

In the following, we will consider a nonlocal multi-component system generalized from Equation (1):

$$i\mathcal{P}_{j,t}(x,t) + \mathcal{P}_{j,xx}(x,t) + 2\left(\sum_{l=1}^{n} \mathcal{P}_l(x,t)\mathcal{P}_l^*(-x,t)\right)\mathcal{P}_j(x,t) = 0, \qquad (j = 1, 2, \cdots, n), \tag{2}$$

where $\mathcal{P}_j(x,t)$ is a complex field of the real variables x and t. In the system (2), the nonlinear term contains a self-induced potential of the form $U(x,t) = \sum_{l=1}^{n} \mathcal{P}_l(x,t)\mathcal{P}_l^*(-x,t)$, which satisfies the PT symmetry condition $U(x,t) = U^*(-x,t)$ for a fixed time, i.e., they are invariant under the parity–time transformation $(x \to -x, t \to t)$.

The Hirota bilinear method is a powerful mathematical method for solving the classical integral system. In this paper, we will investigate the solutions of system (2) using the bilinear method. The current paper is organized as follows. In Section 2, we will apply the Hirota bilinear method to system (2) and present a formula for Nth-order breather solutions. In Section 3, we will discuss breather, periodic, and mixed waves of two- and three-component nonlocal nonlinear Schrödinger equations. Section 4 will give a summary of this paper.

2. Hiirota Bilinear Method for Breather Solutions

2.1. Hirota Bilinear Method

We first need the following transformations [2]:

$$\mathcal{P}_j(x,t) = \mathcal{C}_j \exp(i\theta t)\frac{\mathcal{A}_j(x,t)}{\mathcal{B}(x,t)}, \quad j = 1, 2, \cdots, n, \tag{3}$$

where $\mathcal{C}_j$ and θ are real constants, $\mathcal{A}_j(x,t), j = 1, 2, \cdots, n$ are complex functions, and $\mathcal{B}(x,t)$ is a real function. In addition, $\mathcal{B}(x,t)$ satisfies the following condition:

$$\mathcal{B}^*(-x,t) = \mathcal{B}(x,t). \tag{4}$$

Then, substituting (3) into (2), we obtain

$$i(\mathcal{A}_{j,t}\mathcal{B} - \mathcal{A}_j\mathcal{B}_t)\mathcal{B} - \theta\mathcal{A}_j\mathcal{B}^2 + (\mathcal{A}_{j,xx}\mathcal{B} - \mathcal{A}_j\mathcal{B}_{xx})\mathcal{B} - 2(\mathcal{A}_{j,x}\mathcal{B} - \mathcal{A}_j\mathcal{B}_x)\mathcal{B}_x + 2\mathcal{C}^2\mathcal{A}_j^*\mathcal{A}_j^2 = 0, \qquad j = 1, 2, \cdots, n. \tag{5}$$

Through calculation, we obtain the following bilinear form of the system (5):

$$\begin{aligned} &(iD_t + D_x^2)\mathcal{A}_j \cdot \mathcal{B} = 0, \quad (j = 1, 2, \cdots, n), \\ &D_x^2\mathcal{B} \cdot \mathcal{B} = 2\sum_{r=1}^{n} \mathcal{C}_r^2\mathcal{A}_r^*(-x,t)\mathcal{A}_r(x,t) - \theta\mathcal{B}\mathcal{B}, \end{aligned} \tag{6}$$

where Hirota's bilinear operators D_t and D_x are defined by the following [6]:

$$D_x^{m_1} D_t^{m_2} (\mathcal{A} \cdot \mathcal{B}) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x_1}\right)^{m_1} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t_1}\right)^{m_2} \mathcal{A}(x,t)\mathcal{B}(x_1,t_1)\Big|_{x_1=x,t_1=t}$$
$$= \sum_{j=0}^{m_1} \sum_{r=0}^{m_2} \frac{(-1)^{m_1+m_2-j-r} m_1!}{j!(m_1-j)!} \frac{m_2!}{r!(m_2-r)!} \frac{\partial^{r+j}\mathcal{A}}{\partial t^r \partial x^j} \frac{\partial^{m_1+m_2-r-j}\mathcal{B}}{\partial t^{m_2-r} \partial x^{m_1-j}}.$$

To construct the Nth-order breather solutions of (2) using the bilinear method, we also need the following finite perturbation expansions for functions $\mathcal{A}_j(x,t)$ and $\mathcal{B}(x,t)$:

$$\begin{aligned} &\mathcal{A}_j(x,t) = \mathcal{A}_{j0}(x,t)[1 + \epsilon \mathcal{A}_{j1}(x,t) + \epsilon^2 \mathcal{A}_{j2}(x,t) + \epsilon^3 \mathcal{A}_{j3}(x,t) + \cdots], \qquad (j = 1,2,\cdots,n), \\ &\mathcal{B}(x,t) = 1 + \epsilon \mathcal{B}_1(x,t) + \epsilon^2 \mathcal{B}_2(x,t) + \epsilon^3 \mathcal{B}_3(x,t) + \cdots, \end{aligned} \tag{7}$$

where ϵ is a small perturbation parameter. For the Nth-order breather solutions of (2), the expansions need to stop at ϵ^{2n}. Functions $\mathcal{A}_{jk}$ and $\mathcal{B}_k$, $j = 1,2,\cdots,n$, are specified types of functions with undetermined parameters and $\mathcal{A}_{j0} = \exp(-i\theta t)\mathcal{P}_{j0}(x,t)/\mathcal{C}_j$.

2.2. Breather Solutions

In the following, we first construct the first- and second-order breather solutions of the system (2) and then derive its Nth-order breather solutions. The construction of the first-order breather solutions is similar to that of two soliton solutions. According to the bilinear method, we truncate (7) to $\mathcal{A}_{j2}(x,t)$ and $\mathcal{B}_2(x,t)$, which means that the first-order breather solutions can be obtained by taking

$$\begin{aligned} &\mathcal{A}_{j1}(x,t) = e^{\beta_1(x,t)+2i\omega_1} + e^{\beta_2(x,t)+2i\omega_2}, \mathcal{B}_1(x,t) = e^{\beta_1(x,t)} + e^{\beta_2(x,t)}, \\ &\mathcal{A}_{j2}(x,t) = \mu_{12} e^{\beta_1(x,t)+\beta_2(x,t)+2i\omega_1+2i\omega_2}, \mathcal{B}_2(x,t) = \mu_{12} e^{\beta_1(x,t)+\beta_2(x,t)}, \quad (j = 1,2,\cdots,n), \end{aligned} \tag{8}$$

where $\beta_\ell(x,t) = i\chi_\ell x + \varphi_\ell t + \phi_{0\ell}, (\ell = 1,2)$, and χ_ℓ, φ_ℓ, ϕ_ℓ, ω_ℓ are real parameters.

Then, substituting (8) into (6), we have

$$\chi_\ell = 2\left(\sqrt{\sum_{r=1}^{n} \mathcal{C}_r^2}\right) \sin\omega_\ell, \varphi_\ell = -2\left(\sum_{r=1}^{n} \mathcal{C}_r^2\right) \sin 2\omega_\ell, \theta = 2\sum_{r=1}^{n} \mathcal{C}_r^2,$$

$$\mu_{12} = \left(\sin\frac{\omega_1-\omega_2}{2} \Big/ \sin\frac{\omega_1+\omega_2}{2}\right)^2, \omega_2 = -\omega_1 + \pi.$$

Accordingly, the first-order breather solutions of (2) are given by

$$\mathcal{P}_{j,1}(x,t) = \mathcal{C}_j \exp\left(2i\sum_{r=1}^{n} \mathcal{C}_r^2 t\right) \frac{1 + e^{\beta_1(x,t)+2i\omega_1} + e^{\beta_2(x,t)+2i\omega_2} + \mu_{12} e^{\beta_1(x,t)+\beta_2(x,t)+2i\omega_1+2i\omega_2}}{1 + e^{\beta_1(x,t)} + e^{\beta_2(x,t)} + \mu_{12} e^{\beta_1(x,t)+\beta_2(x,t)}}, \qquad (j = 1,2,\cdots,n). \tag{9}$$

where $\mathcal{B} = 1 + e^{\beta_1(x,t)} + e^{\beta_2(x,t)} + \mu_{12} e^{\beta_1(x,t)+\beta_2(x,t)}$ satisfies Equation (4) and $\epsilon = 1$ in (7).

To construct the second-order breather solutions of (2), we truncate the expressions in (7) to $\mathcal{A}_{j4}(x,t)$ and $\mathcal{B}_4(x,t)$, and take

$$\begin{aligned} \mathcal{A}_{j1}(x,t) =& e^{\beta_1(x,t)+2i\omega_1} + e^{\beta_2(x,t)+2i\omega_2} + e^{\beta_3(x,t)+2i\omega_3} + e^{\beta_4(x,t)+2i\omega_4}, \\ \mathcal{B}_1(x,t) =& e^{\beta_1(x,t)} + e^{\beta_2(x,t)} + e^{\beta_3(x,t)} + e^{\beta_4(x,t)}, \end{aligned} \tag{10}$$

where $\beta_\ell(x,t) = i\chi_\ell x + \varphi_\ell t + \phi_{0\ell}, (\ell = 1,2,3,4)$, and χ_ℓ, φ_ℓ, ϕ_ℓ, ω_ℓ are real constants.

Substituting (10) into (6), we obtian

$$\chi_\ell = 2\left(\sqrt{\sum_{r=1}^{n} \mathcal{C}_r^2}\right)\sin\omega_\ell, \varphi_\ell = -2\left(\sum_{r=1}^{n}\mathcal{C}_r^2\right)\sin 2\omega_\ell, \theta = 2\sum_{r=1}^{n}\mathcal{C}_r^2,$$

$$\mu_{\ell\sigma} = \left(\sin\frac{\omega_\ell - \omega_\sigma}{2} / \sin\frac{\omega_\ell + \omega_\sigma}{2}\right)^2 (\sigma = 1,2,3,4, \ell \neq \sigma, \alpha_\ell = 0,1),$$

$$\omega_2 = -\omega_1 + \pi, \omega_4 = -\omega_3 + \pi.$$

Accordingly, we obtain the second-order breather solutions of the system (2):

$$\mathcal{P}_{j,2}(x,t) = \mathcal{C}_j \exp\left(2i\sum_{r=1}^{n}\mathcal{C}_r^2 t\right)\frac{\sum_{\alpha=0,1}\exp\left(\sum_{\ell=1}^{4}\alpha_\ell(\beta_\ell + 2i\omega_\ell) + \sum_{1\le\ell<\sigma}^{4}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}{\sum_{\alpha=0,1}\exp\left(\sum_{\ell=1}^{4}\alpha_\ell\beta_\ell + \sum_{1\le\ell<\sigma}^{4}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}, \quad (j = 1,2,\cdots,n), \tag{11}$$

where $\mathcal{B} = \sum_{\alpha=0,1}\exp\left(\sum_{\ell=1}^{4}\alpha_\ell\beta_\ell + \sum_{1\le\ell<\sigma}^{4}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)$ satisfies Equation (4).

It is important to note that the above processes for constructing breather solutions can be continued in the same way, i.e., one can construct third- and fourth-order breather solutions, etc. This suggests that a general formula for the Nth-order breather solutions of system (2) can be expressed. Therefore, when $\epsilon = 1$, we take

$$\mathcal{A}_j(x,t) = \sum_{\alpha=0,1}\exp\left(\sum_{\ell=1}^{2N}\alpha_\ell(\beta_\ell + 2i\omega_\ell) + \sum_{1\le\ell<\sigma}^{2N}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}, \qquad \mathcal{B}(x,t) = \sum_{\alpha=0,1}\exp\left(\sum_{\ell=1}^{2N}\alpha_\ell\beta_\ell + \sum_{1\le\ell<\sigma}^{2N}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right),\right. \tag{12}$$

where

$$\mu_{\ell\sigma} = e^{\tau_{\ell\sigma}} = \left(\sin\frac{\omega_\ell - \omega_\sigma}{2} / \sin\frac{\omega_\ell + \omega_\sigma}{2}\right)^2,$$

$$\beta_\ell = \beta_\ell(x,t) = i\chi_\ell x + \varphi_\ell t + \phi_{0\ell}, (\ell = 1,2,3,\cdots,2N),$$

$$\chi_\ell = 2\left(\sqrt{\sum_{r=1}^{n}\mathcal{C}_r^2}\right)\sin\omega_\ell, \varphi_\ell = -2\left(\sum_{r=1}^{n}\mathcal{C}_r^2\right)\sin 2\omega_\ell,$$

and $\ell, \sigma = 1,2,\cdots,2N-1,2N$, but $\ell \neq \sigma$, $\alpha_\ell = 0,1(\ell = 1,2,\cdots,2N-1,2N)$ should take all possible combinations of α.

Thus, when $\omega_{2N} = -\omega_{2N-1} + \pi$, the Nth-order breather solutions of (2) can be written as

$$\mathcal{P}_{j,N}(x,t) = \mathcal{C}_j \exp\left(2i\sum_{r=1}^{n}\mathcal{C}_r^2 t\right)\frac{\sum_{\alpha=0,1}\left(\sum_{\ell=1}^{2n}\alpha_\ell(\beta_\ell + 2i\omega_\ell) + \sum_{1\le\ell<\sigma}^{2n}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}{\sum_{\alpha=0,1}\exp\left(\sum_{\ell=1}^{2n}\alpha_\ell\beta_\ell + \sum_{1\le\ell<\sigma}^{2n}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}, \quad (j = 1,2,\cdots,n). \tag{13}$$

Generally, for the breather wave solutions, their shapes are mainly related to n parameters ω_n. For a given n, the components $\mathcal{P}_j, (j = 1,..,n)$ are distinguished by the coefficients $\mathcal{C}_j$. Under this condition, the components can be proportional; otherwise, they cannot.

3. Examples

3.1. Breather Solutions of the Two-Component NNLS Equations

When $n = 2$ in (2), we obtain the following two-component system:

$$\begin{aligned} i\mathcal{P}_{1,t}(x,t) + \mathcal{P}_{1,xx}(x,t) + 2\left(\sum_{l=1}^{2} \mathcal{P}_l(x,t)\mathcal{P}_l^*(-x,t)\right)\mathcal{P}_1(x,t) = 0, \\ i\mathcal{P}_{2,t}(x,t) + \mathcal{P}_{22,xx}(x,t) + 2\left(\sum_{l=1}^{2} \mathcal{P}_l(x,t)\mathcal{P}_l^*(-x,t)\right)\mathcal{P}_2(x,t) = 0. \end{aligned} \tag{14}$$

The Hirota bilinear forms of (14) are given by

$$\begin{aligned} &(iD_t + D_x^2)\mathcal{A}_1 \cdot \mathcal{B} = 0, \qquad (iD_t + D_x^2)\mathcal{A}_2 \cdot \mathcal{B} = 0, \\ &D_x^2\mathcal{B} \cdot \mathcal{B} = 2\sum_{r=1}^{2} \mathcal{C}_r^2 \mathcal{A}_r^*(-x,t)\mathcal{A}_r(x,t) - \theta\mathcal{B}\mathcal{B}. \end{aligned} \tag{15}$$

3.1.1. The First-Order Breather Solutions

According to (9), the first-order breather solutions of (14) can be written as

$$\begin{aligned} \mathcal{P}_{1,1}(x,t) = \mathcal{C}_1 \exp\left(2i\sum_{r=1}^{2} \mathcal{C}_r^2 t\right) \frac{1 + e^{\beta_1(x,t)+2i\omega_1} + e^{\beta_2(x,t)+2i\omega_2} + \mu_{12}e^{\beta_1(x,t)+\beta_2(x,t)+2i\omega_1+2i\omega_2}}{1 + e^{\beta_1(x,t)} + e^{\beta_2(x,t)} + \mu_{12}e^{\beta_1(x,t)+\beta_2(x,t)}}, \\ \mathcal{P}_{2,1}(x,t) = \mathcal{C}_2 \exp\left(2i\sum_{r=1}^{2} \mathcal{C}_r^2 t\right) \frac{1 + e^{\beta_1(x,t)+2i\omega_1} + e^{\beta_2(x,t)+2i\omega_2} + \mu_{12}e^{\beta_1(x,t)+\beta_2(x,t)+2i\omega_1+2i\omega_2}}{1 + e^{\beta_1(x,t)} + e^{\beta_2(x,t)} + \mu_{12}e^{\beta_1(x,t)+\beta_2(x,t)}}. \end{aligned} \tag{16}$$

Obviously, the denominator $\mathcal{B} = 1 + e^{\beta_1(x,t)} + e^{\beta_2(x,t)} + \mu_{12}e^{\beta_1(x,t)+\beta_2(x,t)}$ of (16) satisfies (4). The corresponding graphs of the first-order breather solutions $\mathcal{P}_{i,1}$ are shown in Figure 1 with $\phi_{01} = \phi_{02} = 0.1$. Figure 1a,c show the number of the peaks determined by C_i at a fixed $\omega_i, i = 1,2$. Figure 1a,b show the number of peaks determined by ω_i at a fixed $C_i, i = 1,2$. In particular, $\mathcal{P}_{1,1}$ is the periodic wave solution at $\omega_1 = \frac{\pi}{2}$. We find that the values of ϕ_{01} and ϕ_{02} have little effect on the changes in peak number and amplitude within a certain range, and this is also true for high-order breather wave solutions.

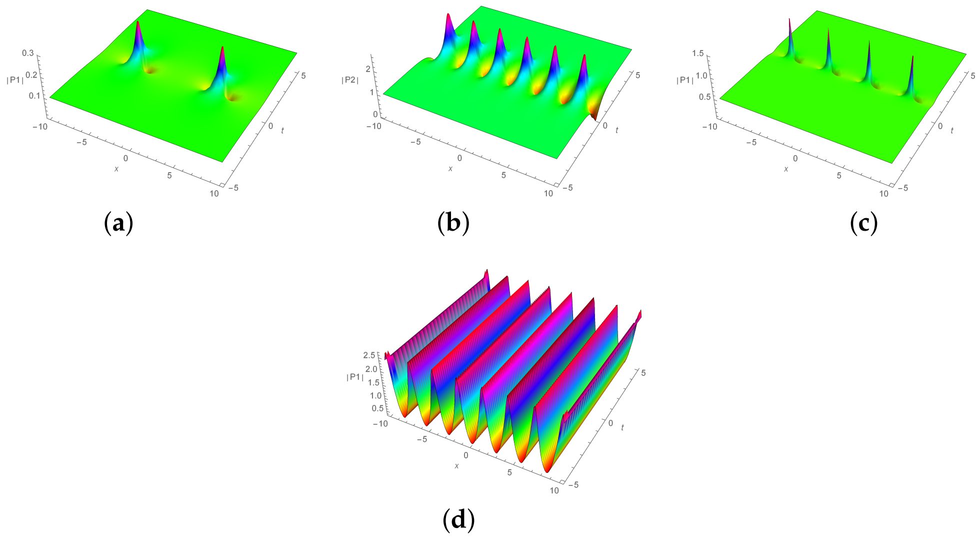


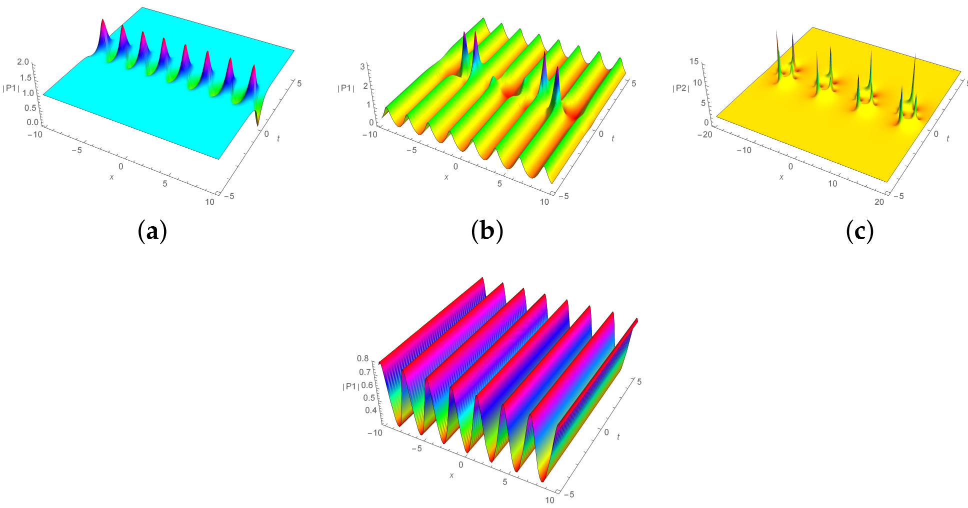
Figure 1. (**a**) First-order breather solution $\mathcal{P}_{1,1}$ with $\omega_1 = \frac{\pi}{10}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1$; (**b**) first-order breather solution $\mathcal{P}_{2,1}$ with $\omega_1 = \frac{\pi}{3}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1$; (**c**) first-order breather solution $\mathcal{P}_{1,1}$ with $\omega_1 = \frac{\pi}{10}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 2$; (**d**) periodic wave solution $\mathcal{P}_{1,1}$ with $\omega_1 = \frac{\pi}{2}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1$.

3.1.2. The Second-Order Breather Solution

According to (11), the second-order breather solutions of (14) can be expressed as

$$\mathcal{P}_{j,2}(x,t) = \mathcal{C}_j \exp\left(2i\sum_{r=1}^{2}\mathcal{C}_r^2 t\right)\frac{\sum\limits_{\alpha=0,1}\exp\left(\sum\limits_{\ell=1}^{4}\alpha_\ell(\beta_\ell+2i\omega_\ell)+\sum\limits_{1\le\ell<\sigma}^{4}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}{\sum\limits_{\alpha=0,1}\exp\left(\sum\limits_{\ell=1}^{4}\alpha_\ell\beta_\ell+\sum\limits_{1\le\ell<\sigma}^{4}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}, \quad (j=1,2). \tag{17}$$

It is clear that the denominator $\mathcal{B} = \sum\limits_{\alpha=0,1}\exp\left(\sum\limits_{\ell=1}^{4}\alpha_\ell\beta_\ell+\sum\limits_{1\le\ell<\sigma}^{4}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)$ of (17) satisfies (4). The corresponding graphs of the second-order breather solutions $\mathcal{P}_{i,2}$ are shown in Figure 2 with $\phi_{0i} = 0.1, i = 1,\dots 4$. Taking $\omega_1 = \omega_3 = \frac{\pi}{10}$ and $\mathcal{C}_1 = \mathcal{C}_2 = 1$, the general breather solution $\mathcal{P}_{1,2}$ is plotted in Figure 2a. Taking $\omega_1 = \frac{\pi}{4}, \omega_3 = \frac{\pi}{2}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1.1$, the mixed-wave solution $\mathcal{P}_{1,2}$ of the breather and periodic wave along the x-axis is plotted in Figure 2b. When $\omega_1 = \frac{\pi}{10}, \omega_3 = \frac{\pi}{20}, \mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 2$, we obtain a complex mixed-wave solution $\mathcal{P}_{2,2}$, namely, a quadrangular breather wave solution (see Figure 2c). In particular, $\mathcal{P}_{1,2}$ is the periodic wave solution with $\omega_1 = \omega_3 = \frac{\pi}{2}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1$ (see Figure 2d). Therefore, we find that the parameter ω_i plays an important role in the structural change of the solutions.



(**a**) (**b**) (**c**) (**d**)

Figure 2. (**a**) Second-order breather solution $\mathcal{P}_{1,2}$ with $\omega_1 = \omega_3 = \frac{\pi}{10}$ and $\mathcal{C}_1 = \mathcal{C}_2 = 1$; (**b**) mixed-wave solution $\mathcal{P}_{1,2}$ with $\omega_1 = \frac{\pi}{4}, \omega_3 = \frac{\pi}{2}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1.1$; (**c**) mixed-wave solution $\mathcal{P}_{2,2}$ with $\omega_1 = \frac{\pi}{10}, \omega_3 = \frac{\pi}{20}, \mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 2$; (**d**) periodic wave solution $\mathcal{P}_{1,2}$ with $\omega_1 = \omega_3 = \frac{\pi}{2}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1$.

3.1.3. The Third-Order Breather Solutions

According to (13), the third-order breather solutions of (14) can be expressed as

$$\mathcal{P}_{j,2}(x,t) = \mathcal{C}_j \exp\left(2i\sum_{r=1}^{2}\mathcal{C}_r^2 t\right)\frac{\sum\limits_{\alpha=0,1}\exp\left(\sum\limits_{\ell=1}^{6}\alpha_\ell(\beta_\ell+2i\omega_\ell)+\sum\limits_{1\le\ell<\sigma}^{6}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}{\sum\limits_{\alpha=0,1}\exp\left(\sum\limits_{\ell=1}^{6}\alpha_\ell\beta_\ell+\sum\limits_{1\le\ell<\sigma}^{6}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}, \quad (j=1,2). \tag{18}$$

The denominator $\mathcal{B} = \sum\limits_{\alpha=0,1}\exp\left(\sum\limits_{\ell=1}^{4}\alpha_\ell\beta_\ell+\sum\limits_{1\le\ell<\sigma}^{4}\alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)$ of (17) satisfies (4). The corresponding graphs of the third-order breather solutions $\mathcal{P}_{j,2}$ are shown in Figure 3 with $\phi_{0i} = 0.i = 1,\dots 6$. Figure 4a shows the third-order breather solution $\mathcal{P}_{2,2}$ with

$\omega_1 = \omega_3 = \omega_5 = \frac{\pi}{3}$, $\mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 2$. Figure 4b shows the periodic solution $\mathcal{P}_{2,2}$ with parameters $\omega_i = \frac{\pi}{2}, i = 1, \ldots 6, \mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 1$ along the x-axis. As shown in Figure 4a,b, the amplitude varies significantly with parameter C_2. Taking $\omega_1 = \frac{\pi}{2}, \omega_3 = \frac{\pi}{20}$, $\omega_5 = \frac{\pi}{30}$, $\mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 1.1$, the mixed-wave solution of the solitary wave and periodic wave along the x-axis is plotted in Figure 3c. When $\omega_1 = \frac{\pi}{10}, \omega_3 = \frac{\pi}{20}$, and $\omega_5 = \frac{\pi}{30}$, Figure 3d shows the mixed waves of four and six solitary wave packets with interleaved periods.

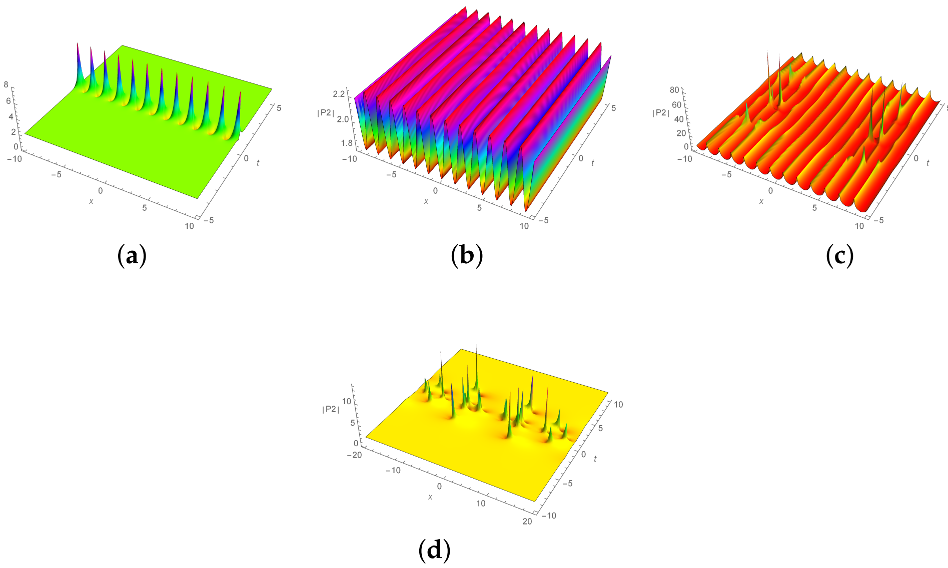


Figure 3. (**a**) Third-order breather solution $\mathcal{P}_{2,2}$ with $\omega_1 = \omega_3 = \omega_5 = \frac{\pi}{3}, \mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 2$; (**b**) periodic wave solution $\mathcal{P}_{2,2}$ with $\omega_i = \frac{\pi}{2}, i = 1, \ldots 6, \mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 1$; (**c**) mixed-wave solution $\mathcal{P}_{2,2}$ with $\omega_1 = \frac{\pi}{2}, \omega_3 = \frac{\pi}{20}, \omega_5 = \frac{\pi}{30}, \mathcal{C}_1 = 0.5$, and $\mathcal{C}_2 = 1.1$; (**d**) mixed-wave solution $\mathcal{P}_{2,2}$ with $\omega_1 = \frac{\pi}{10}, \omega_3 = \frac{\pi}{20}, \omega_5 = \frac{\pi}{30}, \mathcal{C}_1 = 0.1$, and $\mathcal{C}_2 = 2$.

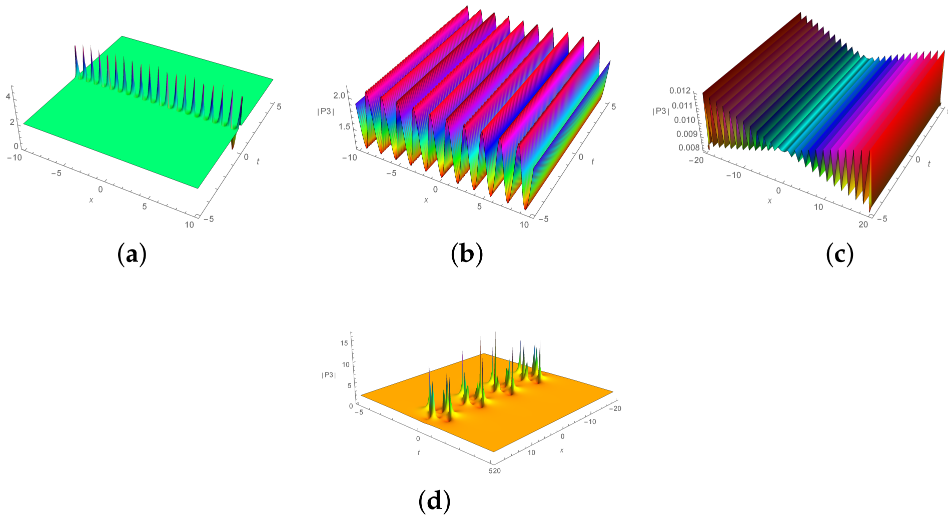


Figure 4. (**a**) Third-order breather solution $\mathcal{P}_{1,3}$ with $\omega_1 = \omega_3 = \omega_5 = \frac{\pi}{3}$ and $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = 2$; (**b**) periodic wave solution $\mathcal{P}_{3,3}$ with $\omega_i = \frac{\pi}{2}, i = 1, \ldots 6, \mathcal{C}_1 = 0.1, \mathcal{C}_2 = 0.5$, and $\mathcal{C}_3 = 1.6$; (**c**) special periodic wave solution $\mathcal{P}_{3,3}$ with $\omega_1 = \frac{\pi}{2}, i = 1, \ldots 6, \mathcal{C}_1 = \mathcal{C}_3 = 0.01$, and $\mathcal{C}_2 = 1.9$; (**d**) mixed-wave solution $\mathcal{P}_{3,3}$ with $\omega_1 = \frac{\pi}{10}, \omega_3 = \frac{\pi}{10}, \omega_5 = \frac{\pi}{20}, \mathcal{C}_1 = 0.1, \mathcal{C}_2 = 0.5$, and $\mathcal{C}_3 = 2$.

3.2. Breather Solutions of the Three-Component NNLS Equations

When $n = 3$ in (2), we obtain the following three-component system:

$$\begin{aligned} i\mathcal{P}_{1,t}(x,t) + \mathcal{P}_{1,xx}(x,t) + 2\left(\sum_{l=1}^{3} \mathcal{P}_l(x,t)\mathcal{P}_l^*(-x,t)\right)\mathcal{P}_1(x,t) = 0, \\ i\mathcal{P}_{2,t}(x,t) + \mathcal{P}_{2,xx}(x,t) + 2\left(\sum_{l=1}^{3} \mathcal{P}_l(x,t)\mathcal{P}_l^*(-x,t)\right)\mathcal{P}_2(x,t) = 0. \\ i\mathcal{P}_{3,t}(x,t) + \mathcal{P}_{3,xx}(x,t) + 2\left(\sum_{l=1}^{2} \mathcal{P}_l(x,t)\mathcal{P}_l^*(-x,t)\right)\mathcal{P}_3(x,t) = 0. \end{aligned} \tag{19}$$

According to (13), the third-order breather solutions of (14) can be expressed as

$$\mathcal{P}_{j,3}(x,t) = \mathcal{C}_j \exp\left(2i\sum_{r=1}^{3} \mathcal{C}_r^2 t\right) \frac{\sum_{\alpha=0,1} \exp\left(\sum_{\ell=1}^{6} \alpha_\ell(\beta_\ell + 2i\omega_\ell) + \sum_{1\le\ell<\sigma}^{6} \alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}{\sum_{\alpha=0,1} \exp\left(\sum_{\ell=1}^{6} \alpha_\ell\beta_\ell + \sum_{1\le\ell<\sigma}^{6} \alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)}, \quad (j = 1,2,3). \tag{20}$$

The denominator $\mathcal{B} = \sum_{\alpha=0,1} \exp\left(\sum_{\ell=1}^{4} \alpha_\ell\beta_\ell + \sum_{1\le\ell<\sigma}^{4} \alpha_\ell\alpha_\sigma\tau_{\ell\sigma}\right)$ of (20) satisfies (4). The corresponding graphs of the third-order breather solutions $\mathcal{P}_{j,3}$ are shown in Figure 4 with $\phi_{0i} = 0, i = 1,\ldots 6$. We find that the structure of the solution varies depending on the parameters it contains. Figure 4a shows the third-order breather solution $\mathcal{P}_{1,3}$ with $\omega_1 = \omega_3 = \omega_5 = \frac{\pi}{3}$ and $\mathcal{C}_i = 2, i = 1,2,3$. Figure 4b shows the periodic solution $\mathcal{P}_{3,3}$ with parameters $\omega_i = \frac{\pi}{2}, i = 1,\ldots 6$, $\mathcal{C}_1 = 0.1, \mathcal{C}_2 = 0.5$, and $\mathcal{C}_3 = 1.6$ along the x-axis. If $\mathcal{C}_1 = \mathcal{C}_3 = 0.01$ and $\mathcal{C}_2 = 1.9$ in Figure 4b, we obtain a special periodic wave solution $\mathcal{P}_{3,3}$ along the x-axis, as shown in Figure 4c. For appropriate values of $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$, we obtain Figure 4c, which is not the case for solutions containing two parameters $\mathcal{C}_1$ and $\mathcal{C}_2$ or only one parameter $\mathcal{C}_1$. When $\omega_1 = \frac{\pi}{10}, \omega_3 = \frac{\pi}{10}$, and $\omega_5 = \frac{\pi}{20}$, Figure 4d shows the mixed-wave solution $\mathcal{P}_{3,3}$ of six solitary wave packets with periods.

4. Conclusions

The nonlinear nonlocal Schrödinger equation and its multicomponent generalizations are important models of nonlocal integrable systems. In this paper, we have presented the Nth-order breather solutions of MNNLS equations by using the Hirota bilinear method. As the first example, the first-order, second-order, and third-order breather solutions, periodic solutions, and mixed-wave solutions of the two-component NNLS equations were illustrated. As a second example, we also illustrated the third-order breather solutions, periodic solutions, and mixed-wave solutions of the three-component NNLS equations. The dynamics of the obtained solutions were analyzed. From the graphical analysis, we found that the wave structures are mainly affected by the parameters ω_i and C_i. In particular, every C_i is related to the amplitude of the waves and the number of peaks. Moreover, the small difference in parameters $\phi_{0i}, i = 1,\ldots n$ has little effect on the wave structure. There are many notable parameters in the general expression of breather solutions of MNNLS equations. Various combinations of multi-parameters have a significant effect on the wave structure. We have only analyzed some representative combinations of parameters in general solutions. However, the results obtained in this article are relatively rich and may have potential implications for research in theoretical physics, applied mathematics, and related fields.

In our future work, we will focus on discovering new exact solutions for multi-component nonlocal Schrödinger equations that are not found in local equations [24–27]. We aim to explore and characterize these solutions, potentially revealing new insights and applications beyond those offered by localized models.

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