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Optimality and Duality of Semi-Preinvariant Convex Multi-Objective Programming Involving Generalized $(F, \alpha, \rho, d) - I-$Type Invex Functions

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Abstract: Multiobjective programming refers to a mathematical problem that requires the simultaneous optimization of multiple independent yet interrelated objective functions when solving a problem. It is widely used in various fields, such as engineering design, financial investment, environmental planning, and transportation planning. Research on the theory and application of convex functions and their generalized convexity in multiobjective programming helps us understand the essence of optimization problems, and promotes the development of optimization algorithms and theories. In this paper, we firstly introduces new classes of generalized $(F, \alpha, \rho, d) - I$ functions for semi-preinvariant convex multiobjective programming. Secondly, based on these generalized functions, we derive several sufficient optimality conditions for a feasible solution to be an efficient or weakly efficient solution. Finally, we prove weak duality theorems for mixed-type duality.

Keywords: semi-preinvariant convexity; multiobjective programming; efficient solution; generalized convexity; mixed-type duality

MSC: 90C26; 90C29; 90C30; 90C34; 90C46

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1. Introduction

Multi-objective programming is a mathematical model for solving decision-making problems with multiple conflicting objective functions [1,2], which has wide applications in various practical fields, including production scheduling [3], resource allocation [4], and portfolio management [5]. This field originated in the 1950s, and was founded by Morgenstern and von Neumann's work. They introduced the concept of cooperative game theory to handle multi-objective decision-making problems [6]. The objective functions in multi-objective programming usually cannot be minimized or maximized simultaneously; therefore, a set of feasible solutions must be found that achieves some balance among all of the objectives [7].

Convexity is an important concept in optimization theory, providing powerful tools for solving various practical problems [8,9]. As an important type of convexity in optimization theory, semi-preinvexity possesses crucial mathematical properties in the field of multi-objective programming, which are particularly useful for handling partially convex or locally convex problems [10]. Research on multi-objective programming and its semi-preinvexity theory can offer theoretical support for understanding and solving complex optimization problems, playing an essential role in modeling and solving practical problems [11–16].

In the past few decades, significant progress has been made in the study of invariant convex multi-objective programming, including advancements in mathematical models, optimization strategies, and new algorithms to effectively address conflicts among multiple objectives [17–20]. These efforts not only provided theoretical proofs, but also validated the effectiveness of the algorithms through examples. Among them, the optimality and duality of multi-objective programming with generalized $(F, \alpha, \rho, d)-$convexity have been further studied [21–24]. The optimality conditions and duality theory under differentiable and non-differentiable conditions are obtained, including weak and strong duality relations, as well as inverse duality relations and mixed duality results. Scholars subsequently expanded their studies to include second-order cases in multi-objective programming problems [25–28], and obtained the corresponding optimality conditions and dual results.

With the increasing complexity of the modern social environment and the diverse range of challenges faced, the effectiveness of the traditional second-order convex multi-objective programming model has decreased. To better address the intricate and evolving needs and challenges of modern society, scholars have expanded their research on higher-order cases related to (F, α, ρ, d) convex multi-objective programming [29–34]. This expansion includes various forms of higher-order $(F, \alpha, \rho, d)-$convexity, and has produced high-order symmetric dual results that incorporate these convexities, as well as non-differentiable multi-objective mixed symmetric dual results. These studies provide more efficient decision support and optimization solutions for dealing with increasingly complex and diverse real-world problems.

Despite significant advancements in the study of generalized (F, α, ρ, d) convexity in multi-objective programming, there has been limited exploration into extending this concept to semi-preinvariant invex functions. Therefore, it is crucial to investigate optimality and duality for semi-preinvariant convex multi-objective programming that involves generalized $(F, \alpha, \rho, d) - I$ invex functions as a means to enhance efficiency and interpretability when solving multi-objective programming problems. Furthermore, this research will contribute to the development of multi-objective programming theory, and provide solutions for complex problems encountered in practical applications.

Motivated by the significance of semi-preinvariant convex multi-objective programming in mathematics and engineering, our investigation focuses on optimality and duality in this field. Specifically, we aim to explore these concepts within the context of semi-preinvariant convex multi-objective programming involving generalized $(F, \alpha, \rho, d) - I-$type invex functions. The main objective of this study is to define new classes of generalized $(F, \alpha, \rho, d) - I$ functions for semi-preinvariant convex multi-objective programming, while deriving several sufficient optimality conditions that determine whether a feasible solution is efficient or weakly efficient. Additionally, we establish weak duality theorems for mixed-type duality.

The contribution of this paper is threefold: (1) We introduce novel classes of generalized $(F, \alpha, \rho, d)-$type I functions that encompass the following: generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions, generalized pseudoquasi $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions, weakly strictly pseudoquasi $(F, \alpha, \rho, d) - I$ invex functions, strongly strictly pseudoquasi $(F, \alpha, \rho, d) - I$ invex functions, sub-strictly pseudoquasi $(F, \alpha, \rho, d)-$type I invex functions, weak quasistrictly pseudo$(F, \alpha, \rho, d) - I$ invex functions, weak quasisemi-pseudo$(F, \alpha, \rho, d) - I$ invex functions, and weak strictly pseudo$(F, \alpha, \rho, d) - I$ invex functions. (2) Based on these generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions, we derive several sufficient optimality conditions for a feasible solution to be an efficient or weakly efficient solution. (3) We investigate the mixed-type duality involving this generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant function, and obtain weak duality theorems.

The structure of this paper is outlined as follows: Section 2 provides an overview of the relevant concepts and notions. Afterwards, we introduce a novel class of generalized convex functions based on the $\alpha - E-$semi-invariant convexity, which forms the basis for our study. In Section 3, we derive several sufficient optimality conditions for $\alpha - E-$semi-preinvariant convex multiobjective programming. Additionally, in Section 4, we explore mixed-type duality involving the generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant function, and establish weak duality theorems. Lastly, concluding remarks are provided in Section 5.

2. Preliminaries and Definitions

In this section, we mainly review some relevant concepts and notions that will be needed throughout the paper. By convention, $\mathbb{R}^n$ is an n-dimensional vector space, and x

and y denote vectors within $\mathbb{R}^n$. First, for ease of reading, Table 1 lists some symbols and their corresponding meanings as follows:

Table 1. Some symbols and meanings.

Symbols	Meanings
$x = y$	If and only if $x_i = y_i, \forall i = 1,2,\ldots,n$.
$x > y$	If and only if $x_i > y_i, \forall i = 1,2,\ldots,n$.
$x \geqq y$	If and only if $x_i \geqq y_i, \forall i = 1,2,\ldots,n$.
$x \geq y$	If and only if $x_i \geqq y_i, x \neq y, \forall i = 1,2,\ldots,n$.
$\mathbb{R}^n$	N-dimensional Euclidean space
F	Sublinear functional
MOP	Multiobjective programming
XMOP	Mixed-type dual of MOP

Next, we will recall the definitions of an invariant convex set, an $\alpha - E-$semi-invariant convex set, and a sublinear functional, which will be needed later.

Definition 1 ([16]). *Let $X \subset \mathbb{R}^n$ be a non-empty subset, if there exists η: $X \times X \to \mathbb{R}^n$, such that for any $x, y \in X, \lambda \in [0,1]$, we have*

$$y + \lambda\eta(x,y) \in X,$$

then, we say that X is an invariant convex set with respect to η.

Definition 2 ([14,15]). *Let $X \subset \mathbb{R}^n$ be a non-empty subset, if there exists η: $X \times X \times [0,1] \to \mathbb{R}^n \setminus \{0\}$, α: $X \times X \to \mathbb{R} \setminus \{0\}$, E: $X \to X$, such that, for any $x, y \in X, \lambda \in [0,1]$, we have*

$$E(y) + \lambda\alpha(E(x),E(y))\eta(E(x),E(y),\lambda) \in X,$$

then, we say that X is an $\alpha - E-$semi-invariant convex set with respect to η and α.

Definition 3 ([14,15]). *Let $X \subset \mathbb{R}^n$ be $\alpha - E$-semi-invariant convex set, f: $X \to \mathbb{R}$, if there exists η: $X \times X \times [0,1] \to \mathbb{R}^n \setminus \{0\}$, α: $X \times X \to \mathbb{R} \setminus \{0\}$, E: $X \to X$, such that, for any $x, y \in X, (E(x) \neq E(y)), \lambda \in [0,1]$, we have*

$$f(E(y) + \lambda\alpha(E(x),E(y))\eta(E(x),E(y),\lambda)) \leq (<)\lambda f(E(x)) + (1-\lambda)f(E(y)),$$

and $\lim\limits_{\lambda\to 0} \lambda\eta(E(x),E(y),\lambda) = 0$, then we say that X is a (strict) $\alpha - E-$semi-invariant convex set with respect to η and α.

Definition 4 ([24]). *Suppose F is a sublinear functional, and let $X \times X \times \mathbb{R}^n \to \mathbb{R}$, for $\forall x, \overline{x} \in X$; we have*

$$F(x,\overline{x};\alpha_1+\alpha_2) \leqq F(x,\overline{x},\alpha_1) + F(x,\overline{x},\alpha_2), \forall \alpha_1,\alpha_2 \in \mathbb{R}^n,$$
$$F(x,\overline{x};\alpha a) = \alpha F(x,\overline{x},a), \forall \alpha \in \mathbb{R}, \alpha \geqq 0, \forall a \in \mathbb{R}^n.$$

In addition, we will introduce a new class of generalized convex functions based on the $\alpha - E-$semi-preinvariant functions and the generalized $(F,\alpha,\rho,d) - I-$type invex functions to investigate the sufficient optimality conditions and duality theorems for multiobjective programming.

Let $X \in \mathbb{R}^n$ be a non-empty $\alpha - E-$semi-invariant convex set. F is a sublinear functional. $f = (f_1, f_2, \ldots, f_q)$: $X \to \mathbb{R}^q$, $f_i(i = 1,2,\ldots,q) \in X$ are local Lipschitz functions, I_{f_i} represents the value of f_i. $G = (G_{f_1,f_2}, \ldots, _{f_q})$: $\mathbb{R} \to \mathbb{R}^q$, G_{f_i}: $I_{f_i} \to \mathbb{R}$ is a strictly monotonically increasing differentiable real valued function. $\forall \xi_i \in G'_{f_i}(f_i(E(\overline{x})\partial f_i(E(\overline{x}))$, and F is a sublinear functional. $\rho = (\rho^1, \rho^2)$, $\rho^1 = (\rho_1, \rho_2, \ldots, \rho_p) \in \mathbb{R}^p$, $\rho^2 = (\rho_{1+p}, \rho_{2+p}, \ldots, \rho_{r+p}) \in \mathbb{R}^r$, $\alpha = (\alpha^1, \alpha^2)$, $\alpha^1, \alpha^2 : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $d(\cdot,\cdot)$: $X \times X \to \mathbb{R}$.

Definition 5 (Generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions)**.** *Let (f, g) be generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions; for $\forall x \in A$, we have*

$$b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) \geqq F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})), \tag{1}$$

$$- b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \geqq F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})). \tag{2}$$

Definition 6 (Generalized pseudoquasi $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions)**.** *Let (f, h) be generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions; for $\forall x \in A$, we have*

$$\begin{aligned} b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) < 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) < 0, \end{aligned} \tag{3}$$

$$\begin{aligned} -b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \leqq 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leqq 0. \end{aligned} \tag{4}$$

If the first inequality in Equation (3) is changed to $\leqq 0$, then (f, g) would be strictly pseudoquasi $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions.

Definition 7. *(f, h) is a weak strictly pseudoquasi $(F, \alpha, \rho, d) - I$ at $\overline{x}$, if for all $x \in A$, we have*

$$\begin{aligned} b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) \leq 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) < 0, \end{aligned} \tag{5}$$

$$\begin{aligned} -b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \leqq 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leqq 0. \end{aligned} \tag{6}$$

Definition 8. *(f, h) is a strong strictly pseudoquasi $(F, \alpha, \rho, d) - I$ at $\overline{x}$, if for all $x \in A$, we have*

$$\begin{aligned} b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) \leq 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) \leq 0, \end{aligned} \tag{7}$$

$$\begin{aligned} -b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \leqq 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leqq 0. \end{aligned} \tag{8}$$

If the inequality (9) given in Definition 7 is satisfied as follows

$$\begin{aligned} b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) < 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) \leq 0, \end{aligned} \tag{9}$$

then we say that (f, g) is weak pseudoquasi $(F, \alpha, \rho, d) - I$ at $\overline{x}$.

Definition 9. *(f, h) is a sub-strictly pseudoquasi $(F, \alpha, \rho, d) - I$ at $\overline{x}$, if for all $x \in A$, we have*

$$\begin{aligned} b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) \leqq 0 \\ \Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) \leq 0, \end{aligned} \tag{10}$$

$$-b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \leqq 0$$
$$\Rightarrow F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leqq 0. \quad (11)$$

Definition 10. *(f, h) is a weak quasi strictly pseudo $(F, \alpha, \rho, d) - I$ at $\overline{x}$, if for all $x \in A$, we have*

$$b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) \leq 0$$
$$\Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) \leqq 0, \quad (12)$$

$$-b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \leqq 0$$
$$\Rightarrow F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leq 0. \quad (13)$$

Definition 11. *(f, h) is a weak quasi semi-pseudo $(F, \alpha, \rho, d) - I$ at $\overline{x}$, if for all $x \in A$ we have*

$$b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) \leq 0$$
$$\Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) \leqq 0, \quad (14)$$

$$-b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \leqq 0$$
$$\Rightarrow F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})) < 0. \quad (15)$$

Definition 12. *(f, h) is a weak strictly pseudo $(F, \alpha, \rho, d) - I$ at $\overline{x}$, if for all $x \in A$ we have*

$$b_i(E(x), E(\overline{x}))\Big(G_{f_i}(f_i(E(x) - G_{f_i}(f_i(E(\overline{x})\Big) \leq 0$$
$$\Rightarrow F(E(x), E(\overline{x}); \alpha^1\xi_i) + \rho_i d^2(E(x), E(\overline{x})) < 0, \quad (16)$$

$$-b_j(E(x), E(\overline{x}))\Big(G_{g_j}(g_j(E(x) - G_{g_j}(g_j(E(\overline{x})\Big) \leqq 0$$
$$\Rightarrow F(E(x), E(\overline{x}); \alpha^2\eta_j) + \rho_j d^2(E(x), E(\overline{x})) < 0. \quad (17)$$

3. Sufficient Optimality Conditions

In this section, we will discuss the $\alpha - E-$semi-preinvariant convex multi-objective programming as follows.

$$(MOP)\begin{cases} \min f(E(x)), \\ s.t. x \in A = \{x \in X | g(E(x)) \leqq 0\}. \end{cases}$$

Let $E : X \to X$, where $X \subset \mathbb{R}^n$, be an $\alpha - E-$semi-preinvariant convex set for η and α, and we have

$$f(E(x)) = \big(f_1(E(x)), f_2(E(x)), \ldots, f_p(E(x))\big)^T, f_i : X \subset \mathbb{R}, i = 1, 2, \ldots, p,$$

$$g(E(x)) = \big(g_1(E(x)), g_2(E(x)), \ldots, g_q(E(x))\big)^T, g_j : X \subset \mathbb{R}, j = 1, 2, \ldots, q,$$

where f_i, g_j is a local Lipschitz function.

Based on different generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions and a multi-objective programming model (MOP), We will derive several sufficient optimality conditions for a feasible solution to be efficient or weakly efficient in the corresponding MOP.

Theorem 1. *Let $\overline{x}$ be a feasible solution to MOP, $\overline{u} \in \mathbb{R}, \overline{v} \in \mathbb{R}^q$, such that*

$$0 \in \sum_{i=1}^{p} \mu_i G'_{f_i}(f_i(E(\overline{x})\partial f_j(E(\overline{x})) + \sum_{j=1}^{q} v_j G'_{g_j}(g_j(E(\overline{x})\partial g_j(E(\overline{x})), \tag{18}$$

$$v_j g_j(E(\overline{x})) = 0, \tag{19}$$

$$\overline{u} > 0, \overline{v} \geqq 0, \tag{20}$$

if (f, g) is a strong pseudoquasi $(F, \alpha, \rho, d)-$type$-E-$semi-preinvariant invex at $\overline{x}$ with

$$\overline{\mu}\rho^1\alpha^1(\cdot, \overline{x})^{-1} + \overline{v}\rho^2\alpha^2(\cdot, \overline{x})^{-1} \geqq 0,$$

then $\overline{x}$ is an efficient solution for (MOP).

Proof. Suppose that $\overline{x}$ is not an efficient solution for (MOP). Then, there exist $x \in A$, such that $f(E(x)) \leq f(E(\overline{x}))$, $g(E(x)) \leqq g(E(\overline{x}))$. There exists at least one i_0, such that $f_{i_0}(E(x)) < f_{i_0}(E(\overline{x}))$, since (f, g) is a strong pseudoquasi $(F, \alpha, \rho, d)-$type$-E-$semi-preinvariant invex at $\overline{x}$, G_{f_i} is increasing, and G_{g_j} is strictly increasing. Hence, we have

$$F(E(x), E(\overline{x}); \alpha^1(E(x), E(\overline{x}))\xi_i) + \rho_i d^2(E(x), E(\overline{x})) \leq 0, \tag{21}$$

$$F(E(x), E(\overline{x}); \alpha^2(E(x), E(\overline{x}))\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leqq 0. \tag{22}$$

There must exist at least one strict $<$; thus, we have

$$\alpha^1(E(x), E(\overline{x}))F(E(x), E(\overline{x}); \xi_i) \leq -\rho_i d^2(E(x), E(\overline{x})), \tag{23}$$

$$\alpha^2(E(x), E(\overline{x}))F(E(x), E(\overline{x}); \eta_j) \leqq -\rho_j d^2(E(x), E(\overline{x})). \tag{24}$$

By multiplying (23) with $\overline{\mu_i}\alpha^1(E(x), E(\overline{x}))^{-1}$, and (24) with $\overline{v_j}\alpha^2(E(x), E(\overline{x}))^{-1}$, we obtain

$$\overline{\mu_i}F(E(x), E(\overline{x}); \xi_i) < -\overline{\mu_i}\rho_i\alpha^1(E(x), E(\overline{x}))^{-1}d^2(E(x), E(\overline{x})), \tag{25}$$

$$\overline{v_j}F(E(x), E(\overline{x}); \eta_j) \leq -\overline{v_j}\rho_j\alpha^2(E(x), E(\overline{x}))^{-1}d^2(E(x), E(\overline{x})). \tag{26}$$

By accumulating and combining with the sublinearity of F, we obtain

$$\sum_{i=1}^{p} \overline{\mu_i}F(E(x), E(\overline{x}); \xi_i) < -\sum_{i=1}^{p} \overline{\mu_i}\rho_i\alpha^1(E(x), E(\overline{x}))^{-1}d^2(E(x), E(\overline{x})), \tag{27}$$

$$\sum_{j=1}^{p} \overline{v_j}F(E(x), E(\overline{x}); \eta_j) \leq -\sum_{i=j}^{p} \overline{v_j}\rho_j\alpha^2(E(x), E(\overline{x}))^{-1}d^2(E(x), E(\overline{x})). \tag{28}$$

From the sublinearity of F, we achieve

$$F(E(x), E(\overline{x}))\left(\sum_{i=1}^{p} \overline{\mu_i}\xi_i + \sum_{j=1}^{q} \overline{v_j}\eta_j\right)$$
$$< -\left(\sum_{i=1}^{p} \overline{\mu_i}\rho_i\alpha^1(E(x), E(\overline{x}))^{-1} + \sum_{j=1}^{q} \overline{v_j}\rho_j\alpha^2(E(x), E(\overline{x}))^{-1}\right)d^2(E(x), E(\overline{x})),$$

Since

$$\sum_{i=1}^{p} \overline{\mu_i}\rho_i\alpha^1(E(x), E(\overline{x}))^{-1} + \sum_{j=1}^{q} \overline{v_j}\rho_j\alpha^2(E(x), E(\overline{x}))^{-1} \geqq 0,$$

from above inequalities, we give

$$F(E(x),E(\overline{x}))\left(\sum_{i=1}^{p}\overline{\mu_i}\xi_i+\sum_{j=1}^{q}\overline{v_j}\eta_j\right)<0,$$

and this contradicts (18). Hence, $\overline{x}$ is an efficient solution for (MOP). □

Theorem 2. *Let $\overline{x}$ be a feasible solution to MOP, $\overline{u}\in\mathbb{R}^p,\overline{v}\in\mathbb{R}^q$, such that*

$$0\in\sum_{i=1}^{p}\mu_iG_{f_i}^{'}(f_i(E(\overline{x})\partial f_j(E(\overline{x}))+\sum_{j=1}^{q}v_jG_{g_j}^{'}(g_j(E(\overline{x})\partial g_j(E(\overline{x})),\tag{29}$$

$$v_jg_j(E(\overline{x}))=0,\tag{30}$$

$$\overline{u}\geq 0,\overline{v}\geqq 0,\tag{31}$$

if (f,g) is a weak strictly pseudoquasi $(F,\alpha,\rho,d)-$type$-E-$semi-preinvariant invex at $\overline{x}$ with

$$\overline{\mu}\rho^1\alpha^1(\cdot,E(\overline{x}))^{-1}+\overline{v}\rho^2\alpha^2(\cdot,E(\overline{x}))^{-1}\geqq 0,$$

then $\overline{x}$ is an efficient solution for (MOP).

Proof. Suppose that $\overline{x}$ is not an efficient solution for (MOP). Then, there exist $x\in A$, such that $f(E(x))\leq f(E(\overline{x}))$, since (f,h) is a weak strictly pseudoquasi $(F,\alpha,\rho,d)-$type$-E-$semi-preinvariant invex at $\overline{x}$, G_{f_i}, and G_{g_j} is strictly increasing. Hence, we have

$$F(E(x),E(\overline{x});\alpha^1(E(x),E(\overline{x}))\xi_i)+\rho_id^2(E(x),E(\overline{x}))<0,\tag{32}$$

$$F(E(x),E(\overline{x});\alpha^2(E(x),E(\overline{x}))\eta_j)+\rho_jd^2(E(x),E(\overline{x}))\leqq 0,\tag{33}$$

The following proof is similar to that of Theorem 1, which completes the proof. □

Theorem 3. *Let $\overline{x}$ be a feasible solution to MOP, $\overline{u}\in\mathbb{R},\overline{v}\in\mathbb{R}^q$, such that*

$$0\in\sum_{i=1}^{p}\mu_iG_{f_i}^{'}\sum_{j=1}^{q}v_jG_{g_j}^{'}(f_i(E(\overline{x})\partial f_j(E(\overline{x}))+\sum_{j=1}^{q}v_jG_{g_j}^{'}(f_i(E(\overline{x})\partial f_j(E(\overline{x})),\tag{34}$$

$$v_jg_j(E(\overline{x}))=0,\tag{35}$$

$$\overline{u}\geqq 0,\overline{v}>0,\tag{36}$$

if (f,g) is a weak quasi strictly pseudo $(F,\alpha,\rho,d)-$type$-E-$semi-preinvariant invex at $\overline{x}$ with

$$\overline{\mu}\rho^1\alpha^1(\cdot,E(\overline{x}))^{-1}+\overline{v}\rho^2\alpha^2(\cdot,E(\overline{x}))^{-1}\geqq 0,$$

then $\overline{x}$ is an efficient solution for (MOP).

Proof. Suppose that $\overline{x}$ is not an efficient solution for (MOP). Then, there exist $x\in A$ such that $f(E(x))\leq f(E(\overline{x}))$, $g(E(x))\leqq g(E(\overline{x}))$, since (f,g) is a weak quasi strictly pseudo $(F,\alpha,\rho,d)-$type$-E-$semi-preinvariant invex at $\overline{x}$, G_{f_i}, and G_{g_j}, is strictly increasing. Hence, we have

$$F(E(x),E(\overline{x});\alpha^1(E(x),E(\overline{x}))\xi_i)+\rho_id^2(E(x),E(\overline{x}))\leqq 0,\tag{37}$$

$$F(E(x),E(\overline{x});\alpha^2(E(x),E(\overline{x}))\eta_j)+\rho_jd^2(E(x),E(\overline{x}))\leq 0.\tag{38}$$

The following proof is similar to that of Theorem 1, which completes the proof. □

Remark 1. *Note that Theorems 1–3 still hold for weak efficient solutions, so convexity can be appropriately reduced for weak efficient solutions.*

Theorem 4. *Let $\overline{x}$ be a feasible solution to MOP, $\overline{u} \in \mathbb{R}, \overline{v} \in \mathbb{R}^q$, such that the triplet $(\overline{x}, \overline{u}, \overline{v})$ satisfies (18)–(20) of Theorem 1. If (f, g) be weak pseudoquasi $(F, \alpha, \rho, d)-$type$-E-$semi-preinvariant invex at $\overline{x}$ with*

$$\overline{\mu}\rho^1\alpha^1(\cdot, \overline{x})^{-1} + \overline{v}\rho^2\alpha^2(\cdot, \overline{x})^{-1} \geqq 0,$$

then $\overline{x}$ is a weak efficient solution for (MOP).

Proof. Suppose that $\overline{x}$ is not a weak efficient solution for (MOP). Then, there exist $x \in A$, such that $f(E(x)) < f(E(\overline{x}))$, since (f, g) is weak pseudoquasi $(F, \alpha, \rho, d)-$type$-E-$semi-preinvariant invex at $\overline{x}$, G_{f_i} and G_{g_j} is strictly increasing. Hence, we have

$$F(E(x), E(\overline{x}); \alpha^1(E(x), E(\overline{x}))\xi_i) + \rho_i d^2(E(x), E(\overline{x})) \leq 0, \tag{39}$$

$$F(E(x), E(\overline{x}); \alpha^2(E(x), E(\overline{x}))\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leqq 0, \tag{40}$$

The following proof is similar to that of Theorem 1, which completes the proof. □

Theorem 5. *Let $\overline{x}$ be a feasible solution to MOP, $\overline{u} \in \mathbb{R}^p, \overline{v} \in \mathbb{R}^q$, such that the triplet $(\overline{x}, \overline{u}, \overline{v})$ satisfies (29)–(31) of Theorem 2. If (f, g) is a pseudoquasi $(F, \alpha, \rho, d)-$type$-E-$semi-preinvariant invex at $\overline{x}$ with*

$$\overline{\mu}\rho^1\alpha^1(\cdot, E(\overline{x}))^{-1} + \overline{v}\rho^2\alpha^2(\cdot, E(\overline{x}))^{-1} \geqq 0,$$

then $\overline{x}$ is a weak efficient solution for (MOP).

Proof. Suppose that $\overline{x}$ is not a weak efficient solution for (MOP). Then, there exist $x \in A$, such that $f(E(x)) < f(E(\overline{x}))$, since (f, g) is a pseudoquasi $(F, \alpha, \rho, d)-$type$-E-$semi-preinvariant invex at $\overline{x}$. Hence, we have

$$F(E(x), E(\overline{x}); \alpha^1(E(x), E(\overline{x}))\xi_i) + \rho_i d^2(E(x), E(\overline{x})) < 0, \tag{41}$$

$$F(E(x), E(\overline{x}); \alpha^2(E(x), E(\overline{x}))\eta_j) + \rho_j d^2(E(x), E(\overline{x})) \leqq 0. \tag{42}$$

The following proof is similar to that of Theorem 2, which completes the proof. □

4. Mixed-Type Duality

In this section, we will investigate the mixed duality problem of $\alpha - E-$semi-invariant convex multiobjective programming and establish the weak duality theorem for the mixed duality model, based on generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions, along with certain sufficient conditions, ensuring that the corresponding feasible solutions of the programming are efficient or weakly efficient.

Let $J_1 \subseteq Q$, $J_2 = Q \setminus \{J_1\}$, and let $e \in \mathbb{R}^p$, whose components are all ones. We consider the following mixed-type dual of (MOP):

$$(XMOP)\begin{cases} \max f(E(y)) + v_{J_1} g_{J_1}(E(y))e, \\ s.t. 0 \in \sum_{i=1}^{p} \mu_i G'_{f_i}(f_i(E(y)\partial f_i(E(y)) + \sum\limits_{j \in J_1} v_j G'_{g_j}(g_j(E(y)\partial g_j(E(y)), & (43) \\ \sum\limits_{j \in J_2} v_j g_j(E(y)) \geq 0, & (44) \\ v \geqq 0, & (45) \\ \mu \geqq 0, \mu^T e = 1, & (46) \end{cases}$$

Remark 2. *For $J_1 = \emptyset$, or $J_2 = \emptyset$ in (XMOP), we can obtain a corresponding Mond–Weir dual or a Wolfe dual, respectively.*

Theorem 6 (Weak duality). *Assume that for all feasible x and (y,u,v) for (MOP) and $(XMOP)$, respectively, that the following holds:*

$(a)\mu > 0, \mu(f(\cdot) + v_{J_1}g_{J_1}(\cdot), v_{J_2}g_{J_2}(\cdot))$ *is pseudoquasi* $(F,\alpha,\rho,d) - type - \alpha - E - semi$-*preinvariant invex at y with* $\rho^1\alpha^1(\cdot,\overline{x})^{-1} + \overline{v}\rho^2\alpha^2(\cdot,\overline{x})^{-1} \geqq 0$,

then, we can obtain

$$f(E(x)) \nleq f(E(y)) + v_{J_1}g_{J_1}(E(y))e. \tag{47}$$

Proof. Suppose $f(E(x)) \leq f(E(y)) + v_{J_1}g_{J_1}(E(y))e$ holds. Since x is feasible for (MOP) and $v \geqq 0$, it implies that

$$f(E(x)) + v_{J_1}g_{J_1}(E(x))e \leq f(E(y)) + v_{J_1}g_{J_1}(E(y))e, \tag{48}$$

hold. Since triplet (y,μ,v) is feasible for (XMOP), it follows that,

$$-v_{J_2}g_{J_2}(E(y)) \leqq 0. \tag{49}$$

According to hypothesis (a), $b_i > 0, b_j > 0$, both G_{f_i} and G_{g_j} are strictly increasing. We have

$$F(E(x),E(y);\alpha^1(E(x),E(y))\xi) + \rho_i d^2(E(x),E(y)) < 0, \tag{50}$$
$$\xi \in \sum_{i=1}^{p} \mu_i G'_{f_i}(f_i(E(y)\partial f_i(E(y)) + \sum_{j\in J_1} v_j G'_{g_j}(g_j(E(y)\partial g_j(E(y)),$$
$$F(E(x),E(y);\alpha^2(E(x),E(y))\eta) + \rho_2 d^2(E(x),E(y)) \leqq 0, \tag{51}$$
$$\eta \in \sum_{j\in J_2} G'_{g_j}(g_j(E(y)\partial g_j(E(y)),$$

and from (48) and $\mu > 0$, we obtain

$$\sum_{i=1}^{p} \mu_i f_i(E(x)) + \sum_{j\in J_1} v_j g_j(E(x)) < \sum_{i=1}^{p} \mu_i f_i(y) + \sum_{j\in J_1} v_j g_j(E(x)).$$

Since $\alpha^1(E(x),E(y)) > 0$ and $\alpha^2(E(x),E(y)) > 0$, by combining (50) and (51), we obtain

$$F(E(x),E(y);\xi) < -\alpha^1(E(x),E(y))^{-1}\rho_1 d^2(E(x),E(y)), \tag{52}$$
$$F(E(x),E(y);\eta) \leqq -\alpha^2(E(x),E(y))^{-1}\rho_2 d^2(E(x),E(y)). \tag{53}$$

By exploiting the sublinearity property of F, we obtain

$$F(E(x),E(y);\xi+\eta) < -\Big(\rho^1\alpha^1(E(x),E(y))^{-1} + \rho^2\alpha^2(E(x),E(y))^{-1}\Big)d^2(E(x),E(y)).$$

Since $\rho^1\alpha^1(\cdot,E(\overline{x}))^{-1} + \rho^2\alpha^2(\cdot,E(\overline{x}))^{-1} \geqq 0$, we have

$$F(E(x),E(y);\xi+\eta) < 0,$$

which contradicts the duality constraint (43). Hence, we have

$$f(E(x)) \nleq f(E(y)) + v_{J_1}g_{J_1}(E(y))e. \tag{54}$$

□

Theorem 7 (Weak duality). *Assume that, for all feasible x and (y,u,v) for (MOP) and $(XMOP)$, respectively, Hypothesis (a) in Theorem 6 and the following point holds:*

$$\mu(f(\cdot)+v_{J_1}g_{J_1}(\cdot),v_{J_2}g_{J_2}(\cdot))\textit{is strictly pseudoquasi}(F,\alpha,\rho,d)-type-\alpha-E-\textit{semi-preinv}$$
$$\textit{-ariant invex at } y \textit{ with } b_i>0, b_j>0, \textit{and } \rho^1\alpha^1(\cdot,E(\overline{x}))^{-1}+\rho^2\alpha^2(\cdot,E(\overline{x}))^{-1}\geqq 0,$$

then, we can obtain

$$f(E(x))\nleq f(E(y))+v_{J_1}g_{J_1}(E(y))e. \tag{55}$$

Proof. Suppose $f(E(x))\le f(E(y))+v_{J_1}g_{J_1}(E(y))e$ holds. Since x is feasible for (MOP) and $v\geqq 0$, it implies that

$$f(E(x))+v_{J_1}g_{J_1}(E(x))e\le f(E(y))+v_{J_1}g_{J_1}(E(y))e, \tag{56}$$

hold. Since triplet (y,μ,v) is feasible for (XMOP), it follows that,

$$-v_{J_2}g_{J_2}(E(y))\geqq 0, \tag{57}$$

by multiplying (56) with μ, we obtain

$$\sum_{i=1}^{p}\mu_i f_i(E(x))+\sum_{j\in J_1}v_j g_j(E(x))\leqq\sum_{i=1}^{p}\mu_i f_i(E(y))+\sum_{j\in J_1}v_j g_j(E(y)). \tag{58}$$

By $b_i>0, b_j>0$, Hypothesis (a), G_{f_i} and G_{g_j} are strictly increasing, and we have

$$F(E(x),E(y);\alpha^1(E(x),E(y)\xi)+\rho_1 d^2(E(x),E(y))<0, \tag{59}$$
$$\xi\in\sum_{i=1}^{p}\mu_i G'_{f_i}(f_i(E(y)\partial f_i(E(y))+\sum_{j\in J_1}v_j G'_{g_j}(g_j(E(y)\partial g_j(E(y)),$$
$$F(E(x),E(y);\alpha^2(E(x),E(y)\eta)+\rho_2 d^2(E(x),E(y))\leqq 0, \tag{60}$$
$$\eta\in\sum_{j\in J_2}G'_{g_j}(g_j(E(y)\partial g_j(E(y)).$$

Since $\alpha^1(E(x),E(y))>0$ and $\alpha^2(E(x),E(y))>0$, by combining (59) and (60), we obtain

$$F(E(x),E(y);\xi)<-\alpha^1(E(x),E(y))^{-1}\rho_1 d^2(E(x),E(y)), \tag{61}$$
$$F(E(x),E(y);\eta)\leqq-\alpha^2(E(x),E(y))^{-1}\rho_2 d^2(E(x),E(y)). \tag{62}$$

By sublinearity of F, we obtain

$$F(E(x),E(y);\xi+\eta)<-\Big(\rho_1\alpha^1(E(x),E(y))^{-1}+\rho_2\alpha^2(E(x),E(y))^{-1}\Big)d^2(E(x),E(y)).$$

Since $\rho^1\alpha^1(\cdot,\overline{x})^{-1}+\rho^2\alpha^2(\cdot,\overline{x})^{-1}\geqq 0$, we have

$$F(E(x),E(y);\xi+\eta)<0,$$

which contradicts the duality constraint (43). Hence, we have

$$f(E(x))\nleq f(E(y))+v_{J_1}g_{J_1}(E(y))e. \tag{63}$$

□

5. Discussion

The generalized $(F,\alpha,\rho,d)-I-E$ semi-invariant convex function is a further generalization of the $(F,\alpha,\rho,d)-I$ invariant convex function. This type of function not only describes general invariant convexity, but also handles more complex optimization problems. Based on the generalized $(F, \alpha, \rho, d) - I - E$ semi-invariant convex function, a series of optimality conditions can be derived to determine the existence and uniqueness of the optimal solution, and provide a theoretical basis for solving such problems. Additionally, the mixed dual results provide a powerful tool for solving multi-objective optimization problems with constraints. Moreover, using a semi-pre-invariant convex condition in this paper serves two purposes: firstly, it highlights that the condition used in this chapter is weaker than that in previous literature; secondly, it ensures that most of the results hold true when studying convexity. Furthermore, the Clarke-directional derivative and Clarke-subdifferentiables are utilized in this paper to define new classes of semi-preinvariant functions. In future studies, the research methodology can be expanded to generalize Clarke-subdifferentiables to K-subdifferentiables, since K-subdifferentiables are defined by the K-directional derivative, which includes most existing directional derivatives. If K takes the Clarke tangent cone, it becomes known as a Clarke-subdifferentiable. Based on these definitions, a more generalized class of semi-pre-invariant convex functions can be defined, and various other types of multi-objective programming problems, such as higher-order $(F, \alpha, \rho, d, E)-$convexity in fractional programming and interval-valued multiobjective optimization problems, can be explored.

6. Conclusions

This paper addresses the optimality and duality of semi-preinvariant convex multi-objective programming involving generalized $(F, \alpha, \rho, d) - I$ invex functions. Firstly, we propose a new class of generalized $(F, \alpha, \rho, d) - I - E$ semi-preinvariant convex functions by utilizing the Clarke-directional derivative and Clarke-subdifferentiables. These functions are more general compared to existing results. Furthermore, based on these new generalized $(F, \alpha, \rho, d) - I - E-$semi-preinvariant invex functions, we derive a series of sufficient optimality conditions. Additionally, this study investigates mixed dual problems involving these generalized convex functions, and obtains corresponding weak dual theorems. These findings can be further extended to nonsmooth semi-infinite programming and nondifferentiable multiobjective programming problems.

The future research directions will further explore additional properties and challenges in semi-invariant convex multi-objective programming, such as investigating strong duality and optimality conditions. Moreover, these theories will be applied to practical problems, including multimodal problems and stochastic problems.

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References

1. Mehrotra, S.P.; Tandon, R. A mathematical model of a single-roll continuous strip caster based on fluid-mechanics considerations. *Steel Res.* **1992**, *63*, 205–211. [CrossRef]
2. Ghosh, A.; Upadhyay, B.B.; Stancu-Minasian, I.M. Pareto efficiency criteria and duality for multiobjective fractional programming problems with equilibrium constraints on Hadamard manifolds. *Mathematics* **2023**, *11*, 3649. [CrossRef]

3. Cao, W.; Wang, X. A multiobjective multiperiod mixed-integer programming optimization model for integrated scheduling of supply chain under demand uncertainty. *IEEE Access* **2022**, *10*, 63958–63970. [CrossRef]
4. Wu, J.Y.; Luo, J.A.; Du, X.Z.; Zhang, H.; Qin, S. Optimizing water allocation in an inter-basin water diversion project with equity-efficiency tradeoff: A bi-level multiobjective programming model under uncertainty. *J. Clean Prod.* **2022**, *371*, 133606. [CrossRef]
5. Gulia, P.; Kumar, R.; Viriyasitavat, W.; Aledaily, A.N.; Yadav, K.; Kaur, A.; Dhiman, G. A systematic review on Fuzzy-based multi-objective linear programming methodologies: Concepts, challenges and applications. *Arch. Comput. Method Eng.* **2023**, *30*, 4983–5022. [CrossRef]
6. Von Neumann, J.; Morgenstern, O. *Theory of Games and Economic Behavior*; Princeton University Press: New York, NY, USA, 1944.
7. Baatar, D.; Hamacher, H.W.; Ehrgott, M.; Woeginger, G.J. Decomposition of integer matrices and multileaf collimator sequencing. *Discret Appl. Math.* **2005**, *152*, 6–34. [CrossRef]
8. Upadhyaya, S.; Sharma, R.; Agarwal, D.; Malik, G. Convexity analysis and cost optimization of a retrial queue with Bernoulli vacation and delayed phase mending. *Int. J. Syst. Assur. Eng. Manag.* **2023**, *14*, 1671–1690. [CrossRef]
9. Yang, H.; Feng, Q.; Wang, X.; Urynbassarova, D.; Teali, A.A. Reduced Biquaternion Windowed Linear Canonical Transform: Properties and Applications. *Mathematics* **2024**, *12*, 743. [CrossRef]
10. Mangasarian, O.L. Unconstrained Lagrangians in nonlinear programming. *SIAM J. Control* **1975**, *13*, 772–791. [CrossRef]
11. Elbrolosy, M.E. Semi-(E, F)-convexity in complex programming problems. *AIMS Math.* **2022**, *7*, 11119–11131. [CrossRef]
12. Wang, S.L.; Mei, Y.; Zhang, M.J. A multi-objective genetic programming algorithm with α dominance and archive for uncertain capacitated arc routing problem. *IEEE Trans. Evol. Comput.* **2023**, *27*, 1633–1647. [CrossRef]
13. Du, T.; Awan, M.U.; Kashuri, A.; Zhao, S. Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m, h)-preinvexity. *Appl. Anal.* **2021** , *100*, 642–662. [CrossRef]
14. Luo, Z.M.; Jian, J.B. Some properties of semi-E-preinvex maps in Banach spaces. *Nonlinear Anal.-Real World Appl.* **2011**, *12*, 1243–1249. [CrossRef]
15. Long, X.J.; Peng, J.W. Semi-B-preinvex functions. *J. Optim. Theory Appl.* **2006**, *131*, 301–305. [CrossRef]
16. Yang, X.M. On E-convex sets, E-convex functions, and E-convex programming. *J. Optim. Theory Appl.* **2001**, *109*, 699–704. [CrossRef]
17. Supian, S.; Subiyanto; Megantara, T.R.; Bon, A.T. Ride-Hailing Matching with Uncertain Travel Time: A Novel Interval-Valued Fuzzy Multi-Objective Linear Programming Approach. *Mathematics* **2024**, *12*, 1355. [CrossRef]
18. Khakzad, N. A goal programming approach to multi-objective optimization of firefighting strategies in the event of domino effects. *Reliab. Eng. Syst. Saf.* **2023**, *239*, 109523. [CrossRef]
19. Forget, N.; Gadegaard, S.L.; Nielsen, L.R. Warm-starting lower bound set computations for branch-and-bound algorithms for multi objective integer linear programs. *Eur. J. Oper. Res.* **2022**, *302*, 909–924. [CrossRef]
20. Bai, Z.Z.; Zhou, H.Y.; Shi, J.M.; Xing, L.; Wang, J. A hybrid multi-objective evolutionary algorithm with high solving efficiency for UAV defense programming. *Swarm Evol. Comput.* **2024**, *87*, 101572. [CrossRef]
21. Liang, Z.A.; Huang, H.X.; Pardalos, P.M. Optimality conditions and duality for a class of nonlinear fractional programming problems. *J. Optim. Theory Appl.* **2001**, *110*, 611–619. [CrossRef]
22. Hachimi, M.; Aghezzaf, B. Sufficiency and duality in nondifferentiable multiobjective programming involving generalized type I functions. *J. Math. Anal. Appl.* **2006**, *319*, 110–123. [CrossRef]
23. Kailey, N.; Gupta, S.K. Duality for a class of symmetric nondifferentiable multiobjective fractional variational problems with generalized (F, α, ρ, d)-convexity. *Math. Comput. Model.* **2013**, *57*, 1453–1465. [CrossRef]
24. Hachimi, M.; Aghezzaf, B. Sufficiency and duality in differentiable multiobjective programming involving generalized type I functions. *J. Math. Anal. Appl.* **2004**, *296*, 382–392. [CrossRef]
25. Jayswal, A.; Stancu-Minasian, I.; Ahmad, I. Second order duality for minmax fractional programming problem involving (F, α, ρ, d)-type I functions. *Bull. Malays. Math. Sci. Soc.* **2014**, *37*, 893–905.
26. Gupta, S.K.; Dangar, D. Generalized multiobjective symmetric duality under second-order (F, α, ρ, d)-convexity. *Acta Math. Appl. Sin.-Engl. Ser.* **2015**, *31*, 529–542. [CrossRef]
27. Dubey, R.; Gupta, S.K. On duality for a second-order multiobjective fractional programming problem involving type-I functions. *Georgian Math. J.* **2019**, *26*, 393–404. [CrossRef]
28. Kassem, M.A.E.H. Second-order symmetric duality in vector optimization involving (K, η)-pseudobonvexity. *Bull. Sci. Math.* **2022**, *175*, 103109. [CrossRef]
29. Kharbanda, P.; Agarwal, D. Non-smooth multi-objective fractional programming problem involving higher order functions. *Int. J. Comput. Sci. Math.* **2019**, *10*, 351–363. [CrossRef]
30. Kailey, N.; Sethi, S.; Dhingra, V. Correspondence between a new pair of nondifferentiable mixed dual vector programs and higher-order generalized convexity. *Opsearch* **2024** , *61*, 1507–1540. [CrossRef]
31. Tiwari, H.; Meena S.; Kumar, D.; Ojha, D.B. Higher-Order $(F, \alpha, \beta, \rho, d, E)$-Convexity in Fractional Programming. *J. Math. Res. Appl.* **2023**, *43*, 303–312.
32. Kaur, A.; Sharma, M.K.; Ahmad, I. Multiobjective symmetric duality in higher-order fractional variational programming. *Asia Pac. J. Oper. Res.* **2023**, *40*, 2250008. [CrossRef]

33. Dhingra, V.; Kailey, N. Duality results for fractional variational problems and its application. *Bull. Malays. Math. Sci. Soc.* **2022**, *45*, 2195–2223. [CrossRef]
34. Yadav, T.; Gupta, S.K. On duality theory for multiobjective semi-infinite fractional optimization model using higher order convexity. *Rairo-Oper. Res.* **2021**, *55*, 1343–1370. [CrossRef]

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