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Analytically Pricing a Vulnerable Option under a Stochastic Liquidity Risk Model with Stochastic Volatility

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Abstract: This paper considers the valuation of a vulnerable option when underlying stock is subject to liquidity risks. That is, it is assumed that the underlying stock is not perfectly liquid. We establish a framework where the stock price follows the stochastic volatility model and the option contains the default risk of the option issuer. In addition, we assume that liquidity risks are caused by stochastic market liquidity, and the default occurs at the first jump time of a stochastic Poisson process, which has a stochastic default intensity process consisting of both idiosyncratic and systematic components. By employing a change of measure, we derive an analytical formula for the value of a vulnerable option. Finally, we present several numerical examples to illustrate the sensitivity of significant parameters.

Keywords: vulnerable option; default risk; stochastic volatility; liquidity risk

MSC: 91G20; 91G40

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1. Introduction

While the pioneering Black–Scholes model is still commonly employed in practice, its idealized assumptions limit its ability to accurately describe real markets. One of the examples is the assumption of complete liquidity in the underlying financial market, which means that investors can trade immediately and at a low or no cost. However, it is widely known that all securities are not completely liquid, even in widely recognized financial markets. That is, every investment is exposed to liquidity risks, which have emerged as one of the most significant challenges in financial markets. Since liquidity risks are multifaceted, there have been many studies on their effects in finance. In this paper, we focus on a reliable model that effectively captures the influence of liquidity risks on derivative pricing.

Liquidity risks have been extensively investigated in the option pricing literature. Fang et al. [1] first adopted stochastic market liquidity to develop the option pricing model. They considered a liquidity discount factor which follows the mean-reversion process. In recent years, many researchers have investigated the extension on the valuation of options, based on the work of Fang et al. [1]. Xu et al. [2] proposed an approach for valuing variance and volatility swaps with stochastic liquidity risk. Pasricha et al. [3] extended the work of [1] by constructing a general correlation structure among random variables. Pasricha and He [4] considered the liquidity risk in the underlying assets when the exchange option is priced. They provided the pricing formula using the characteristic function approach. In addition, Cai et al. [5] presented the approximated pricing formula of vulnerable spread options with a stochastic liquidity risk under levy processes.

Vulnerable options, which are financial derivatives that consider the counterparty's credit risk, have garnered significant research interest. These options are derivatives that capture the risk of a counterparty default. A credit risk model is required to appropriately value these options. Traditionally, two main frameworks were used: the structural model and the reduced-form model. The structural model, pioneered by Merton [6], establishes a direct correlation between credit events and a firm's value path. In this framework, a credit event occurs when the firm's value falls below its liabilities at maturity. The reduced-form model, developed by Jarrow and Turnbull [7] and improved by Jarrow and Yu [8], separates credit events from the firm's underlying value. Instead, it claims that a credit event is triggered by the first jump of a Poisson process with a specified intensity, which operates independently of the firm's value dynamics. In this paper, we consider a reduced-form model to model a credit risk.

Recently, many researchers have conducted research on vulnerable options using the reduced-form models. Fard [9] developed a pricing method for vulnerable options based on a generalized jump model, using a reduced-form technique for determining a counterparty's credit risk. Koo and Kim [10] used a reduced-form model to consider the option issuer's credit event and presented an explicit analytical valuation for a catastrophe put option with a default risk that employed the multidimensional Girsanov theorem. Pasricha and Goel [11] investigated a vulnerable power exchange option with two underlying assets using a reduced-form approach, modeling the counterparty's credit event as a doubly stochastic Poisson process and positing correlation among the three assets in both continuous and jump components. Wang [12,13] completed a study on the valuation of European, Asian, and Fader options with a default risk and stochastic volatility using a reduced-form model. This research also considers the vulnerable option using a stochastic volatility model. In addition, Jeon and Kim [14] investigated the valuation of a commodity-linked bond with a credit risk, stochastic volatility model, and stochastic convenience yield model in a reduced-form model.

This paper studies the value of vulnerable European options with market liquidity risk using a reduced-form model, motivated by the above results. In fact, the vulnerable options with market liquidity risk have been studied by many researchers in more recent years [15–20]. However, there is no research on vulnerable options with stochastic liquidity risk when the underlying asset follows stochastic volatility models. Among the literature, Pan et al. [20] studied a vulnerable option with stochastic liquidity risk. They assumed that the volatility of the underlying asset is constant. In fact, the financial market contains both liquidity risk and a counterparty's default risk. Moreover, it is well known that the volatility of the underlying asset is not constant. In this paper, we extended the work of Pan et al. [20] and aimed to derive the explicit pricing formula for a vulnerable option when the underlying asset is illiquid and follows a stochastic volatility model. As a result, the proposed model in this study is practical and should be of significant help for investors and practitioners.

The rest of this paper is organized as follows. In Section 2, we introduce a framework for option pricing based on a physical measure. In Section 3, we establish the equivalent martingale measure and explicitly derive the pricing formula of a vulnerable option with stochastic liquidity risk and stochastic volatility. Section 4 provides numerical examples, and Section 5 outlines the concluding remarks.

2. Model

We consider a filtered probability space $(\Omega, \mathcal{F}, P)$, where P is the physical measure. In this section, we describe the model for pricing the vulnerable option with liquidity risk and stochastic volatility. We construct the model based on the stochastic volatility model in the works of Heston [21] and He and Lin [22]. Under the measure P, the dynamics of a perfectly liquid stock and a stochastic volatility are presented by

$$\begin{aligned} dS_L(t) &= \mu S_L(t)dt + \sqrt{v_1(t)}S_L(t)dW(t), \\ dv_1(t) &= a_1(b_1 - v_1(t))dt + \sigma_1\sqrt{v_1(t)}dW_1(t), \end{aligned}$$

where σ_1 is constant volatility; a_1 and b_1 in the variance dynamic are constants; and $W(t)$ and $W_1(t)$ are the standard Brownian motions, and their correlation is $dW(t)dW_1(t) = \rho_1 dt$.

To allow for liquidity risk, we use the liquidity discount factor, assuming stocks are not perfectly liquid. Following Brunetti and Caldarera [23] and Fang et al. [1], we assume that the liquidity discount factor is used in the demand function to calculate the imperfectly liquid stock price, which is the price at which demand equals supply. We denote D as the demand function in the form of

$$D(S(t), \gamma(t), I(t)) = g\left(\frac{I_v(t)}{\gamma(t)S(t)}\right),$$

where $S(t)$ is the liquidity risk-adjusted stock price, $\gamma(t)$ is the liquidity discount factor, $I(t)$ is the information process, g is a smooth, strictly increasing function, and $v > 0$ is a constant. As in Brunetti and Caldarera [23] and He and Lin [22], we assume that the process $\gamma(t)$ is defined by

$$\gamma(t) = \exp\left(-\beta\left(\int_0^t L(s)ds + \int_0^t L_(s)dW_\gamma(s)\right)\right),$$

where $L(t)$ is market liquidity, β is the sensitivity of the stock to the level of market illiquidity, and $W_\gamma(t)$ is the standard Brownian motion with $dW_\gamma(t)dW(t) = 0,\ dW_\gamma(t)dW_1(t) = 0$. We assume that $\overline{S}$ is the fixed supply of the stock. Thus, the imperfectly liquidity stock price $S(t)$ is given by

$$S(t) = \frac{1}{\gamma(t)}\left(\frac{I^v(t)}{g^{-1}(\overline{S})}\right).$$

If the market is perfectly liquid, $\gamma(t) = 1$. That is, $S_L(t) = \frac{I^v(t)}{g^{-1}(\overline{S})}$. This also yields the underlying stock's price adjusted by the liquidity risk.

$$S(t) = \frac{1}{\gamma(t)}S^L(t). \tag{1}$$

According to Feng et al. [1], market liquidity has a mean-reverting property. That is, we choose a mean-reverting process for the modeling of market liquidity.

$$dL(t) = a_L(b_L - L(t))dt + \sigma_L dW_L(t), \tag{2}$$

where a_L, b_L and σ_L are constants, and $W_L(t)$ is the standard Brownian motion with with $dW_L(t)dW_1(t) = 0,\ dW_L(t)dW(t) = 0,\ dW_\gamma(t)dW_L(t) = \rho_2 dt$. Using Ito's lemma and $L(t)$, the dynamics of the liquidity discount factor $\gamma(t)$ are represented by

$$\frac{d\gamma(t)}{\gamma(t)} = \left(\frac{1}{2}\beta^2 L(t)^2 - \beta L(t)\right)dt - \beta L(t)dW_\gamma(t), \quad \gamma(0) = 1. \tag{3}$$

Next, we consider the dynamics of liquidity risk-adjusted stock price. Applying the product rule to (1), we have the following:

$$dS(t) = d\left(\frac{1}{\gamma(t)}\right)S_L(t) + \frac{1}{\gamma(t)}S_L(t) + d\left(\frac{1}{\gamma(t)}\right)dS_L(t). \tag{4}$$

Since

$$\frac{d\left(\frac{1}{\gamma(t)}\right)}{\left(\frac{1}{\gamma(t)}\right)} = \left(\beta L(t) + \frac{1}{2}\beta^2 L(t)^2\right)dt + \beta L(t)dW_\gamma(t),$$

we obtain the price $S(t)$ of the imperfectly liquid stock as

$$\frac{dS(t)}{S(t)} = (\mu + \beta L(t) + \frac{1}{2}\beta^2 L(t)^2)dt + \sqrt{v_1(t)}dW(t) + \beta L(t)dW_\gamma(t). \tag{5}$$

In (5), we can find that the process has a two-factor stochastic volatility model after accounting for liquidity effects, with market liquidity as one of the two factors under the measure P.

We now introduce the reduced-for model for modeling of the counterparty's credit risk. In the reduced-for model, if $N(t)$ is a doubly Poisson process with intensity $\lambda(t)$ and the first jump time of $N(t)$ is τ, then τ is assumed to be the default time. Following Wang [12], the default time τ satisfies the following:

$$\mathrm{P}(\tau > T) = \mathrm{E}\left[e^{-\int_0^T \lambda(s)ds}\right],$$

where T is the maturity. Following the works of Wang [12,13], we assume that the intensity process is given by

$$\lambda(t) = \kappa v_1(t) + v(t), \tag{6}$$

where $v_1(t)$ represents a systematic risk, $\kappa > 0$, and $v(t)$ represents idiosyncratic risk, which is defined by

$$dv(t) = a(b - v(t))dt + \sigma\sqrt{v(t)}dW_v(t), \tag{7}$$

where a is the rate of mean reversion, b is the long-run level of the process, σ is the volatility of the idiosyncratic risk, and $W_v(t)$ is the standard Brownian motion that is independent of all other Brownian motions. We additionally notice that a positive value κ guarantees that the process $\lambda(t)$ has positive values.

We consider the vulnerable European option under the reduced-form model. As in Fard [9] and Wang [12,13], the value of the vulnerable European call option at time 0 in the reduced-form model is represented by

$$\begin{aligned} C &= \mathrm{E}^Q\left[we^{-r\tau}\mathbf{1}_{\{0<\tau\le T\}}\mathrm{E}^Q\left[e^{-r(T-\tau)}(S(T)-K)^+|\mathcal{F}(\tau)\right]\right] + e^{-rT}\mathrm{E}^Q\left[(S(T)-K)^+\mathbf{1}_{\{\tau>T\}}\right] \\ &= (1-w)e^{-rT}\mathrm{E}^Q\left[e^{-\int_0^T \lambda(s)ds}(S(T)-K)^+\right] + we^{-rT}\mathrm{E}^Q\left[(S(T)-K)^+\right], \end{aligned} \tag{8}$$

where r is the interest rate, K is the strike, w is the recovery rate of the option, and $\mathrm{E}^Q[\cdot]$ denotes the expectation under the risk neutral measure Q.

3. The Valuation of the Vulnerable European Option

In the previous section, we represent the dynamics under the physical measure P. The dynamics under the measure P cannot be directly used to derive the option pricing formula. Therefore, to obtain the pricing formula, we should determine an equivalent martingale measure.

Considering the correlations of several Brownian motions, the stock process $S(t)$ can be rewritten as

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \left(\mu + \beta L(t) + \frac{1}{2}\beta^2 L(t)^2\right)dt + \sqrt{v_1(t)}\left(\rho_1 dW_1(t) + \sqrt{1-\rho_1^2}d\widehat{W}(t)\right) \\ &+ \beta L(t)\left(\rho_2 dW_L(t) + \sqrt{1-\rho_2^2}d\widehat{W}_\gamma(t)\right), \end{aligned} \tag{9}$$

where $\widehat{W}(t), W_1(t), W_L(t)$ and $\widehat{W}_\gamma(t)$ are independent standard Brownian motions. To define the risk neutral measure Q, we use the following Radon–Nikodym derivative.

$$\frac{dQ}{dP} = \exp\bigg\{ -\int_0^t \eta_1(s)dW_1(s) - \int_0^t \eta_2(s)dW_L(s) - \int_0^t \eta_3(s)d\widehat{W}(s) - \int_0^t \eta_4(s)d\widehat{W}_\gamma(s) \\ -\int_0^t \frac{1}{2}\eta_1^2(s)ds - \int_0^t \frac{1}{2}\eta_2^2(s)ds - \int_0^t \frac{1}{2}\eta_3^2(s)ds - \int_0^t \frac{1}{2}\eta_4^2(s)ds \bigg\}. \tag{10}$$

Using Girsanov's theorem,

$$\begin{aligned} W_1^Q(t) &= W_1(t) + \int_0^t \eta_1(s)ds, \\ W_L^Q(t) &= W_L(t) + \int_0^t \eta_2(s)ds, \\ W^Q(t) &= \widehat{W}(t) + \int_0^t \eta_3(s)ds, \\ W_\gamma^Q(t) &= \widehat{W}_\gamma(t) + \int_0^t \eta_4(s)ds, \end{aligned}$$

are independent standard Brownian motions under the risk neutral measure Q. Using these Brownian motions, the processes $v_1(t)$ and $L(t)$ are represented as

$$\begin{aligned} dv_1(t) &= \Big(a_1(b_1 - v_1(t)) - \sigma_1\sqrt{v_1(t)}\eta_1(t)\Big)dt + \sigma_1\sqrt{v_1(t)}dW_1^Q(t), \\ dL(t) &= (a_L(b_L - L(t)) - \sigma_L\eta_2(t))dt + \sigma_L dW_L^Q(t). \end{aligned}$$

We note that $\sigma_1\sqrt{v_1(t)}\eta_1(t)$ is the market liquidity risk premium and $\sigma_L\eta_2(t)$ is the volatility risk premium. The market prices are set by considering volatility and liquidity levels. This assumption has been verified by two studies [1,21] and is generally accepted in the field of research. That is, to achieve tractability, we assume that the liquidity risk premium in proportion to market liquidity.

$$\begin{aligned} \sqrt{v_1(t)}\eta_1(t) &= \frac{\eta_1 v_1(t)}{\sigma_1}, \\ \eta_2(t) &= \frac{\eta_2 L(t)}{\sigma_L}, \end{aligned}$$

where η_1 and η_2 are constants that satisfy the above equations. Thus, we can rewrite the processes of volatility and market liquidity under the measure Q.

$$dv_1(t) = \widehat{a}_1(\widehat{b}_1 - v_1(t))dt + \sigma_1\sqrt{v_1(t)}dW_1^Q(t), \tag{11}$$

$$dL(t) = \widehat{a}_L(\widehat{b}_L - L(t))dt + \sigma_L dW_L^Q(t), \tag{12}$$

where $\widehat{a}_1 = a_1 + \eta_1$, $\widehat{b}_1 = \frac{a_1 b_1}{a_1+\eta_1}$, $\widehat{a}_L = a_L + \eta_2$, and $\widehat{b}_L = \frac{a_L b_L}{a_L+b_L}$.

Using the standard Brownian motions under the risk neutral measure Q, the process $S(t)$ is given by

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \widehat{\mu}dt + \sqrt{v_1(t)}\Big(\rho_1 dW_1^Q(t) + \sqrt{1-\rho_1^2}dW^Q(t)\Big) \\ &+ \beta L(t)\Big(\rho_2 dW_L^Q(t) + \sqrt{1-\rho_2^2}dW_\gamma^Q(t)\Big). \end{aligned} \tag{13}$$

Recall the Radon–Nikodym derivative (10). Since $\widehat{\mu} - r = 0$ is an equivalent martingale measure, $\eta_1(t), \eta_2(t), \eta_3(t)$ and $\eta_4(t)$ satisfy

$$\mu + \beta L(t) + \frac{1}{2}\beta^2 L(t)^2 - \sqrt{v_1(t)}\left(\rho_1\eta_1(t) + \sqrt{1-\rho_1^2}\eta_3(t)\right)$$
$$-\beta L(t)\left(\rho_2\eta_2(t) + \sqrt{1-\rho_2^2}\eta_4(t)\right) - r = 0.$$

Therefore, the price $S(t)$ of the imperfectly liquid stock under the risk neutral measure Q is represented by

$$\begin{aligned} \frac{dS(t)}{S(t)} &= rdt + \sqrt{v_1(t)}\left(\rho_1 dW_1^Q(t) + \sqrt{1-\rho_1^2}dW^Q(t)\right) \\ &+ \beta L(t)\left(\rho_2 dW_L^Q(t) + \sqrt{1-\rho_2^2}dW_\gamma^Q(t)\right). \end{aligned} \tag{14}$$

To obtain an analytic pricing formula in the proposed model, we need the characteristic function

$$f(\phi_1, \phi_2) := \mathrm{E}^Q\left[e^{\phi_1 x(T) + \phi_2 \int_0^T \lambda(s)ds}\right], \tag{15}$$

where ϕ_1 and ϕ_2 are complex variables and $x(T) = \ln S(T)$. To obtain the closed-form expression of $f(\phi_1, \phi_2)$, we need some of the characteristics of the given processes. We now introduce the Lemmas for deriving the characteristic function.

Lemma 1. *For any complex numbers s_1, s_2 and s_3, the following holds:*

$$\begin{aligned} \widehat{P}(s_1, s_2, s_3; t, T) &= E\left[e^{-s_1\int_t^T L(s)^2 ds - s_2\int_t^T L(s)ds + s_3 L(T)^2}|L(t)\right] \\ &= e^{\frac{1}{2}H_1(s_1,s_3;t,T)L(t)^2 + H_2(s_1,s_2,s_3;t,T)L(t) + H_3(s_1,s_2,s_3;t,T)}, \end{aligned} \tag{16}$$

with terminal condition $\widehat{P}(s_1, s_2, s_3; T, T) = e^{s_3 L(T)^2}$, where $L(t)$ is defined in (26), $0 \le t \le T$, and

$$\begin{aligned} H_1(s_1, s_3; t, T) &= \frac{1}{\sigma_L^2}\left(\widehat{a}_L - \delta_1 \frac{\sinh(\delta_1(T-t)) + \delta_1\cosh(\delta_2(T-t))}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))}\right), \\ H_2(s_1, s_2, s_3; t, T) &= \frac{1}{\sigma_L^2\delta_1}\left(\frac{(\widehat{a}_L\widehat{b}_L\delta_1 - \delta_2\delta_3)}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))} - \widehat{a}_L\widehat{b}_L\delta_1\right) \\ &+ \frac{\delta_3}{\sigma_L^2\delta_1}\left(\frac{\sinh(\delta_1(T-t)) + \delta_2\cosh(\delta_1(T-t))}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))}\right), \\ H_3(s_1, s_2, s_3; t, T) &= -\frac{1}{2}\ln(\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))) + \frac{1}{2}\widehat{a}_L(T-t) \\ &+ \frac{(\widehat{a}_L^2\widehat{b}_L^2\delta_1^2 - \delta_3^2)}{2\sigma_L^2\delta_1^3}\left(\frac{\sinh(\delta_1(T-t))}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))} - \delta_1(T-t)\right) \\ &+ \frac{(\widehat{a}_L\widehat{b}_L\delta_1 - \delta_2\delta_3)\delta_3}{\sigma_L^2\delta_1^3}\left(\frac{\cosh(\delta_1(T-t)) - 1}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))}\right), \end{aligned}$$

with

$$\delta_1 = \sqrt{2\sigma_L^2 s_1 + \widehat{a}_L^2},\ \delta_2 = \frac{(\widehat{a}_L - 2\sigma_L^2 s_3)}{\delta_1},\ \delta_3 = \widehat{a}_L^2\widehat{b}_L - \sigma_L^2 s_2.$$

Proof. Applying the Feynman–Kac theorem to Equation (16), $\widehat{P}(s_1, s_2, s_3; t, T)$ satisfies the following differential equation:

$$\frac{\partial \widehat{P}}{\partial t} + \frac{\partial \widehat{P}}{\partial L}\widehat{a}_L(\widehat{b}_L - L(t)) + \frac{1}{2}\frac{\partial^2 \widehat{P}}{\partial L^2}\sigma_L^2 - (s_1 L(t)^2 + s_2 L(t))\widehat{P} = 0. \tag{17}$$

Then, it is assumed that the solution of $\widehat{P}(s_1, s_2, s_3; t, T)$ has the following form:

$$\widehat{P}(s_1, s_2, s_3; t, T) = e^{\frac{1}{2}H_1(t,T)L(t)^2 + H_2(t,T)L(t) + H_3(t,T)}, \tag{18}$$

with terminal conditions $H_1(s_1, s_3; T, T) = 2s_3$, $H_2(s_1, s_2, s_3; T, T) = 0$, and $H_3(s_1, s_2, s_3; T, T) = 0$. Thus, we have the following system of ordinary differential equations (ODEs):

$$\frac{\partial H_1}{\partial t} + \sigma_L^2 H_1^2 - 2\widehat{a}_L H_1 - 2s_1 = 0, \tag{19}$$

$$\frac{\partial H_2}{\partial t} - (\widehat{a}_L - \sigma_L^2 H_1)H_2 + \widehat{a}_L\widehat{b}_L H_1 - s_2 = 0, \tag{20}$$

$$\frac{\partial H_3}{\partial t} + \frac{1}{2}\sigma_L^2 H_2^2 - \widehat{a}_L\widehat{b}_L H_2 - \frac{1}{2}\sigma_L^2 H_1 = 0. \tag{21}$$

Solving the above equations, we can obtain H_1, H_2, and H_3. This completes $\widehat{P}(s_1, s_2, s_3; t, T)$. □

The following Lemma is well known. For more details, see Cox et al. [24].

Lemma 2. *For any complex numbers s_1 and s_2, the joint characteristic function of $(\int_t^T v(s)ds, v(T))$ is given by*

$$\begin{aligned} P(s_1, s_2; t, T) &= E\left[e^{s_1 \int_t^T v(s)ds + s_2 v(T)} | v(t) = v\right] \\ &= e^{A(s_1,s_2;t,T) - B(s_1,s_2;t,T)v} \end{aligned} \tag{22}$$

where $v(t)$ is defined in (7), $0 \le t \le T$, and

$$\begin{aligned} A(s_1, s_2) &= \frac{2ab}{\sigma^2}\ln\left(\frac{2m(s_1)e^{(m(s_1)+a)(T-t)/2}}{(m(s_1) + a - \sigma^2 s_2)(e^{m(s_1)(T-t)} - 1) + 2m(s_1)}\right), \\ m(s_1) &= \sqrt{a^2 - 2\sigma^2 s_1}, \\ B(s_1, s_2) &= \frac{2s_1(1 - e^{m(s_1)(T-t)}) - s_2\Big(2m(s_1) + (m(s_1) - a)(e^{m(s_1)(T-t)} - 1)\Big)}{(m(s_1) + a - s_2\sigma^2)(e^{m(s_1)(T-t)} - 1) + 2m(s_1)}. \end{aligned}$$

The closed-form expression of $f(\phi_1, \phi_2)$ defined in (15) is presented in the following proposition.

Proposition 1. *In the proposed model, the characteristic function $f(\phi_1, \phi_2)$ is expressed as*

$$\begin{aligned} &f(\phi_1, \phi_2) \\ &= \exp[A_1(\phi_1, \phi_2) + A_2(\phi_2) - B_1(\phi_1, \phi_2)v_1(0) - B_2(\phi_2)v(0)] \\ &\times \exp\left[\frac{1}{2}D_1(\phi_1)L(0)^2 + D_2(\phi_1)L(0) + D_3(\phi_1) + G(\phi_1)\right] \end{aligned} \tag{23}$$

where

$$A_1(\phi_1,\phi_2)=\frac{2\widehat{a}_1\widehat{b}_1}{\sigma_1^2}\ln\left(\frac{2m_1(\phi_1,\phi_2)e^{\frac{1}{2}(m_1(\phi_1,\phi_2)+\widehat{a}_1)(T)}}{(m_1(\phi_1,\phi_2)+\widehat{a}_1-\sigma_1^2\zeta_2(\phi_1))(e^{m_1(\phi_1,\phi_2)(T)}-1)+2m_1(\phi_1,\phi_2)}\right),$$

$$B_1(\phi_1,\phi_2)=\frac{2\zeta_1(\phi_1,\phi_2)(1-e^{m_1(\phi_1,\phi_2)(T)})-\zeta_2(\phi_1)\Big(2m_1(\phi_1,\phi_2)+(m_1(\phi_1,\phi_2)-\widehat{a}_1)(e^{m_1(\phi_1,\phi_2)(T)}-1)\Big)}{(m_1(\phi_1,\phi_2)+\widehat{a}_1-\sigma_1^2\zeta_2(\phi_1))(e^{m_1(\phi_1,\phi_2)(T)}-1)+2m_1(\phi_1,\phi_2)},$$

$$m_1(\phi_1,\phi_2)=\sqrt{\widehat{a}_1^2-2\sigma_1^2\zeta_1(\phi_1,\phi_2)},\;\;\zeta_1(\phi_1,\phi_2)=\phi_1^2\frac{1-\rho_1^2}{2}+\phi_2\kappa-\frac{\phi_1\widehat{a}_1\rho_1}{\sigma_1}-\frac{\phi_1}{2},\;\;\zeta_2(\phi_1)=\frac{\phi_1\rho_1}{\sigma_1},$$

$$A_2(\phi_2)=\frac{2ab}{\sigma^2}\ln\left(\frac{2m_2(\phi_2)e^{\frac{1}{2}(m_2(\phi_2)+a)(T)}}{(m_2(\phi_2)+a)(e^{m_2(\phi_2)(T)}-1)+2m_2(\phi_2)}\right),B_2(\phi_2)=\frac{2\phi_2(1-e^{(m_2(\phi_2)(T))})}{(m_2(\phi_2)+a)(e^{m_2(\phi_2)(T)}-1)+2m_2(\phi_2)},$$

$$m_2(\phi_2)=\sqrt{a^2-2\sigma^2\phi_2},$$

$$D_1(\phi_1)=\frac{1}{\sigma_L^2}\left(\widehat{a}_L-\delta_1\frac{\sinh(\delta_1T)+\delta_2\cosh(\delta_1T)}{\cosh(\delta_1T)+\delta_2\sinh(\delta_1T)}\right),$$

$$D_2(\phi_1)=\frac{1}{\sigma_L^2\delta_1}\left(\frac{(\widehat{a}_L\widehat{b}_L\delta_1-\delta_2\delta_3)}{\cosh(\delta_1T)+\delta_2\sinh(\delta_1T)}-\widehat{a}_L\widehat{b}_L\delta_1\right)+\frac{\delta_3}{\sigma_L^2\delta_1}\left(\frac{\sinh(\delta_1T)+\delta_2\cosh(\delta_1T)}{\cosh(\delta_1T)+\delta_2\sinh(\delta_1T)}\right),$$

$$D_3(\phi_1)=-\frac{1}{2}\ln(\cosh(\delta_1T)+\delta_2\sinh(\delta_1T))+\frac{1}{2}\widehat{a}_LT+\frac{(\widehat{a}_L^2\widehat{b}_L^2\delta_1^2-\delta_3^2)}{2\sigma_L^2\delta_1^3}\left(\frac{\sinh(\delta_1T)}{\cosh(\delta_1T)+\delta_2\sinh(\delta_1T)}-\delta_1T\right),$$
$$+\frac{(\widehat{a}_L\widehat{b}_L\delta_1-\delta_2\delta_3)\delta_3}{\sigma_L^2\delta_1^3}\left(\frac{\cosh(\delta_1T)-1}{\cosh(\delta_1T)+\delta_2\sinh(\delta_1T)}\right),$$

$$\delta_1=\sqrt{2\sigma_L^2\vartheta_1(\phi_1)+\widehat{a}_L^2},\;\delta_2=\frac{(\widehat{a}_L-2\sigma_L^2\vartheta_3(\phi_1))}{\delta_1},\;\delta_3=\widehat{a}_L^2\widehat{b}_L-\sigma_L^2\vartheta_2(\phi_1),$$

$$\vartheta_1(\phi_1)=\phi_1\left(\frac{\beta^2}{2}-\frac{\phi_1\beta^2(1-\rho_2^2)}{2}-\frac{\beta\rho_2\widehat{a}_L}{\sigma_L}\right),\vartheta_2(\phi_1)=\phi_1\frac{\beta\rho_2\widehat{a}_L\widehat{b}_L}{\sigma_L},\vartheta_3(\phi_1)=\phi_1\frac{\beta\rho_2}{2\sigma_L},$$

$$G(\phi_1)=\phi_1\left(\ln S(0)+rT-\frac{\rho_1}{\sigma_1}(v_1(0)+\widehat{a}_1\widehat{b}_1T)-\frac{\beta\rho_2}{2\sigma_L}(L(0)^2+\sigma_LT)\right).$$

Proof. Let us consider the dynamics in (14). Thus, we have

$$\begin{aligned}x(T)=\ln S(T)&=\ln S(0)+rT-\frac{1}{2}\left(\int_0^T v_1(s)ds+\beta^2\int_0^T L(s)^2ds\right)\\&+\rho_1\int_0^T\sqrt{v_1(s)}dW_1^Q(s)+\sqrt{1-\rho_1^2}\int_0^T\sqrt{v_1(s)}dW^Q(s)\\&+\beta\rho_2\int_0^T L(s)dW_L^Q(s)+\beta\sqrt{1-\rho_2^2}\int_0^T L(s)dW_\gamma^Q(s).\end{aligned}\tag{24}$$

Note that

$$\begin{aligned}d\Big(L(t)^2\Big)&=2L(t)dL(t)+\sigma_L^2dt\\&=\Big(2\widehat{a}_L(\widehat{b}_L-L(t))L(t)+\sigma_L^2\Big)dt+2\sigma_LL(t)dW_L^Q(t),\end{aligned}\tag{25}$$

which implies that

$$\int_0^T L(s)dW_L^Q(s)=\frac{1}{2\sigma_L}\left(L(T)^2-L(0)^2-2\widehat{a}_L\int_0^T L(s)(\widehat{b}_L-L(s))ds-\sigma_L^2T\right).\tag{26}$$

In addition, from the dynamics $v_1(t)$ under the measure Q,

$$\int_0^T \sqrt{v_1(s)}dW_1^Q(s) = \frac{1}{\sigma_1}\left(v_1(T) - v_1(0) - \widehat{a}_1\widehat{b}_1 T - \widehat{a}_1 \int_0^T v_1(s)ds\right). \tag{27}$$

Note that $W^Q(t)$, $W_\gamma^Q(t)$, $W_1^Q(t)$, and $W_v(t)$ are independent. Using the results in (26) and (27), we have the following:

$$\begin{aligned}
f(\phi_1,\phi_2) &= \mathrm{E}^Q\left[e^{\phi_1 x(T)+\phi_2\int_0^T\lambda(s)ds}\right] \\
&= \exp\left\{\phi_1\left(\ln S(0) + rT - \frac{\rho_1}{\sigma_1}(v_1(0)+\widehat{a}_1\widehat{b}_1T) - \frac{\beta\rho_2}{2\sigma_L}(L(0)^2+\sigma_L T)\right)\right\} \\
&\times \mathrm{E}^Q\left[\exp\left\{\phi_1\left(\frac{\rho_1}{\sigma_1}v_1(T) - \left(\frac{\widehat{a}_1\rho_1}{\sigma_1}+\frac{1}{2}\right)\int_0^T v_1(s)ds + \sqrt{1-\rho_1^2}\int_0^T\sqrt{v_1(s)}dW^Q(s)\right)\right\}\right] \\
&\times \mathrm{E}^Q\left[\exp\left\{\phi_1\left(\frac{\beta\rho_2}{2\sigma_L}L(T)^2 - \frac{\beta\rho_2\widehat{a}_L\widehat{b}_L}{\sigma_L}\int_0^T L(s)ds + \left(\frac{\beta\rho_2\widehat{a}_L}{\sigma_L}-\frac{\beta^2}{2}\right)\int_0^T L(s)^2ds + \beta\sqrt{1-\rho_2^2}\int_0^T L(s)dW_\gamma^Q(s)\right)\right\}\right] \\
&\times \mathrm{E}^Q\left[\exp\left\{\phi_2\kappa\int_0^T v_1(s)ds\right\}\right] \times \mathrm{E}^Q\left[\exp\left\{\phi_2\int_0^T v(s)ds\right\}\right]
\end{aligned} \tag{28}$$

Let us apply the law of iterated expectations in (28). Thus, we can obtain the following:

$$\begin{aligned}
f(\phi_1,\phi_2) &= \exp\left\{\phi_1\left(\ln S(0) + rT - \frac{\rho_1}{\sigma_1}(v_1(0)+\widehat{a}_1\widehat{b}_1T) - \frac{\beta\rho_2}{2\sigma_L}(L(0)^2+\sigma_L T)\right)\right\} \\
&\times \mathrm{E}^Q\left[\exp\left\{\frac{\phi_1\rho_1}{\sigma_1}v_1(T) + \left(\phi_1^2\frac{1-\rho_1^2}{2} + \phi_2\kappa - \frac{\phi_1\widehat{a}_1\rho_1}{\sigma_1} - \frac{\phi_1}{2}\right)\int_0^T v_1(s)ds\right\}\right] \\
&\times \mathrm{E}^Q\left[\exp\left\{-\phi_1\left(\frac{\beta^2}{2} - \frac{\phi_1\beta^2(1-\rho_2^2)}{2} - \frac{\beta\rho_2\widehat{a}_L}{\sigma_L}\right)\int_0^T L(s)^2ds - \phi_1\frac{\beta\rho_2\widehat{a}_L\widehat{b}_L}{\sigma_L}\int_0^T L(s)ds + \phi_1\frac{\beta\rho_2}{2\sigma_L}L(T)^2\right\}\right] \\
&\times \mathrm{E}^Q\left[\exp\left\{\phi_2\int_0^T v(s)ds\right\}\right] \\
&:= \exp\{G(\phi_1)\} \times \mathrm{E}^Q\left[\exp\left\{\zeta_1(\phi_1,\phi_2)\int_0^T v_1(s)ds + \zeta_2(\phi_1)v_1(T)\right\}\right] \times \mathrm{E}^Q\left[\exp\left\{\phi_2\int_0^T v(s)ds\right\}\right] \\
&\times \mathrm{E}^Q\left[\exp\left\{-\vartheta_1(\phi_1)\int_0^T L(s)^2ds - \vartheta_2(\phi_1)\int_0^T L(s)ds + \vartheta_3(\phi_1)L(T)^2\right\}\right]
\end{aligned}$$

Then, by using Lemma 2 and Lemma 1, the proof is completed. □

We now are ready to obtain the solution for the valuation of a vulnerable option in (8). Specifically, we derive the pricing formula by inverting the characteristic function of the logarithm of the underlying asset and the measure change technique. The vulnerable option value is presented in the following proposition.

Proposition 2. *Under the proposed model, the value at time 0 of the vulnerable option is given by*

$$C = (1-w)e^{-rT}(E_1 - K\cdot E_2) + we^{-rT}(E_3 - K\cdot E_4)$$

where

$$\begin{aligned}
E_1 &= \frac{1}{2}f(1,-1) + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1\ln K}f(1+i\phi_1,-1)}{i\phi_1}\right]d\phi_1 \\
E_2 &= \frac{1}{2}f(0,-1) + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1\ln K}f(i\phi_1,-1)}{i\phi_1}\right]d\phi_1 \\
E_3 &= \frac{1}{2}f(1,0) + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1\ln K}f(1+i\phi_1,0)}{i\phi_1}\right]d\phi_1 \\
E_4 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1\ln K}f(i\phi_1,0)}{i\phi_1}\right]d\phi_1
\end{aligned}$$

Proof. Recall the value of the vulnerable option at time 0 in (8).

$$\begin{aligned} C &= (1-w)e^{-rT}\mathrm{E}^{Q}\left[e^{-\int_0^T \lambda(s)ds}(S(T)-K)^+\right] + we^{-rT}\mathrm{E}^{Q}\left[(S(T)-K)^+\right] \\ &:= (1-w)e^{-rT}(E_1 - K\cdot E_2) + we^{-rT}(E_3 - K\cdot E_4), \end{aligned} \tag{29}$$

where

$$\begin{aligned} E_1 &= \mathrm{E}^{Q}\left[e^{x(T)-\int_0^T \lambda(s)ds}\cdot \mathbf{1}_{\{S(T)>K\}}\right], \\ E_2 &= \mathrm{E}^{Q}\left[e^{-\int_0^T \lambda(s)ds}\cdot \mathbf{1}_{\{S(T)>K\}}\right], \\ E_3 &= \mathrm{E}^{Q}\left[e^{x(T)}\cdot \mathbf{1}_{\{S(T)>K\}}\right], \\ E_4 &= \mathrm{E}^{Q}\left[\mathbf{1}_{\{S(T)>K\}}\right]. \end{aligned}$$

To calculate E_1, E_2, E_3 and E_4, we adopt the measure change technique. In fact, E_1, E_2, E_3 and E_4 are calculated using the Fourier inversion formula by noting that $f(i\phi_1, i\phi_2)$ is the joint characteristic function of $\ln S(T)$ and $\int_0^T \lambda(s)ds$ under the measure Q.

We simplify the calculation for E_1 by introducing a new measure, Q_1, defined by

$$\frac{dQ_1}{dQ} := \frac{e^{x(T)-\int_0^T \lambda(u)du}}{\mathrm{E}\left[e^{x(T)-\int_0^T \lambda(u)du}\right]}.$$

Thus, the characteristic function of $x(T)$ under the measure Q_1 is given by

$$\begin{aligned} \mathrm{E}^{Q_1}\left[e^{i\phi_1 x(T)}\right] &= \mathrm{E}^{Q}\left[\frac{e^{x(T)-\int_0^T \lambda(s)ds}}{\mathrm{E}^{Q}\left[e^{x(T)-\int_0^T \lambda(s)ds}\right]}e^{i\phi_1 x(T)}\right] \\ &= \frac{f(1+i\phi_1,-1)}{f(1,-1)}. \end{aligned}$$

Then, by employing the Fourier inversion formula and the measure change technique, we have

$$\begin{aligned} E_1 &= \mathrm{E}^{Q}\left[e^{x(T)-\int_0^T \lambda(s)ds}\cdot \mathbf{1}_{\{x(T)>\ln K\}}\right] \\ &= \mathrm{E}^{Q}\left[e^{x(T)-\int_0^T \lambda(s)ds}\right]\mathrm{E}^{Q_1}\left[\frac{dQ_1}{dQ}\cdot \mathbf{1}_{\{x(T)>\ln K\}}\right] \\ &= f(1,-1)\times\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1\ln K}f(1+i\phi_1,-1)/f(1,-1)}{i\phi_1}\right]d\phi_1\right) \\ &= \frac{1}{2}f(1,\,-1)+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1\ln K}f(1+i\phi_1,-1)}{i\phi_1}\right]d\phi_1, \end{aligned}$$

Similarly, we deal with E_2 by defining another measure, Q_2, as follows:

$$\frac{dQ_2}{dQ} := \frac{e^{-\int_0^T \lambda(u)du}}{\mathrm{E}^{Q}\left[e^{-\int_0^T \lambda(u)du}\right]}.$$

Under the measure Q_2, the characteristic function of $x(T)$ is given by

$$\begin{aligned}
\mathrm{E}^{Q_2}\left[e^{i\phi_1 x(T)}\right] &= \mathrm{E}^{Q}\left[\frac{e^{-\int_0^T \lambda(s)ds}}{\mathrm{E}^{Q}\left[e^{-\int_0^T \lambda(s)ds}\right]} e^{i\phi_1 x(T)}\right] \\
&= \frac{f(i\phi_1,-1)}{f(0,-1)}.
\end{aligned}$$

Thus, we can obtain the following:

$$\begin{aligned}
E_2 &= \mathrm{E}^{Q}\left[e^{-\int_0^T \lambda(s)ds} \cdot \mathbf{1}_{\{x(T)>\ln K\}}\right] \\
&= \mathrm{E}^{Q}\left[e^{-\int_0^T \lambda(s)ds}\right]\mathrm{E}^{Q_2}\left[\frac{dQ_2}{dQ} \cdot \mathbf{1}_{\{x(T)>\ln K\}}\right] \\
&= f(0,-1) \times \left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1 \ln K} f(i\phi_1,-1)/f(0,-1)}{i\phi_1}\right] d\phi_1\right) \\
&= \frac{1}{2} f(0,\,-1) + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1 \ln K} f(i\phi_1,-1)}{i\phi_1}\right] d\phi_1.
\end{aligned}$$

In a similar way, we can calculate E_3 and E_4 by defining new measures. Thus, the proof is completed. □

4. Numerical Examples

In this section, we provide numerical examples to illustrate the effects of market liquidity risk and default risk of the option prices with stochastic volatility. We particularly focus on the behavior of prices with and without liquidity risks. For the experiments, we use the following parameters: $S(0) = 100, K = 100, T = 1, r = 0,04, \widehat{a}_1 = 1.1, \widehat{b}_1 = 0.05, \sigma_1 = 0.15, \rho_1 = -0.7, v_1(0) = 0.03, \widehat{a}_L = 3, \widehat{b}_L = 0.2, \sigma_L = 0.1, \rho_2 = -0.4, L(0) = 0.1, a = 2, b = 0.01, \sigma = 0.1, v(0) = 0.03, \beta = 0.3, \kappa = 1.2, w = 0.4$. We also adopt the quadrature method to obtain the value of the integrals in Proposition 2.

We first observe the impact of stochastic liquidity risk on the values of vulnerable European options. Figure 1 illustrates that the option value decreases as the strike price K increases. By analyzing the distance between the two lines, we can see that the liquidity risk has a rather large impact on out-the-money options. In addition, as the strike prices increase, the impact of the liquidity risk remains significant. Figure 2 shows the option values against β, illustrating the sensitivity of stock prices with respect to market liquidity. Obviously, the value of an option without a liquidity risk under the proposed model maintains a constant option because no liquidity risk means that $\beta = 0$. However, the option value increases as the stock becomes more dependent on market illiquidity. This is due to the fact that the values of β affect the overall volatility of the underlying asset. In Figure 3, we find that the values increase as the maturities T increase. Higher values of β correspond to higher option values and have a larger increasing slope.

Figure 4 presents the option values against the maturity T for three different values of recovery rates. In Figure 4, we can see that there is little difference between the values for short maturities. Clearly, larger recovery rates have larger option values. However, we can also see that it they are less sensitive than liquidity risks. Figure 5 shows the behavior of the option values for values of κ in the intensity process (6). Higher values of κ indicate a higher default probability, resulting in lower pricing for options with a default risk. In addition, by comparing the distance between the two lines, we can see that the impact of a default risk is relatively small compared to the impact of a liquidity risk. Figure 6 illustrates option values with respect to the initial values of the market liquidity measure $L(t)$. In other words, what is shown in Figure 6 is how a change in $L(0)$ can affect option values. An interesting feature that can be seen is that the value of an option with a liquidity risk increases exponentially as the value of $L(0)$ increases. In (14), we can see that $L(0)$ is volatile. Since the volatility is in the exponent of the underlying asset, we can observe the phenomenon that occurs due to it, as shown in Figure 6. In other words, this is explained by the fact that higher initial values of the market liquidity measure lead to a higher volatility of the underlying asset. Clearly, if there is no liquidity risk, the option value has a constant value regardless of the value of $L(0)$. Additionally, we can observe that the values of an option without a default risk are higher than values with a default risk.

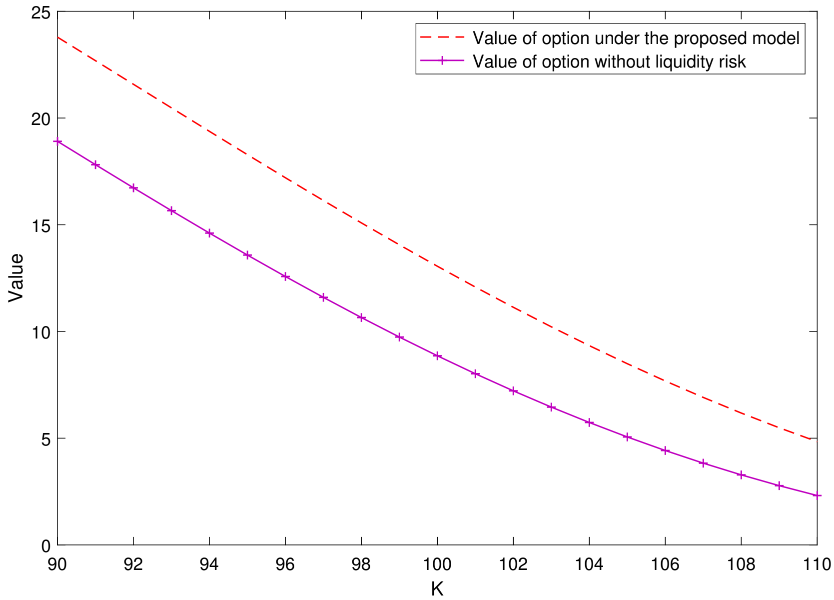

Figure 1. Values of option against K for two cases (without liquidity risk ($\beta = 0$) and with liquidity risk ($\beta = 0.3$)).

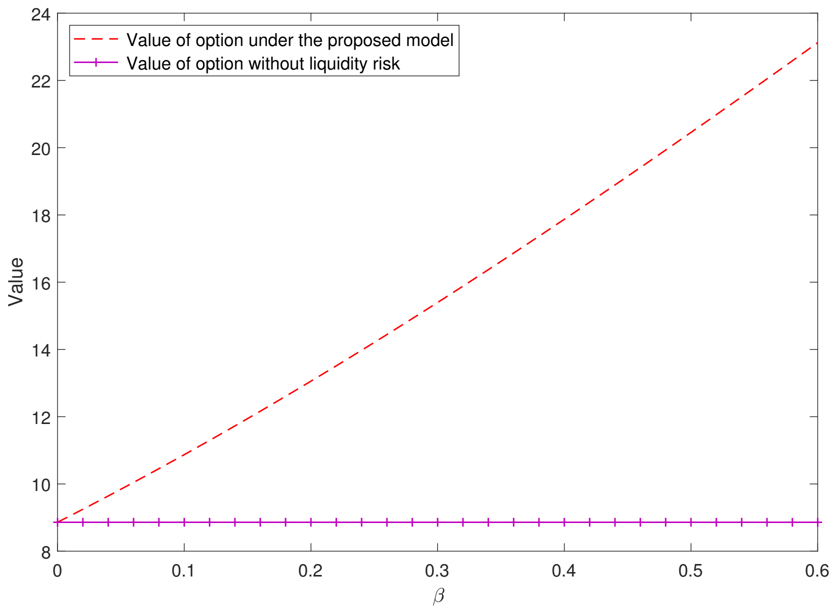

Figure 2. Values of option against β for two cases (without liquidity risk ($\beta = 0$) and with liquidity risk ($\beta = 0.3$)).

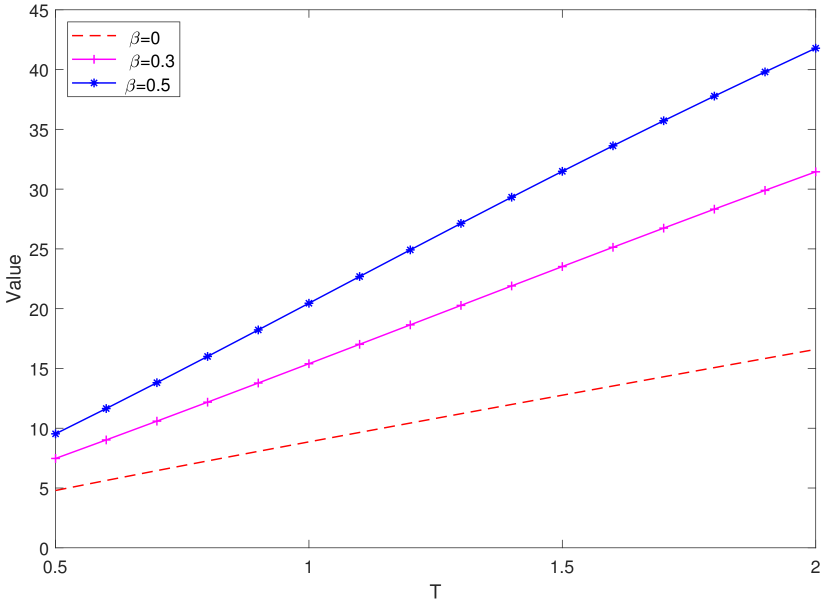

Figure 3. Values of option against T for $\beta = 0, 0.3, 0.5$.

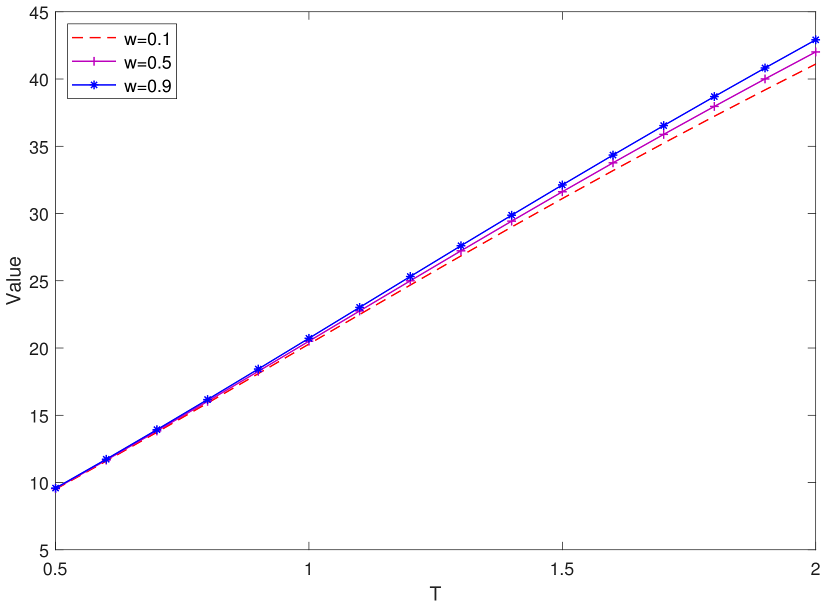

Figure 4. Values of option against T for recovery rates $w = 0.1, 0.5, 0.9$.

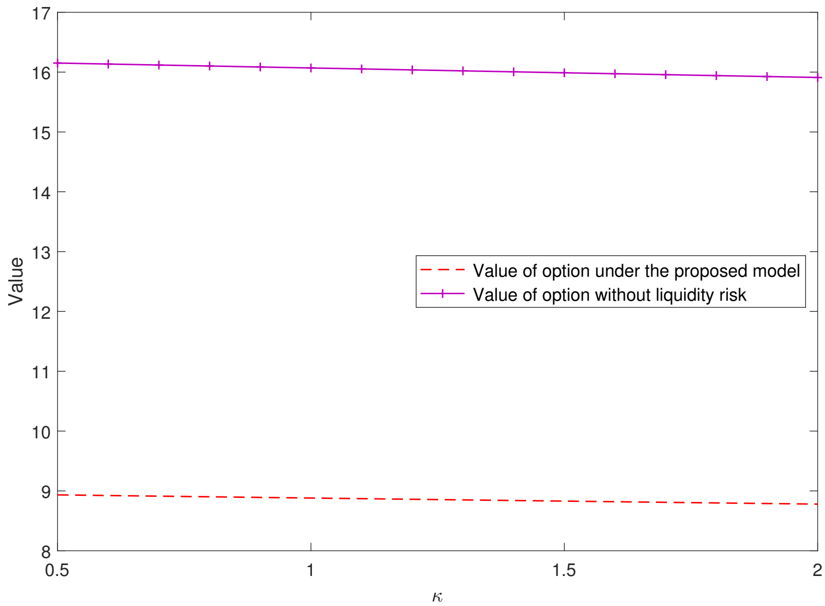

Figure 5. Values of option against κ for two cases (without liquidity risk ($\beta = 0$) and with liquidity risk ($\beta = 0.3$)).

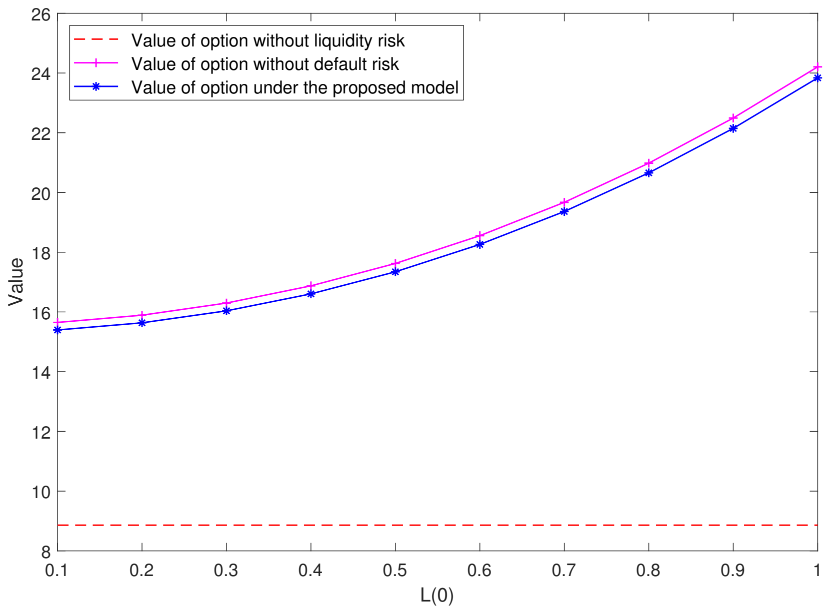

Figure 6. Values of option against $L(0)$ for three cases (without and with liquidity risk ($\beta = 0$,$\beta = 0.3$), and without default risk ($w = 1$)).

5. Concluding Remarks

In this study, we extend the existing research on the option pricing model by incorporating stochastic volatility, counterparty default risk, and market liquidity risk. We propose a framework for a vulnerable option that takes into account both the liquidity discount factor and the default intensity process when the underlying asset follows the Heston stochastic volatility model. This framework is studied based on a probabilistic approach as an extension of the work of Pan et al. [20]. That is, to facilitate option pricing, we determine a risk-neutral measure and use the characteristic function and the measure change technique. We then present the analytical pricing formula for a vulnerable European call option with stochastic liquidity risk. Finally, we conclude with numerical examples that illustrate how stochastic volatility, default risk, and stochastic liquidity risk impact vulnerable option prices.

The main contribution of this paper is that it developed a frameworks for pricing options with liquidity risk, stochastic volatility, and default risk. In contrast to Pan et al. [20], we used the stochastic volatility model to describe the stochastic process for an underlying asset. However, we have some limitations. Our research does not take into account jumps in the underlying asset, as well as structural models for the credit risk model. These are important areas which we aim to investigate in the future.

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