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Weak ψ-Contractions on Directed Graphs with Applications to Integral Equations

Doaa Filali [1], Mohammad Dilshad [2] and Mohammad Akram [3,*]

[1] Department of Mathematical Science, College of Sciences, Princess Nourah Bint Abdulrahman University, Riyadh 11671, Saudi Arabia; dkfilali@pnu.edu.sa
[2] Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia; mdilshaad@gmail.com
[3] Department of Mathematics, Islamic University of Madinah, Madinah 42351, Saudi Arabia
* Correspondence: akramkhan_20@rediffmail.com

Abstract: This article deals with a few outcomes ensuring the fixed points of a weak (G, ψ)-contraction map of metric spaces comprised with a reflexive and transitive digraph G. To validate our findings, we furnish several examples. The findings we obtain enable us to seek out the unique solution of a nonlinear integral equation. The outcomes presented herewith sharpen, subsume, unify, improve, enrich, and compile a number of existing theorems.

Keywords: fixed points; (C)-graph; Picard mappings; nonlinear integral equations

MSC: 47H10; 54H25; 45G10

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1. Introduction

Within this text, the following notations and abbreviations are adopted:

- $\mathbb{N}$—the set of natural numbers;
- $\mathbb{R}$—the set of real numbers;
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
- $\mathbb{R}^+ := [0, \infty)$;
- BCP—Banach contraction principle;
- MS—metric space;
- CMS—complete metric space;
- fpt—fixed-point theorem;
- NIE—nonlinear integral equation;
- Fix$(\mathcal{H})$—the set of fixed points of a self-map $\mathcal{H}$.

The classical BCP, proved in 1922, pioneered the discipline of metric fixed-point theory. This standard outcome continues to be the primary finding in nonlinear functional analysis. Later on, to generalize the BCP, a number of investigators expanded the class of contraction mappings. One of the noted generalizations of contraction mapping is nonlinear contraction or ϕ-contraction given as

$$\varpi(\mathcal{H}t, \mathcal{H}s) \leq \phi(\varpi(t, s)).$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ remains an appropriate function. The idea of ϕ-contractions was first proposed by Browder [1] and developed by Boyd and Wong [2] and Matkowski [3]. Rhoades [4] developed a comparison of several contractive mapping types, assessing and contrasting 149 distinct conditions. A large portion of Rhoades's work [4] was absorbed by Browder [5] using an understandable and straightforward style of argumentation.

In 1972, Krasnosel'skii et al. [6] investigated the concept of weak ψ-contraction depending on an auxiliary function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, which is defined as follows:

$$\varpi(\mathcal{H}t, \mathcal{H}s) \leq \varpi(t, s) - \psi(\varpi(t, s)).$$

Following the idea of weak contraction, Alber and Guerre-Delabriere [7] presented certain fpt in the setup of Hilbert space. Subsequently, Rhoades [8] generalized the BCP for weak contractions and pointed out that the fpts contained in [7] also remain accurate in CMSs.

In recent years, various researchers extended many existing fpts employing the structure of a directed graph, e.g., [9–14]. This trend was investigated by Jachymski [15], who obtained the graphical formulation of the BCP. In such outcomes, the contraction inequality requires a hold on merely the edges of the graph. Consequently, graphical contractions remain weaker than the corresponding usual contractions. Owing to such limitations, these outcomes can be employed to solve special kinds of NIEs wherein the usual fpt cannot be applied.

The outcomes presented in this manuscript are fixed-point theorems under a weak (G, ψ)-contraction in the setup of metric spaces equipped with a reflexive and transitive directed graph. The findings proved herewith sharpen, subsume, unify, improve, enrich, and compile a number of existing theorems, especially thanks to Rhoades [4], Jachymski [15], Samreen and Kamran [9], Filali et al. [10], Geraghty [16], and Harjani and Sadarangani [17]. Our findings are illustrated by several examples. As an application, we utilize our outcomes to seek out the unique solution of an NIE prescribed with some additional hypotheses.

2. Preliminaries

This section aims to summarize allied essential notions and supplementary findings needed to prove our main results.

Definition 1 ([18]). *A graph G is interpreted by pair $(\mathsf{v}(G), \mathsf{e}(G))$, whereas $\mathsf{v}(G)$ is a nonempty set and $\mathsf{e}(G)$ remains a binary relation on $\mathsf{v}(G)$. Elements of $\mathsf{v}(G)$ and $\mathsf{e}(G)$ are known as the vertices and the edges, respectively.*

Definition 2 ([18]). *A graph in which each edge remains an ordered pair of vertices is referred to as a digraph (or directed graph).*

Definition 3 ([18]). *Given a graph G, the induced graph G^{-1} described by*

$$\mathsf{v}(G^{-1}) = \mathsf{v}(G) \quad \text{and} \quad \mathsf{e}(G^{-1}) = \{(t, s) \in \mathsf{v}(G)^2 : (s, t) \in \mathsf{e}(G)\}$$

is named as the transpose of G.

Definition 4 ([18]). *Every digraph $G = (\mathsf{v}(G), \mathsf{e}(G))$ induces an undirected graph $\tilde{G}$, which is described by*

$$\mathsf{v}(\tilde{G}) = \mathsf{v}(G) \quad \text{and} \quad \mathsf{e}(\tilde{G}) = \mathsf{e}(G) \cup \mathsf{e}(G^{-1}).$$

Definition 5 ([13]). *A digraph G is named as transitive if for every $t, s, r \in \mathsf{v}(G)$ with*

$$(t, s) \in \mathsf{e}(G) \quad \text{and} \quad (s, r) \in \mathsf{e}(G) \Longrightarrow (t, r) \in \mathsf{e}(G).$$

Definition 6 ([18]). *Given $t, s \in \mathsf{v}(G)$, the finite set $\{r_0, r_1, r_2, \dots r_\ell\}$ of vertices is referred as a path in G from t to s of length ℓ if $r_0 = t$, $r_\ell = s$, and $(r_{j-1}, r_j) \in \mathsf{e}(G)$, $\forall\, j \in \{1, 2, \dots \ell\}$.*

Definition 7 ([18]). *A graph G is named as connected if any two vertices in G admit a path. One says that G is weakly connected when $\tilde{G}$ is connected.*

Definition 8 ([15]). *An MS $(\mathbf{P}, \varpi)$ is named as an MS endowed with a graph G if*

- $\mathsf{v}(G) = \mathbf{P}$;

- *$e(G)$ contains all loops;*
- *G admits no parallel edge.*

Definition 9 ([15]). *Let $(\mathbf{P}, \varpi)$ be an MS endowed with a digraph G. A map $\mathcal{H} : \mathbf{P} \to \mathbf{P}$ is named as an orbitally G-continuous if, for all $t, \bar{t} \in \mathbf{P}$ and for each sequence $\{n_j\} \subset \mathbb{N}$, we have*

$$\lim_{j\to\infty} \mathcal{H}^{n_j}(t) = \bar{t} \text{ and } (\mathcal{H}^{n_j}t, \mathcal{H}^{n_j+1}t) \in e(G) \Longrightarrow \lim_{n\to\infty} \mathcal{H}(\mathcal{H}^{n_j}t) = \mathcal{H}(\bar{t}).$$

Definition 10 ([11]). *Let $(\mathbf{P}, \varpi)$ be an MS endowed with a digraph G. Then, G is referred as a (C)-graph if each sequence $\{t_n\} \subset \mathbf{P}$ enjoying the properties $t_n \to t$ and $(t_n, t_{n+1}) \in e(G)$ for each $n \in \mathbb{N}$ admits a subsequence $\{t_{n_k}\}$ verifying $(t_{n_k}, t) \in e(G)$ for each $k \in \mathbb{N}$.*

Following Hossain et al. [19], Ψ refers to the collection of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ verifying the following:

$\Psi_1 : \ \psi(r) > 0, \quad \forall\, r > 0,$
$\Psi_2 : \ \liminf_{p\to r} \psi(p) > 0, \quad \forall\, r > 0.$

Remark 1 ([19]). *Ψ_1 is equivalent to*

$\Psi_1' : \ \psi(r) = 0 \Longrightarrow r = 0.$

Proposition 1 ([19]). *If $\psi \in \Psi$ satisfies $r \le r' - \psi(r')$ for every $r, r' \in \mathbb{R}^+$ with $r' \neq 0$, then*

$$r < r'.$$

Definition 11. *Let $(\mathbf{P}, \varpi)$ be an MS endowed with a digraph G and $\psi \in \Psi$. A map $\mathcal{H} : \mathbf{P} \to \mathbf{P}$ is named as a weak (G, ψ)-contraction if*

(i) $(t, s) \in e(G) \Longrightarrow (\mathcal{H}t, \mathcal{H}s) \in e(G)$;
(ii) $(t, s) \in e(G) \Longrightarrow \varpi(\mathcal{H}t, \mathcal{H}s) \le \varpi(t, s) - \psi(\varpi(t, s))$.

Lemma 1 ([20]). *If a sequence $\{t_n\}$ in an MS $(\mathbf{P}, \varpi)$ is not Cauchy, then there exist $\epsilon_0 > 0$ and $\exists$ subsequences $\{t_{n_k}\}$ and $\{t_{l_k}\}$ of $\{t_n\}$ verifying*

- $k \le l_k < n_k, \quad \forall\, k \in \mathbb{N}$;
- $\varpi(t_{l_k}, t_{n_k}) > \epsilon_0, \quad \forall\, k \in \mathbb{N}$;
- $\varpi(t_{l_k}, t_{n_{k-1}}) \le \epsilon_0, \quad \forall\, k \in \mathbb{N}$.

Furthermore, if $\lim_{n\to\infty} \varpi(t_n, t_{n+1}) = 0$, then

- $\lim_{k\to\infty} \varpi(t_{l_k}, t_{n_k}) = \epsilon_0$;
- $\lim_{k\to\infty} \varpi(t_{l_k}, t_{n_k+1}) = \epsilon_0$;
- $\lim_{k\to\infty} \varpi(t_{l_k+1}, t_{n_k}) = \epsilon_0$;
- $\lim_{k\to\infty} \varpi(t_{l_k+1}, t_{n_k+1}) = \epsilon_0$.

3. Main Results

For a digraph G with $v(G) = \mathbf{P}$ and a map $\mathcal{H} : \mathbf{P} \to \mathbf{P}$, we use the following notation:

$$\mathbf{P}_{\mathcal{H}} = \{t \in \mathbf{P} : (t, \mathcal{H}t) \in e(G)\}.$$

We are now going to demonstrate the following fpt in an MS over a weak (G, ψ)-contractivity condition.

Theorem 1. *Let $(\mathbf{P}, \varpi)$ be a CMS endowed with a transitive digraph G. Let $\mathcal{H} : \mathbf{P} \to \mathbf{P}$ be a weak (G, ψ)-contraction map. If either $\mathcal{H}$ is orbitally G-continuous or G is a (C)-graph, then $\mathcal{H}$ owns a fixed point provided $\mathbf{P}_{\mathcal{H}} \neq \emptyset$.*

Proof. Take $t_0 \in \mathbf{P}_{\mathcal{H}}$ so that $(t_0, \mathcal{H}t_0) \in \mathsf{e}(G)$. Construct a sequence $\{t_n\}$ in the following way:

$$t_{n+1} = \mathcal{H}^n(t_0) = \mathcal{H}(t_n), \quad \forall\, n \in \mathbb{N}_0. \tag{1}$$

Since $(t_0, \mathcal{H}t_0) \in \mathsf{e}(G)$, using (i) of weak (G, ψ)-contractivity condition inductively, we have

$$(\mathcal{H}^n t_0, \mathcal{H}^{n+1} t_0) \in \mathsf{e}(G),$$

which through (1) simplifies to

$$(t_n, t_{n+1}) \in \mathsf{e}(G) \quad \forall\, n \in \mathbb{N}_0. \tag{2}$$

Define $\varpi_n := \varpi(t_n, t_{n+1})$. If there is some $n_0 \in \mathbb{N}_0$ with $\varpi_{n_0} = 0$, then using (1), we find $t_{n_0} = t_{n_0+1} = \mathcal{H}(t_{n_0})$; so, $t_{n_0} \in \mathrm{Fix}(\mathcal{H})$ unless we have $\varpi_n > 0$ for every $n \in \mathbb{N}_0$. Upon implementing a weak (G, ψ)-contractivity condition, we find

$$\varpi(t_n, t_{n+1}) = \varpi(\mathcal{H}t_{n-1}, \mathcal{H}t_n) \leq \varpi(t_{n-1}, t_n) - \psi(\varpi(t_{n-1}, t_n))$$

so that

$$\varpi_{n+1} \quad \leq \quad \varpi_n - \psi(\varpi_n). \tag{3}$$

In view of Proposition 1, Equation (3) gives rise to

$$\varpi_n < \varpi_{n-1}, \quad \forall\, n \in \mathbb{N}_0$$

which follows that $\{\varpi_n\} \subset \mathbb{R}^+ - \{0\}$ remains a decreasing sequence. Consequently, there is an element $\delta \geq 0$ verifying

$$\lim_{n\to\infty} \varpi_n = \delta. \tag{4}$$

We now argue that $\delta = 0$. In contrast, assume that $\delta > 0$. From (3), we find

$$\begin{aligned} \limsup_{n\to\infty} \varpi_{n+1} \quad &\leq \quad \limsup_{n\to\infty} \varpi_n + \limsup_{n\to\infty}[-\psi(\varpi_n)] \\ &\leq \quad \limsup_{n\to\infty} \varpi_n - \liminf_{n\to\infty} \psi(\varpi_n) \end{aligned}$$

Employing (4), the inequality shown above simplifies to

$$\delta \leq \delta - \liminf_{n\to\infty} \psi(\varpi_n)$$

leading to, in turn,

$$\liminf_{\varpi_n \to \delta} \psi(\varpi_n) = \liminf_{n\to\infty} \psi(\varpi_n) \leq 0$$

which contradicts Ψ_2. Thus, we have

$$\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} \varpi(t_n, t_{n+1}) = 0. \tag{5}$$

We now argue that $\{t_n\}$ remains Cauchy. In contrast, assuming that $\{t_\mathsf{n}\}$ is not Cauchy, we apply Lemma 1, $\exists\, \epsilon_0 > 0$ and $\exists$ subsequences $\{t_{\mathsf{n}_k}\}$ and $\{t_{l_k}\}$ of $\{t_\mathsf{n}\}$ for which

$$k \leq l_k < \mathsf{n}_k,\ \varpi(t_{l_k}, t_{\mathsf{n}_k}) > \epsilon_0 \geq \varpi(t_{l_k}, t_{\mathsf{n}_{k-1}}), \quad \forall\, k \in \mathbb{N}.$$

Define $\delta_k := \varpi(t_{l_k}, t_{\mathsf{n}_k})$. Employing (2) and the transitivity of G, we find $(t_{l_k}, t_{\mathsf{n}_k}) \in \mathsf{e}(G)$. Using a weak (G, ψ)-contractivity condition, we obtain

$$\varpi(t_{l_k+1}, t_{\mathsf{n}_k+1}) = \varpi(\mathbf{P}t_{l_k}, \mathbf{P}t_{\mathsf{n}_k}) \leq \varpi(t_{l_k}, t_{\mathsf{n}_k}) - \psi(\varpi(t_{l_k}, t_{\mathsf{n}_k}))$$

so that

$$\varpi(t_{l_k+1}, t_{n_k+1}) \leq \delta_k - \psi(\delta_k). \tag{6}$$

From (6) and Lemma 1, one obtains

$$\limsup_{k\to\infty} \varpi(t_{l_k+1}, t_{n_k+1}) \leq \limsup_{k\to\infty} \delta_k + \limsup_{k\to\infty}[-\psi(\delta_k)],$$

yielding thereby

$$\epsilon_0 \leq \epsilon_0 - \liminf_{k\to\infty} \psi(\delta_k)$$

so that

$$\liminf_{\delta_k\to\epsilon} \psi(\delta_k) = \liminf_{k\to\infty} \psi(\delta_k) \leq 0,$$

which contradicts Ψ_2. Consequently, $\{t_n\}$ is a Cauchy. Through the completeness of $(\mathbf{P}, \varpi)$, there exists $t \in \mathbf{P}$ whereby $t_n \xrightarrow{\varpi} t$.

Suppose that $\mathcal{H}$ is orbitally G-continuous. Then, one finds

$$t_{n+1} = \mathcal{H}(t_n) \xrightarrow{\varpi} \mathcal{H}(t),$$

leading to in turn $\mathcal{H}(t) = t$. Therefore, t is a fixed point of $\mathcal{H}$. Otherwise, if G is a (C)-graph, then a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ can be determined that satisfies $(t_{n_k}, t) \in \mathsf{e}(G)$ for every $k \in \mathbb{N}_0$. Based on the weak (G, ψ)-contractivity condition, we have

$$\begin{aligned} \varpi(t_{n_k+1}, \mathcal{H}t) &= \varpi(\mathcal{H}t_{n_k}, \mathcal{H}t) \\ &\leq \varpi(t_{n_k}, t) - \psi(\varpi(t_{n_k}, t)) \\ &\leq \varpi(t_{n_k}, t), \quad \forall\, k \in \mathbb{N}_0. \end{aligned}$$

Taking $k \to \infty$ in the above inequality and using $t_{n_k} \xrightarrow{\varpi} t$, we obtain

$$t_{n_k+1} \xrightarrow{\varpi} \mathcal{H}(t),$$

leading to in turn $\mathcal{H}(t) = t$. Hence, t is a fixed point of $\mathcal{H}$. □

Theorem 2. *Irrespective of the presumptions laid out in Theorem 1, if G is weakly connected, then $\mathcal{H}$ owns a unique fixed point.*

Proof. Regarding Theorem 1, if $t, s \in \mathrm{Fix}(\mathcal{H})$, then for every $n \in \mathbb{N}_0$ we find

$$\mathcal{H}^n(t) = t, \mathcal{H}^n(s) = s.$$

With the weak connectedness of G, there is a path $\{r_0, r_1, r_2, \ldots r_\ell\}$ between t and s, i.e.,

$$r_0 = t,\ r_\ell = s \text{ and } (r_{j-1}, r_j) \in \mathsf{e}(G), \quad \forall\, j \in \{1, 2, \ldots \ell\}. \tag{7}$$

Through the utilisation of (i) of the weak (G, ψ)-contractivity condition, we find for each $0 \leq j \leq \ell - 1$ that

$$(\mathcal{H}^n r_j, \mathcal{H}^n r_{j+1}) \in \mathsf{e}(\tilde{G}), \quad \forall\, n \in \mathbb{N}_0. \tag{8}$$

The application of the triangle inequality reveals

$$\varpi(t, s) = \varpi(\mathcal{H}^n r_0, \mathcal{H}^n r_\ell) \leq \sum_{j=0}^{\ell-1} \varpi(\mathcal{H}^n r_j, \mathcal{H}^n r_{j+1}). \tag{9}$$

For every $j(0 \leq j \leq \ell - 1)$, denote $\mu_n^j := \varpi(\mathcal{H}^n r_j, \mathcal{H}^n r_{j+1})$, where $n \in \mathbb{N}_0$. Now, claim that

$$\lim_{n\to\infty} \mu_n^j = 0.$$

To substantiate this, upon fixing j, assuming first that $\mu_{n_0}^j = 0$ for some $n_0 \in \mathbb{N}_0$, then $\mathcal{H}^{n_0+1}(r_j) = \mathcal{H}^{n_0+1}(r_{j+1})$. Thus, we find $\mu_{n_0+1}^j = \varpi(\mathcal{H}^{n_0+1} r_j, \mathcal{H}^{n_0+1} r_{j+1}) = 0$; so, inductively, we find $\mu_n^j = 0$ for every $n \geq n_0$ so that $\lim_{n\to\infty} \mu_n^j = 0$. In contrast, if $\mu_n^j > 0$ for every $n \in \mathbb{N}_0$, then by (8) and (ii) of the weak (G, ψ)-contractivity condition, we obtain

$$\begin{aligned} \mu_{n+1}^j &= \varpi(\mathcal{H}^{n+1} r_j, \mathcal{H}^{n+1} r_{j+1}) \\ &\leq \varpi(\mathcal{H}^n r_j, \mathcal{H}^n r_{j+1}) - \psi(\varpi(\mathcal{H}^n r_j, \mathcal{H}^n r_{j+1})) \\ &= \mu_n^j - \psi(\mu_n^j) \\ &\leq \mu_n^j. \end{aligned} \tag{10}$$

Consequently, $\{\mu_n^j\}$ is a decreasing sequence in $\mathbb{R}^+$; so $\mu_n^j \to \mu \in \mathbb{R}^+$. In contrast, if $\mu \neq 0$, then from (10), we obtain

$$\limsup_{n\to\infty} \mu_{n+1}^j \leq \limsup_{n\to\infty} \mu_n^j + \limsup_{n\to\infty}(-\psi(\mu_n^j))$$

$$\mu \leq \mu - \liminf_{n\to\infty}(\psi(\mu_n^j))$$

implying thereby $\liminf_{n\to\infty} \psi(\mu_n^j) \leq 0$, which contradicts Ψ_2. Thus, $\lim_{n\to\infty} \mu_n^j = 0$. Furthermore, (9) can be written as

$$\begin{aligned} \varpi(t,s) = \varpi(\mathcal{H}^n r_0, \mathcal{H}^n r_\ell) &\leq \sum_{j=0}^{\ell-1} \varpi(\mathcal{H}^n r_j, \mathcal{H}^n r_{j+1}) \\ &\leq \mu_n^0 + \mu_n^1 + \cdots + \mu_n^{\ell-1} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

which yields $t = s$; so, $\mathcal{H}$ owns a unique fixed point. □

4. Special Cases

Note that Theorems 1 and 2 deduce a number of existing results in the following respects:

Case-1. Under the complete digraph G_0 defined by $\mathsf{e}(G_0) = \mathbf{P}^2$, Theorem 2 reduces to a sharpened version of classical fixed-point theorem of Rhoades [4], which again by setting $\psi(r) = r - r\beta(r)$, whereas $\beta : \mathbb{R}^+ \to [0, 1)$ verifying $\beta(r_n) \to 1 \Rightarrow r_n \to 0$, reduces to the classical fixed-point theorem of Geraghty [16].

Case-2. Let $(\mathbf{P}, \preceq)$ be a poset. Define a digraph G_1 on $\mathbf{P}$ by $\mathsf{e}(G_1) = \{(t, s) \in \mathbf{P}^2 : t \preceq s\}$; our main results reduce to Theorems 2 and 3 of Harjani and Sadarangani [17]. Condition (i) of Definition 11 means that $\mathcal{H}$ is $\preceq$-nondecreasing, and Condition (ii) of Definition 11 means that the weak ψ-contraction inequality holds for merely $\preceq$-preserving elements of $\mathbf{P}$.

Case-3. Let $(\mathbf{P}, \preceq)$ be a poset. Define a digraph G_2 on $\mathbf{P}$ by $\mathsf{e}(G_2) = \{(t, s) \in \mathbf{P}^2 : t \preceq s \vee s \preceq t\}$. Theorem 2 reduces to Theorem 6 of Harjani and Sadarangani [17]. Condition (i) of Definition 11 means that $\mathcal{H}$ map two $\preceq$-comparable elements to two $\preceq$-comparable elements, and Condition (ii) of Definition 11 means that the weak ψ-contraction inequality holds for merely $\preceq$-comparable elements of $\mathbf{P}$.

Case-4. Let $e > 0$ be fixed. Define a digraph G_3 on $\mathbf{P}$ by $\mathsf{e}(G_3) = \{(t, s) \in \mathbf{P}^2 : \varpi(t, s) \leq e\}$. Elements of $\mathsf{e}(G_3)$ are termed e-closed (c.f. [14]). We see that $\mathsf{e}(G_3)$ contains all loops. Under this setting, our results deduce certain new results wherein the map $\mathcal{H}$ takes e-closed elements of $\mathbf{P}$ to e-closed elements, and the weak ψ-contraction inequality must be satisfied for merely e-closed elements of $\mathbf{P}$.

Case-5. Particularly for $\psi(r) = (1-\lambda)r$ $(0 \leq \lambda \leq 1)$, (after removing the transitivity requirement on G), Theorems 1 and 2 reduce to the corresponding results of Jachymski [15].

Case-6. If $\psi : \mathbb{R}^+ \to [0, \infty)$ is continuous and increasing such that $\psi(r) > 0$ for $r > 0$, $\psi(0) = 0$ and $\lim_{p\to\infty} \psi(p) = \infty$, then $\psi \in \Psi$. Under this substitution, Theorems 1 and 2 deduce the corresponding results of Samreen and Kamran [9].

Case-7. If $\psi(r) = r - \varphi(r)$, where φ is a right upper semi-continuous function such that $\varphi(r) < r$ for $r > 0$, then $\psi \in \Psi$. Under this substitution, Theorems 1 and 2 deduce the recent results because of Filali et al. [10].

5. Examples

To illuminate our results, we offer the following examples.

Example 1. *Suppose* $\mathbf{P} = [0,1]$ *is an MS with Euclidean metric* ϖ*. It is clear that* $(\mathbf{P}, \varpi)$ *is a CMS. Define a directed graph by* $v(G) = \mathbf{P}$ *and* $e(G) = \{t, s \in \mathbf{P} :\ 0 \leq t < s \leq \frac{1}{2}\}$*. Then, G is transitive.*

Consider the self-map

$$\mathcal{H}(t) = \begin{cases} \frac{1}{5} + \frac{t}{3}, & \text{if } t \in [0, \frac{1}{2}] \\ \frac{1}{4}, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Then, $\mathcal{H}$ *is orbitally G-continuous. Now, take* $(t,s) \in e(G)$*; then,* $(\mathcal{H}t, \mathcal{H}s) \in e(G)$ *and*

$$\begin{aligned} \varpi(\mathcal{H}t, \mathcal{H}s) &= \left|\frac{1}{5} + \frac{t}{3} - \frac{1}{5} - \frac{s}{3}\right| \\ &= \left|\frac{t}{3} - \frac{s}{3}\right| \\ &= \frac{1}{3}|t-s| \\ &\leq \frac{1}{6} \quad \textit{as} \quad (t,s) \in e(G) \quad \textit{then} \quad |t-s| \leq \frac{1}{2} \\ &< |t-s|(1-|t-s|) \\ &= |t-s| - |t-s|^2 \\ &= \varpi(t,s) - \psi(\varpi(t,s)), \end{aligned}$$

which implies that $\mathcal{H}$ *satisfies the weak* (G, ψ)*-contractivity condition for* $\psi(r) = r^2$*. Henceforth, all requirements of Theorem 1 are met; consequently,* $\mathcal{H}$ *owns a fixed point. Meanwhile, the fixed point remains unique because it fulfills every assumption contained in Theorem 2. Note that the fixed point of* $\mathcal{H}$ *in the present instance is* $t = \frac{3}{10}$*.*

Example 2. *Suppose* $\mathbf{P} = \mathbb{R}^2$ *is a MS with the following metric:*

$$\varpi((t,s)(p,q)) = \frac{|t-p| + |s-q|}{2} \quad \forall (t,s), (p,q) \in \mathbf{P}.$$

It is clear that $(\mathbf{P}, \varpi)$ *is a CMS. Define a directed graph by* $v(G) = \mathbf{P}$ *and* $e(G) = \{((t,s),(p,q)) \in \mathbb{M}^2 \times \mathbb{M}^2 : p - t \geq 0, s - q \geq 0\}$*. Then, G is transitive.*

Consider the self-map

$$\mathcal{H}(t,s) = \left(\frac{t-2s}{4}, \frac{s-2t}{4}\right) \forall (t,s) \in \mathbf{P}.$$

Then, $(-2,3) \in \mathbb{M}(\mathcal{H}, e(G))$. Define $\psi(r) = \frac{r}{8}$. Then, $\psi \in \Psi$. Take $(t,s),(p,q)) \in e(G)$. Then, we have $(\mathcal{H}t, \mathcal{H}s),(p,q)) \in e(G)$. Also, we have

$$\begin{aligned} \varpi(\mathcal{H}(t,s), \mathcal{H}(p,q)) &= \frac{1}{4}(|(t-p)+2(q-s)| + |(q-s)+2(t-p)|) \\ &= \frac{3}{8}(t-p+q-s). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \varpi((t,s),(p,q))) - \psi(\varpi((t,s),(p,q))) &= \varpi((t,s),(p,q)) - \frac{\varpi((t,s),(p,q))}{8} \\ &= \frac{7}{8}\varpi((t,s),(p,q)) \\ &= \frac{7}{16}(t-p+q-s). \end{aligned}$$

Thus, the weak (G,ψ)-contractivity condition holds. It can be easily verified that G remains a (C)-graph. Hence, by Theorem 1, $\mathcal{H}$ possesses a fixed point. Meanwhile, the fixed point remains unique because it fulfills every assumption contained in Theorem 2. Note that the fixed point of $\mathcal{H}$ in the present instance is $\mathbf{t} = (0,0)$.

Example 3. *Suppose $\mathbf{P} = [0,1]$ is a MS with Euclidean metric ϖ. It is clear that $(\mathbf{P}, \varpi)$ is a CMS. Define a directed graph by $v(G) = \mathbf{P}$ and $e(G) = \{(t,s) \in \mathbf{P}^2 : ts \in \{t,s\}\}$. Then G is transitive.*

Let $\zeta : \mathbf{V} \to \mathbf{V}$ be a map defined by

$$\mathcal{H}(t) = \begin{cases} \frac{t}{2}, & \text{if } t \in \mathbb{Q} \cap \mathbf{P} \\ 0, & \text{if } t \in \mathbb{Q}^c \cap \mathbf{P}. \end{cases}$$

Then $\mathcal{H}$ is orbitally G-continuous. Also, it can be easily verified that $\mathcal{H}$ satisfies the weak (G,ψ)-contractivity condition for $\psi(r) = \frac{1}{2}r$. Consequently, in lieu of Theorem 1, $\mathcal{H}$ owns a fixed point. Meanwhile, the fixed point remains unique because it fulfills every assumption contained in Theorem 2. Take note that the fixed point of $\mathcal{H}$ in the present instance is $\mathbf{t} = 0$.

The involved map in the above example is not a weak ψ-contraction, as, in particular, for the pair $t = 0$ and $s = \frac{1}{\sqrt{2}}$, we have

$$\varpi(\mathcal{H}t, \mathcal{H}s) = 0 > \frac{1}{2\sqrt{2}} = \varpi(t,s) - \psi(\varpi(t,s)).$$

Thus far, Example 3 cannot work in the context of an ordinary MS, which substantiates the utility of fixed-point outcomes in an MS endowed with a graph over the corresponding outcomes in an ordinary MS.

6. An Application to NIEs

This section addresses an application of earlier fpts to compute a unique solution of the following NIE:

$$v(\omega) = \digamma(\omega) + \int_a^b \mathbf{L}(\omega,\tau)\Theta(\tau, v(\tau))d\tau, \qquad \omega \in S := [a,b] \tag{11}$$

where $\digamma : S \to \mathbb{R}$, $\mathbf{L} : S^2 \to \mathbb{R}$ and $\Theta : S \times \mathbb{R} \to \mathbb{R}$ remain functions.

Definition 12. $\overline{v} \in \mathcal{C}(S)$ *is named as a lower solution of* (11) *if*

$$\overline{v}(\omega) \leq \digamma(\omega) + \int_a^b \mathbf{L}(\omega,\tau)\Theta(\tau,\overline{v}(\tau))d\tau, \quad \forall\, \omega \in S.$$

Definition 13. $\underline{v} \in \mathcal{C}(S)$ *is named as an upper solution of* (11) *if*

$$\underline{v}(\omega) \geq \digamma(\omega) + \int_a^b \mathbf{L}(\omega,\tau)\Theta(\tau,\underline{v}(\tau))d\tau, \quad \forall\, \omega \in S.$$

Below are the essential results of this section.

Theorem 3. *In addition to Problem* (11), *suppose that*

(a) $\digamma$, Θ *and* $\mathbf{L}$ *are continuous;*
(b) $\mathbf{L}(\omega,\tau) > 0$, *for all* $\omega,\tau \in S$;
(c) *There exists* $\varepsilon > 0$ *that satisfies for any* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq \beta$ *that*

$$0 \leq [\Theta(\omega,\alpha) + \varepsilon\alpha] - [\Theta(\omega,\beta) + \varepsilon\beta] \leq \varepsilon \ln(\alpha - \beta + 1), \tag{12}$$

(d) $\sup\limits_{\omega\in S} \int_a^b \mathbf{L}(\omega,\tau)d\tau \leq \dfrac{1}{\varepsilon}$.

If Problem (11) *has a lower solution, then it enjoys a unique solution.*

Proof. Define $\mathbf{P} := \mathcal{C}(S)$. Consider metric ϖ on $\mathbf{P}$ by

$$\varpi(v,u) = \sup_{\omega\in S} |v(\omega) - u(\omega)|, \quad \text{for all } v,u \in \mathbf{P}. \tag{13}$$

Clearly $(\mathbf{P},\varpi)$ is a CMS. Define a digraph G by $\mathsf{v}(G) = \mathbf{P}$ and $\mathsf{e}(G) = \{(v,u) \in \mathbf{P}^2 : v(\omega) \leq u(\omega),\ \forall\, \omega \in S\}$. Then, G is transitive. Consider a map $\mathcal{H} : \mathbf{P} \to \mathbf{P}$ defined by

$$(\mathcal{H}v)(\omega) = \digamma(\omega) + \int_a^b \mathbf{L}(\omega,\tau)\Theta(\tau,v(\tau))d\tau, \quad \text{for all } \omega \in \mathbf{P}. \tag{14}$$

Naturally, $v_0 \in \mathbf{P}$ forms a solution to Problem (11) iff v_0 is a fixed point of $\mathcal{H}$.

Define the auxiliary function $\psi \in \Phi$ by $\psi(r) = r - \ln(r+1)$. In order to verify that $\mathcal{H}$ satisfies a weak (G,ψ)-contractivity condition, let us take $v,u \in \mathbf{P}$ such that $(v,u) \in \mathsf{e}(G)$. For a coat of (c), we arrive at

$$\Theta(\omega,v(\tau)) - \Theta(\omega,u(\tau)) \leq 0, \quad \forall\, \omega,\tau \in S. \tag{15}$$

Through (14), (15), and (b), we conclude

$$(\mathcal{H}v)(\omega) - (\mathcal{H}u)(\omega) = \int_a^b \mathbf{L}(\omega,\tau)[\Theta(\tau,v(\tau)) - \Theta(\tau,u(\tau))]d\tau \leq 0,$$

following that $(\mathcal{H}v)(\omega) \leq (\mathcal{H}u)(\omega)$. Consequently, we have $(\mathcal{H}v,\mathcal{H}u) \in \mathsf{e}(G)$. Furthermore, using item (c), (13) and (14), we find

$$\begin{aligned}
\varpi(\mathcal{H}v,\mathcal{H}u) &= \sup_{\omega\in S} |(\mathcal{H}v)(\omega) - (\mathcal{H}u)(\omega)| = \sup_{\omega\in S}[(\mathcal{H}u)(\omega) - (\mathcal{H}v)(\omega)] \\
&\leq \sup_{\omega\in S} \int_a^b \mathbf{L}(\omega,\tau)[\Theta(\omega,u(\tau)) + \varepsilon u(\tau) - \Theta(\omega,v(\tau)) - \varepsilon v(\tau)]d\tau \\
&\leq \sup_{\omega\in S} \int_a^b \mathbf{L}(\omega,\tau)\varepsilon \ln(u(\tau) - v(\tau) + 1)d\tau. \qquad (16)
\end{aligned}$$

But $\theta(r) = \ln(r+1)$ is increasing, and $0 \le u(\tau) - v(\tau) \le \varpi(v,u)$; so, we find

$$\ln(u(\tau) - v(\tau) + 1) \le \ln(\varpi(v,u) + 1).$$

Hence, (16) becomes

$$\begin{aligned} \varpi(\mathcal{H}v, \mathcal{H}u) &\le \varepsilon \ln(\varpi(v,u)+1) \sup_{\omega \in S} \int_a^b \mathbf{L}(\omega,\tau) d\tau \\ &\le \varepsilon \ln(\varpi(v,u)+1) \cdot \frac{1}{\varepsilon} \\ &= \varpi(v,u) - [\varpi(v,u) - \ln(\varpi(v,u)+1)] \end{aligned}$$

so that

$$\varpi(\mathcal{H}v, \mathcal{H}u) \le \varpi(v,u) - \psi(\varpi(v,u)), \quad \forall\, (v,u) \in \mathsf{e}(G).$$

Assuming that $\{v_n\} \subset \mathbf{P}$ is a sequence, which verifies $(v_n, v_{n+1}) \in \mathsf{e}(G)$ and $v_n \to \tilde{v} \in \mathbf{P}$. Then, for every $\omega \in S$, $\{v_n(\omega)\} \subset \mathbb{R}$ is increasing, which will converge to $\tilde{v}(\omega)$, indicating that $v_n(\omega) \le \tilde{v}(\omega)$ for all $\omega \in S$, and so, $(v_n, \tilde{v}) \in \mathsf{e}(G)$. Thus, G is a (C)-graph.

Let $\overline{v} \in \mathbf{P}$ be a lower solution of (11). Then,

$$\overline{v}(\omega) \le F(\omega) + \int_a^b \mathbf{L}(\omega,\tau)\Theta(\tau, \overline{v}(\tau)) d\tau = (\mathcal{H}\overline{v})(\omega)$$

yielding thereby $(\overline{v}, \mathcal{H}\overline{v}) \in \mathsf{e}(G)$ so that $\mathbf{P}_{\mathcal{H}} \ne \emptyset$. Thus, all premises of Theorem 1 are validated, and hence, $\mathcal{H}$ is WPM. Finally, we verify the assumption of Theorem 2. Let $v, u \in$ be chosen arbitrarily. Set $w := \max\{v,u\} \in \mathbf{P}$. We have $(v,w) \in \mathsf{e}(G)$ and $(u,w) \in \mathsf{e}(G)$. Consequently, G is weakly connected; so, using Theorem 2, $\mathcal{H}$ admits a unique fixed point, which remains a unique solution for (11). □

Theorem 4. *In the conjunction with accusations* (a)–(d) *of Theorem 3, if there is an upper solution of* (11), *then the problem admits a unique solution.*

Proof. Let $\mathbf{P} := \mathcal{C}(S)$ with a metric ϖ. Clearly, $(\mathbf{P}, \varpi)$ is a CMS. Consider a map $\mathcal{H} : \mathbf{P} \to \mathbf{P}$ defined the same as the proof of Theorem 3. Define a directed graph G' by $\mathsf{v}(G') = \mathbf{P}$ and $\mathsf{e}(G)' = \{(v,u) \in \mathbf{P}^2 : v(\omega) \ge u(\omega),\ \forall\, \omega \in S\}$. Then, G is transitive. If $\underline{v} \in \mathbf{P}$ is an upper solution of (11), then we obtain

$$\underline{v}(\omega) \ge F(\omega) + \int_a^b \mathbf{L}(\omega,\tau)\Theta(\tau, \underline{v}(\tau)) d\tau = (\mathcal{H}\underline{v})(\omega)$$

which concludes that $(\underline{v}, \mathcal{H}\underline{v}) \in \mathsf{e}(G)'$ so that $\mathbf{P}_{\mathcal{H}} \ne \emptyset$.

Take $v, u \in \mathbf{P}$ such that $(v,u) \in \mathsf{e}(G)'$. Using (c), we obtain

$$\Theta(\omega, v(\tau)) - \Theta(\omega, u(\tau)) \ge 0, \quad \text{for all } \omega, \tau \in S. \tag{17}$$

Making use of (14), (17) and (b), we get

$$(\mathcal{H}v)(\omega) - (\mathcal{H}u)(\omega) = \int_a^b \mathbf{L}(\omega,\tau)[\Theta(\tau, v(\tau)) - \Theta(\tau, u(\tau))] d\tau \ge 0.$$

Thus, $(\mathcal{H}v)(\omega) \ge (\mathcal{H}u)(\omega)$ so that $(\mathcal{H}v, \mathcal{H}u) \in \mathsf{e}(G')$.

Let $\{v_n\} \subset \mathbf{P}$ be a sequence, which verifies $(v_n, v_{n+1}) \in \mathsf{e}(G')$ and $v_n \to \tilde{v} \in \mathbf{P}$. Then, for every $\omega \in S$, $\{v_n(\omega)\}$ is a decreasing sequence (in $\mathbb{R}$) that converges to $v(\omega)$. Consequently, for all $n \in \mathbb{N}$, we have $v_n(\omega) \ge v(\omega)$ for all $\omega \in S$; so, $(v_n, v) \in \mathsf{e}(G')$. Thus, G' is a (C)-graph.

Overall, we have therefore confirmed all claims of Theorems 1 and 2 for the MS $(\mathbf{P}, \varpi)$, the map $\mathcal{H}$ and the graph G'. The proof is now finalized. □

7. Conclusions

We demonstrate some fpts employing a weak (G, ψ)-contraction map of metric spaces comprising a reflexive and transitive digraph G. We also observe that our findings deduce a number of existing results especially contained in Harjani and Sadarangani [17], Rhoades [4], Geraghty [16], Jachymski [15], Samreen and Kamran [9], and Filali et al. [10]. To validate our findings, we furnish several examples. The findings we obtain enable us to seek out the unique solution of a nonlinear integral equation.

The idea of weak ψ-contractions was further generalized and developed by Dutta and Choudhury [21], Đorić [22], Popescu [23], Fallahi et al. [24] and similar others by employing the concept of (ψ, ϕ)-contractions that depend on two auxiliary functions. Thus far, as a future work, we can further extend Theorem 1 for (ψ, ϕ)-contractions in a metric space equipped with a reflexive and transitive directed graph. In a short while, scholars may also adapt the fpt we found to two or more maps or to various generalized distance spaces, such as b-metric space, quasi-metric space, cone metric space, etc., or we can apply the same results to obtain a unique solution of certain boundary value problems.

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