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Cross-ValidatedFunctional Generalized Partially Linear Single-Functional Index Model Supplementary Materials

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In this supplementary file, we will present (i) the rest of the simulation presented in the manuscript, in particular the comparison of our model with other methods in the literature, (ii) the proof of Theorem 1, (iii) the proofs of Lemmas 1, 2 and 3 allowing to show Theorem 1, (iv) the proof of Lemmas **A5** and **A6** that are used to show Theorem 2., and (v) the proof of Theorem 3.

Simulation study

In order to evaluate the effectiveness of the CVGPLFSIM model, we employ a random splitting of the sample into two subsets: a training subset, denoted as I_1, comprising 80% of the observations, and a test subset, denoted as I_2, comprising 20% of the observations. The purpose of the training subset is to estimate the model parameters, while the test subset is used to assess the accuracy of the predictors. We utilize the Mean Square Error of Prediction (MSEP), as defined in Aneiros et al. [1] and given by

$$\mathtt{MSEP} = \frac{1}{\#I_2}\sum_{i\in I_2}(Y_i - \widehat{Y}_i)^2/\mathrm{var}_{I_2}(Y_i),$$

where $\widehat{Y}_i$ represents the predicted value based on the training subset, $\#I_2$ denotes the number of observations in the test subset I_2, and $\mathrm{var}_{I_2}(Y_i)$ denotes the variance of the response variables in the test subset. This indicator allows us to assess the accuracy of our predictions with respect to the variability in the test dataset.

The performance comparison of the CVGPLFSIM model with other models is presented in Tables S1 and S2 when $n = 500$ and in Tables S3 and S4 when $n = 1000$. Based on the obtained results, we can infer that the CVGPLFSIM model demonstrates competitiveness and effectiveness in analyzing the given dataset.

Tables S1,S2, S3 and S4 show the performance of the CVGPLFSIM model by comparing it with other models when n increases from $n = 500$ to $n = 1000$. The CVGPLFSIM model is a competitive one for such data.

Table S1. MSEP for different models in the Gaussian case when $n = 500$.

Functional models		MSEP
M-1 (CVGPLFSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(\langle\beta, Z_i\rangle) + \varepsilon_i$	0.062
M-2 (GNP-FPLSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.079
M-3 (FPLSIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.098
M-4 (SIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + \varepsilon_i$	1.113

Table S2. MSEP for different models in the logistic case when $n = 500$.

Functional models		MSEP
M-1 (CVGPLFSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(\langle\beta, Z_i\rangle) + \varepsilon_i$	0.068
M-2 (GNP-FPLSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.071
M-3 (FPLSIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.088
M-4 (SIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + \varepsilon_i$	1.005

Table S3. MSEP for different models in the Gaussian case when $n = 1000$.

Functional models		MSEP
M-1 (CVGPLFSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(\langle\beta, Z_i\rangle) + \varepsilon_i$	0.043
M-2 (GNP-FPLSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.068
M-3 (FPLSIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.872
M-4 (SIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + \varepsilon_i$	1.034

Table S4. MSEP for different models in the logistic case when $n = 1000$.

Functional models		MSEP
M-1 (CVGPLFSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(\langle\beta, Z_i\rangle) + \varepsilon_i$	0.051
M-2 (GNP-FPLSIM)	$g(\mu_i(X_i, Z_i)) = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.063
M-3 (FPLSIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + R(Z_i) + \varepsilon_i$	0.097
M-4 (SIM)	$Y_i = \eta(\alpha_1 X_{1,i} + \alpha_2 X_{2,i}) + \varepsilon_i$	1.091

Proof of Theorem 1

For more details, we refer the reader to the paper by Ait-Saïdi et al. [2].

Proofs of Lemma 1 **and Lemma** 2**.**

Set $\tilde{a} = \sqrt{n}(\tilde{\tau} - \tau_0)$, $\tilde{b} = \sqrt{n}(\tilde{\delta} - \delta_0)$ and $\bar{W} = B_1(W)$. Then $(\tilde{a}, \tilde{b}) = \arg\max_{a,b} \tilde{l}(a, b)$, where

$$
\begin{aligned}
\tilde{l}(a,b) &= \frac{1}{n}\sum_{i=1}^{n} Q\Big(g^{-1}\Big\{\tilde{\eta}\Big(\alpha^\top(\tau)X_i\Big) + \delta^\top \bar{W}_i\Big\}, Y_i\Big) - \frac{1}{n}\sum_{i=1}^{n} Q\Big(g^{-1}\{\tilde{m}_{0i}\}, Y_i\Big) \\
&= \frac{1}{n}\sum_{i=1}^{n} Q\left(g^{-1}\left\{\tilde{\eta}\left(\alpha\left(\tau_0 + \frac{1}{\sqrt{n}}a\right)^\top X_i\right) + \left(\delta_0 + \frac{1}{\sqrt{n}}b\right)^\top \bar{W}_i\right\}, Y_i\right) \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} Q\Big(g^{-1}\{\tilde{m}_{0i}\}, Y_i\Big) \\
&= \frac{1}{n}\sum_{i=1}^{n} Q\left(g^{-1}\left\{\tilde{\eta}\left(\alpha^\top(\tau_0)X_i + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\right) + \delta_0^\top \bar{W}_i + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\right\}, Y_i\right) \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} Q\Big(g^{-1}\{\tilde{m}_{0i}\}, Y_i\Big) \\
&= \frac{1}{n}\sum_{i=1}^{n} Q\left(g^{-1}\left\{\tilde{m}_{0i} + \tilde{\eta}\left(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\right) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\right\}, Y_i\right) \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} Q\Big(g^{-1}\{\tilde{m}_{0i}\}, Y_i\Big), \qquad \text{(S1)}
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{m}_i &= \tilde{m}_i(X_i, \bar{W}_i) = \tilde{\gamma}^\top B\Big(\alpha(\tilde{\tau})^\top X_i\Big) + \delta^\top \bar{W}_i, \\
T_i &= \Big(X_i^\top, \bar{W}_i^\top\Big)^\top, \\
\tilde{m}_{0i} &= \tilde{m}_{0i}(X_i, \bar{W}_i) = \gamma_0^\top B\Big(\alpha^\top(\tau_0) X_i\Big) + \delta_0^\top \bar{W}_i = \gamma_0^\top B(U_{\tau,0i}) + \delta_0^\top \bar{W}_i,
\end{aligned}
$$

where $U_{\tau,0i} = \alpha^\top(\tau_0)X_i$.
Thus

$$
\begin{aligned}
\tilde{m}_{0i} &= \gamma_0^\top B(U_{\tau,0i}) + \delta_0^\top \bar{W}_i = \tilde{\eta}(U_{\tau,0i}) + \delta_0^\top \bar{W}_i, \\
\delta_0^\top \bar{W}_i &= \tilde{m}_{0i} - \tilde{\eta}(U_{\tau,0i}), \\
m_0(T) &= \gamma_0^\top B\Big(\alpha^\top(\tau_0) X\Big) + \delta_0^\top \bar{W}, \\
\bar{W} &= \gamma_0^\top B(U_{\tau,0}) + \delta_0^\top \bar{W}_i \text{ where } U_{\tau,0} = \alpha^\top(\tau_0)X.
\end{aligned}
$$

So

$$
\Big(\tilde{a}^\top, \tilde{b}^\top\Big)^\top = \arg\max_{a,b} \ \tilde{l}(a,b),
$$

where

$$
\begin{aligned}
\tilde{l}(a,b) &= \frac{1}{n}\sum_{i=1}^{n} Q\bigg(g^{-1}\bigg\{\tilde{m}_{0i} + \tilde{\eta}\bigg(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\bigg) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\bigg\}, Y_i\bigg) \\
&\quad -\frac{1}{n}\sum_{i=1}^{n} Q\Big(g^{-1}\{\tilde{m}_{0i}\}, Y_i\Big) \qquad \text{(S2)}
\end{aligned}
$$

By the Taylor expansion, it exists ζ_{ni} between 0 and $\tilde{\eta}\Big(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\Big) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i$, independent of Y_i such that

$$
\begin{aligned}
&Q\bigg(g^{-1}\bigg\{\tilde{m}_{0i} + \tilde{\eta}\bigg(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\bigg) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\bigg\}, Y_i\bigg) - Q\Big(g^{-1}\{\tilde{m}_{oi}\}, Y_i\Big) \\
&\quad = q_1(m_{oi}, Y_i)\bigg[\tilde{\eta}\bigg(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\bigg) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\bigg] \\
&\qquad + \frac{1}{2}q_2(m_{oi} + \zeta_{ni}, Y_i)\bigg[\tilde{\eta}\bigg(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\bigg) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\bigg]^2,
\end{aligned}
$$

where

$$
q_1(m,y) = \frac{\partial}{\partial m}\Big[Q\Big(g^{-1}(x), y\Big)\Big] \text{ and } q_2(m,y) = \frac{\partial^2}{\partial m^2}\Big[Q\Big(g^{-1}(x), y\Big)\Big].
$$

Then

$$
\begin{aligned}
\tilde{l}(a,b) &= \sum_{i=1}^{n} q_1(m_{oi}, Y_i)\bigg[\tilde{\eta}\bigg(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\bigg) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\bigg] \\
&\quad + \frac{1}{2}\sum_{i=1}^{n} q_2(m_{oi} + \zeta_{ni}, Y_i)\bigg[\tilde{\eta}\bigg(U_{\tau,oi} + \frac{1}{\sqrt{n}}a^\top J^\top(\tau_0)X_i\bigg) - \tilde{\eta}(U_{\tau,oi}) + \frac{1}{\sqrt{n}}b^\top \bar{W}_i\bigg]^2.
\end{aligned}
$$

Applying again the Taylor's formula to the function $\tilde{\eta}(\cdot)$, we have the existence of $\overline{U}_{\tau,i}$ between $U_{\tau,0i}$ and $U_{\tau,0i} + \frac{1}{\sqrt{n}} a^\top J^\top(\tau_0) X_i$ independent of Y_i such that

$$\begin{aligned} &\tilde{\eta}\left(U_{\tau,0i} + \frac{1}{\sqrt{n}} a^\top J^\top(\tau_0) X_i\right) - \tilde{\eta}(U_{\tau,0i}) \\ = \ & \frac{1}{\sqrt{n}} \tilde{\eta}'(U_{\tau,0i}) a^\top J^\top(\tau_0) X_i + \frac{1}{2} \tilde{\eta}''(\overline{U}_{\tau,i}) \left(\frac{1}{\sqrt{n}} a^\top J^\top(\tau_0) X_i\right) \left(\frac{1}{\sqrt{n}} a^\top J^\top(\tau_0) X_i\right)^\top \\ = \ & \frac{1}{\sqrt{n}} \tilde{\eta}'(U_{\tau,0i}) a^\top J^\top(\tau_0) X_i + \frac{1}{2n} \tilde{\eta}''(\overline{U}\tau_{\tau,i}) a^\top J^\top(\tau_0) X_i X_i^\top J(\tau_0) a. \end{aligned}$$

Consequently

$$\begin{aligned} \widetilde{l}(a,b) \ = \ & \sum_{i=1}^n q_1(m_{oi}, Y_i) \left[\frac{1}{\sqrt{n}} \tilde{\eta}(U_{\tau,0i}) a^\top J^\top(\tau_0) X_i + \frac{1}{2n} \tilde{\eta}''(\overline{U}_{\tau,i}) a^\top J^\top(\tau_0) X_i X_i^\top J(\tau_0) a\right] \\ & + \frac{1}{\sqrt{n}} \sum_{i=1}^n q_1(m_{oi}, Y_i) b^\top \bar{W}_i + \frac{1}{2} \sum_{i=1}^n q_2(m_{bi} + \zeta_{ni}, Y_i) K_i, \end{aligned}$$

where

$$K_i = \left[\frac{1}{\sqrt{n}} \tilde{\eta}'(U_{\tau,\alpha}) a^\top J^\top(\tau_0) X_i + \frac{1}{2n} \tilde{\eta}''(\overline{U}_{\tau_i}) a^\top J^\top(\tau_0) X_i X_i^\top J(\tau_0) a_+ \frac{1}{\sqrt{n}} b^\top \bar{W}_i\right]^2.$$

Therefore

$$\begin{aligned} \widetilde{l}(a,b) \ = \ & \frac{1}{\sqrt{n}} \sum_{i=1}^n q_1(\tilde{m}_{oi}, Y_i) \tilde{\eta}'(U_{\tau,0i}) a^\top J^\top(\tau_0) X_i + \frac{1}{\sqrt{n}} \sum_{i=1}^n q_1(\tilde{m}_{oi}, Y_i) b^\top \bar{W}_i \\ & + \frac{1}{2n} \sum_{i=1}^n q_1(\tilde{m}_{oi}, Y_i) \tilde{\eta}''(\overline{U}_{\tau,i}) a^\top J^\top(\tau_0) X_i X_i^\top J(\tau_0) a + \frac{1}{2n} \sum_{i=1}^n q_2(\tilde{m}_{oi} + \zeta_{ni}, Y_i) M_i \\ & + \frac{1}{2n} \sum_{i=1}^n q_2(\tilde{m}_{oi} + \zeta_{ni}, Y_i) b^\top \bar{W}_i \bar{W}_i^\top b + \frac{1}{n} \sum_{i=1}^n q_2(\tilde{m}_{oi} + \zeta_{ni}, Y_i) L_i, \end{aligned}$$

where

$$\begin{aligned} M_i \ &= \ \left[\tilde{\eta}'(U_{\tau,0i}) a^\top J^\top(\tau_0) X_i + \frac{1}{2\sqrt{n}} \tilde{\eta}''(\overline{U}_{\tau,i}) a^\top J^\top(\tau_0) X_i X_i^\top J(\tau_0) a\right]^2 \\ L_i \ &= \ \left[\gamma^\top B'(U_{\tau,0i}) a^\top J^\top(\tau_0) X_i + \frac{1}{2\sqrt{n}} \gamma^\top B''(\overline{U}\tau_{i,j}) a^\top J^\top(\tau_0) X_i X_i^\top J(\tau_0) a\right] \bar{W}_i^\top b. \end{aligned}$$

Thus

$$\begin{aligned} \tilde{l}(a,b) = \ & \frac{1}{\sqrt{n}} \sum_{i=1}^n q_1(\tilde{m}_{oi}, Y_i) \tilde{\eta}'(U_{\tau,0i}) a^\top J^\top(\tau_0) X_i + \frac{1}{\sqrt{n}} \sum_{i=1}^n q_1(\tilde{m}_{oi}, Y_i) b^\top \bar{W}_i \\ & + \frac{1}{2} a^\top A_{n,11} a \{1 + O_{\mathbb{P}}(1)\} + \frac{1}{2} b^\top A_{n,22} b + a^\top A_{n,12} b \{1 + O_{\mathbb{P}}(1)\}, \end{aligned} \tag{S3}$$

with

$$\begin{aligned} A_{n,11} \ &= \ \frac{1}{n} \sum_{i=1}^n \left[q_2(\tilde{m}_{oi} + \zeta_{ni}, Y_i) \{\tilde{\eta}'(U_{\tau,0i})\}^2 + q_1(\tilde{m}_{oi}, Y_i) \tilde{\eta}''(\overline{U}_{\tau,i})\right] J^\top(\tau_0) X_i X_i^\top J(\tau_0) \\ A_{n,22} \ &= \ \frac{1}{n} \sum_{i=1}^n q_2(\tilde{m}_{oi} + \zeta_{ni}, Y_i) \bar{W}_i \bar{W}_i^\top \\ A_{n,12} \ &= \ \frac{1}{n} \sum_{i=1}^n q_2(\tilde{m}_{oi} + \zeta_{ni}, Y_i) \tilde{\eta}'(U_{\tau,0i}) J^\top(\tilde{\tau}) X_i \bar{W}_i^\top \end{aligned}$$

From [3], there exist large enough constants c_1 and c_2 such that

$$\left\| A_{n,11} - \frac{1}{n}\sum_{i=1}^{n}\Big[q_2(m_{oi},Y_i)\{\eta_0'(U_{\tau,0i})\}^2 + q_1(m_{oi},Y_i)\tilde{\eta}''(U_{\tau,0i})\Big]J^\top(\tau_0)X_iX_i^\top J(\tau_0)\right\|$$

$$= \mathrm{O}_{\mathbb{P}}\left\{\frac{1}{n}\sum_{i=1}^{n}(c_1|Y_i|+c_2)\right\} = O_{\mathbb{P}}(1).$$

Moreover, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\Big[q_2(m_{oi})\{\eta_0'(U_{\tau,0i})\}^2 + q_1(m_{oi},Y_i)\tilde{\eta}\prime\prime(U_{\tau,0i})\Big]J^\top(\tau_0)X_iX_i^\top J(\tau_0)\\
\overset{L.L.N.^1}{=}\quad & \mathbb{E}\Big[\Big[q_2(m_{0i})\{\eta_0'(U_{\tau,0i})\}^2 + q_1(m_{0i},Y_i)\tilde{\eta}''(U_{\tau,0i})\Big]J^\top(\tau_0)X_iX_i^\top J(\tau_0)\Big]\\
& +O_{\mathbb{P}}(1)\\
\overset{i.i.d^2}{=}\quad & \mathbb{E}\Big[q_2(m_0(T),Y)\{\eta_0'(U_{\tau,0})\}^2 + q_1(m_0(T),Y)\tilde{\eta}''(U_{\tau,0})\Big]J^\top(\tau_0)XX^\top J(\tau_0)]\\
& +O_{\mathbb{P}}(1)\\
=\quad & \mathbb{E}\Big[\Big[\Big\{\Big(Y-g^{-1}(m_0(T))\Big)\rho_1'(m_0(T)) - \rho_2(m_0(T))\Big\}\{\eta_0'(U_{\tau,0})\}^2\\
& +\Big(Y-g^{-1}(m_0(T))\Big)\rho_1(m_0(T))\tilde{\eta}''(U_{\tau,0})\Big]J^\top(\tau_0)XX^\top J(\tau_0)\Big] + O_{\mathbb{P}}(1)\\
=\quad & -\mathbb{E}\Big[\rho_2(m_0(T))\{\eta_0'(U_{\tau,0})\}^2 J^\top(\tau_0)XX^\top J(\tau_0)\Big] + O_{\mathbb{P}}(1).
\end{aligned}$$

By the same arguments, we get

$$\left\| A_{n,22} - \frac{1}{n}\sum_{i=1}^{n} q_2(m_{oi},Y_i)\bar{W}_i\bar{W}_i^\top \right\| = O_{\mathbb{P}}\left\{\frac{1}{n}\sum_{i=1}^{n}(c_1|Y_i|+c_2)\right\} = O_{\mathbb{P}}(1)$$

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} q_2(m_{oi},Y_i)\bar{W}_i\bar{W}_i^\top \overset{L.L.N.}{=}\ & \mathbb{E}\Big[q_2(m_{oi},Y_i)\bar{W}_i\bar{W}_i^\top\Big] + O_{\mathbb{P}}(1)\\
\overset{i.i.d.}{=}\ & \mathbb{E}\Big[q_2(m_o(T),Y)\bar{W}\bar{W}^\top\Big] + O_{\mathbb{P}}(1)\\
=\ & \mathbb{E}\Big[\Big\{\Big(Y-g^{-1}(m_o(T))\Big)\rho_1'(m_o(T)) - \rho_2(m_o(T))\Big\}\bar{W}\bar{W}^\top\Big] + O_{\mathbb{P}}(1)\\
=\ & -\mathbb{E}\Big[\rho_2(m_o(T))\bar{W}\bar{W}^\top\Big] + O_{\mathbb{P}}(1).
\end{aligned}$$

By De Boor [4], $\|\tilde{\eta}' - \eta_0'\|_\infty = O(h^{p-1})$, so

$$\left\| A_{n,12} - \frac{1}{n}\sum_{i=1}^{n} q_2(m_{oi},Y_i)\eta_0'(U_{\tau,0i})J^\top(\tau_0)X_i\bar{W}_i^\top \right\| = O_{\mathbb{P}}\left\{\frac{1}{n}\sum_{i=1}^{n}(c_1|Y_i|+c_2)\right\} = O_{\mathbb{P}}(1)$$

and, from the law of large numbers, we have

$$\frac{1}{n}\sum_{i=1}^{n} q_2(m_{oi},Y_i)\eta_0'(U_{\tau,0i})J^\top(\tau_0)X_i\bar{W}_i^\top = \mathbb{E}\Big[q_2(m_{oi},Y_i)\eta_0'(U_{\tau,0i})J^\top(\tau_0)X_i\bar{W}_i^\top\Big] + O_{\mathbb{P}}(1).$$

Then, we obtain

$$\begin{aligned}A_{n,11} &= -\mathbb{E}\Big[\rho_2(m_o(T))\{\eta_0'(U_{\tau,0})\}^2 J^\top(\tau_0)X\bar{W}^\top J(\tau_0)\Big] + op(1) = -A_{11} + O_{\mathbb{P}}(1)\\ A_{n,22} &= -\mathbb{E}\Big[\rho_2(m_o(T))\bar{W}\bar{W}^{\mathrm{T}}\Big] + O_{\mathbb{P}}(1) = -A_{22} + O_{\mathbb{P}}(1)\\ A_{n,12} &= -\mathbb{E}\Big[\rho_2(m_o(T))\eta_0'(U_{\tau,0})J^\top(\tau_0)X\bar{W}^\top\Big] + O_{\mathbb{P}}(1) = -A_{12} + O_{\mathbb{P}}(1),\end{aligned}$$

with

$$\begin{aligned}A_{11} &= \mathbb{E}\Big[\rho_2(m_o(T))\{\eta_0'(U_{\tau,0})\}^2 J^\top(\tau_0)X\bar{W}^\top J(\tau_0)\Big]\\ A_{22} &= \mathbb{E}\Big[\rho_2(m_0(T))\bar{W}\bar{W}^\top\Big]\\ A_{12} &= \mathbb{E}\Big[\rho_2(m_o(T))\eta_0'(U_{\tau,0})J^\top(\tau_0)X\bar{W}^\top\Big].\end{aligned}$$

We deduce that

$$\begin{aligned}\tilde{l}(a,b) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n q_1(m_{oi},Y_i)\eta_0'(U_{\tau,0i})a^\top J^\top(\tau_0)X_i + \frac{1}{\sqrt{n}}\sum_{i=1}^n q_1(m_{oi},Y_i)\, b^\top\bar{W}_i\\ &\quad + \frac{1}{2}a^\top A_{11}a\{1+O_{\mathbb{P}}(1)\} + \frac{1}{2}b^\top A_{22}b + a^\top A_{12}b\{1+O_{\mathbb{P}}(1)\}\\ &= \begin{pmatrix} a\\ b\end{pmatrix}^\top \frac{1}{\sqrt{n}}\sum_{i=1}^n q_1(m_{oi},Y_i)\begin{pmatrix}\eta_0' U_{\tau,0i})J^\top(\tau_0)X_i\\ \bar{W}_i\end{pmatrix} + \frac{1}{2}\Big(a^\top,b^\top\Big)A\begin{pmatrix} a\\ b\end{pmatrix} + O_{\mathbb{P}}(1).\end{aligned}$$

From the Pollard's convexity lemma [5], we have

$$\begin{pmatrix}\tilde{a}\\ \tilde{b}\end{pmatrix} = \frac{1}{\sqrt{n}}A^{-1}\sum_{i=1}^n q_1(m_{oi},Y_i)\begin{pmatrix}\eta_0'(U_{\tau,0i})J^\top(\tau_0)X_i\\ \bar{W}_i\end{pmatrix} + O_{\mathbb{P}}(1), \tag{S4}$$

where $A = \begin{pmatrix} A_{11} & A_{12}\\ A_{12}^\top & A_{22}\end{pmatrix}$.
Elsewhere, we have

$$\begin{aligned}&\mathrm{Var}\Big[q_1(m_0(T),Y)\begin{pmatrix}\eta_0'(U_{\tau,0})J^\top(\tau_0)X\\ \bar{W}\end{pmatrix}\Big]\\ &= \mathbb{E}\Big[q_1^2(m_0(T),Y)\begin{pmatrix}\eta_0'(U_{\tau,0})J^\top(\tau_0)X\\ \bar{W}\end{pmatrix}\begin{pmatrix}\eta_0'(U_{\tau,0})J^\top(\tau_0)X\\ \bar{W}\end{pmatrix}^\top\Big]\\ &:= \Sigma_1.\end{aligned} \tag{S5}$$

Then by applying the δ-method, we obtain

$$\sqrt{n}\begin{pmatrix}\alpha(\tilde{\tau}) - \alpha(\tau_0)\\ \tilde{\delta} - \delta_0\end{pmatrix} \xrightarrow{D} \mathcal{N}\Big(0, R(\tau_0)A^{-1}\Sigma_1 A^{-1}R^\top(\tau_0)\Big).$$

Thus $\alpha(\tilde{\tau}) - \alpha(\tau_0) = O_{\mathbb{P}}(1/\sqrt{n})$ and $\tilde{\delta} - \delta_0 = O_{\mathbb{P}}(1/\sqrt{n})$.
Proof of Lemma 3
We set

$$\begin{aligned}&\tilde{\theta} = \Big(\tilde{\tau}^\top,\tilde{\beta}^\top,\tilde{\gamma}^\top\Big)^\top, \widehat{\theta} = \Big(\widehat{\tau}^\top,\widehat{\beta}^\top,\widehat{\gamma}^\top\Big)^\top, T_i = \Big(X_i^\top,\bar{W}_i^\top\Big)^\top,\\ &\tilde{m}_i = \tilde{\eta}\Big(\alpha^\top(\tilde{\tau})X_i\Big) + \tilde{\delta}^\top\bar{W}_i = \tilde{\gamma}^\top B\Big(\alpha^\top(\tilde{\tau})X_i\Big) + \tilde{\delta}^\top\bar{W}_i\\ \text{and}\quad &m_{0i} = m(T_i) = \eta_0\Big(\alpha^\top(\tau_0)X_i\Big) + \tilde{\delta}^\top\bar{W}_i = \eta_0(U_{\tau,0i}) + \tilde{\delta}^\top\bar{W}_i,\end{aligned}$$

with $U_{\tau,0i} = \alpha^\top(\tau_0)X_i$. Thus

$$\tilde{m}_i - m_{0i} = \left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \eta_0\left(\alpha^\top(\tau_0)X_i\right)\right\} + \bar{W}_i^\top(\delta - \delta_0). \tag{S6}$$

There exists ξ_i between $\tilde{m}_i$ and m_{0i} such that

$$\begin{aligned} q_1(\tilde{m}_i, Y_i) &= q_1(m_{0i}, Y_i) + q_2(\xi_i, Y_i)(\tilde{m}_i - m_{0i}) \\ &= q_1(m_{0i}, Y_i) + q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \eta_0\left(\alpha^\top(\tau_0)X_i\right)\right\} \\ &\quad + q_2(\xi_i, Y_i)\bar{W}_i^\top(\delta - \delta_0), \text{ from (S6)}. \end{aligned} \tag{S7}$$

There exists $\overline{\theta}$ between $\widehat{\theta}$ and $\tilde{\theta}$ such that

$$0 = \frac{\partial l}{\partial \theta}(\widehat{\theta}) = \frac{\partial l}{\partial \theta}(\tilde{\theta}) + \frac{\partial^2 l}{\partial \theta^\top \partial \theta}(\overline{\theta})(\widehat{\theta} - \tilde{\theta}),$$

thus $\widehat{\theta} - \tilde{\theta} = -\left\{\frac{\partial^2 l}{\partial \theta^\top \partial \theta}(\overline{\theta})\right\}^{-1}\frac{\partial l}{\partial \theta}(\tilde{\theta})$.
We know that

$$\begin{aligned} S(\tilde{\theta}) &= \frac{\partial l}{\partial \theta}(\tilde{\theta}) \\ &= \frac{1}{n}\sum_{i=1}^n q_1(m_i, Y_i)\xi_i(\tau, \gamma, \delta) \\ &= \frac{1}{n}\sum_{i=1}^n q_1(m_i, Y_i)\begin{pmatrix} \gamma^\top B'(\alpha^\top(\tau)X_i)J^\top(\tau)X_i \\ B(\alpha^\top(\tau)X_i) \\ \bar{W}_i \end{pmatrix}. \end{aligned} \tag{S8}$$

Then, from (S7), we obtain successively

$$\begin{aligned} \frac{\partial l}{\partial \gamma}(\tilde{\theta}) &= \frac{1}{n}\sum_{i=1}^n q_1(\tilde{m}_i, Y_i)B\left(\alpha^\top(\tilde{\tau})X_i\right) \\ &= \frac{1}{n}\sum_{i=1}^n q_1(m_{0i}, Y_i)B\left(\alpha^\top(\tilde{\tau})X_i\right) \\ &\quad + \frac{1}{n}\sum_{i=1}^n q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \eta_0\left(\alpha^\top(\tilde{\tau})X_i\right)\right\}B\left(\alpha^\top(\tilde{\tau})X_i\right) \\ &\quad + \frac{1}{n}\sum_{i=1}^n q_2(\xi_i, Y_i)\bar{W}_i^\top(\tilde{\delta} - \delta_0)B\left(\alpha^\top(\tilde{\tau})X_i\right). \end{aligned} \tag{S9}$$

For the first sum in (S9), we obtain

$$\begin{aligned} \frac{1}{n}\left|\sum_{i=1}^n q_1(m_{0i}, Y_i)B\left(\alpha^\top(\tilde{\tau})X_i\right)\right| &\leq \frac{1}{n}\left|\sum_{i=1}^n q_1(m_{0i}, Y_i)B\left(\alpha^\top(\tau_0)X_i\right)\right| \\ &\quad + \frac{1}{n}\left|\sum_{i=1}^n q_1(m_{0i}, Y_i)\left\{B\left(\alpha^\top(\tilde{\tau})X_i\right) - B\left(\alpha^\top(\tau_0)X_i\right)\right\}\right|. \end{aligned}$$

But, since $\alpha(\tilde{\tau}) - \alpha(\tau_0) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$, we have

$$\begin{aligned} B\left(\alpha^\top(\tilde{\tau})X_i\right) - B\left(\alpha^\top(\tau_0)X_i\right) &= B'\left(\alpha^\top(\tau_0)X_i\right)J^\top(\tau_0)X_i\{\alpha(\tilde{\tau}) - \alpha(\tau_0)\}^\top X_i \\ &\quad + \{\alpha(\tilde{\tau}) - \alpha(\tau_0)\}^\top X_i\, o(1) \\ &= B'\left(\alpha^\top(\tau_0)X_i\right)J^\top(\tau_0)X_i + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

We know that

$$q_1(m_{0i}, Y_i) = \Big(Y_i - g^{-1}(m_{0i})\Big)\rho_1(m_{0i}) = \rho_1(m_{0i})\varepsilon_i. \tag{S10}$$

From the Bernstein's inequality, we have

$$\frac{1}{n}\left|\sum_{i=1}^{n}\rho_1(m_{0i})\varepsilon_i B\Big(\alpha^\top(\tau_0)X_i\Big)\right| = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{n}}\right).$$

Since $\frac{1}{n}\left|\sum_{i=1}^{n} q_1(m_{0i}, Y_i)X_i^\top\right| \stackrel{L.L.N.}{=} O_{\mathbb{P}}(1)$, $\alpha(\tilde{\tau}) - \alpha(\tau_0) = O_{\mathbb{P}}\Big(\frac{1}{\sqrt{n}}\Big)$ and $B'\big(\alpha^\top(\tau_0)X_i\big) = O(1/h)$ and from the weak law of large numbers we obtain

$$\begin{aligned}
&\frac{1}{n}\left|\sum_{i=1}^{n} q_1(m_{0i}, Y_i)B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right| \\
&\quad\leq \frac{1}{n}\left|\sum_{i=1}^{n}\rho_1(m_{0i})\varepsilon_i B\Big(\alpha^\top(\tau_0)X_i\Big)\right| \\
&\qquad+\frac{1}{n}\left|\sum_{i=1}^{n} q_1(m_{0i}, Y_i)B'\Big(\alpha^\top(\tau_0)X_i\Big)J^\top(\tau_0)X_iX_i^\top\{\alpha(\tilde{\tau}) - \alpha(\tau_0)\}\right| + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&\quad= O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}\right) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&\quad= O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}\right).
\end{aligned}$$

Then

$$\frac{1}{n}\left|\sum_{i=1}^{n} q_1(m_{0i}, Y_i)B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right| = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}\right). \tag{S11}$$

For the third term in (S9), since $\tilde{\delta} - \delta_0 = O_{\mathbb{P}}(1/\sqrt{n})$ and from the weak law of large numbers, we obtain

$$\begin{aligned}
\frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)\bar{W}_i^\top(\tilde{\delta} - \delta_0)B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right| &= \frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)B\Big(\alpha^\top(\tilde{\tau})X_i\Big)W_i^\top(\tilde{\delta} - \delta_0)\right| \\
&= O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

For the second term in (S9), we have

$$\begin{aligned}
&\frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)\Big\{\tilde{\eta}\Big(\alpha^\top(\tilde{\tau})X_i\Big) - \eta_0\Big(\alpha^\top(\tau_0)X_i\Big)\Big\}B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right| \\
&\quad\leq \frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)\Big\{\tilde{\eta}\Big(\alpha^\top(\tilde{\tau})X_i\Big) - \tilde{\eta}\Big(\alpha^\top(\tau_0)X_i\Big)\Big\}B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right| \\
&\qquad+\frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)\Big\{\tilde{\eta}\Big(\alpha^\top(\tau_0)X_i\Big) - \eta_0\Big(\alpha^\top(\tau_0)X_i\Big)\Big\}B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right| \\
&\quad\leq \frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)\tilde{\gamma}^\top\Big\{B\Big(\alpha^\top(\tilde{\tau})X_i\Big) - B\Big(\alpha^\top(\tau_0)X_i\Big)\Big\}B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right| \qquad \text{(S12)} \\
&\qquad+\frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)B\Big(\alpha^\top(\tilde{\tau})X_i\Big)\right|\|\tilde{\eta} - \eta_0\|_\infty. \qquad \text{(S13)}
\end{aligned}$$

Since $\|\tilde{\eta} - \eta_0\|_\infty = O\left(h^d\right)$ and from the weak law of large numbers, we obtain (S13) $= O_{\mathbb{P}}(h^p)$.
As previously, we have $\alpha(\tilde{\tau}) - \alpha(\tau_0) = O_{\mathbb{P}}(1/\sqrt{n})$, and $\tilde{\gamma}^\top B'(\alpha(\tau_0)X_i) = O\left(h^{-1}\right)$. Then, from the weak law of large numbers, we obtain

$$\begin{aligned}
(S12) &= \frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i, Y_i)\tilde{\gamma}^\top \left\{B\left(\alpha^\top(\tilde{\tau})X_i\right) - B\left(\delta^\top(\tau_0)X_i\right)\right\}B\left(\delta^\top(\tilde{\tau})X_i\right)\right| \\
&\quad \frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i, Y_i)\tilde{\gamma}^\top B'\left(\alpha^\top(\tau_0)X_i\right)J^\top(\tau_0)X_i^\top\{\alpha(\tilde{\tau}) - \alpha(\tau_0)\}B\left(\alpha^\top(\tilde{\tau})X_i\right)\right| \\
&\quad + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}\right).
\end{aligned} \tag{S14}$$

We deduce that

$$\frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \eta_0\left(\alpha^\top(\tau_0)X_i\right)\right\}B\left(\alpha^\top(\tilde{\tau})X_i\right)\right| = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}} + h^p\right). \tag{S15}$$

Then

$$\left\|\frac{\partial l}{\partial \gamma}(\tilde{\theta})\right\|_\infty = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}} + h^p\right). \tag{S16}$$

Elsewhere, since (S7) we have

$$\begin{aligned}
\frac{\partial l}{\partial \tau}(\tilde{\theta}) &= \frac{1}{n}\sum_{i=1}^n q_1(\tilde{m}_i, Y_i)\tilde{\gamma}^\top B_1'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\
&= \frac{1}{n}\sum_{i=1}^n q_1(\tilde{m}_i, Y_i)\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\
&= \frac{1}{n}\sum_{i=1}^n q_1(m_{0i}, Y_i)\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\
&\quad + \frac{1}{n}\sum_{i=1}^n q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \eta_0(\alpha(\tau_0)X_i)\right\}\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\
&\quad + \frac{1}{n}\sum_{i=1}^n q_2(\xi_i, Y_i)\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i\bar{W}_i^\top(\delta - \delta_0).
\end{aligned} \tag{S17}$$

For the first term in (S17): since $\|\tilde{\eta} - \eta_0\|_\infty = O(h^p)$, $\|\tilde{\eta}' - \eta_0'\|_\infty = O\left(h^{p-1}\right)$, $\eta_0 \in S_n$ and $\eta \in \mathcal{H}(p)$, we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n q_1(m_{0i}, Y_i)\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\
&= \frac{1}{n}\sum_{i=1}^n q_1(m_{0i}, Y_i)\eta_0'\left(\alpha^\top(\tau_0)X_i\right)J^\top(\tau_0)X_i \\
&\quad + \frac{1}{n}\sum_{i=1}^n q_1(m_{0i}, Y_i)\left\{\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tilde{\tau}) - \tilde{\eta}'\left(\alpha^\top(\tau_0)X_i\right)J^\top(\tau_0)\right\}X_i \\
&\quad + \frac{1}{n}\sum_{i=1}^n q_1(m_{0i}, Y_i)\left\{\tilde{\eta}'\left(\alpha^\top(\tau_0)X_i\right) - \eta_0'\left(\alpha^\top(\tau_0)X_i\right)\right\}J^\top(\tau_0)X_i \\
&= \frac{1}{n}\sum_{i=1}^n q_1(m_{0i}, Y_i)\eta_0'\left(\alpha^\top(\tau_0)X_i\right)J^\top(\tau_0)X_i\{1 + O_{\mathbb{P}}(1)\}.
\end{aligned}$$

On the other hand $q_1(m_{0i}, Y_i) = (Y_i - g^{-1}(m_{0i}))\rho_1(m_{0i}) = \rho_1(m_{0i})\varepsilon_i$, with $\varepsilon_i = Y_i - g^{-1}(m_{0i})$ is the ith residue.
From the Bernstein's inequality, we have

$$\begin{aligned}\frac{1}{n}\left|\sum_{i=1}^{n} q_1(m_{0i}, Y_i)\eta_0'\left(\alpha^\top(\tau_0)X_i\right)J^\top(\tau_0)X_i\right| &= \frac{1}{n}\left|\sum_{i=1}^{n}\rho_1(m_{0i})\varepsilon_i\eta_0'\left(\delta^\top(\tau_0)X_i\right)J^\top(\tau_0)X_i\right| \\ &= O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{n}}\right).\end{aligned}$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n} q_1(m_{0i}, Y_i)\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{n}}\right). \tag{S18}$$

For the second term in (S17), since $\|\tilde{\eta} - \eta_0\|_\infty = O(h^p)$, $\alpha(\tilde{\tau}) - \alpha(\tau_0) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$, $\tilde{\eta} \in \mathcal{H}(p)$ and $\left|\tilde{\eta}(\alpha^\top(\tilde{\tau})X_i) - \tilde{\eta}(\alpha^\top(\tau_0)X_i) \le C\left|X_i^\top\{\alpha(\tilde{\tau}) - \alpha(\tau_0)\}\right|\right.$, we get

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n} q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \eta_0\left(\delta^\top(\tau_0)X_i\right)\right\}\tilde{\eta}'\left(\delta^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\ &= \frac{1}{n}\sum_{i=1}^{n} q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \tilde{\eta}\left(\alpha^\top(\tau_0)X_i\right)\right\}\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\ &\quad + \frac{1}{n}\sum_{i=1}^{n} q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tau_0)X_i\right) - \eta_0\left(\alpha^\top(\tau_0)X_i\right)\right\}\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i \\ &= O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} + h^p\right).\end{aligned}$$

For the third term in (S17), we have $\left|\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)\right| = \left|\tilde{\gamma}^\top B'\left(\alpha^\top(\tilde{\tau})X_i\right)\right| = O(1/h)$.
Since $\left\|B'\left(\alpha^\top(\tilde{\tau})X_i\right)\right\| = O(1/h)$ and $\tilde{\delta} - \delta_0 = O_{\mathbb{P}}(1/\sqrt{n})$, we obtain

$$\frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)\tilde{\eta}'\left(\alpha^\top(\tilde{\tau})X_i\right)J^\top(\tau_0)X_i\bar{W}_i^\top(\tilde{\delta} - \delta_0)\right| = O_{\mathbb{P}}\left(\frac{1}{h\sqrt{n}}\right). \tag{S19}$$

Thus

$$\left\|\frac{\partial l}{\partial \tau}(\tilde{\theta})\right\|_\infty = O_{\mathbb{P}}\left(\frac{1}{h\sqrt{n}} + h^p\right) \tag{S20}$$

$$\begin{aligned}\frac{\partial l}{\partial \delta}(\tilde{\theta}) &= \frac{1}{n}\sum_{i=1}^{n} q_1(\tilde{m}_i, Y_i)\bar{W}_i \\ &= \frac{1}{n}\sum_{i=1}^{n} q_1(m_{0i}, Y_i)\bar{W}_i + \frac{1}{n}\sum_{i=1}^{n} q_2(\xi_i, Y_i)(\tilde{\delta} - \delta_0)\bar{W}_i^\top\bar{W}_i \\ &\quad + \frac{1}{n}\sum_{i=1}^{n} q_2(\xi_i, Y_i)\left\{\tilde{\eta}\left(\alpha^\top(\tilde{\tau})X_i\right) - \eta_0\left(\alpha^\top(\tau_0)X_i\right)\right\}\bar{W}_i.\end{aligned} \tag{S21}$$

For the first term in (S21), according to the Bernstein's inequality, we obtain

$$\frac{1}{n}\left|\sum_{i=1}^{n} q_1(m_{0i}, Y_i)W_i\right| = \frac{1}{n}\left|\sum_{i=1}^{n}\rho_1(m_{0i})\varepsilon_i\bar{W}_i\right| = O_{\mathbb{P}}\left(\sqrt{\frac{\log n}{n}}\right). \tag{S22}$$

Since $\tilde{\delta} - \delta_0 = O_{\mathbb{P}}(1/\sqrt{n})$ and from the law of large numbers, we get

$$\frac{1}{n}\left|\sum_{i=1}^{n} q_2(\xi_i, Y_i)(\tilde{\delta} - \delta_0)\bar{W}_i^\top\bar{W}_i\right| = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \tag{S23}$$

From the law of large numbers, we obtain, $\|\tilde{\eta}-\eta_0\|_\infty = O(h^p)$, $\alpha(\tilde{\tau})-\alpha(\tau_0) = O_{\mathbb{P}}(1/\sqrt{n})$ and $|\tilde{\eta}'(\alpha^\top(\tilde{\tau})X_i) = |\tilde{\gamma}^\top B'(\alpha^\top(\tilde{\tau})X_i) = O(1/h)$, we have

$$\begin{aligned}
&\frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i,Y_i)\Big\{\tilde{\eta}\Big(\alpha^\top(\tilde{\tau})X_i\Big)-\eta_0\Big(\alpha^\top(\tau_0)X_i\Big)\Big\}\bar{W}_i\right| \\
&= \frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i,Y_i)\Big\{\tilde{\gamma}^\top B\Big(\alpha^\top(\tilde{\tau})X_i\Big)-\eta_0\Big(\alpha^\top(\tau_0)X_i\Big)\Big\}\bar{W}_i\right| \\
&\le \frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i,Y_i)\tilde{\gamma}^\top\Big\{B\Big(\alpha^\top(\tilde{\tau})X_i\Big)-B\Big(\alpha^\top(\tau_0)X_i\Big)\Big\}\bar{W}_i\right| \\
&\quad +\frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i,Y_i)\bar{W}_i\right|\|\tilde{\eta}-\eta_0\|_\infty \\
&= \frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i,Y_i)\tilde{\gamma}^\top B'\Big(\alpha^\top(\tau_0)X_i\Big)\{\alpha(\tilde{\tau})-\alpha(\tau_0)\}^\top J^\top(\tau_0)X_i\bar{W}_i\right| \\
&\quad +\frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i,Y_i)\bar{W}_i\right|\|\tilde{\eta}-\eta_0\|_\infty + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{n}\left|\sum_{i=1}^n q_2(\xi_i,Y_i)\tilde{\gamma}^\top B'\Big(\delta^\top(\tau_0)W_i\Big)\{\delta(\tilde{\tau})-\delta(\tau_0)\}^\top J^\top(\tau_0)X_i\bar{W}_i\right| \\
&\quad +O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}+h^p\right) \\
&= O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}\right)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}+h^p\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}+h^p\right).
\end{aligned} \tag{S24}$$

From (S22),(S23) and (S24), we deduce

$$\left\|\frac{\partial l}{\partial \delta}(\tilde{\theta})\right\|_\infty = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}+h^p\right). \tag{S25}$$

From (S16),(S20) and (S25), we deduce

$$\left\|\frac{\partial l}{\partial \theta}(\tilde{\theta})\right\|_\infty = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}}+h^p\right). \tag{S26}$$

For the second order derivative, from (S1), one has

$$\begin{aligned}
V_{n,\theta} = &\frac{1}{n}\sum_{i=1}^n q_2(m_i,Y_i)\xi_i(\tau,\gamma,\delta)\xi_i^\top(\tau,\gamma,\delta) \\
&+\frac{1}{n}\sum_{i=1}^n q_1(m_i,Y_i)\begin{pmatrix} \gamma^\top B''(\alpha^\top(\tau)X_i)J^\top(\tau)X_iJ^\top(\tau)X_i & B'(\alpha^\top(\tau)X_i)J^\top(\tau)X_i & 0 \\ B'(\alpha^\top(\tau)X_i)J^\top(\tau)X_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+\frac{1}{n}\sum_{i=1}^n q_1(m_i,Y_i)\begin{pmatrix} \gamma^\top B'(\alpha^\top(\tau)X_i)\frac{\partial}{\partial\tau}\{J^\top(\tau)\}X_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

so, by posing $D_{n,\theta} = \begin{pmatrix} \gamma^\top B'\big(\alpha^\top(\tilde{\tau})X_i\big)J^\top(\tau) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & B\big(\alpha^\top(\tilde{\tau})X_i\big) \end{pmatrix}$,

we get $D_{n,\theta}\begin{pmatrix} T_i \\ 1 \end{pmatrix}\begin{pmatrix} T_i \\ 1 \end{pmatrix}^\top D_{n,\theta}^\top = \xi_i(\tau,\beta,\gamma)\xi_i^\top(\tau,\beta,\gamma)$. As a result

$$\begin{aligned} V_{n,\theta} \; &= \frac{1}{n}\sum_{i=1}^n q_2(m_i,Y_i)D_{n,\theta}\begin{pmatrix} T_i \\ 1 \end{pmatrix}\begin{pmatrix} T_i \\ 1 \end{pmatrix}^\top D_{n,\theta}^\top \\ &+\frac{1}{n}\sum_{i=1}^n q_1(m_i,Y_i)\begin{pmatrix} \gamma^\top B''\big(\alpha^\top(\tau)X_i\big)J^\top(\tau)X_iJ^\top(\tau)X_i & B'\big(\alpha^\top(\tau)X_i\big)J^\top(\tau)X_i & 0 \\ B'\big(\alpha^\top(\tau)X_i\big)J^\top(\tau)X_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+\frac{1}{n}\sum_{i=1}^n q_1(m_i,Y_i)\begin{pmatrix} \gamma^\top B'\big(\alpha^\top(\tau)X_i\big)\frac{\partial}{\partial\tau}\big\{J^\top(\tau)\big\}X_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

According to Lemma **A1** and the condition (*C*2), we have

$$\sup_\theta \left\| V_{n,\theta}^{-1} \right\|_2 = O(\sqrt{N_n}) \text{ almost-surely}$$

and

$$\left\| \frac{\partial l}{\partial \theta}(\tilde{\theta}) \right\|_2 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh}} + h^p \right).$$

Then

$$\|\widehat{\theta}-\tilde{\theta}\| = \left\| -\left\{ \frac{\partial^2 l}{\partial\rho^\top\partial\rho}(\tilde{\theta}) \right\}^{-1} \frac{\partial l}{\partial\rho}(\tilde{\theta}) \right\| \le \sup_\theta \left\| V_{n,\theta}^{-1} \right\|_2 \left\| \frac{\partial l}{\partial\theta}(\tilde{\theta}) \right\|_2 = O_{\mathbb{P}}\left\{ \sqrt{N_n}\left(\frac{1}{\sqrt{nh}} + h^p \right) \right\}.$$

Proof of Theorem 2
By De Boor [4], we know that $\left\|\tilde{R}-R\right\|_2 = O(h)$ and, from Lemma **A1**, we get

$$\begin{aligned} \|\widehat{R}(t)-\tilde{R}(t)\|_2^2 \; &= \; \left\| (\widehat{\delta}-\tilde{\delta})^\top B_1(t) \right\|_2^2 \\ &= \; (\widehat{\delta}-\tilde{\delta})^\top B_1(t)B_1{}^\top(t)(\widehat{\delta}-\tilde{\delta}) \\ &= \; (\widehat{\delta}-\tilde{\delta})^\top \left\{ \frac{1}{n}\sum_{i=1}^n B_1(t)B_1{}^\top(t) \right\}(\widehat{\delta}-\tilde{\delta}) \\ &= \; C\left\| (\widehat{\delta}-\tilde{\delta})^\top \right\|_2^2, \end{aligned}$$

and from Lemma 3, we obtain $\|\widehat{R}(t)-\tilde{R}(t)\|_2 = O_{\mathbb{P}}\left\{\sqrt{N_n}\left(1/\sqrt{nh}+h^p\right)\right\}$. Then

$$\begin{aligned} \left\|\widehat{R}-R\right\|_2 \; &\le \; \|\widehat{R}(t)-\tilde{R}(t)\|_2 + \left\|\tilde{R}-R\right\|_2 \\ &= \; O_{\mathbb{P}}\left\{ \sqrt{N_n}\left(\frac{1}{\sqrt{nh}} + h^p \right) \right\} + O(h) = O_{\mathbb{P}}\left\{ \sqrt{N_n}\left(\frac{1}{\sqrt{nh}} + h^p \right) \right\}. \end{aligned}$$

Proof of Theorem 3
From Lemma **A1**, we have

$$\begin{aligned} \|\widehat{\eta}(t)-\tilde{\eta}(t)\|_2^2 \; &= \; \left\| (\widehat{\gamma}-\tilde{\gamma})^\top B(t) \right\|_2^2 = (\widehat{\gamma}-\tilde{\gamma})^\top B(t)B^\top(t)(\widehat{\gamma}-\tilde{\gamma}) \\ &= \; (\widehat{\gamma}-\tilde{\gamma})^\top \left\{ \frac{1}{n}\sum_{i=1}^n B(t)B^\top(t) \right\}(\widehat{\gamma}-\tilde{\gamma}) = C\left\| (\widehat{\gamma}-\tilde{\gamma})^\top \right\|_2^2, \end{aligned}$$

and from Lemma 3, we have $\|\widehat{\eta}(t)-\tilde{\eta}(t)\|_2 = O_{\mathbb{P}}\Big\{\sqrt{N_n}\Big(\frac{1}{\sqrt{nh}}+h^p\Big)\Big\}$. Then

$$\begin{aligned}\|\widehat{\eta}-\eta_0\|_2 &\leq \|\widehat{\eta}(t)-\tilde{\eta}(t)\|_2+\|\tilde{\eta}-\eta_0\|_2\\ &= O_{\mathbb{P}}\left\{\sqrt{N_n}\left(\frac{1}{\sqrt{nh}}+h^p\right)\right\}+O_{\mathbb{P}}\left(h^d\right)\\ &= O_{\mathbb{P}}\left\{\sqrt{N_n}\left(\frac{1}{\sqrt{nh}}+h^p\right)\right\}.\end{aligned}$$

On the other hand, from Lemma 3, we have

$$\sup_{\eta_1,\eta_2\in S_n}\left|\frac{\langle\eta_1,\eta_2\rangle_n-\langle\eta_1,\eta_2\rangle}{\|\eta_1\|_2\|\eta_2\|_2}\right| = O_{a.s.}\left\{\sqrt{\frac{\log n}{nh}}\right\}.$$

Thus

$$\|\widehat{\eta}-\tilde{\eta}\|_n = O_{\mathbb{P}}\left\{\sqrt{N_n}\left(\frac{1}{\sqrt{nh}}+h^p\right)\right\}.$$

Finally

$$\begin{aligned}\|\widehat{\eta}-\eta_0\|_n \leq \|\widehat{\eta}-\tilde{\eta}\|_n+\|\tilde{\eta}-\eta_0\|_n &= O_{\mathbb{P}}\left\{\sqrt{N_n}\left(\frac{1}{\sqrt{nh}}+h^p\right)\right\}+O_{\mathbb{P}}(h^p)\\ &= O_{\mathbb{P}}\left\{\sqrt{N_n}\left(\frac{1}{\sqrt{nh}}+h^p\right).\right.\end{aligned}$$

Proof of Lemma A2
Note that according to the condition $(C8)$, ρ_2 is a fixed bounded function.
According to Lemma 2, the logarithm of the ϵ-bracketing of the class of functions $\mathcal{A}_2(\delta) := \{\rho_2(m(t)) : m\in\mathcal{M}_n, \|m-m_0\|\leq\delta\}$ is $c\{N_n\log(\delta/\epsilon)+\log(1/\delta)\}$, so the corresponding entropy integral is given by

$$\begin{aligned}J_{[\,]}(\epsilon,\mathcal{A}_2(\delta),\|\|) &= \int_0^\delta\left\{1+\log N_{[\,]}(\epsilon,\mathcal{A}_2(\delta),\|\|)\right\}^{\frac{1}{2}}d\epsilon\\ &\leq \int_0^\delta\left\{1+cN_n\log\left(\frac{\delta}{\epsilon}\right)\right\}^{\frac{1}{2}}d\varepsilon \leq c\,\delta\left[\sqrt{N_n}+\sqrt{\log\left(\frac{1}{\delta}\right)}\right].\end{aligned}$$

According to Lemma 7 of [6] and Theorem 3,

$$\|\hat{\eta}-\eta_0\|_\infty \leq c\sqrt{N_n}\|\hat{\eta}-\eta_0\| = O_{\mathbb{P}}\left\{N_n\left(h^p+\frac{1}{\sqrt{nh}}\right)\right\}. \tag{S27}$$

By the Lemma 1 and Theorem 3, one has

$$\begin{aligned}&\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n\rho_2(m_{oi})\{\hat{\eta}(U_{\tau,oi})-\eta_0(U_{\tau,oi})\}\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\right.\\ &\qquad\left.-\mathbb{E}\Big[\rho_2(m_o)\{\hat{\eta}(U_{\tau,o})-\eta_0(U_{\tau,o})\}\eta_0'(U_{\tau,o})J^\top(\tau_0)\Phi(X)\Big]\right|\\ &\qquad\leq \frac{c_0}{\sqrt{n}}\times O_{\mathbb{P}}\left\{N_n\left(h^p+\frac{1}{\sqrt{nh}}\right)\right\}\times\left\{\delta\left(\sqrt{N_n}+\sqrt{\log\left(\frac{1}{\delta}\right)}\right)\right\}.\end{aligned}$$

By conditions $(C1)-(C5)$, we obtain

$$O_{\mathbb{P}}\left\{N_n\left(h^p+\frac{1}{\sqrt{nh}}\right)\right\}\times\left\{\delta\left(\sqrt{N_n}+\sqrt{\log\left(\frac{1}{\delta}\right)}\right)\right\} = O_{\mathbb{P}}(1),$$

which implies that

$$\begin{aligned}\mathbb{E}\Bigg|\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\{\hat{\eta}(U_{\tau,oi})-\eta_0(U_{\tau,oi})\}\eta_0'(U_{\tau,oi})J^{\top}(\tau_0)\Phi(X_i)\\ -\mathbb{E}\Big[\rho_2(m_o)\{\hat{\eta}(U_{\tau,o})-\eta_0(U_{\tau,o})\}\eta_0'(U_{\tau,o})J^{\top}(\tau_0)\Phi(X)\Big]\Bigg|\\ = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}\big[\rho_2(m_0)\{\hat{\eta}(U_{\tau,0})-\eta_0(U_{\tau,0})\}\eta_0'(U_{\tau,0})\Phi(X)\big]\\ &= \mathbb{E}\big[\rho_2(m_0)\{\eta^*(U_{\tau,0})-\eta_0(U_{\tau,0})\}\eta_0'(U_{\tau,0})\Phi(X)\big]\\ &\quad+\mathbb{E}\big[\rho_2(m_0)\{\hat{\eta}(U_{\tau,0})-\eta^*(U_{\tau,0})\}\eta_0'(U_{\tau,0})\Phi(X)\big]\\ &= \mathbb{E}\big[\rho_2(m_0)\{\eta^*(U_{\tau,0})-\eta_0(U_{\tau,0})\}\eta_0'(U_{\tau,0})\Phi(X)\big]+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right),\end{aligned}$$

where η^* is the projection of η onto S_n^0 which is B-Spline space with the normalized basis, relative to the theoretical inner products.
By the definition of Φ, for any measurable function g, we have

$$\mathbb{E}\Big[g(U_{\tau,0})\rho_2(m_0(T))J^{\top}(\tau_0)\Phi(X)\Big]=\mathbb{E}\Big[g(U_{\tau,0})\rho_2(m_0(T))J^{\top}(\tau_0)(X-\Upsilon(U_{\tau,0}))\Big]=0.$$

Hence the result about (26) in Lemma **A2** holds. Similarly, (27) and (28), in Lemma **A2**, follow from Lemma 1.
Proof of Lemma A3
Under the condition (*C*6), the functions Ψ and ρ_2 are fixed and bounded. Set

$$\mathcal{A}_2(\delta)=\{\rho_2(m(t))\Psi(t)\text{ such that } m\in\mathcal{M}_n,\ \|m-m_0\|\le\delta\}.$$

We have $\log N_{[\,]}(\varepsilon,\mathcal{A}_2(\delta),\|\|)\le c\left\{N_n\log\left(\frac{\delta}{\varepsilon}\right)+\log\left(\frac{1}{\delta}\right)\right\}$.
So, the corresponding entropy integral is bounded by Lemma 6 since

$$\begin{aligned}J_{[\,]}(\varepsilon,\mathcal{A}_2(\delta),\|\|\|) &= \int_0^{\delta}\Big\{1+\log N_{[\,]}(\varepsilon,\mathcal{A}_2(\delta),\|\|)\Big\}^{\frac{1}{2}}d\varepsilon\\ &\le c\delta\left\{\sqrt{N_n}+\sqrt{\log\left(\frac{1}{\delta}\right)}\right\}.\end{aligned}$$

According to (26) in Lemma **A2**, Lemma 3, Theorem 3 and the condition (C1), we obtain

$$\begin{aligned}\mathbb{E}\Bigg|\frac{1}{n}\sum_{i=1}^{n}\{\hat{\eta}(U_{\tau,oi})-\eta_0(U_{\tau,oi})\}\rho_2(m_{0i})\Psi(T_i)\\ -\mathbb{E}[\{\hat{\eta}(U_{\tau,o})-\eta_0(U_{\tau,o})\}\rho_2(m_0(T))\Psi(T)]\Bigg| = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right),\end{aligned}$$

which implies that

$$\begin{aligned}\mathbb{E}\Bigg|\frac{1}{n}\sum_{i=1}^{n}\{\hat{\eta}(U_{\tau,oi})-\eta_0(U_{\tau,oi})\}\rho_2(m_{0i})\Psi(T_i)\\ -\mathbb{E}[\{\hat{\eta}(U_{\tau,o})-\eta_0(U_{\tau,o})\}\rho_2(m_0(T))\Psi(T)]\Bigg| = o\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

By the definition of ψ, for any measurable function $S(.)$, we have $\mathbb{E}[S(U_{\tau,0})\rho_2(m_0(T))\Psi(T)] = 0$. Hence (29) in Lemma **A3** holds. Similarly (30) and (31) in Lemma **A3** follow from Lemma 1.

Proof of Theorem 4

Let us denote

i. $\hat{U}_\tau = \alpha^\top(\hat{\tau})X$.

ii. For all $v = \left(v_1^\top, v_2^\top\right)^\top$, where $v_1 \in \mathbb{R}^{N_1}$ and $v_2 \in \mathbb{R}^{N_2}$, we define

$$\hat{m}(v;\alpha(\hat{\tau}),\hat{\delta}) = \hat{\eta}\left(\hat{U}_\tau + v_1 J^\top(\hat{\tau})\Phi(x);\alpha(\hat{\tau}),\hat{\delta}\right) + v_2^\top \Psi(\bar{W}).$$

iii. $\mathcal{M}_n = \left\{m(x,w) = \eta\left(\alpha^\top(\tau)x\right) + \delta^\top w; \eta \in \mathcal{H}(p)\right\}$.

$\hat{m}(v;\alpha(\hat{\tau}),\hat{\delta})$ maximizes $l(m) = \dfrac{1}{n}\sum_{i=1}^{n} Q\Big[g^{-1}\{m(X_i,W_i)\}, Y_i\Big]$ for all $m \in \mathcal{M}_n$ when $v = 0$, then $\dfrac{\partial}{\partial v} l\left(\hat{m}\right)|_{v=0} = 0$. i.e.

$$\frac{\partial}{\partial v_1} l\left(\hat{m}\right)|_{v=0} = 0 \;\text{ and }\; \frac{\partial}{\partial v_2} l\left(\hat{m}\right)|_{v=0} = 0,$$

we use the notations,

iv. $\hat{m}_i = \hat{\eta}\left(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\right) + \hat{\delta}^\top \bar{W}_i = \hat{\gamma}^\top B_1\left(\hat{U}_{\tau,i}\right) + \hat{\delta}^\top \bar{W}$.

v. $\hat{U}_{\tau,i} = \alpha^\top(\hat{\tau})X_i$ et $\hat{U}_{\tau,0i} = \alpha^\top(\tau_0)X_i$.

Thus

$$\begin{aligned}
0 &= \frac{\partial}{\partial v_1} l\left(\hat{m}\right)|_{v=0} \\
&= \frac{1}{n}\sum_{i=1}^{n} q_1(\hat{m}_i, Y_i)\hat{\eta}'\left(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\right)J^\top(\hat{\tau})\Phi(X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\Big[Y_i - g^{-1}(\hat{m}_i)\Big]\rho_1(\hat{m}_i)\hat{\eta}'\left(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\right)J^\top(\hat{\tau})\Phi(X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\Big[Y_i - g^{-1}(\hat{m}_i)\Big]\rho_1(\hat{m}_i)\hat{\eta}_0'(U_{\tau,0i})J^\top(\hat{\tau})\Phi(X_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\Big[Y_i - g^{-1}(\hat{m}_i)\Big]\rho_1(\hat{m}_i)\{\hat{\eta}'\left(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\right) - \hat{\eta}_0'(U_{\tau,0i})\}J^\top(\tau)\Phi(X_i).
\end{aligned}$$

We have

$$\begin{aligned}
&\hat{\eta}'\left(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\right) - \hat{\eta}_0'(U_{\tau,0i}) \\
&= \Big\{\hat{\eta}'\left(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\right) - \hat{\eta}'\left(\alpha^\top(\tau_0)X_i;\alpha(\hat{\tau}),\hat{\delta}\right)\Big\} + \left\{\hat{\eta}'\left(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\right) - \hat{\eta}_0'(U_{\tau,0i})\right\} \\
&= \hat{\eta}''\left(\alpha^\top(\tau_0)X_i;\alpha(\hat{\tau}),\hat{\delta}\right)\{\alpha(\hat{\tau}) - \alpha(\tau_0)\}^\top X_i + \left\{\hat{\eta}'\left(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\right) - \hat{\eta}_0'(U_{\tau,0i})\right\} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= \hat{\eta}''\left(\alpha^\top(\tau_0)X_i;\alpha(\hat{\tau}),\hat{\delta}\right)X_i^\top J(\tau_0)(\hat{\tau} - \tau_0) + \left\{\hat{\eta}'\left(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\right) - \hat{\eta}_0'(U_{\tau,0i})\right\} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

because $\hat{\tau} - \tau_0 = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$, then

$$\{\alpha(\hat{\tau}) - \alpha(\tau_0)\}^\top X_i \;=\; X_i^\top\{\alpha(\hat{\tau}) - \alpha(\tau_0)\} = X_i^\top J(\tau_0)(\hat{\tau} - \tau_0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

Moreover, we have

$$\begin{aligned} Y_i - g^{-1}(\hat{m}_i) &= \Big\{Y_i - g^{-1}(m_{0i})\Big\}Y_i - \Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\} \\ &= \varepsilon_i - \Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}. \end{aligned}$$

Then, using conditions (C3), (C4) and (C9), we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\Big[Y_i - g^{-1}(\hat{m}_i)\Big]\rho_1(\hat{m}_i)\{\hat{\eta}'\big(\hat{U}_{\tau,j}, \alpha(\hat{\tau}), \hat{\delta}\big) - \eta_0'(U_{\tau,0i})\}J^\top(\hat{\tau})\Phi(X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(\hat{m}_i)\{\hat{\eta}'\big(\hat{U}_{\tau,i}; \alpha(\hat{\tau}), \hat{\delta}\big) - \eta_0'(U_{\tau,0i})\}J^\top(\hat{\tau})\Phi(X_i) \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big]\rho_1(\hat{m}_i)\{\hat{\eta}'\big(\hat{U}_{\tau,i}; \alpha(\hat{\tau}), \hat{\delta}\big) - \eta_0'(U_{\tau,0i})\}J^\top(\hat{\tau})\Phi(X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(\hat{m}_i)\{\hat{\eta}'\big(\hat{U}_{\tau,i}; \alpha(\hat{\tau}), \hat{\delta}\big) - \hat{\eta}'\big(U_{\tau,0i}; \alpha(\hat{\tau}), \hat{\delta}\big)\}J^\top(\hat{\tau})\Phi(X_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(\hat{m}_i)\{\hat{\eta}'\big(U_{\tau,0i}; \alpha(\hat{\tau}), \hat{\delta}\big) - \eta_0'(U_{\tau,0i})\}J^\top(\hat{\tau})\Phi(X_i) \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big]\rho_1(\hat{m}_i)\{\hat{\eta}'\big(\hat{U}_{\tau,i}; \alpha(\hat{\tau}), \hat{\delta}\big) - \eta_0'(U_{\tau,0i})\}J^\top(\hat{\tau})\Phi(X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(\hat{m}_i)\hat{\eta}''\big(U_{\tau,0i}; \alpha(\hat{\tau}), \hat{\delta}\big)J^\top(\hat{\tau})\Phi(X_i)X_i^\top J(\tau_0)(\hat{\tau} - \tau_0) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(\hat{m}_i)\{\hat{\eta}'\big(U_{\tau,0i}; \alpha(\hat{\tau}), \hat{\delta}\big) - \eta_0'(U_{\tau,0i})\}J^\top(\hat{\tau})\Phi(X_i) \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big]\rho_1(\hat{m}_i)\hat{\eta}''\big(U_{\tau,0;}; \alpha(\hat{\tau}), \hat{\delta}\big)J^\top(\hat{\tau})\Phi(X_i)X_i^\top J(\tau_0)(\hat{\tau} - \tau_0) \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big]\rho_1(\hat{m}_i)\{\hat{\eta}'\big(\hat{U}_{\tau,i}; \alpha(\hat{\tau}), \hat{\delta}\big) - \eta_0'(U_{\tau,0i})\}J^\top(\hat{\tau})\Phi(X_i) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Then

$$\begin{aligned}
0 &= \frac{1}{n}\sum_{i=1}^{n}\Big[Y_i - g^{-1}(\hat{m}_i)\Big]\rho_1(\hat{m}_i)\eta_0'(U_{\tau,oi})J^\top(\hat{\tau})\Phi(X_i) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(\hat{m}_i)\eta_0'(U_{\tau,oi})J^\top(\hat{\tau})\Phi(X_i) \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big]\rho_1(\hat{m}_i)\eta_0'(U_{\tau,oi})J^\top(\hat{\tau})\Phi(X_i) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(m_{0i})\eta_0'(U_{\tau,oi})J^\top(\hat{\tau})\Phi(X_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\}\eta_0'(U_{\tau,oi})J^\top(\hat{\tau})\Phi(X_i) \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big]\rho_1(\hat{m}_i)\eta_0'(U_{\tau,oi})J^\top(\hat{\tau})\Phi(X_i) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= I + II - III + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \qquad \text{(S28)}
\end{aligned}$$

According to (S27), we have $\|\hat{m} - m_0\|_\infty = O_{\mathbb{P}}\left\{N_n\left(h^p + \frac{1}{\sqrt{nh}}\right)\right\}$, thus,

$$\begin{aligned} II &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\}\eta_0'(U_{\tau,0i})J^\top(\hat{\tau})\Phi(X_i) \\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{0i})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\ &= II^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

since $J(\hat{\tau}) - J(\tau_0) = \dfrac{\partial}{\partial\tau}\{J(\tau)\}|_{\tau=\tau_0}(\hat{\tau} - \tau_0) + O_{\mathbb{P}}\left(\dfrac{1}{\sqrt{n}}\right)$.
According to the condition $(C10)$, the expectation of the square of the kth column of II^* is $\mathbb{E}\left[\left\{\dfrac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{oi})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)e_k\right\}^2\right] = o\left(\dfrac{1}{n}\right)$.
Indeed, since $(C10)$, weak law of large numbers and $\|\hat{m} - m_0\|_\infty = O_{\mathbb{P}}\left\{N_n\left(h^p + \frac{1}{\sqrt{nh}}\right)\right\}$, we obtain

$$\begin{aligned} &\mathbb{E}\left[\left\{\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{oi})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)e_k\right\}^2\right] \\ &= \frac{1}{n}\mathbb{E}\left[\frac{1}{n}\left\{\sum_{i=1}^{n}\varepsilon_i^2\rho_1'(m_{oi})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)e_k\right\}^2\right] \\ &\leq \frac{C_0}{n}\mathbb{E}\left[\frac{1}{n}\left\{\sum_{i=1}^{n}\rho_1'(m_{oi})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)e_k\right\}^2\right] \\ &= o\left(\frac{1}{n}\right). \end{aligned}$$

By the Markov's inequality

$$II^* = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{oi})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

Since, for all $v > 0$, we have

$$\begin{aligned} &\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{oi})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\right| \geq \frac{v}{\sqrt{n}}\right] \\ &\leq \frac{n}{v^2}\mathbb{E}\left[\left\{\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{oi})(\hat{m}_i - m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\right\}^2\right] \\ &= \frac{1}{v^2}o(1) \underset{n\to+\infty}{\longrightarrow} 0\ . \end{aligned}$$

For the third term, it exists $\overline{m_i}$ between $\hat{m}_i$ and m_{0i} such that

$$g^{-1}(\hat{m}_i) - g^{-1}(m_{oi}) = (\hat{m}_i - m_{oi})\frac{d}{dm}g^{-1}(m)|_{m=m_{oi}} + \frac{d^2}{dm^2}g^{-1}(m)|_{m=\overline{m_i}}(\hat{m}_i - m_{oi})^2.$$

Then

$$\begin{aligned}
III &= \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i)-g^{-1}(m_{0i})\Big]\rho_1(\hat{m}_i)\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i)\\
&= \frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i)-g^{-1}(m_{0i})\Big]\rho_1(m_{0i})\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i)\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i)-g^{-1}(m_{0i})\Big][\rho_1(\hat{m}_i)-\rho_1(m_{0i})]\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i)\\
&= \frac{1}{n}\sum_{i=1}^{n}(\hat{m}_i-m_{0i})\rho_2(m_{0i})\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i)\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\frac{d^2g^{-1}(m)}{dm^2}\bigg|_{m=\overline{m}_i}(\hat{m}_i-m_{0i})^2\rho_1(m_{0i})\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i)\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\Big[g^{-1}(\hat{m}_i)-g^{-1}(m_{0i})\Big][\rho_1(\hat{m}_i)-\rho_1(m_{0i})]\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i)\\
&= III_1+III_2+III_3.
\end{aligned}$$

In addition, since $\|\hat{\eta}-\eta_0\|_\infty = O_{\mathbb{P}}\Big\{\sqrt{N_n}\Big(h^p+\frac{1}{\sqrt{nh}}\Big)\Big\}$, we have

$$\begin{aligned}
\hat{m}_i-m_{0i} &= \hat{\eta}\big(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\big)-\eta_0(U_{\tau,oi})+\big(\hat{\delta}-\delta_0\big)^\top\bar{W}_i\\
&= \hat{\eta}\big(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\big)-\hat{\eta}\big(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\big)+\hat{\eta}\big(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\big)-\eta_0(U_{\tau,oi})+\big(\hat{\delta}-\delta_0\big)^\top\bar{W}_i\\
&= \eta_0'(U_{\tau,0i})\{\alpha(\hat{\tau})-\alpha(\tau_0)\}^\top X_i+\hat{\eta}\big(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\big)-\eta_0(U_{\tau,oi})\big(\hat{\delta}-\delta_0\big)^\top\bar{W}_i\\
&\quad+O_{\mathbb{P}}\Big(\frac{1}{\sqrt{n}}\Big).
\end{aligned}$$

Since $\Phi(x)=x-\Upsilon(u_{\tau,0})$,
$\Psi(w)=w-\Gamma(u_{\tau,0})$ and $\{\alpha(\hat{\tau})-\alpha(\tau_0)\}^\top\Phi(X_i)=\Phi(X_i)^\top J(\tau_0)(\hat{\tau}-\tau_0)+O_{\mathbb{P}}\Big(\frac{1}{\sqrt{n}}\Big)$,
then

$$\begin{aligned}
\hat{m}_i-m_{0i} &= \eta'\big(U_{\tau,\rho i}\big)\{\alpha(\hat{\tau})-\alpha(\tau_0)\}^\top\{\Phi(X_i)+\Upsilon(u_{\tau,0i})\}\\
&\quad+\hat{\eta}\big(U_{\tau,oi};\alpha(\hat{\tau}),\hat{\delta}\big)-\eta_0\big(U_{\tau,\rho i}\big)+\big(\hat{\delta}-\delta_0\big)^\top\big\{\Psi(\bar{W}_i)+\Gamma(U_{\tau,oi})\big\}+O_{\mathbb{P}}\bigg(\frac{1}{\sqrt{n}}\bigg)\\
&= \eta'(U_{\tau,oi})\Phi(X_i)^\top J(\tau_0)(\hat{\tau}-\tau_0)+\hat{\eta}\big(U_{\tau,oi};\alpha(\hat{\tau}),\hat{\delta}\big)-\eta_0(U_{\tau,oi})\\
&\quad+\eta'(U_{\tau,oi})\Upsilon(u_{\tau,0i})^\top J(\tau_0)(\hat{\tau}-\tau_0)+\big(\hat{\delta}-\delta_0\big)^\top\Psi(\bar{W}_i)+\big(\hat{\delta}-\delta_0\big)^\top\Gamma(U_{\tau,oi})\\
&\quad+O_{\mathbb{P}}\Big(\frac{1}{\sqrt{n}}\Big).
\end{aligned}$$

Then

$$
\begin{aligned}
III_1 &= \frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})[\hat{m}-m_{oi}]\eta_0'(U_{r,oi})J^\top(\tau_0)\Phi(X_i)\\
&= \frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\{\eta_0'(U_{\tau,oi})\}^2J^\top(\tau_0)\Phi(X_i)\Phi(X_i)^\top J(\tau_0)(\hat{\tau}-\tau_0)\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\Psi(W_i)^\top J(\tau_0)\left(\hat{\delta}-\delta_0\right)\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\left\{\hat{\eta}\left(U_{\tau,oi};\alpha(\hat{\tau}),\hat{\delta}\right)-\eta_0(U_{\tau,oi})\right\}\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\{\eta_0'(U_{\tau,oi})\}^2J^\top(\tau_0)\Phi(X_i)\Upsilon(U_{r,oi})^\top J(\tau_0)(\hat{\tau}-\tau_0)\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\Gamma(U_{\tau,oi})^\top\left(\hat{\delta}-\delta_0\right)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

From Lemma **A2**, we have

$$
\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\left\{\hat{\eta}\left(U_{\tau,oi};\alpha(\hat{\tau}),\hat{\delta}\right)-\eta_0(U_{\tau,oi})\right\}_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)
$$

$$
\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\{\eta_0'(U_{\tau,oi})\}^2J^\top(\tau_0)\Phi(X_i)\Upsilon(U_{\tau,oi})^\top J(\tau_0)(\hat{\tau}-\tau_0)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)
$$

$$
\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\Gamma(U_{\tau,oi})^\top\left(\hat{\delta}-\delta_0\right)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
$$

Then

$$
\begin{aligned}
III_1 &= \frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})[\hat{m}-m_{oi}]\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\\
&= \frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\{\eta_0'(U_{\tau,oi})\}^2J^\top(\tau_0)\Phi(X_i)\Phi(X_i)^\top J(\tau_0)(\hat{\tau}-\tau_0)\\
&= +\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\Psi(\bar{W}_i)^\top J(\tau_0)\left(\hat{\delta}-\delta_0\right)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

According to conditions $(C5)-(C10)$, we have

$$
\begin{aligned}
III_2 &= \frac{1}{n}\sum_{i=1}^{n}\frac{d^2g^{-1}(m)}{dm^2}\bigg|_{m=\overline{m}_i}(\hat{m}_i-m_{oi})^2\rho_1(m_{oi})\eta_0'(U_{\tau,oi})J^\top(\tau_0)\Phi(X_i)\\
&\leq c\|\hat{m}-m_o\|_\infty^2=O_{\mathbb{P}}\left(N_n^2\left(h^p+\frac{1}{\sqrt{nh}}\right)^2\right)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

Likewise, since $|\rho_1(\hat{m}_i)-\rho_1(m_{0i})|\leq C_\rho^*|\hat{m}_i-m_{0i}|\leq C_\rho^*\|\hat{m}-m_0\|_\infty$ for all $|\hat{m}_i-m_{0i}|\leq M_0$ and, $\left|g^{-1}(\hat{m}_i)-g^{-1}(m_{0i})\right| = \left|\frac{dg^{-1}(m)}{dm}\Big|_{m=\overline{m}_i}(\hat{m}_i-m_{0i})\right|\leq C_g^*\left|\hat{m}_i-m_{0i}\right| \leq C_g^*\|\hat{m}-m_0\|_\infty$ for all $|\hat{m}_i-m_{0i}|\leq M_1$, we obtain

$$
\begin{aligned}
III_3 &= \frac{1}{n}\sum_{i=1}^{n}\left[g^{-1}(\hat{m}_i)-g^{-1}(m_{0i})\right][\rho_1(\hat{m}_i)-\rho_1(m_{0i})]J^\top(\tau_0)\Phi(X_i)\\
&\leq c\|\hat{m}-m_0\|_\infty^2=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

Thus

$$\begin{aligned} III &= \frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\big\{\eta_0'(U_{\tau,0i})\big\}^2 J^\top(\tau_0)\Phi(X_i)\Phi(X_i)^\top J(\tau_0)(\hat{\tau}-\tau_0) \\ &+\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{oi})\eta_0'(U_{\tau,0i})J^\top(\tau_0)\Phi(X_i)\Psi(W_i)^\top(\hat{\delta}-\delta_0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

Moreover, we have,

$$\begin{aligned} 0 &= \frac{\partial}{\partial v_2} l(\hat{m})|_{v_2=0} = \frac{1}{n}\sum_{i=1}^{n} q_1(\hat{m}_i, Y_i)\rho_1(\hat{m}_i)\Psi(\bar{W}_i) \\ &= \frac{1}{n}\sum_{i=1}^{n}\Big[Y_i - g^{-1}(\hat{m}_i)\Big]\rho_1(\hat{m}_i)\Psi(\bar{W}_i). \end{aligned}$$

As $Y_i - g^{-1}(\hat{m}_i) = \{Y_i - g^{-1}(m_{0i})\} - \{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\} = \varepsilon_i - \{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\}$, then

$$\begin{aligned} 0 &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(\hat{m}_i)\Psi(W_i) - \frac{1}{n}\sum_{i=1}^{n}\Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\rho_1(\hat{m}_i)\Psi(\bar{W}_i) \\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1(m_{0i})\Psi(W_i) + \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\}\Psi(\bar{W}_i) \\ &-\frac{1}{n}\sum_{i=1}^{n}\Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\rho_1(\hat{m}_i)\Psi(\bar{W}_i) \\ &= IV + V - VI. \end{aligned}$$

As in the above, we show that

$$V = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\}\Psi(\bar{W}_i) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

Indeed, we have, $\|\hat{m} - m_0\|_\infty = O_{\mathbb{P}}\Big\{N_n\Big(h^p + \frac{1}{\sqrt{nh}}\Big)\Big\}$, then

$$\begin{aligned} V &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\}\Psi(\bar{W}_i) \\ &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{0i})(\hat{m}_i - m_{0i})\Psi(\bar{W}_i) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\ &= V^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

From the condition $(C10)$, we have

$$\mathbb{E}\left[\left\{\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{0i})(\hat{m}_i - m_{0i})\Psi(\bar{W}_i)e_k\right\}^2\right] = O\left(\frac{1}{n}\right).$$

By Markov's inequality, we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\rho_1'(m_{0i})(\hat{m}_i - m_{0i})\Psi(\bar{W}_i) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right),$$

so

$$V = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

It exists $\overline{m}_i$ between $\hat{m}_i$ and m_{0i} such that

$$
\begin{aligned}
&\Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\rho_1(\hat{m}_i) \\
&= \Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\rho_1(m_{0_i}) + \Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\} \\
&= (\hat{m}_i - m_{0i})\frac{d}{dm}\Big\{g^{-1}(m)\Big\}\Big|_{m=m_{0i}}\rho_1(m_{0i}) + (\hat{m}_i - m_{0i})^2\frac{d^2}{dm^2}\Big\{g^{-1}(m)\Big\}\Big|_{m=\overline{m}_i}\rho_1(m_{0i}) \\
&\quad + \Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\} \\
&= (\hat{m}_i - m_{0i})\rho_2(m_{0i}) + (\hat{m}_i - m_{0i})^2\frac{d^2}{dm^2}\Big\{g^{-1}(m)\Big\}\Big|_{m=\overline{m}_i}\rho_1(m_{0i}) \\
&\quad + \Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
VI &= \frac{1}{n}\sum_{i=1}^{n}\Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\rho_1(\hat{m}_i)\Psi(\bar{W}_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}(\hat{m}_i - m_{0i})\rho_2(m_{0i})\Psi(\bar{W}_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}(\hat{m}_i - m_{0i})^2\frac{d^2}{dm^2}\Big\{g^{-1}(m)\Big\}\Big|_{m=\overline{m}_i}\rho_1(m_{0i})\Psi(\bar{W}_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\Big\{g^{-1}(\hat{m}_i) - g^{-1}(m_{0i})\Big\}\{\rho_1(\hat{m}_i) - \rho_1(m_{0i})\}\Psi(\bar{W}_i) \\
&= VI_1 + VI_2 + VI_3,
\end{aligned}
$$

we know that,

$$
\begin{aligned}
\hat{m}_i - m_{0i} &= \hat{\eta}\big(\hat{U}_{\tau,i};\alpha(\hat{\tau}),\hat{\delta}\big) - \eta_0(U_{\tau,0i}) + \big(\hat{\delta} - \delta_0\big)^\top \bar{W}_i \\
&= \eta'(U_{\tau,0i})\Phi(X_i)^\top(\alpha(\hat{\tau}) - \alpha(\tau_0)) + \hat{\eta}\big(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\big) - \eta_0(U_{\tau,0i}) \\
&\quad + \eta'(U_{\tau,0i})\Upsilon^\top(U_{\tau,0i})(\alpha(\hat{\tau}) - \alpha(\tau_0)) + \Psi^\top(\bar{W}_i)\big(\hat{\delta} - \delta_0\big) \\
&\quad + \Gamma^\top(U_{\tau,0i})\big(\hat{\delta} - \delta_0\big) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \\
&= \eta'(U_{\tau,0i})\Phi^\top(X_i)J(\tau_0)(\hat{\tau} - \tau_0) + \hat{\eta}\big(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\big) - \eta_0(U_{\tau,0i}) \\
&\quad + \Big[\Psi^\top(\bar{W}_i) + \Gamma^\top(U_{\tau,0i})\Big]\big(\hat{\delta} - \delta_0\big) + \eta'(U_{\tau,0i})\Upsilon^\top(U_{\tau,0i})J(\tau_0)(\hat{\tau} - \tau_0) \\
&\quad + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
VI_1 &= \frac{1}{n}\sum_{i=1}^{n}(\hat{m}_i - m_{0i})\rho_2(m_{0i})\Psi(\bar{W}_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\eta'(U_{\tau,0i})\rho_2(m_{0i})\Psi(\bar{W}_i)\Phi^\top(X_i)J(\tau_0)(\hat{\tau} - \tau_0) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{0i})\Psi(\bar{W}_i)\Big[\Psi^\top(\bar{W}_i) + \Gamma^\top(U_{\tau,0i})\Big]\big(\hat{\delta} - \delta_0\big) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\big[\hat{\eta}\big(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\big) - \eta_0(U_{\tau,0i})\big]\rho_2(m_{0i})\Psi(\bar{W}_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\eta'(U_{\tau,0i})\rho_2(m_{0i})\Psi(\bar{W}_i)\Upsilon^\top(U_{\tau,0i})J(\tau_0)(\hat{\tau} - \tau_0) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

From Lemma **A3**, we have

$$\frac{1}{n}\sum_{i=1}^{n}\left[\hat{\eta}\left(U_{\tau,0i};\alpha(\hat{\tau}),\hat{\delta}\right)-\eta_0(U_{\tau,0i})\right]\rho_2(m_{0i})\Psi(\bar{W}_i)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$$

$$\frac{1}{n}\sum_{i=1}^{n}\eta'(U_{\tau,0i})\rho_2(m_{0i})\Psi(\bar{W}_i)\Upsilon^{\top}(U_{\tau,0i})J(\tau_0)(\hat{\tau}-\tau_0)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$$

$$\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{0i})\Psi(\bar{W}_i)\Gamma^{\top}(U_{\tau,0i})(\hat{\delta}-\delta_0)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

Then

$$\begin{aligned} VI_1 &= \frac{1}{n}\sum_{i=1}^{n}\eta'(U_{\tau,0i})\rho_2(m_{0i})\Psi(\bar{W}_i)\Phi^{\top}(X_i)J(\tau_0)(\hat{\tau}-\tau_0) \\ &\quad +\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{0i})\Psi(\bar{W}_i)\Psi^{\top}(\bar{W}_i)(\hat{\delta}-\delta_0)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

As in III_2 and III_3, we show that, $|VI_2|=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$ and $|VI_3|=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$, then

$$\begin{aligned} VI &= \frac{1}{n}\sum_{i=1}^{n}\eta'(U_{\tau,0i})\rho_2(m_{0i})\Psi(\bar{W}_i)\Phi^{\top}(X_i)J(\tau_0)(\hat{\tau}-\tau_0) \\ &\quad +\left\{\frac{1}{n}\sum_{i=1}^{n}\rho_2(m_{0i})\Psi(\bar{W}_i)\Psi^{\top}(\bar{W}_i)\right\}(\hat{\delta}-\delta_0)+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{S29}$$

By (S21) and (S22), we deduce that,

$$\begin{aligned} 0 &= \frac{1}{n}\sum_{i=1}^{n}q_1(m_{0i},Y_i)\begin{pmatrix}\eta_0'(U_{\tau,0i})J^{\top}(\tau_0)\Phi(X_i)\\ \Psi(\bar{W}_i)\end{pmatrix} \\ &+\left\{\mathbb{E}\left[\rho_2(m_0(T))\begin{pmatrix}\eta_0'(U_{\tau,0})J^{\top}(\tau_0)\Phi(X)\\ \Psi(\bar{W})\end{pmatrix}\times\left(\eta_0'(U_{\tau,0})\Phi^{\top}(X)J(\tau_0)\Psi^{\top}(\bar{W})\right)\right]+O_{\mathbb{P}}(1)\right\} \\ &\begin{pmatrix}\hat{\tau}-\tau_0\\ \hat{\delta}-\delta_0\end{pmatrix}+O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

By applying the central limit theorem and the δ-method, we obtain the distribution of $\begin{pmatrix}\hat{\tau}-\tau_0\\ \hat{\delta}-\delta_0\end{pmatrix}$.

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