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Fixed-Point Theorems Using α-Series in F-Metric Spaces

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Abstract: Fixed-point theory, which has been developing since 1922, is widely used. Various contraction principles have been defined in the literature. In this work, we define rational-type contraction and weak Choudhury type contraction using α-series in F-metric spaces and prove common fixed-point theorems for sequences of self-mappings. This method is based on the convergence series of coefficients. Our results are the generalized version of the results in the literature. Finally, we apply our main results to solve an integral equation and a differential equation.

Keywords: α-series; fixed point; contraction; F-metric

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1. Introduction

Metric spaces, dating back about 100 years ago, are an important tool in the transition from classical analysis to modern analysis. Metrics and distances are used in many areas of mathematics and other sciences. Fixed-point and common-fixed-point theorems in metric spaces have become a popular research topic since 1922, including in generalized metric spaces. In 1922, Banach stated his famous theorem called the Banach contraction principle.

Theorem 1 ([1])**.** *Let (Γ, d) be a complete metric space and $k : \Gamma \to \Gamma$ be a mapping. Suppose that $\gamma \in (0, 1)$ exists, such that*

$$d(kx, ky) \leq \gamma d(x, y) \tag{1}$$

for all $x, y \in \Gamma$. Then, k has a unique fixed point.

This result is very powerful since it not only guarantees the presence and inimitableness of the fixed points of the exact self-maps of metric spaces, but it also ensures a constitutive technique to discover such fixed points. Researchers generalized this principle and proved fixed-point theorems with various conditions for contractions, mappings, or sets. In 1968, Kannan proved a more general contraction principle from Banach's principle.

Theorem 2 ([2])**.** *Let (Γ, d) be a complete metric space and $k : \Gamma \to \Gamma$ be a mapping. Suppose that $\gamma \in (0, \frac{1}{2})$ exists, such that*

$$d(kx, ky) \leq \gamma[d(kx, x) + d(ky, y)] \tag{2}$$

for all $x, y \in \Gamma$. Then, k has a unique fixed point.

Chatterjee [3], Reich [4], and Hardy and Roger [5] generalized (1) and (2) as the following contractions:

1. Chatterjee type: $d(kx,ky) \leq \gamma[d(ky,x)+d(kx,y)]$ where $\gamma \in (0,\frac{1}{2})$.
2. Reich type: $d(kx,ky) \leq \gamma d(x,y)+\delta d(kx,x)+\lambda d(ky,y)$ where $\gamma,\delta,\lambda \in (0,1)$ with $\gamma+\delta+\lambda<1$.
3. Hardy–Rogers type: $d(kx,ky) \leq \gamma d(x,y)+\delta d(kx,x)+\lambda d(ky,y)+\mu d(kx,y)+\nu d(ky,x)$ where $\gamma,\delta,\lambda \in [0,1)$ with $\gamma+\delta+\lambda+\mu+\nu<1$.

In 2009, Choudhury introduced a weak contraction principle by using Chatterjee's theorem.

Theorem 3 ([6])**.** *Let (Γ,d) be a complete metric space and $k:\Gamma\to\Gamma$ be a mapping. Suppose that $\gamma \in [0,\frac{1}{2})$ exists such that*

$$d(kx,ky) \leq \gamma[d(ky,x)+d(kx,y)]-\psi(d(ky,x),d(kx,y))$$

for all $x,y\in\Gamma$, where $\psi:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous mapping with $\psi(0,0)=0$. Then, k has a unique fixed point.

Later, a rational-type contraction was defined by Das and Gupta [7]. Some well-known results in this direction are involved [8–10].

Theorem 4 ([7])**.** *Let (Γ,d) be a complete metric space and $k:\Gamma\to\Gamma$ be a mapping. Suppose that $\gamma,\delta\in[0,1)$, where $\gamma+\delta<1$, such that*

$$d(kx,ky) \leq \gamma d(x,y)+\delta\frac{d(y,ky)[1+d(x,kx)]}{1+d(x,y)}$$

for all $x,y\in\Gamma$. Then, k has a unique fixed point.

Recently, researchers proved fixed-point theorems with various conditions for contractions, mappings, or sets. In 2014, authors generalized the contraction principle using the coefficients of α-series, which are a larger class of convergent series, with a new approach [11]. Vats et al. [12] developed tripled-fixed-point results using the coefficients of an α-series. Gaba [13,14] defined a λ-sequence of $(0,\infty)$, specifically endowed with a max metric, which is also an α-sequence.

In this work, we present a generalized rational-type contraction and weak Choudhury type contraction using the coefficients of an α-series for a sequence of mappings and prove common fixed-point theorems in F-metric spaces. This method is based on the convergence of an appropriate series of coefficients instead of real coefficients.

Definition 1 ([11])**.** *Let $\{x_n\}$ be a sequence of non-negative real numbers. The series $\sum_{n=1}^{\infty}x_n$ is named an α-series if there exists $0<\alpha<1$ and $n(\alpha)\in\mathbb{N}$ such that*

$$\sum_{n=1}^{L}x_n \leq \alpha L$$

for each $L\geq n(\alpha)$.

Example 1 ([15])**.** *Series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ are α-series. Note that all convergent series with non-negative real sentences are α-series.*

Recently, Bonab et al. [15] gave the following contraction using the coefficients of an α-series.

Theorem 5. *Let $\{k_n\}$ be a sequence of self-mappings for a complete metric space, (Γ,d). Assume that for all $x,y\in\Gamma$*

$$d\big(k_i(x),k_j(y)\big) \leq \vartheta_{i,j}^2\left[d(x,k_i(x))+d(y,k_j(y))+\frac{d(y,k_i(x))+d(x,k_j(y))}{2}\right]$$

where $\vartheta_{i,j} \in [0,1)$, $(i,j = 1,2,...)$. If $\sum_{i=1}^{\infty} \frac{2\vartheta_{i,i+1}}{1-\vartheta_{i,i+1}}$ is an α-series, then $\{k_n\}$ has a unique common fixed point.

2. Materials and Methods

Metric spaces have many generalizations due to their widespread use. Zhukovskiy [16] generalized Banach's contraction mapping theorem to complete f-quasimetric spaces [17]. A.V. Arutyunov and A.V. Greshnov proved fixed-point theorems for $(q1 - q2)$-quasimetric spaces [18,19]. Similarly in 2018, Jleli and Samet [20] introduced F-metric spaces, which have generalized triangular inequality by a special F-function, and proved Banach's contraction mapping theorem. Bera et al. proved their topological properties [21]. F-metric spaces have gained significance due to the development of the metric fixed-point theory. Most fixed-point results were given in F-metric spaces. In F-metric spaces, Hussain and Kanwal [22] proved some coupled-fixed-point theorems. Mitrovic et al. [23] and Jahangir et al. [24] proved some generalized fixed-point results. Lateefa [25] and Zhou et al. [26] gave the best proximity results in F-metric spaces, and in [27], the authors gave some fixed-point theorems. Alansari et al. [28] proved some fuzzy fixed-point theorems, Al-Mezel et al. [29] and Faraji et al. [30] defined (α, β)-admissible-type contractions. Faraji et al. [30] defined $(\alpha, \beta$-)-admissible-type contractions and proved some fixed-point theorems. Mudhesh et al. [31] introduced Geraghty type contractions, and Ozturk [32] proved some fixed-point theorems for Ciric–Presic type contractions. Kanwal et al. [33] defined orthogonal F-metric spaces and gave some fixed-point results. Acar and Ozturk [34] introduced almost contractions in these spaces.

Definition 2 ([16])**.** *Let Γ be a non-empty set and $q : \Gamma \times \Gamma \to \mathbb{R}^+$ be a mapping. Consider the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f(r_1, r_2) \to 0$ as $(r_1, r_2) \to (0,0)$. Consider the following conditions*

- (q_1) $x, y \in \Gamma, q(x,y) = 0 \Leftrightarrow x = y;$
- (q_2) $x, y \in \Gamma, q(x,y) = q(y,x);$
- (q_3) *There exists $t > 0$ such that for all $x, y, z \in \Gamma$, $q(x,y) < t$, $q(y,z) < t \Rightarrow q(x,z) \leq f(q(x,y), q(y,z))$.*

If the mapping q satisfy (q1) and (q3), then q is called an f-quasimetric. If q satisfy (q1), (q2), and (q3), then q is called an f-metric. If in (q3), we have $t = \infty$ and $f(r_1, r_2) = q_1 r_1 + q_2 r_2$ with $q_1, q_2 \geq 1$, then the f-quasimetric is called a (q_1, q_2)-quasimetric.

Now, we give some of the properties of F-metric spaces and two examples.

Let Ω be the family of functions $F : (0, \infty) \to \mathbb{R}$ satisfying

- (F_1) F is increasing;
- (F_2) For each sequence $\{x_n\}_{n\in\mathbb{N}}$, $\lim_{s\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} F(x_n) = -\infty$.

Definition 3 ([20])**.** *Let Γ be a non-empty set and let $D : \Gamma \times \Gamma \to [0, \infty)$ be a given mapping. Suppose that there exists $(F, t) \in \Omega \times [0, +\infty)$ such that*

- (d_1) $x, y \in \Gamma, D(x,y) = 0 \Leftrightarrow x = y;$
- (d_2) $x, y \in \Gamma, D(x,y) = D(y,x);$
- (d_3) *For all $x, y \in \Gamma$, for every $n \in \mathbb{N}$, $n \geq 2$, and for every $(u_i)_{i=1}^{n} \subset \Gamma$ with $(u_1, u_n) = (x, y)$, we have*

$$D(x,y) > 0 \Longrightarrow F(D(x,y)) \leq F\left(\Sigma_{i=1}^{n-1} D(u_i, u_{i+1})\right) + t.$$

Then, (Γ, D) is called an F-metric space (or F-ms for short).

Example 2 ([35])**.** *Let $\Gamma = \mathbb{N}$ and $D : \Gamma \times \Gamma \to [0, \infty)$ be defined by*

$$D(x,\mu) = \begin{cases} (x-\mu)^2, & \text{if } x, \mu \in \{1,2,3\} \\ 2|x-\mu|, & \text{other.} \end{cases}$$

Then, (Γ, D) is an F-complete F-ms with $F(x) = \frac{-1}{x}$ and $t = \ln 3$. However, (Γ, D) is not a metric space. If we take $x = 1$, $\mu = 3$, and $y = 2$, then we have

$$\begin{aligned} D(x,\mu) &= (3-1)^2 = 4 \\ &\nleq (2-1)^2 + (3-2)^2 = D(x,y) + D(y,\mu). \end{aligned}$$

Thus, D does not satisfy the triangular inequality of a classical metric.

Example 3 ([36])**.** *Let $\Gamma = [0,\infty)$ and $D : \Gamma \times \Gamma \to [0,\infty)$ be defined by*

$$D(x,\mu) = \begin{cases} (x-\mu)^2, & \text{if } (x,\mu) \in [0,1] \times [0,1] \\ |x-\mu|, & \text{if } (x,\mu) \notin [0,1] \times [0,1] \end{cases}$$

Then, (Γ, D) is an F-complete F-ms with $F(x) = \ln x$ and $t = \frac{1}{2}$. However, (Γ, D) is not a metric space. If we take $x = 0$, $\mu = 1$, and $y = \frac{1}{2}$, we have

$$\begin{aligned} D(x,\mu) &= (1-0)^2 = 1 \\ &\nleq \left(0 - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - 1\right)^2 = D(x,y) + D(y,\mu). \end{aligned}$$

Hence, D does not satisfy the triangular inequality of a classical metric.

Definition 4. *Let (Γ, D) be an F-ms and $\{x_n\} \subset \Gamma$.*

i. *$\{x_n\}$ is named F-convergent if there is a $\mu \in \Gamma$ such that $D(x_n, \mu) \to 0$ as $n \to \infty$.*
ii. *$\{x_n\}$ is named an F-Cauchy sequence if $D(x_n, x_v) \to 0$ as $n, v \to \infty$.*
iii. *(Γ, D) is named F-complete if each F-Cauchy sequence is F-convergent.*

3. Results

Definition 5. *Let (Γ, D) be an F-ms and $\{k_n\}$ be a sequence of self-mappings in Γ. Assume that there exist sequences $\{a_n\}$, $\{c_n\}$, $\{d_n\}$, and $\{e_n\}$ of Γ, such that*

$$D\big(k_i(x), k_j(y)\big) \le \vartheta_{i,j} D(x,y) + \beta_{i,j} D(x, k_i(x)) + \gamma_{i,j} D(y, k_j(y)) + \zeta_{i,j} \frac{D(x,k_i(x)) D(y,k_j(y))}{1 + D(x,y)} \tag{3}$$

for all distinct $x, y \in \Gamma$, where $\vartheta_{i,j}, \beta_{i,j}, \gamma_{i,j}, \zeta_{i,j} \in [0,1)$, $(i,j = 1,2,\ldots)$ and $\vartheta_{i,j} = D\big(a_i, a_j\big)$, $\beta_{i,j} = D\big(c_i, c_j\big)$, $\gamma_{i,j} = D\big(d_i, d_j\big)$, $\zeta_{i,j} = D\big(e_i, e_j\big)$. If $\sum_{i=1}^{\infty} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})}$ is an α-series, then $\{k_n\}$ is named a rational-type contraction mapping.

Theorem 6. *Let (Γ, D) be an F-complete F-ms and $\{k_n\}$ be a sequence of self-mappings for Γ. If $\{k_n\}$ is a rational-type contraction mapping, then $\{k_n\}$ has a unique common fixed point.*

Proof. Let $x_0 \in \Gamma$ be any arbitrary element. We introduce a sequence $x_n = k_n x_{n-1}$ for all $n \ge 1$. From (3), we obtain

$$\begin{aligned} D(x_1, x_2) &= D(k_1(x_0), k_2(x_1)) \\ &\le \vartheta_{1,2} D(x_0, x_1) + \beta_{1,2} D(x_0, k_1(x_0)) + \gamma_{1,2} D(x_1, k_2(x_1)) \\ &\quad + \zeta_{1,2} \frac{D(x_0, k_1(x_0)) D(x_1, k_2(x_1))}{1 + D(x_0, x_1)} \\ &= \vartheta_{1,2} D(x_0, x_1) + \beta_{1,2} D(x_0, x_1) + \gamma_{1,2} D(x_1, x_2) \\ &\quad + \zeta_{1,2} \frac{D(x_0, x_1) D(x_1, x_2)}{1 + D(x_0, x_1)} \\ &\le (\vartheta_{1,2} + \beta_{1,2}) D(x_0, x_1) + (\gamma_{1,2} + \zeta_{1,2}) D(x_1, x_2). \end{aligned}$$

Thus, we see that
$$D(x_1, x_2) \leq \frac{(\vartheta_{1,2} + \beta_{1,2})}{1 - (\gamma_{1,2} + \zeta_{1,2})} D(x_0, x_1).$$
Therefore, we obtain
$$D(x_2, x_3) \leq \frac{(\vartheta_{2,3} + \beta_{2,3})}{1 - (\gamma_{2,3} + \zeta_{2,3})} \frac{(\vartheta_{1,2} + \beta_{1,2})}{1 - (\gamma_{1,2} + \zeta_{1,2})} D(x_0, x_1).$$
By continuing this process, we have
$$D(x_n, x_{n+1}) \leq \Pi_{i=1}^{n} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})} D(x_0, x_1).$$
Hence, for $p > 0$, we have
$$\begin{aligned}
\sum_{k=n}^{n+p-1} D(x_k, x_{k+1}) \quad &\leq \quad \Pi_{i=1}^{n} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})} D(x_0, x_1) \\
&\quad + \Pi_{i=1}^{n+1} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})} D(x_0, x_1) \\
&\quad + \ldots \\
&\quad + \Pi_{i=1}^{n+p-1} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})} D(x_0, x_1) \\
&= \quad \sum_{k=n}^{n+p-1} \Pi_{i=1}^{n+k} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})} D(x_0, x_1).
\end{aligned}$$
Therefore,
$$\begin{aligned}
\sum_{k=n}^{n+p-1} \Pi_{i=1}^{n+k} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})} D(x_0, x_1) \quad &\leq \quad \sum_{k=n}^{n+p-1} \lambda^k D(x_0, x_1) \\
&\leq \quad \frac{\lambda^n}{1 - \lambda} D(x_0, x_1).
\end{aligned}$$
Since $0 < \lambda < 1$, for all $\delta > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$,
$$0 < \frac{\lambda^n}{1 - \lambda} D(x_0, x_1) < \delta.$$
Let $(F, t) \in \Omega \times [0, +\infty)$, such that ($d_3$) is satisfied. Let $s > 0$ be fixed; then, according to (F2), there exists
$$0 < v < l \text{ implies } F(v) < F(s) - t.$$
By considering δ as l, we obtain
$$F\left(\frac{\lambda^n}{1 - \lambda} D(x_0, x_1)\right) < F(s) - t.$$
According to (F1), we have
$$F\left(\sum_{i=n}^{n+p-1} D(x_i, x_{i+1})\right) \leq F\left(\frac{\lambda^n}{1 - \lambda} D(x_0, x_1).\right) < F(s) - t.$$
From (d_3) and the last inequality,

$$
\begin{aligned}
F(D(x_n, x_{n+p})) &\leq F\left(\sum_{i=n}^{n+p-1} D(x_i, x_{i+1})\right) + t \\
&< F(s).
\end{aligned}
$$

According to (F1), we have $D(x_n, x_{n+p}) < s$. Hence, $\{x_n\}$ is an F-Cauchy sequence in Γ. Since Γ is F-complete, there exists $y \in \Gamma$, such that $x_n \to y$.
For any $m \in \mathbb{N}$,

$$
\begin{aligned}
D(x_n, k_m(y)) &= D(k_n(x_{n-1}), k_m(y)) \\
&\leq \vartheta_{n,m} D(x_{n-1}, y) + \beta_{n,m} D(x_{n-1}, k_n(x_{n-1})) + \gamma_{n,m} D(y, k_m(y)) \\
&\quad + \zeta_{n,m} \frac{D(x_{n-1}, k_n(x_{n-1})) D(y, k_m(y))}{1 + D(x_{n-1}, y)}.
\end{aligned}
$$

As $n \to \infty$,

$$
\begin{aligned}
D(y, k_m(y)) &\leq \vartheta_{n,m} D(y, y) + \beta_{n,m} D(y, y) + \gamma_{n,m} D(y, k_m(y)) + \zeta_{n,m} \frac{D(y, y) D(y, k_m(y))}{1 + D(y, y)} \\
&= \gamma_{n,m} D(y, k_m(y)).
\end{aligned}
$$

Thus, according to $\gamma \in [0, 1)$, we have $D(y, k_m(y)) = 0$, so $y = k_m(y)$, i.e., y is a common fixed point of $\{k_m\}$.

Here, we show the uniqueness of the fixed point. Suppose $\{k_m\}$ has different fixed points, y, z. According to (3),

$$
\begin{aligned}
D(y, z) &= D(k_m(y), k_m(z))) \\
&\leq \vartheta_{n,m} D(y, z) + \beta_{n,m} D(y, k_m(y)) + \gamma_{n,m} D(z, k_m(z)) \\
&\quad + \zeta_{n,m} \frac{D(y, k_m(y)) D(z, k_m(z))}{1 + D(y, z)} \\
&= \vartheta_{n,m} D(y, z)
\end{aligned}
$$

which is a contradiction. Thus, $\{k_m\}$ has a unique fixed point. □

Example 4. *Let* $\Gamma = [0, 1]$ *and* $D : \Gamma \times \Gamma \to [0, \infty)$ *be defined by*

$$
D(x, y) = \begin{cases} \max\{x, y\}, & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}
$$

Then, (Γ, D) *is an F-complete F-ms with* $F(x) = \frac{-1}{x}$ *and* $t = 2$.

For the sequences $(a_i) = (\frac{1}{i})$, $(c_i) = (d_i) = (\frac{1}{(1+4^i)})$, *and* $(e_i) = (\frac{1}{1+i^2})$, *we have*

$$
\begin{aligned}
\vartheta_{i,j} &= \max\left\{\frac{1}{i}, \frac{1}{j}\right\}, \ \beta_{i,j} = \gamma_{i,j} = \max\left\{\frac{1}{1+4^i}, \frac{1}{1+4^j}\right\} \\
\zeta_{i,j} &= \max\left\{\frac{1}{1+i^2} \frac{1}{1+j^2}\right\}, \quad i, j \geq 1.
\end{aligned}
$$

Let $k_i : \Gamma \to \Gamma$ *be defined by* $k_i(x) = \frac{x}{i}$. *Then, for* $\min\{i, j\} = n$, *we have*

$$\begin{aligned} D(k_i(x),k_j(y)) &= \max\left\{\frac{x}{4i},\frac{y}{4j}\right\} = \frac{1}{4}\max\left\{\frac{x}{i},\frac{y}{j}\right\} \\ &\leq \max\left\{\frac{1}{i},\frac{1}{j}\right\}.\max\{x,y\} \\ &= \frac{1}{n}\max\{x,y\} \\ &= \vartheta_{i,j}D(x,y) \\ &\leq \begin{array}{c}\vartheta_{i,j}D(x,y)+\beta_{i,j}D(x,k_i(x))+\gamma_{i,j}D(y,k_j(y)) \\ +\zeta_{i,j}\frac{D(x,k_i(x))D(y,k_j(y))}{1+D(x,y)}\end{array} \end{aligned}$$

all conditions of Theorem 6 are satisfied. Hence, 0 *is a common fixed point of* $\{k_i\}$.

Theorem 7. *Let* (Γ, D) *be an F-complete F-ms and* $\{k_n\}$ *be a sequence of self-mappings in* Γ*. Assume that there exist sequences* $\{a_n\}$, $\{c_n\}$, $\{d_n\}$, *and* $\{e_n\}$ *of* Γ*, such that*

$$D(k_i(x),k_j(y)) \leq \vartheta_{i,j}D(x,y)+\beta_{i,j}D(x,k_i(x))+\gamma_{i,j}D(y,k_j(y))+\zeta_{i,j}D(y,k_i(x)) \quad (4)$$

for all distinct $x,y \in \Gamma$, *where* $\vartheta_{i,j},\beta_{i,j},\gamma_{i,j},\zeta_{i,j} \in [0,1)$ $(\,i,j=1,2,\ldots)$ *and* $\vartheta_{i,l}=D(a_i,a_j)$, $\beta_{i,l}=D(c_i,c_j)$, $\gamma_{i,l}=D(d_i,d_j)$, $\zeta_{i,j}=D(e_i,e_j)$. *If* $\sum_{i=1}^{\infty}\frac{(\vartheta_{i,i+1}+\beta_{i,i+1})}{1-\gamma_{i,i+1}}$ *is an* α*-series, then* $\{k_n\}$ *has a unique common fixed point.*

Proof. Let $x_0 \in \Gamma$ be any arbitrary element. We introduce a sequence $x_n = k_n x_{n-1}$ for all $n \geq 1$.

By using (4), we have

$$\begin{aligned} D(x_1,x_2) &= D(k_1(x_0),k_2(x_1)) \\ &\leq \vartheta_{1,2}D(x_0,x_1)+\beta_{1,2}D(x_0,k_1(x_0))+\gamma_{1,2}D(x_1,k_2(x_1))+\zeta_{1,2}D(x_1,k_1(x_0)) \\ &= \vartheta_{1,2}D(x_0,x_1)+\beta_{1,2}D(x_0,x_1)+\gamma_{1,2}D(x_1,x_2)+\zeta_{1,2}D(x_1,x_1) \\ &\leq (\vartheta_{1,2}+\beta_{1,2})D(x_0,x_1)+(\gamma_{1,2})D(x_1,x_2). \end{aligned} \quad (5)$$

Thus, according to (5),

$$D(x_1,x_2) \leq \frac{(\vartheta_{1,2}+\beta_{1,2})}{1-\gamma_{1,2}}D(x_0,x_1).$$

Therefore, we obtain

$$D(x_2,x_3) \leq \frac{(\vartheta_{2,3}+\beta_{2,3})}{1-\gamma_{2,3}}\frac{(\vartheta_{1,2}+\beta_{1,2})}{1-\gamma_{1,2}}D(x_0,x_1).$$

By continuing this process, we have

$$D(x_n,x_{n+1}) \leq \Pi_{i=1}^{n}\frac{(\vartheta_{i,i+1}+\beta_{i,i+1})}{1-\gamma_{i,i+1}}D(x_0,x_1). \quad (6)$$

For $p>0$, according to (6),

$$\begin{aligned} \sum_{k=n}^{n+p-1}D(x_k,x_{k+1}) &\leq \Pi_{i=1}^{n}\frac{(\vartheta_{i,i+1}+\beta_{i,i+1})}{1-\gamma_{i,i+1}}D(x_0,x_1) \\ &\quad +\Pi_{i=1}^{n+1}\frac{(\vartheta_{i,i+1}+\beta_{i,i+1})}{1-\gamma_{i,i+1}}D(x_0,x_1) \\ &\quad +\ldots \\ &\quad +\Pi_{i=1}^{n+p-1}\frac{(\vartheta_{i,i+1}+\beta_{i,i+1})}{1-\gamma_{i,i+1}}D(x_0,x_1) \\ &= \sum_{k=n}^{n+p-1}\Pi_{i=1}^{n+k}\frac{(\vartheta_{i,i+1}+\beta_{i,i+1})}{1-\gamma_{i,i+1}}D(x_0,x_1). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\sum_{k=n}^{n+p-1} \Pi_{i=1}^{n+k} \frac{(\vartheta_{i,i+1}+\beta_{i,i+1})}{1-\gamma_{i,i+1}} D(x_0,x_1) &\leq \sum_{k=n}^{n+p-1} \lambda^k D(x_0,x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} D(x_0,x_1).\end{aligned}$$

Since $0<\lambda<1$, for all $\delta>0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$0 < \frac{\lambda^n}{1-\lambda} D(x_0,x_1) < \delta.$$

Let $(F,t) \in \Omega \times [0,+\infty)$ be such that (d_3) is satisfied. Let $s>0$ be fixed; then, according to (F2) there exits

$$0 < v < l \text{ implies } F(v) < F(s) - t.$$

By considering δ as l, we obtain

$$F\left(\frac{\lambda^n}{1-\lambda} D(x_0,x_1)\right) < F(s) - t.$$

According to (F1), we have

$$F\left(\sum_{i=n}^{n+p-1} D(x_i,x_{i+1})\right) \leq F\left(\frac{\lambda^n}{1-\lambda} D(x_0,x_1).\right) < F(s) - t.$$

From (d_3), and by using the last inequality,

$$\begin{aligned}F(D(x_n,x_{n+p})) &\leq F\left(\sum_{i=n}^{n+p-1} D(x_i,x_{i+1})\right) + t \\ &< F(s).\end{aligned}$$

By using (F1), we have $D(x_n,x_{n+p}) < s$. Hence, $\{x_n\}$ is an F-Cauchy sequence in Γ. Since Γ is F-complete, there exists $y \in \Gamma$, such that $x_n \to y$. For any $m \in \mathbb{N}$,

$$\begin{aligned}D(x_n,k_m(y)) &= D(k_n(x_{n-1}),k_m(y)) \\ &\leq \vartheta_{n,m} D(x_{n-1},y) + \beta_{n,m} D(x_{n-1},k_n(x_{n-1})) + \gamma_{n,m} D(y,k_m(y)) \\ &\quad + \zeta_{n,m} D(y,k_n(x_{n-1}))\end{aligned}$$

As $n \to \infty$,

$$\begin{aligned}D(y,k_m(y)) &\leq \vartheta_{n,m} D(y,y) + \beta_{n,m} D(y,y) + \gamma_{n,m} D(y,k_m(y)) + \zeta_{n,m} D(y,y) \\ &= \gamma_{n,m} D(y,k_m(y)).\end{aligned}$$

Thus, by using $\gamma_{n,m} \in [0,1)$, we have $D(y,k_m(y)) = 0$. Thus, $y = k_m(y)$, i.e., y is a common fixed point of $\{k_m\}$.

Here, we show the uniqueness of the fixed point. Suppose $\{k_m\}$ has different fixed points, y,z. According to (4),

$$\begin{aligned}D(y,z) &= D(k_m(y),k_m(z))) \\ &\leq \vartheta_{n,m} D(y,z) + \beta_{n,m} D(y,k_m(y)) + \gamma_{n,m} D(z,k_m(z)) + \zeta_{n,m} D(z,k_m(y)) \\ &= \vartheta_{n,m} D(y,z) + \zeta_{n,m} D(z,y)\end{aligned}$$

which is a contradiction with $\vartheta_{n,m} + \zeta_{n,m} < 1$. Thus, $\{k_m\}$ has a unique fixed point. □

Corollary 1. *Let (Γ, D) be an F-complete F-ms and $\{k_n\}$ be a sequence of self-mappings for Γ. Assume that there exists a sequence $\{a_n\} \subset \Gamma$, such that*

$$D\big(k_i(x), k_j(y)\big) \leq \vartheta_{i,j}[D(x,y) + D(x, k_i(x)) + D(y, k_j(y))]$$

for all distinct $x, y \in \Gamma$, where $0 \leq \vartheta_{i,j} < \frac{1}{3}$, ($i, j = 1, 2, \ldots$) and $\vartheta_{i,j} = D\big(a_i, a_j\big)$. If $\sum_{i=1}^{\infty} \frac{2\vartheta_{i,i+1}}{1-\vartheta_{i,i+1}}$ is an α-series. Then, $\{k_n\}$ has a unique common fixed point.

Corollary 2. *Let (Γ, D) be an F-complete F-ms and $\{k_n\}$ be a sequence of self-mappings on Γ. Assume the sequences $\{a_n\}, \{c_n\} \subset \Gamma$ such that*

$$D\big(k_i(x), k_j(y)\big) \leq \vartheta_{i,j}[D(x, k_i(x)) + D(y, k_j(y)] + \zeta_{i,j} D(x,y)$$

$x, y \in B$, with $x \neq y$ and $\vartheta_{i,j}, \zeta_{i,j} \in [0,1)$, $\vartheta_{i,j} = b\big(a_i, a_j\big)$, and $\zeta_{i,j} = b\big(c_i, c_j\big)$ ($i, j = 1, 2, \ldots$). If $\sum_{i=1}^{\infty} \frac{\vartheta_{i,i+1}+\zeta_{i,i+1}}{1-\vartheta_{i,i+1}}$ is an α-series, then $\{k_n\}$ has a unique common fixed point.

Here, we give a common fixed-point theorem for weak Choudhury type contraction.

Definition 6. *Let (Γ, D) be an F-ms and $\{k_n\}$ be a sequence of self-mappings for Γ. Assume that there exist two sequences $\{a_n\}$, $\{c_n\}$ of elements of Γ, such that*

$$D\big(k_i(x), k_j(y)\big) \leq \vartheta_{i,j}\big[D(x, k_i(x)) + D(y, k_j(y))\big] - \zeta_{i,j}\psi\big(D(x, k_i(x)), D(y, k_j(y))\big) \quad (7)$$

for all distinct $x, y \in \Gamma$, where $\vartheta_{i,j}, \zeta_{i,j} \in [0,1)$, $i, j = 1, 2, \ldots$, $\vartheta_{i,j} = D\big(a_i, a_j\big)$, $\zeta_{i,j} = D\big(c_i, c_j\big)$, and $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous mapping with $\psi(0,0) = 0$. If $\sum_{i=1}^{\infty} \frac{\vartheta_{i,i+1}}{1-\vartheta_{i,i+1}}$ is an α-series, then $\{k_n\}$ is named weak Choudhury type contraction mapping.

Theorem 8. *Let (Γ, D) be an F-complete F-ms and $\{k_n\}$ be a sequence of self-mappings for Γ. If $\{k_n\}$ is a weak Choudhury type contraction mapping, then $\{k_n\}$ has a unique common fixed point.*

Proof. Let $x_0 \in \Gamma$ be any arbitrary element. We introduce a sequence $x_n = k_n x_{n-1}$ for all $n \geq 1$.

From (7), we have

$$\begin{aligned} D(x_1, x_2) &= D(k_1(x_0), k_2(x_1)) \\ &\leq \vartheta_{1,2}[D(x_0, k_1(x_0)) + D(x_1, k_2(x_1))] - \zeta_{1,2}\psi(D(x_0, k_1(x_0)), D(x_1, k_2(x_1)) \\ &= \vartheta_{1,2}[D(x_0, x_1) + D(x_1, x_2)] - \zeta_{1,2}\psi(D(x_0, x_1), D(x_1, x_2)) \\ &\leq \vartheta_{1,2}[D(x_0, x_1) + D(x_1, x_2)] \end{aligned} \quad (8)$$

Thus, we obtain

$$D(x_1, x_2) \leq \frac{\vartheta_{1,2}}{1-\vartheta_{1,2}} D(x_0, x_1).$$

Therefore,

$$D(x_n, x_{n+1}) \leq \Pi_{i=1}^{n} \frac{\vartheta_{i,i+1}}{1-\vartheta_{i,i+1}} D(x_0, x_1). \quad (9)$$

Moreover, similar to Theorem 1, we obtain $\{x_n\}$, which is an F-Cauchy sequence in Γ. Since Γ is F-complete, there exists $y \in \Gamma$, such that $x_n \to y$.

For any $m \in \mathbb{N}$,

$$\begin{aligned} D(x_n, k_m(y)) &= D(k_n(x_{n-1}), k_m(y)) \\ &\leq \begin{array}{l} \vartheta_{n,m}[D(x_{n-1}, k_n(x_{n-1})) + D(y, k_m(y))] \\ -\zeta_{n,m}\psi(D(x_{n-1}, k_n(x_{n-1})), D(y, k_m(y))) \end{array} \end{aligned}$$

As $n \to \infty$, by using the continuity of ψ, we obtain

$$\begin{aligned} D(y,k_m(y)) &\leq \vartheta_{n,m}[D(y,y)+D(y,k_m(y))]-\zeta_{n,m}\psi(D(y,y),D(y,k_m(y))) \\ &= \vartheta_{n,m}D(y,k_m(y))-\zeta_{n,m}\psi(0,D(y,k_m(y))) \\ &\leq \vartheta_{n,m}D(y,k_m(y)) \end{aligned}$$

which is a contradiction. Thus, $D(y,k_m(y))=0$. Thus, $y=k_m(y)$, i.e., y is a common fixed point of $\{k_m\}$.

Here, we show the uniqueness of the fixed point. Suppose $\{k_m\}$ has different fixed points y,z. According to (7),

$$\begin{aligned} D(y,z) &= D(k_m(y),k_m(z))) \\ &\leq \begin{array}{l} \vartheta_{n,m}[D(y,k_m(y))+D(z,k_m(z))] \\ -\zeta_{n,m}(D(y,k_m(y))+D(z,k_m(z))) \end{array} \\ &= \vartheta_{n,m}[D(y,y)+D(z,z)]-\zeta_{n,m}(0,0) \\ &= 0 \end{aligned}$$

which is a contradiction. Thus, $\{k_m\}$ has a unique fixed point. □

Example 5. *Let* $\Gamma=[0,1]$ *and* $D:\Gamma\times\Gamma\to[0,\infty)$ *be defined by*

$$D(x,y)=\begin{cases} e^{|x-y|}, & \text{if } x\neq y \\ 0 & \text{if } x=y \end{cases}$$

Then, (Γ,D) *is an F-complete F-ms with* $F(u)=\frac{-1}{u}$ *and* $t=\ln 3$.

For the sequences $a_i=\ln\left|\frac{1}{i}+1\right|$ *and* $c_i=\ln\left|\frac{1}{2^i}+1\right|$, *we get*

$$\begin{aligned} \vartheta_{i,j} &= e^{\left|\ln\left|\frac{1}{i}+1\right|-\ln\left|\frac{1}{j}+1\right|\right|}=e^{\ln\left|\frac{\frac{1}{i}+1}{\frac{1}{j}+1}\right|}=\left|\frac{\frac{1}{i}+1}{\frac{1}{j}+1}\right| \text{ and} \\ \zeta_{i,j} &= e^{\left|\ln\left(\frac{1}{2^i}+\frac{1}{3}\right)-\ln\left(\frac{1}{2^i}+\frac{1}{3}\right)\right|}=e^{\ln\left|\frac{\frac{1}{2^i}+\frac{1}{3}}{\frac{1}{2^j}+\frac{1}{3}}\right|}=\left|\frac{\frac{1}{2^i}+\frac{1}{3}}{\frac{1}{2^j}+\frac{1}{3}}\right|, \text{ for } i>j. \end{aligned}$$

Assume that $k_i:\Gamma\to\Gamma$ *is defined by* $k_i(x)=\begin{cases} 3x, & \text{if } 0\leq x\leq\frac{1}{3} \\ 0 & \text{other} \end{cases}\Bigg|$ *and* $\psi:[0,\infty)\times[0,\infty)\to[0,\infty)$ *is defined by* $\psi(t,s)=t+s$.

If $x,y\in\left[0,\frac{1}{3}\right]$, *we have*

$$\begin{aligned} D\big(k_i(x),k_j(y)\big) &= e^{|3x-3y|} \\ &\leq \left(\left|\frac{\frac{1}{i}+1}{\frac{1}{j}+1}\right|-\left|\frac{\frac{1}{2^i}+\frac{1}{3}}{\frac{1}{2^j}+\frac{1}{3}}\right|\right)\left[e^{|x-3x|}+e^{|y-3y|}\right] \\ &= \big(\vartheta_{i,j}-\zeta_{i,j}\big)\left[D(x,k_i(x))+D(y,k_j(y))\right] \\ &= \vartheta_{i,j}\left[D(x,k_i(x))+D(y,k_j(y))\right]-\zeta_{i,j}\psi\big(D(x,k_i(x)),D(y,k_j(y))\big). \end{aligned}$$

If $x \in \left[0, \frac{1}{3}\right]$ *and* $y \notin \left[0, \frac{1}{3}\right]$, *we have*

$$\begin{aligned}
D\big(k_i(x), k_j(y)\big) &= e^{|3x|} \\
&\leq \left(\left| \frac{\frac{1}{i}+1}{\frac{1}{j}+1} \right| - \left| \frac{\frac{1}{2^i}+\frac{1}{3}}{\frac{1}{2^j}+\frac{1}{3}} \right| \right) \left[e^{|x-3x|} + e^{|y|} \right] \\
&= \big(\vartheta_{i,j} - \zeta_{i,j}\big)\left[D(x, k_i(x)) + D(y, k_j(y))\right] \\
&= \vartheta_{i,j}\left[D(x, k_i(x)) + D(y, k_j(y))\right] - \zeta_{i,j}\psi\big(D(x, k_i(x)), D(y, k_j(y))\big).
\end{aligned}$$

If $x, y \notin \left[0, \frac{1}{3}\right]$, *we have*

$$\begin{aligned}
D\big(k_i(x), k_j(y)\big) &= e^{0} = 1 \\
&\leq \left(\left| \frac{\frac{1}{i}+1}{\frac{1}{j}+1} \right| - \left| \frac{\frac{1}{2^i}+\frac{1}{3}}{\frac{1}{2^j}+\frac{1}{3}} \right| \right) \left[e^{|x|} + e^{|y|} \right] \\
&= \big(\vartheta_{i,j} - \zeta_{i,j}\big)\left[D(x, k_i(x)) + D(y, k_j(y))\right] \\
&= \vartheta_{i,j}\left[D(x, k_i(x)) + D(y, k_j(y))\right] - \zeta_{i,j}\psi\big(D(x, k_i(x)), D(y, k_j(y))\big).
\end{aligned}$$

Hence, all conditions of Theorem 8 are satisfied, and 0 *is common fixed point of* $\{k_i\}$.

Remark 1. *In the previous example, a Banach contraction is not applicable. Indeed, for* $x = 0$ *and* $y = \frac{1}{3}$, *we have*

$$D\big(k_i(x), k_j(y)\big) = e^{1} > \left| \frac{\frac{1}{i}+1}{\frac{1}{j}+1} \right| e^{\frac{1}{3}} = \vartheta_{i,j} D(x, y)$$

Corollary 3. *Let* (Γ, D) *be an F-complete F-ms and* $\{k_n\}$ *be a sequence of self-mappings for* Γ. *Assume that there exists a sequence* $\{a_n\} \subset \Gamma$, *such that*

$$D\big(k_i(x), k_j(y)\big) \leq \vartheta_{i,j}[D(x, k_i(x)) + D(y, k_j(y)]$$

$x, y \in \Gamma$ *with* $x \neq y$ *and* $0 \leq \vartheta_{i,j} < \frac{1}{2}$, $\vartheta_{i,j} = D\big(a_i, a_j\big)$ $(i, j = 1, 2, \ldots)$. *Then,* $\{k_n\}$ *has a unique common fixed point.*

4. Application

4.1. An Application Using an Integral Equation

Let us consider the following integral equation:

$$x(t) = \int_0^t G_i(t, r, x(r))dr, \quad t \in [0, a] \tag{10}$$

where $a > 0$, $G_i : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$.

Let us consider that $\Gamma = (C[0, a], \mathbb{R})$ is the set of all continuous functions $x : [0, a] \to \mathbb{R}$, and the sequence of mappings $k_i : \Gamma \to \Gamma$ is defined by

$$k_i(x)(t) = \int_0^t G_i(t, r, x(r))]dr, \quad t \in [0, a].$$

Let us consider the following F-metric on Γ,

$$D(x, y) = \sup_{t \in [0, a]} \left\{ |x(t) - y(t)|^2 \right\}$$

for all $x, y \in \Gamma$. Then, (Γ, D) is an F-complete F-metric space with $F(x) = x$ and $t = 0$. Also, (Γ, D) is a $(q1 - q2)$-quasimetric space.

Theorem 9. *Suppose the following hypotheses hold:*

(i) *The functions G_i are continuous;*

(ii) *For all $t, r \in [0,a]$ and $x, y \in \mathbb{R}$,*

$$|G_i(t,r,x) - G_i(t,r,y)| \leq \left(\frac{1}{t}.\frac{1}{2^{i+j+2}}\right)^{1/2} \begin{pmatrix} |x(t)-y(t)|^2 + |x(t)-k_i(x)(t)|^2 \\ +\left|v(t)-k_j(y)(t)\right|^2 + \frac{|x(t)-k_i(x)(t)|^2\left|v(t)-k_j(y)(t)\right|^2}{1+|x(t)-y(t)|^2} \end{pmatrix}^{1/2}$$

$i, j = 1, 2, ...$ Then, the integral Equation (10) has a unique solution.

Proof. For all $x, y \in \Gamma$, by using the Cauchy–Schwarz inequality and (ii), we have

$$\begin{aligned} \left|k_i(x(t)) - k_j(y(t))\right|^2 &= \left|\int_0^t G_i(t,r,x(r))dr - \int_0^t G_j(t,r,y(r))dr\right|^2 \\ &\leq \left(\int_0^t 1dr\right)^2 \left(\int_0^t \left|G_i(t,r,x(r)) - G_j(t,r,y(r))\right|dr\right)^2 \\ &\leq \left(\frac{1}{t}\frac{1}{2^{i+j+2}}\right)\left(\int_0^t \begin{pmatrix} |x(t)-y(t)|^2 + |x(t)-k_i(x)(t)|^2 \\ +\left|y(t)-k_j(y)(t)\right|^2 \\ +\frac{|x(t)-k_i(x)(t)|^2\left|v(t)-k_j(y)(t)\right|^2}{1+|x(t)-y(t)|^2} \end{pmatrix}^{1/2} dr\right)^2 \\ &\leq \left(\frac{1}{t}\frac{1}{2^{i+j+2}}\right)\begin{pmatrix} |x(t)-y(t)|^2 + |x(t)-k_i(x)(t)|^2 \\ +\left|y(t)-k_j(y)(t)\right|^2 \\ +\frac{|x(t)-k_i(x)(t)|^2\left|y(t)-k_j(y)(t)\right|^2}{1+|x(t)-y(t)|^2} \end{pmatrix}\left(\int_0^t dr\right)^2 \end{aligned}$$

Thus, we obtain

$$\begin{aligned} D\big(k_i(x), k_j(y)\big) &\leq \vartheta_{i,j} D(x,y) + \beta_{i,j} D(x, k_i(x)) + \gamma_{i,j} D(y, k_j(y)) \\ &\quad + \zeta_{i,j} \frac{D(x, k_i(x)) D\big(y, k_j(y)\big)}{1 + D(x,y)}. \end{aligned}$$

for $\vartheta_{i,j}, \beta_{i,j}, \gamma_{i,j}, \zeta_{i,j} = \frac{1}{2^{i+j+2}}$. Thus, according to Theorem 6, (10) has a unique solution. □

4.2. An Application Using Differential Equations

We consider a Cauchy problem involving a nonlinear fractional differential equation with the following condition given by

$$^cD^q z(t) = g(t, z(t)) \quad 0 < t < 1,\ 1 < q \leq 2 \tag{11}$$

via $z(0) = 0$, $Iz(1) = z'(0)$, where $z \in C([0,1], \mathbb{R})$. Here $^cD^q$ represents the Caputo fractional derivative of order q defined by

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-r)^{n-q-1} g^n(r)dr, \ \ n-1 < q < n,\ n = [q]+1$$

and $I^n g$ represents the Riemann–Liouville fractional integral of order q of a continuous function $g : \mathbb{R}^+ \to \mathbb{R}$, given by

$$I^n g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-r)^{q-1} g(r)dr, \quad q > 0.$$

The F-metric on $\Gamma = (C[0,1], \mathbb{R})$ is given by $D(z,v) = \max_{t\in[0,1]}\left\{|z(t)-v(t)|^2\right\}$ for all $z, v \in C[0,1]$ with $F(u) = \ln u$ and $t = \ln 2$.

Theorem 10. *Consider the nonlinear fractional differential Equation (11). Assume that these assertions hold:*

(i) $k_i : \Gamma \to \Gamma$ *is a continuous operator defined by*

$$\begin{aligned} k_i z(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-r)^{q-1} g(r, z(r)) dr \\ &+ \frac{2t}{\Gamma(q)} \int_0^1 \left(\int_0^r (r-m)^{q-1} g(m, z(m)) dm \right) dr, \end{aligned}$$

(ii) $g(r, z(r)) - g(r, v(r)) \leq \left(\frac{1}{3i} - \frac{1}{3j}\right)^{1/2} \left[|z - k_i(z)| + |v - k_j(v)| \right]$.

Then, (11) has a unique solution.

Proof. From the definition of D,

$$\begin{aligned}
&\left(D\left(k_i(z(t)), k_j(v(t))\right)\right)^{1/2} \\
= &\left(\max_{t\in[0,1]} \left\{ \left|k_i(z(t)) - k_j(v(t))\right|^2 \right\} \right)^{1/2} \\
= &\left(\max_{t\in[0,1]} \left\{ \left| \begin{array}{c} \frac{1}{\Gamma(q)} \int_0^t (t-r)^{q-1} g(r, z(r)) dr \\ -\frac{1}{\Gamma(q)} \int_0^t (t-r)^{q-1} g(r, v(r)) dr \\ +\frac{2t}{\Gamma(q)} \int_0^1 \left(\int_0^r (r-m)^{q-1} g(m, z(m)) dm \right) dr \\ -\frac{2t}{\Gamma(q)} \int_0^1 \left(\int_0^r (r-m)^{q-1} g(m, v(m)) dm \right) dr \end{array} \right|^2 \right\} \right)^{1/2} \\
\leq &\left(\max_{t\in[0,1]} \left\{ \left| \begin{array}{c} \frac{1}{\Gamma(q)} \int_0^t (t-r)^{q-1} (g(r, z(r)) - g(r, v(r))) dr \\ +\frac{2t}{\Gamma(q)} \int_0^1 \left(\int_0^r (r-m)^{q-1} \left(\begin{array}{c} g(m, z(m)) \\ -g(m, v(m)) \end{array} \right) dm \right) dr \end{array} \right|^2 \right\} \right)^{1/2} \\
\leq &\left(\max_{t\in[0,1]} \left\{ \left| \begin{array}{c} \frac{\Gamma(q+1)}{\Gamma(q)} \cdot \left(\frac{1}{3i} - \frac{1}{3j}\right) \left(\begin{array}{c} |z(t) - k_i(z(t))| \\ +\left|v(t) - k_j(v(t))\right| \end{array} \right) \int_0^t (t-r)^{q-1} dr \\ +\frac{2t.\Gamma(q+1)}{\Gamma(q)} \left(\frac{1}{3i} - \frac{1}{3j}\right) \left(\begin{array}{c} |z(t) - k_i(z(t))| \\ +\left|v(t) - k_j(v(t))\right| \end{array} \right) \int_0^1 \left(\int_0^r (r-m)^{q-1} dm \right) dr \end{array} \right|^2 \right\} \right)^{1/2} \\
\leq &\left(\max_{t\in[0,1]} \left\{ \left| \begin{array}{c} \frac{\Gamma(q)\Gamma(q+1)}{\Gamma(q)\Gamma(q+1)} \cdot \left(\frac{1}{3i} - \frac{1}{3j}\right)^{1/2} \left(\begin{array}{c} |z(t) - k_i(z(t))| \\ +\left|v(t) - k_j(v(t))\right| \end{array} \right) \\ +B(q+1, 1) \frac{\Gamma(q)\Gamma(q+1)}{\Gamma(q)\Gamma(q+1)} 2t \left(\frac{1}{3i} - \frac{1}{3j}\right)^{1/2} \left(\begin{array}{c} |z(t) - k_i(z(t))| \\ +\left|v(t) - k_j(v(t))\right| \end{array} \right) \end{array} \right|^2 \right\} \right)^{1/2} \\
\leq &\left(\max_{t\in[0,1]} \left\{ \left| \begin{array}{c} \left(\frac{1}{3i} - \frac{1}{3j}\right)^{1/2} \left(|z(t) - k_i(z(t))| + \left|v(t) - k_j(v(t))\right|\right) \\ +2\left(\frac{1}{3i} - \frac{1}{3j}\right)^{1/2} \left(|z(t) - k_i(z(t))| + \left|v(t) - k_j(v(t))\right|\right) \end{array} \right|^2 \right\} \right)^{1/2} \\
= &\left(\max_{t\in[0,1]} \left\{ \left| 3\left(\frac{1}{3i} - \frac{1}{3j}\right)^{1/2} \left(|z(t) - k_i(z(t))| + \left|v(t) - k_j(v(t))\right|\right) \right|^2 \right\} \right)^{1/2} \\
\leq &\; 3^{1/2} \left(\frac{1}{i} - \frac{1}{j}\right)^{1/2} \max_{t\in[0,1]} \left\{ \left(|z(t) - k_i(z(t))| + \left|v(t) - k_j(v(t))\right|\right) \right\} \\
\leq &\; 3^{1/2} \left(\frac{1}{i} - \frac{1}{j}\right)^{1/2} \left(\max_{t\in[0,1]} \{|z(t) - k_i(z(t))|\} + \max_{t\in[0,1]} \left\{\left|v(t) - k_j(v(t))\right|\right\} \right)
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
D\big(k_i(z(t)), k_j(v(t))\big) &= 3\left(\frac{1}{i} - \frac{1}{j}\right)\big[D(z, k_i(z)) + D(v, k_j(v))\big] \\
&= \vartheta_{i,j}\big[D(z, k_i(z)) + D(v, k_j(v))\big] - \zeta_{i,j}\psi\big(D(z, k_i(z)), D(v, k_j(v))\big)
\end{aligned}
$$

Hence, by Theorem 8, taking $\psi(z, v) = 2z + 2v$ and $\vartheta_{i,j} = \frac{1}{i}$, $\zeta_{i,j} = \frac{1}{j}$, k_i has a unique common fixed point which implies that system (11) has a unique solution. □

5. Discussion

F-metric spaces are defined as a generalization of metric spaces. Every metric is an F-metric, but the converse is not true. In this work, we introduced new types of contractions by using an α-series in F-metric spaces, and we proved more general fixed-point theorems than the results in the literature. Consequently, applications of fixed-point results are useful in many sciences, and they can make solving problems easier. In future studies, a weak-type contractive condition can be defined, and coupled- or tripled-fixed-point theorems, common-fixed-point results, and best-proximity theorems can be proven in F-metric spaces. In addition, applications to Fredholm and Volterra integral equations and differential equations can be given. There will be several useful applications, especially in mathematics and engineering.

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