mathematics

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Defining and Analyzing New Classes Associated with (λ, γ)-Symmetrical Functions and Quantum Calculus

Hanen Louati [1,2], Afrah Y. Al-Rezami [3,4,*], Abdulbasit A. Darem [5] and Fuad Alsarari [6]

[1] Department of Mathematics, College of Science, Northeren Border University, Arar 73222, Saudi Arabia; hanen.louati@nbu.edu.sa
[2] Laboratory of PDEs and Applications (LR03ES04), Faculty of Science of Tunis, University of Tunis El Manar, Tunis 1068, Tunisia
[3] Mathematics Department, Prince Sattam Bin Abdulaziz University, Al-Kharj 16278, Saudi Arabia
[4] Department of Statistics and Information, Sana'a University, Sana'a 1247, Yemen
[5] Department of Computer Science, College of Science, Northern Border University, Arar 73222, Saudi Arabia; basit.darem@nbu.edu.sa
[6] Department of Mathematics and Statistics, College of Sciences, Taibah University, Yanbu 46423, Saudi Arabia; alsrary@gmail.com or fsarari@taibahu.edu.sa
* Correspondence: a.alrezamee@psau.edu.sa or razwanniaz11@gmail.com

Abstract: In this paper, we introduce new classes of functions defined within the open unit disk by integrating the concepts of (λ, γ)-symmetrical functions, generalized Janowski functions, and quantum calculus. We derive a structural formula and a representation theorem for the class $\mathcal{S}_q^{\lambda,\gamma}(x, y, z)$. Utilizing convolution techniques and quantum calculus, we investigate convolution conditions supported by examples and corollary, establishing sufficient conditions. Additionally, we derive properties related to coefficient estimates, which further elucidate the characteristics of the defined function classes.

Keywords: convolution; Janowski functions; q-calculus; (λ, γ)-symmetric points

MSC: 81Q12; 30C45

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Citation: Louati, H.; Al-Rezami, A.Y.; Darem, A.A.; Alsarari, F. Defining and Analyzing New Classes Associated with (λ, γ)-Symmetrical Functions and Quantum Calculus. *Mathematics* **2024**, *12*, 2603. https://doi.org/10.3390/math12162603

Academic Editor: Michael M. Tung

Received: 5 June 2024
Revised: 17 August 2024
Accepted: 21 August 2024
Published: 22 August 2024

1. Introduction

Geometric function theory (GFT) is a branch of complex analysis that studies holomorphic functions by exploring their geometric properties and behaviors. This field combines techniques from complex analysis, topology, and differential geometry to investigate mappings in the complex plane and higher-dimensional complex spaces. This work focuses on the space analytic functions. $\widehat{\mathcal{A}}(\Psi)$ denotes the set of analytic functions within the open unit disk $\Psi = \{\mu \in \mathbb{C} : |\mu| < 1\}$, and $\widehat{\mathcal{A}}$ represents a specific subset, characterized by a class $h \in \widehat{\mathcal{A}}(\Psi)$, and expressed using the following form:

$$h(\mu) = \mu + \sum_{k=2}^{\infty} a_k \mu^k. \tag{1}$$

Let $\mathcal{S}$ denote the subclass of $\widehat{\mathcal{A}}$ consisting of all functions which are univalent in Ψ. Let h and g be analytic in Ψ. We say that the function h is subordinate to g in Ψ, denoted by $h(\mu) \prec g(\mu)$, if there exists an analytic function ω in Ψ, such that $|\omega(\mu)| < 1$ with $\omega(0) = 0$, and $h(\mu) = g(\omega(\mu))$. If g is univalent in Ψ, then the subordination is equivalent to $h(0) = g(0)$ and $h(\Psi) \subset g(\Psi)$. Let h and g be analytic in Ψ. The convolution (or Hadamard product) of h and g, denoted by $(h * g)(\mu)$, which has the following definition.

If h is given by (1) and $g(\mu) = \sum_{n=0}^{\infty} b_n \mu^n$, then the following is true:

$$(h * g)(\mu) = \mu + \sum_{k=2}^{\infty} a_k b_k \mu^k.$$

Let us utilize the concept of subordination to define the well-known Carathéodory class. The Carathéodory class $\mathcal{P}$ is defined as follows:

$$\mathcal{P} = \left\{ p \in \widehat{\mathcal{A}}(\Psi) \mid p(0) = 1, \operatorname{Re} p(\mu) > 0 \text{ for all } \mu \in \Psi \right\}.$$

Any function p in $\mathcal{P}$ has the representation $p(\mu) = \dfrac{1+\omega(\mu)}{1-\omega(\mu)}$, where $\omega \in \Omega$, and the following is true:

$$\Omega = \{\omega \in \widehat{\mathcal{A}} : \omega(0) = 0, |\omega(\mu)| < 1\}. \tag{2}$$

In reference [1], Janowski introduced a specific subclass of $\mathcal{P}$, denoted as $\mathcal{P}[\widetilde{x}, y]$ $-1 \leq y < \widetilde{x} \leq 1$. For this class, an analytic function p in Ψ with $p(0) = 1$ belongs to $\mathcal{P}[\widetilde{x}, y]$, if $p(\mu) = \frac{1+\widetilde{x}\omega(\mu)}{1-y\omega(\mu)}$. The class $\mathcal{P}[x, y]$ of generalized Janowski functions was introduced in [2] for $x = (1-z)\widetilde{x} + zy$ with $0 \leq z < 1$.

This work introduces a novel class of q-Janowski symmetrical functions defined within the open unit disk Ψ. These functions are intrinsically linked to the q-derivative operator. Before delving into the specifics of this new class, we begin by providing a concise overview of the fundamental concepts of (λ, γ)-symmetrical functions and q-calculus. This foundational background is essential for understanding the subsequent development and analysis of our proposed class of functions.

We first recall the concept of γ-fold symmetric functions defined within a γ-fold symmetric domain, where γ is any positive integer. A domain $\mathcal{Q}$ is considered γ-fold symmetric if a rotation of $\mathcal{Q}$ around the origin by an angle of $\frac{2\pi}{\gamma}$ maps $\mathcal{Q}$ onto itself. A function h is deemed γ-fold symmetric in $\mathcal{Q}$ if, for every μ in $\mathcal{Q}$, the following holds:

$$h(\varepsilon\mu) = \varepsilon h(\mu), \ (\varepsilon = e^{\frac{2\pi i}{\gamma}}), \ \mu \in \mathcal{Q}.$$

The family of all γ-fold symmetric functions is denoted by $\widetilde{\mathcal{S}}^b$. Notably, when $\gamma = 2$, we obtain the class of odd univalent functions. Liczberski and Polubinski, in [3], extended this notion by developing the theory of (λ, γ)-symmetrical functions, where $\lambda = 0, 1, 2, \ldots, \gamma - 1$ and $\gamma = 2, 3, \ldots$. In a γ-fold symmetric domain $\mathcal{Q}$, a function $h : \mathcal{Q} \to \mathbb{C}$ is termed (λ, γ)-symmetrical if, for every $\mu \in \mathcal{Q}$, $h(\varepsilon\mu) = \varepsilon^{\lambda} h(\mu)$. It is important to note that (λ, γ)-symmetrical functions generalize the concepts of even, odd, and γ-symmetrical functions. We observe that Ψ is a γ-fold symmetric domain with the γ of any integer. We use the unique decomposition [3] of every mapping $h : \Psi \mapsto \mathbb{C}$, as follows:

$$h(\mu) = \sum_{\lambda=0}^{\gamma-1} h_{\lambda,\gamma}(\mu), \qquad \text{where} \qquad h_{\lambda,\gamma}(\mu) = \gamma^{-1} \sum_{r=0}^{\gamma-1} \varepsilon^{-r\lambda} h(\varepsilon^r \mu), \ \mu \in \Psi. \tag{3}$$

Equivalently, (3) may be written as follows:

$$h_{\lambda,\gamma}(\mu) = \sum_{v=1}^{\infty} a_v \alpha_{\lambda}^{v} \mu^v, \ \ a_1 = 1, \tag{4}$$

where

$$\alpha_{\lambda}^{k} = \frac{1}{\gamma} \sum_{r=0}^{\gamma-1} \varepsilon^{(k-\lambda)r} = \begin{cases} 1, & k = l\gamma + \lambda; \\ 0, & k \neq l\gamma + \lambda; \end{cases}, \tag{5}$$

$$(l \in \mathbb{N}, \ \gamma = 1, 2, \ldots, \ \lambda = 0, 1, 2, \ldots, \gamma - 1).$$

The family of all starlike functions, with respect to (λ, γ)-symmetric points, is denoted by $\widetilde{\mathcal{S}}^{(\lambda,\gamma)}$, which generalizes several well-known subclasses of starlike functions, such as

$\widetilde{\mathcal{S}}^{(0,2)}$, $\widetilde{\mathcal{S}}^{(1,2)}$, and $\widetilde{\mathcal{S}}^{(1,\gamma)}$. These correspond to the classes of even, odd, and γ-symmetric functions, respectively. The study of starlike functions is a significant area in the field of geometric function theory, due to its applications in complex analysis and mathematical modeling. Traditional starlike functions map the unit disk onto starlike domains with respect to the origin. However, recent research has extended this concept to starlike functions with respect to (λ,γ)-symmetric points. These studies have investigated various properties of the class $\widetilde{\mathcal{S}}^{(\lambda,\gamma)}$, including coefficient estimates, distortion theorems, and subordination results; please see [4–6].

In [7], Jackson introduced and studied the concept of the q-derivative operator $\mathcal{D}_q h(\mu)$, where q satisfies the condition $0 < q < 1$, as follows:

$$\mathcal{D}_q h(\mu) = \begin{cases} \frac{h(\mu)-h(q\mu)}{\mu(1-q)}, & \mu \neq 0, \\ h'(0), & \mu = 0. \end{cases} \tag{6}$$

Equivalently (6), may be written as follows:

$$\mathcal{D}_q h(\mu) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \mu^{k-1} \quad \mu \neq 0,$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + \ldots + q^{k-1}. \tag{7}$$

Note that, as $q \to 1^-$, $[k]_q \to k$. For the function $h(\mu) = \mu^k$, we note that the following is true:

$$\mathcal{D}_q h(\mu) = \mathcal{D}_q(\mu^k) = \frac{1-q^k}{1-q}\mu^{k-1} = [k]_q \mu^{k-1}.$$

Then, the following is also true:

$$\lim_{q\to 1^-} \mathcal{D}_q h(\mu) = \lim_{q\to 1^-} [k]_q \mu^{k-1} = k\mu^{k-1} = h'(\mu),$$

where $h'(\mu)$ is the ordinary derivative.

Assuming the definition of the q-difference operator, the following rules hold:

- $\mathcal{D}_q(ah(\mu) \pm bg(\mu)) = a\mathcal{D}_q h(\mu) \pm b\mathcal{D}_q g(\mu)$, where a and b are real (or complex) constants.
- $\mathcal{D}_q(h(\mu)g(\mu)) = h(q\mu)\mathcal{D}_q g(\mu) + g(\mu)\mathcal{D}_q h(\mu) = h(\mu)\mathcal{D}_q g(\mu) + g(q\mu)\mathcal{D}_q h(\mu)$
- $\mathcal{D}_q\left(\frac{h(\mu)}{g(\mu)}\right) = \frac{g(\mu)\mathcal{D}_q h(\mu) - h(\mu)\mathcal{D}_q g(\mu)}{g(q\mu)g(\mu)}$.
- $\mu\mathcal{D}_q h(\mu) * g(\mu) = h(\mu) * \mu\mathcal{D}_q g(\mu)$.

In [8], Jackson introduced the q-integral of a function h as a right inverse, expressed as follows:

$$\int_0^{\mu} h(w) d_q w = \mu(1-q) \sum_{k=0}^{\infty} q^k h(\mu q^k),$$

provided the q-series converges. The connection between quantum calculus and geometric function theory was first established by Ismail et al. [9]. This groundbreaking work opened a new avenue for exploring the geometric properties of analytic functions using the powerful tools of quantum calculus. In recent years, there has been a surge of interest in applying quantum calculus to investigate various subclasses of analytic functions. For example, Naeem et al. [10] delved into the properties of q-convex functions, while Srivastava et al. [11] explored subclasses of q-starlike functions. Alsarari et al. [12] analyzed convolution conditions for q-Janowski symmetrical function classes. Ovindaraj and Sivasubramanian [13] identified subclasses associated with q-conic domains. Khan et al. [14] utilized the symmetric q-derivative operator to further expand the field. Srivastava's [15] comprehensive survey-cum-expository review paper has been instrumental in guiding researchers in this burgeoning area.

By leveraging the powerful tools of generalized Janowski functions and (λ, γ)-symmetrical functions, in conjunction with the concept of the q-calculus, we embark on defining a novel set of classes.

Definition 1. *The function h in $\widehat{\mathcal{A}}$ is said to belong to the class $\mathcal{S}_q^{\lambda,\gamma}(x,y)$, $(-1 \leq y < x \leq 1)$, if the following holds:*

$$\frac{\mu \mathcal{D}_q h(\mu)}{h_{\lambda,\gamma}(\mu)} \prec \frac{1+x\mu}{1+y\mu}, \quad \mu \in \Psi,$$

where $h_{\lambda,\gamma}(\mu)$ is given by (3).

The general framework we propose encompasses various existing classes. By selecting particular values for q, x, y, λ and γ, we can retrieve specific classes as special cases. We will list some of these recovered classes for illustration.

1. $\mathcal{S}_1^{1,\gamma}(x,y) := \mathcal{S}_\gamma(x,y)$, introduced and studied by Darus et al. [16].
2. $\mathcal{S}_q^{1,1}(1-2\kappa,-1) = \mathcal{S}_q(\kappa)$, the class motivated by Agrawal and Sahoo in [17].
3. $\mathcal{S}_1^{1,1}(1-2\kappa,-1) = \mathcal{S}(\kappa)$, the well-known class of starlike function of order κ by Robertson [18].
4. $\mathcal{S}_1^{1,\gamma}(1,-1) := \mathcal{S}_y$, motivated by Sakaguchi [19].
5. $\mathcal{S}_q^{1,1}(1,-1) = \mathcal{S}_q$, which was first introduced by Ismail et al. [9].
6. $\mathcal{S}_1^{1,1}(x,y) := \mathcal{S}[x,y]$, which reduces to the well-known class defined by Janowski [1].
7. $\mathcal{S}_1^{1,1}(1,-1) = \mathcal{S}^*$, the class introduced by Nevanlinna [20].

We denote, using $\mathcal{K}_q^{\lambda,\gamma}(x,y)$, the subclass of $\widehat{\mathcal{A}}$, which consists of all functions h, such that the following is true:

$$\mu \mathcal{D}_q h(\mu) \in \mathcal{S}_q^{\lambda,\gamma}(x,y). \tag{8}$$

The class $\mathcal{S}_q^{\lambda,\gamma}(x,y)$ consists of functions with specific properties. Here are some examples of functions belonging to this class, using the following parameter values:

Example 1.

1. *In the basic case with $\gamma = 1$, the definition simplifies, since there are no roots of unity involved. Let $\lambda = 0$, $x = -0.25$, $y = -0.5$. In this case, the condition becomes the following expression:*

$$\frac{\mu \mathcal{D}_q h(\mu)}{h(\mu)} \prec \frac{1-0.25\mu}{1-0.5\mu}.$$

Consider the following function:

$$h(\mu) = \frac{\mu}{(1-\mu)^2}.$$

We compute its q-derivative as follows:

$$\mathcal{D}_q h(\mu) = \frac{h(q\mu)-h(\mu)}{(q-1)\mu} = \frac{\frac{q\mu}{(1-q\mu)^2} - \frac{\mu}{(1-\mu)^2}}{(q-1)\mu}.$$

Simplifying the above expression can verify the condition. For this specific example, if the subordination holds, $h(\mu)$ belongs to $\mathcal{S}_q^{0,1}(-0.25,-0.5)$.

2. *In the case of symmetric points with $\gamma = 2$, we use the primitive 2nd roots of unity $\varepsilon = -1$. Let $\lambda = 1$, $x = -0.25$, and $y = -0.5$. In this case, the condition becomes the following expression:*

$$\frac{\mu \mathcal{D}_q h(\mu)}{h_{1,2}(\mu)} \prec \frac{1+0.25\mu}{1-0.5\mu}.$$

Consider the following function:

$$h(\mu) = \frac{\mu}{1-\mu}.$$

We construct $h_{1,2}(\mu)$ as follows:

$$h_{1,2}(\mu) = \frac{1}{2}(h(\mu) - h(-\mu)) = \frac{1}{2}\left(\frac{\mu}{1-\mu} - \frac{-\mu}{1+\mu}\right) = \frac{\mu(1+\mu) + \mu(1-\mu)}{2(1-\mu)(1+\mu)} = \frac{\mu}{(1-\mu^2)}.$$

Then, the following is true:

$$\mu\mathcal{D}_q h(\mu) = \mu \cdot \frac{h(q\mu) - h(\mu)}{(q-1)\mu} = \frac{\mu\left(\frac{q\mu}{1-q\mu} - \frac{\mu}{1-\mu}\right)}{(q-1)\mu}.$$

Simplifying this and checking the subordination condition verifies that $h(\mu)$ satisfies the condition for $\mathcal{S}_q^{1,2}(0.5, -0.5)$.

3. *In the general case with $\gamma = 3$, we use the primitive 3rd roots of unity $\varepsilon = e^{2\pi i/3}$ Let $\lambda = 1$, $x = 0.5$, and $y = 0$. In this case, the condition becomes the following expression:*

$$\frac{\mu\mathcal{D}_q h(\mu)}{h_{1,3}(\mu)} \prec 1 + 0.5\mu.$$

Consider the following function:

$$h(\mu) = \frac{\mu}{(1-\mu)^3}. \tag{9}$$

We construct $h_{1,3}(\mu)$ as follows:

$$h_{1,3}(\mu) = \frac{1}{3}\Big(h(\mu) + \varepsilon^{-1}h(\varepsilon\mu) + \varepsilon^{-2}h(\varepsilon^2\mu)\Big).$$

Computing $h(\varepsilon\mu)$ and $h(\varepsilon^2\mu)$ and averaging them can verify the function's symmetry properties.

The examples illustrate how to verify functions belonging to the class $\mathcal{S}_q^{\lambda,\gamma}(x,y)$ by constructing symmetric points and checking the subordination condition. These specific functions and parameters help demonstrate the membership in the defined class, ensuring that the functions are starlike with respect to symmetric points.

In this work, we derive a structural formula and a representation theorem for the class $\mathcal{S}_q^{\lambda,\gamma}(x,y)$. Utilizing convolution techniques and quantum calculus, we investigate convolution conditions, providing supporting examples and corollaries to establish sufficient conditions. Additionally, we derive properties related to coefficient estimates, further elucidating the characteristics of the defined function classes.

We need the following lemma to prove our main results.

Lemma 1 ([21] (Lemma 2.1)). *Let $p(\mu) = 1 + \sum_{k=1}^{\infty} c_k\mu^k \in \mathcal{P}[x,y]$; then, for $k \geq 1$, the following is true:*

$$|c_k| \leq x - y.$$

2. Main Results

Theorem 1. *The function h belongs to the class $\mathcal{S}_q^{\lambda,\gamma}(x,y)$ if and only if*

$$h(\mu) = \int_0^{\mu} p(\omega)f(\omega)d\omega, \tag{10}$$

where $f(\omega) = \exp\Big\{\int_0^{\omega} \frac{1}{\lambda u}\Big(\sum_{v=0}^{\lambda-1} p(\varepsilon^v u) - \lambda\Big)d_q u\Big\}.$

Proof. For the arbitrary function $h \in \mathcal{S}_q^{\lambda,\gamma}(x,y,z)$, we have the following expression:

$$\frac{\mu \mathcal{D}_q h(\mu)}{h_{\lambda,\gamma}(\mu)} = p(\mu), \qquad p \in \mathcal{P}[x,y]. \tag{11}$$

Replacing μ with $\varepsilon^v \mu$ in (11), we obtain the following:

$$\frac{\varepsilon^{v(1-\lambda)} \mu \mathcal{D}_q h(\varepsilon^v \mu)}{h_{\lambda,\gamma}(\mu)} = p(\varepsilon^v \mu), \tag{12}$$

From (11) and (12), we obtain the following:

$$\mathcal{D}_q h(\varepsilon^v \mu) = p(\varepsilon^v \mu) \frac{\varepsilon^{v(\lambda-1)} \mathcal{D}_q h(\mu)}{p(\mu)} \tag{13}$$

Through the q-differentiation of (11), we obtain the following:

$$\mathcal{D}_q h_{\lambda,\gamma}(\mu) = \frac{q\mu \mathcal{D}_q(\mathcal{D}_q h(\mu)) + \mathcal{D}_q h(\mu)}{p(\mu)} - q\mu \mathcal{D}_q h(\mu) \frac{\mathcal{D}_q p(\mu)}{p(\mu) p(q\mu)}. \tag{14}$$

From (5) and (13), we obtain the following:

$$\mathcal{D}_q h_{\lambda,\gamma}(\mu) = \frac{1}{\gamma} \frac{\mathcal{D}_q h(\mu)}{p(\mu)} \sum_{v=0}^{\lambda-1} p(\varepsilon^v \mu), \tag{15}$$

From (14) and (15), we obtain the following:

$$\frac{\mathcal{D}_q(\mathcal{D}_q h(\mu))}{\mathcal{D}_q h(\mu)} = \frac{\mathcal{D}_q p(\mu)}{p(\mu)} + \frac{1}{\lambda \mu} \left(\sum_{v=0}^{\lambda-1} p(\varepsilon^v \mu) - \lambda \right).$$

By repeatedly q-integrating the above equation, we obtain the required structural formula, as follows:

$$h(\mu) = \int_0^\mu p(\omega) f(\omega) d_q \omega.$$

This proves the necessity. To prove the sufficiency of (10), we suppose that (10) holds with $p \in \mathcal{P}[x,y]$. The function h defined by (10) is obviously in $\mathcal{A}$ with $h(0) = 0$ and $h(0) = 1$. To confirm the validity of (10), we proceed by verifying the given identity through $q-$differentiation, as follows:

$$\mu f(\mu) = \int_0^{\varepsilon^v \mu} \left[\frac{1}{\lambda} \sum_{v=0}^{\lambda-1} \varepsilon^{-\eta v} p(\omega) f(\omega) \right] d_q \omega, \tag{16}$$

where f is given in (10). Furthermore, using (10), we obtain the following:

$$\mathcal{D}_q h(\mu) = p(\mu) f(\mu), \tag{17}$$

which shows that $\mathcal{D}_q f \neq 0$ in Ψ.

From (10), since ε is the root of unity, we conclude that the following holds:

$$h_{\lambda,\gamma}(\mu) = \int_0^{\varepsilon^v \mu} \left[\frac{1}{\lambda} \sum_{v=0}^{\lambda-1} \varepsilon^{-\lambda v} p(\omega) f(\omega) \right] d_q \omega. \tag{18}$$

Using (16)–(18), we arrive at the following result:

$$h_{\lambda,\gamma}(\mu) = \frac{t \mathcal{D}_q h(\mu)}{p(\mu)}.$$

In this way, we have demonstrated the sufficiency of Equation (10). □

Remark 1. *For specific selections of λ, γ, x, y, and $q \to 1$, we obtain the structural formula, which was previously derived for classes cited in [22,23].*

Theorem 2. *The function $h \in \mathcal{S}_q^{\lambda,\gamma}(x,y)$ if and only if*

$$\frac{1}{\mu}\left[h * \left\{\frac{\mu(1+ye^{i\theta})}{(1-\mu)(1-q\mu)} - \frac{1+\tilde{x}\mu e^{i\theta}}{(1-\alpha_\lambda \mu)}\right\}\right] \neq 0, \tag{19}$$

where $q \in (0,1), -1 \leq y < x \leq 1$, and $\theta \in [0, 2\pi)$.

Proof. If we suppose that $h \in \mathcal{S}_q^{\lambda,\gamma}(x,y)$, then the following holds true:

$$\frac{\mu \mathcal{D}_q h(\mu)}{h_{\lambda,\gamma}(\mu)} = p(\mu), \qquad p \in \mathcal{P}[x,y],$$

if and only if

$$\frac{\mu \mathcal{D}_q h(\mu)}{h_{\lambda,\gamma}(\mu)} \neq \frac{1+xe^{i\theta}}{1+ye^{i\theta}}, \tag{20}$$

for all $\mu \in \Psi$ and $0 \leq \theta < 2\pi$. The condition (20) can be written as follows:

$$\frac{1}{\mu}[\mu \mathcal{D}_q h(\mu)(1+ye^{i\theta}) - h_{\lambda,\gamma}(\mu)(1+xe^{i\theta})] \neq 0. \tag{21}$$

Setting $h(\mu) = \mu + \sum_{k=2}^{\infty} a_k \mu^k$, we obtain the following:

$$\mathcal{D}_q h(\mu) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \mu^{k-1}, = h * \frac{\mu}{(1-\mu)(1-q\mu)}. \tag{22}$$

$$h_{\lambda,\gamma}(\mu) = h * \frac{\mu}{(1-\alpha_\lambda^1 \mu)} = \sum_{k=1}^{\infty} \alpha_\lambda^k a_k \mu^k, \tag{23}$$

where α_λ^k is given by (5).

We derive (19) from (22) and (23), leading to (21).

□

Example 2. *Now, we apply Theorem 2 to explain that $h(\mu) = \frac{\mu}{(1-\mu)^3}$, given by (9), belongs to the class $\in \mathcal{S}_q^{\lambda,\gamma}(x,y)$; we only need to prove that $h(\mu) * G(\mu) \neq 0$.*

$G(\mu) = \frac{\mu(1+ye^{i\theta})}{(1-\mu)(1-q\mu)} - \frac{1+x\mu e^{i\theta}}{(1-\alpha_\lambda \mu)}$ *is given by Equation (19).*

First, we expand $h(\mu)$ and $G(\mu)$ into their respective power series, as follows:

$$h(\mu) = \frac{\mu}{(1-\mu)^3} = \mu \sum_{k=0}^{\infty} \binom{k+2}{2} \mu^k = \sum_{k=1}^{\infty} \binom{k+1}{2} \mu^k.$$

To expand $G(\mu)$, we need to break it into simpler parts, as follows:

$$G(\mu) = \frac{\mu(1+ye^{i\theta})}{(1-\mu)(1-q\mu)} - \frac{1+x\mu e^{i\theta}}{(1-\alpha_\lambda \mu)}$$

Using geometric series expansions for each term, we obtain the following:

$$\frac{\mu(1+ye^{i\theta})}{(1-\mu)(1-q\mu)} = \mu(1+ye^{i\theta})\left(\sum_{n=0}^{\infty} \mu^n\right)\left(\sum_{m=0}^{\infty} (q\mu)^m\right) = \mu(1+ye^{i\theta}) \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} q^{k-j}\right) \mu^k,$$

$$\frac{1 + x\mu e^{i\theta}}{(1 - \alpha_\lambda \mu)} = \left(1 + x\mu e^{i\theta}\right) \sum_{k=0}^{\infty} (\alpha_\lambda \mu)^k$$

Combining these into a single series expansion for $G(\mu)$, we can write the following:

$$G(\mu) = \sum_{n=0}^{\infty} b_n \mu^n,$$

where b_n are the coefficients obtained from the above series expansions.

Since $\binom{k+1}{2} \neq 0$ for all $k \geq 1$, and the coefficients b_k in $G(\mu)$ are derived from a combination of geometric series expansions which are typically non-zero unless specifically designed to cancel out, it is clear that not all b_k are zero.

Thus, there exist some k, such that $\binom{k+1}{2} b_k \neq 0$.

Note that, from Theorem 2, we can easily derive the equivalent condition for a function $h \in \mathcal{S}_q^{\lambda,\gamma}(x,y)$, as stated in the following corollary.

Corollary 1. *For $q \in (0,1)$, $-1 \leq y < x \leq 1$, and $\phi \in [0, 2\pi)$, the following is true:*

$$h \in \mathcal{S}_q^{\lambda,\gamma}(x,y) \Leftrightarrow \frac{(h * T)(\mu)}{\mu} \neq 0, \quad, \mu \in \Psi, \tag{24}$$

where $T(\mu)$ has the following form:

$$\begin{aligned} T(\mu) &= \mu + \sum_{k=2}^{\infty} t_k \mu^k, \\ t_k &= \frac{[k]_q - \alpha_\lambda^k + ([k]_q y - \alpha_\lambda^k x e^{i\phi})}{(y - x)e^{i\phi}}. \end{aligned} \tag{25}$$

By applying Corollary 1, we derive the sufficient condition stated in the theorem.

Theorem 3. *Let $h(\mu) = \mu + \sum_{k=2}^{\infty} a_k \mu^k$, be analytic in Ψ, for $-1 \leq y < x \leq 1$ and $0 < q < 1$; if*

$$\sum_{k=2}^{\infty} \left\{ \frac{[k]_q - \alpha_\lambda^k + |[k]_q y - \alpha_\lambda^k x|}{|(x - y)|} \right\} |a_v| \leq 1, \tag{26}$$

then $h(\mu) \in \mathcal{S}_q^{\lambda,\gamma}(x,y)$.

Proof. To prove Theorem 3, it is sufficient to demonstrate that $\frac{(h*T)(\mu)}{\mu} \neq 0$, where T is defined as in Equation (25). Let us assume that $h(\mu) = \mu + \sum_{k=2}^{\infty} a_k \mu^v$. Then, we can consider the convolution

$$\frac{(h * T)(\mu)}{\mu} = 1 + \sum_{k=2}^{\infty} t_k a_k \mu^{k-1}, \mu \in \Psi.$$

Through Corollary 1, we know that $h(\mu)$ belongs to $\mathcal{S}_q^{\lambda,\gamma}(x,y)$ if and only if $(h * T)(\mu)/\mu$ is non-zero, where T is defined by (25). Using (25) and (26), we can derive the following inequality:

$$\left| \frac{(h * T)(\mu)}{\mu} \right| \geq 1 - \sum_{k=2}^{\infty} \frac{[k]_q - \alpha_\lambda^k + |[k]_q y - \alpha_\lambda^k x|}{|(x - y)|} |a_k| |\mu|^{k-1} > 0, \quad \text{for all } \mu \in \Psi.$$

This implies that $(h * T)(\mu)/\mu$ is non-zero, and, hence, $h(\mu) \in \mathcal{S}_q^{\lambda,\gamma}(x,y)$. □

Theorem 4. *Let $h(\mu) \in \mathcal{S}_q^{\lambda,\gamma}(x,y)$. For $-1 \leq y < x \leq 1$ and $q \in (0,1)$, then the following holds:*

$$|a_k| \leq \prod_{m=0}^{k-1} \frac{[(x-y)-1]\alpha_\lambda^k + [m]_q}{[m+1]_q - \alpha_\lambda^{m+1}}, \;\; k \geq 2, \tag{27}$$

where α_λ^m is given by (3).

Proof. According to definition (1), we know that the following is true:

$$\frac{\mu \mathcal{D}_q h(\mu)}{h_{\lambda,\gamma}(\mu)} = p(\mu), \;\; \text{where} \;\; p(\mu) = 1 + \sum_{k=1}^{\infty} c_k \mu^k \in \mathcal{P}[x,y].$$

This gives the following expression:

$$\mu \mathcal{D}_q h(\mu) = \left[\sum_{k=1}^{\infty} c_k \mu^k\right] h_{\lambda,\gamma}(\mu).$$

after simplifying the expression below:

$$(1 - \alpha_\lambda^1) + \sum_{k=2}^{\infty} ([k]_q - \alpha_\lambda^k) a_k \mu^k = \left[\sum_{k=1}^{\infty} c_k \mu^k\right]\left[\sum_{k=1}^{\infty} \alpha_\lambda^k a_k \mu^k\right].$$

Applying the Cauchy product formula to the inequality above and equating coefficients of $\mu^k, \; k \geq 2$, we obtain the following:

$$a_k = \frac{1}{[k]_q - \alpha_\lambda^k} \sum_{m=1}^{k-1} c_m \alpha_\lambda^{k-m} a_{k-m}, \tag{28}$$

Using Lemma 1, we obtain the following:

$$|a_k| \leq \frac{(x-y)}{[k]_q - \alpha_\lambda^k} \sum_{m=1}^{k-1} \alpha_\lambda^m |a_m|. \tag{29}$$

The proof is completed by showing the following:

$$\frac{(x-y)}{[k]_q - \alpha_\lambda^k} \sum_{m=1}^{k-1} \alpha_\lambda^m |a_m| \leq \prod_{m=0}^{k-2} \frac{[(x-y)-1]\alpha_\lambda^k + [m]_q}{[m+1]_q - \alpha_\lambda^{m+1}}. \tag{30}$$

To prove this, we will employ the method of mathematical induction. It is clear that (30) is true for $k=2$ and 3.

Let the hypothesis be true for $k = m$; in this case, we obtain the following:

$$\frac{(x-y)}{[m]_q - \alpha_\lambda^m} \sum_{r=1}^{m-1} \alpha_\lambda^r |a_r| \leq \prod_{r=0}^{m-1} \frac{[(x-y)-1]\alpha_\lambda^r + [r]_q}{[r+1]_q - \alpha_\lambda^{r+1}}.$$

Multiplying both sides by $\frac{[(x-y)-1]\alpha_\lambda^m + [m]_q}{[m+1]_q - \alpha_\lambda^{m+1}}$, we then obtain the following:

$$\prod_{r=0}^{m} \frac{[(x-y)-1]\alpha_\lambda^r + [r]_q}{[r+1]_q - \alpha_\lambda^{r+1}} \geq \frac{[(x-y)-1]\alpha_\lambda^m + [m]_q}{[m+1]_q - \alpha_\lambda^{m+1}} . \frac{(x-y)}{[m]_q - \alpha_\lambda^m} \sum_{r=1}^{m-1} \alpha_\lambda^r |a_r|,$$

$$= \frac{(x-y)}{[m+1]_q - \alpha_\lambda^{m+1}} . \left[1 + \frac{(x-y)}{[m]_q - \alpha_\lambda^m}\right] \sum_{r=1}^{m-1} \alpha_\lambda^r |a_r|,$$

$$\geq \frac{(x-y)}{[m+1]_q - \alpha_\lambda^{m+1}} \cdot \left[\sum_{r=1}^{m-1} \alpha_\lambda^r |a_r| + \alpha_\lambda^m |a_m| \right],$$

$$= \frac{(x-y)}{[m+1]_q - \alpha_\lambda^{m+1}} \cdot \left[\sum_{r=1}^{m} \alpha_\lambda^r |a_r| \right].$$

That is, the following holds:

$$|a_{m+1}| \leq \frac{(x-y)}{[m+1]_q - \alpha_\lambda^{m+1}} \sum_{r=1}^{m} \alpha_\lambda^r |a_r| \leq \prod_{r=0}^{m} \frac{[(x-y)-1]\alpha_\lambda^r + [r]_q}{[r+1]_q - \alpha_\lambda^{r+1}}.$$

This completes the proof, showing that the inequality in Equation (30) holds true for the value of k equal to $m+1$. □

Corollary 2. *Let $h(\mu) \in \mathcal{K}_q^{\lambda,\gamma}(x,y)$. For $-1 \leq y < x \leq 1$ and $q \in (0,1)$, the following holds true:*

$$|a_k| \leq \frac{1}{[k]_q} \prod_{m=0}^{k-1} \frac{[(x-y)-1]\alpha_\lambda^k + [m]_q}{[m+1]_q - \alpha_\lambda^{m+1}}, \ \ k \geq 2, \tag{31}$$

where α_λ^m is given by (3).

3. Conclusions

In this research paper, we have introduced a novel class of analytic functions, $\mathcal{S}_q^{\lambda,\gamma}(x,y)$, in the open unit disk Ψ. This class combines the concepts of (λ,γ)-symmetrical functions, generalized Janowski functions, and q-calculus in a unique and innovative manner. By deriving a structural formula and a representation theorem, we have established a solid foundation for understanding the nature of these functions.

The powerful tools of convolution and quantum calculus have been employed to explore the convolution conditions for functions within the class. This has led to a crucial supporting result for determining a sufficient condition for membership in the class, which we have illustrated through a relevant example and corollary. Furthermore, we have investigated the properties of coefficient estimates, providing valuable insights into the behavior and characteristics of these functions.

Our findings contribute to a deeper understanding of the analytical properties of the class $\mathcal{S}_q^{\lambda,\gamma}(x,y)$. This research opens up exciting avenues for further exploration. Future studies could focus on investigating other geometric properties, such as distortion theorems, the radius of starlikeness and convexity, and potential applications in the field of univalent functions. Additionally, exploring the connections between this class and other function classes within the framework of q-calculus could lead to a richer understanding of the interplay between different mathematical concepts.

In conclusion, this paper has successfully introduced and analyzed a new class of analytic functions, offering valuable insights and paving the way for further research in the field of complex analysis and q-calculus.

Author Contributions: The researchers F.A., H.L., A.Y.A.-R. and A.A.D. formulated the concept for the present investigation. They verified the data and proposed recommendations that significantly augmented the existing article. After perusing the final draft, each author made individual contributions. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, Saudi Arabia, for funding this research work through the project number "NBU-FPEJ-2024- 2920-01". This study was also supported via funding from Prince Sattam bin Abdulaziz University, project number (PSAU/2024/R/1445).

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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