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Investigating Statistical Submersions and Their Properties

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Abstract: This research aims to prove a sharp relationship between statistical submersions and doubly minimal immersions. We consider a non-trivial statistical submersion on a statistical manifold with isometric fibers. Then, we investigate that it cannot be isometrically immersed as a doubly minimal manifold into any statistical manifold of non-positive sectional curvature using a submersion invariant for dual connections and additional conditions.

Keywords: statistical manifolds; statistical submersions; doubly minimal immersions; scalar curvature; isometric fibers; submersion invariants; dual affine connections

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1. Introduction

In the context of differential geometry and statistics, a statistical manifold refers to a semi-Riemannian manifold (see [1]) equipped with additional structures. Specifically, a statistical manifold is defined as a pair (M, g), where M is a smooth manifold and g is a semi-Riemannian metric on M.

The introduction of Riemannian submersions by Gray and O'Neill has had a significant impact on the field of differential geometry. It has provided a powerful tool for constructing and analyzing Riemannian manifolds with desired curvature properties, as well as facilitating comparisons and investigations of geometric structures. The theory of Riemannian submersions continues to be actively studied and utilized in various areas of mathematics and physics. The theory of Riemannian submersions has been extended over the last three decades by many geometers [2,3].

M. Noguchi [4] conducted a study on statistical manifolds. On a statistical manifold, an alternative connection, referred to as the conjugate (or dual) connection, is established [5,6]. This notion has been extensively explored in information geometry [5,7]. The concept of statistical submersions between statistical manifolds is a specialized topic in mathematical statistics and differential geometry. By generalizing some of the foundational results of B. O'Neill on Riemannian submersions and geodesics [8], Abe and Hasegawa [9] extended the framework to the context of statistical manifolds. Since then, many geometers have contributed to this area (see [10–12]). In recent years, various types of statistical submersions have been explored, such as cosymplectic-like statistical submersions [13], quaternionic Kähler-like statistical submersions [14], and para-Kähler-like statistical submersions [15]. Building on Takano's work, M.D. Siddiqi et al. [16] presented and comprehensively discussed Kenmotsu-like statistical submersions. Recent research by S. Kazan et al. [17] investigated holomorphic statistical submersions, unveiling anti-invariant statistical submersions from holomorphic statistical manifolds. Additionally, a comprehensive exploration of locally product-like statistical submersions was undertaken in [18]. Indeed, numerous submersions are discussed in the survey article [19]. Also, the Chen–Ricci inequality has been established for the statistical submersion (see [20,21]). Moreover, other inequalities have been derived for constant curvature submanifolds within statistical manifolds [22].

We observe that these structures hold immense significance not only in the field of differential geometry but also across a diverse array of scientific and engineering domains. For instance, in string theory, these structures play a crucial role in understanding the fundamental nature of particles and their interactions. In integrable systems, they help solve complex equations that describe physical phenomena. Quantum systems benefit from these structures in modeling and analyzing the behavior of particles at the quantum level. Additionally, in statistical mechanics, these structures aid in studying the behavior of systems with a large number of particles, providing insights into phase transitions and critical phenomena. In the field of motion planning, these structures are essential for devising algorithms that enable autonomous agents, such as robots, to navigate and perform tasks efficiently. In robotic control and sensing, they enhance the precision and reliability of robotic movements and the interpretation of sensory data. Furthermore, in sensor networks, these structures facilitate the efficient organization and communication of data across multiple sensors, improving the overall performance of the network. Lastly, in digital signal processing, they contribute to the development of advanced techniques for filtering, compressing, and analyzing signals, leading to better performance in applications such as audio and image processing.

In summary, the importance of these structures extends far beyond differential geometry, influencing a wide range of fields that are fundamental to both theoretical research and practical applications in science and engineering.

Constructing Riemannian manifolds with positive or non-negative sectional curvature is a fundamental and classic challenge in Riemannian geometry. Riemannian submersions serve as one method for this, and they have also been instrumental in developing many known Einstein manifolds. The versatility of Riemannian submersions is demonstrated by their application in various fields, including Kaluza–Klein theory, statistical machine learning, medical imaging, statistical analysis on manifolds, and robotics theory. Additionally, this research aims to establish a simple, optimal connection between statistical submersions and minimal immersions, with a discussion of some related findings.

This research primarily employs the following lemma from [23] to demonstrate that a statistical manifold, which admits a non-trivial statistical submersion with isometric fibers, cannot be isometrically immersed as a doubly minimal manifold in any statistical manifold of non-positive sectional curvature.

Lemma 1. *Let $A_1, A_2, \cdots, A_p, C$ be $p+1$ $(p \geq 2)$ real numbers such that*

$$\left(\sum_i A_i\right)^2 = (p-1)\left(\sum_i A_i^2 + C\right),$$

where i runs from i to p. Then, we have $2A_1A_2 \geq C$. The equality case holds if and only if $A_1 + A_2 = A_3 = \cdots = A_p$.

This paper is structured as follows. Section 1 provides an introduction. In Section 2, we present essential notions related to statistical submersions. In Section 3, we investigate statistical submersions with isometric fibers. Section 4 explores the sharp relationship between statistical submersions and doubly minimal immersions. This paper concludes with some final remarks.

2. Statistical Submersions

Definition 1. *Consider two semi-Riemannian manifolds $(\mathbb{M}, G)$ and $(\mathbb{N}, g)$. Then, a surjective mapping $\vartheta : \mathbb{M} \to \mathbb{N}$ is called a semi-Riemannian submersion if it satisfies the following conditions:*

1. *ϑ has maximal rank;*
2. *ϑ_* preserves the lengths of horizontal vectors.*

Now, we take the dimensions of $\mathbb{M}$ and $\mathbb{N}$ as $r > 0$ and $s > 0$, respectively, with $r > s$. For each point $x \in \mathbb{N}$, the submanifold $\vartheta^{-1}(x)$ of $\mathbb{M}$, with the induced metric $\tilde{G}$ and dimension $r - s > 0$, is called a fiber and symbolized by $\mathbb{B}$. A vector field on $\mathbb{M}$ is either tangent to fibers (called vertical) or orthogonal to fibers (called horizontal). Also, a vector field $\mathbb{X}$ on $\mathbb{M}$ is called basic if it satisfies the following conditions:

1. It is ϑ-related to the vector field $\mathbb{X}_*$ on $\mathbb{N}$;
2. It is horizontal.

For each point $y \in \mathbb{M}$, let the tangent space of the total space $\mathbb{M}$ be $T_y\mathbb{M}$ and the vertical and horizontal subspaces in $T_y\mathbb{M}$ be $V_y(\mathbb{M})$ and $H_y(\mathbb{M})$, respectively. Let the tangent bundle denoted by $T\mathbb{M}$ of $\mathbb{M}$ be expressed as

$$T\mathbb{M} = V(\mathbb{M}) \oplus H(\mathbb{M}),$$

where $V(\mathbb{M})$ and $H(\mathbb{M})$ are the vertical and horizontal distributions, respectively. The projection mappings are denoted as $V : T\mathbb{M} \to V(\mathbb{M})$ and $H : T\mathbb{M} \to H(\mathbb{M})$, respectively.

For a torsion-free affine connection D and a metric G on a (semi-)Riemannian manifold $(\mathbb{M}, G)$, we say that $(\mathbb{M}, D, G)$ is a statistical manifold if DG is a symmetric $(0, 3)$-tensor. Any torsion-free affine connection D always has a dual (or conjugate) connection given by $2D^0 = D + D'$, where D' denotes the conjugate connection of D on $\mathbb{M}$ and D^0 is the Levi–Civita connection on $\mathbb{M}$.

Let us take a statistical manifold $(\mathbb{M}, D, G)$ and a semi-Riemannian submersion $\vartheta : \mathbb{M} \to \mathbb{N}$. We denote the affine connections of $\mathbb{B}$ as ∇ and ∇', which are torsion-free and conjugate to each other for $\tilde{G}$. The triple $(\mathbb{B}, \nabla, \tilde{G})$ is a statistical manifold and so is $(\mathbb{B}, \nabla', \tilde{G})$.

Definition 2. *Let $(\mathbb{M}, D, G)$ and $(\mathbb{N}, \mathcal{D}, g)$ be two statistical manifolds, where $\mathcal{D}$ is the affine connection on $\mathbb{N}$. Then a semi-Riemannian submersion $\vartheta : (\mathbb{M}, D, G) \to (\mathbb{N}, \mathcal{D}, g)$ is said to be a statistical submersion [11] if ϑ satisfies $\vartheta_*(D_{\mathbb{X}}\mathbb{Y})_y = (\mathcal{D}_{\mathbb{X}_*}\mathbb{Y}_*)_{\vartheta(y)}$ for basic vector fields $\mathbb{X}, \mathbb{Y}$.*

In classical semi-Riemannian geometry, B. O'Neill defined two $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ in [8]. So, in statistical geometry, $\mathcal{T}$ and $\mathcal{A}$ on $\mathbb{M}$ with respect to D are defined by the following formulas [11,12]:

$$\mathcal{T}_X Y = H D_{VX} VY + V D_{VX} HY, \quad (1)$$

$$\mathcal{A}_X Y = V D_{HX} HY + H D_{HX} VY, \quad (2)$$

for $X, Y \in \Gamma(T\mathbb{M})$.

Similarly, the tensor fields $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$ on $\mathbb{M}$ can also be defined by replacing D with D' in the above equations.

For vertical vector fields $\mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{W}'$ and horizontal vector fields $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{Z}'$, we have the following formulas:

$$R_{\mathbb{U},\mathbb{V};\mathbb{W},\mathbb{W}'} = R^F_{\mathbb{U},\mathbb{V};\mathbb{W},\mathbb{W}'} + G(\mathcal{T}_{\mathbb{U}}\mathbb{W}, \tilde{\mathcal{T}}_{\mathbb{V}}\mathbb{W}') - G(\mathcal{T}_{\mathbb{V}}\mathbb{W}, \tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{W}'), \quad (3)$$

$$\begin{aligned} R_{\mathbb{X},\mathbb{Y};\mathbb{Z},\mathbb{Z}'} = {} & R^{\mathbb{N}}_{\mathbb{X},\mathbb{Y};\mathbb{Z},\mathbb{Z}'} - G(\mathcal{A}_{\mathbb{Y}}\mathbb{Z}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{Z}') + G(\mathcal{A}_{\mathbb{X}}\mathbb{Z}, \tilde{\mathcal{A}}_{\mathbb{Y}}\mathbb{Z}') \\ & + G((\mathcal{A}_{\mathbb{X}} + \tilde{\mathcal{A}}_{\mathbb{X}})\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{Z}}\mathbb{Z}'), \end{aligned} \quad (4)$$

$$\begin{aligned} R_{\mathbb{X},\mathbb{U};\mathbb{V},\mathbb{Y}} = {} & G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{V}, \mathbb{Y}) - G((D_{\mathbb{U}}\mathcal{A})_{\mathbb{X}}\mathbb{V}, \mathbb{Y}) \\ & + G(\mathcal{A}_{\mathbb{X}}\mathbb{U}, \tilde{\mathcal{A}}_{\mathbb{Y}}\mathbb{V}) - G(\mathcal{T}_{\mathbb{U}}\mathbb{X}, \tilde{\mathcal{T}}_{\mathbb{V}}\mathbb{Y}), \end{aligned} \quad (5)$$

$$\begin{aligned} R_{\mathbb{X},\mathbb{U};\mathbb{Y},\mathbb{V}} = {} & G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{Y}, \mathbb{V}) - G((D_{\mathbb{U}}\mathcal{A})_{\mathbb{X}}\mathbb{Y}, \mathbb{V}) \\ & - G(\mathcal{A}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{Y}}\mathbb{V}) + G(\mathcal{T}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{V}}\mathbb{Y}), \end{aligned} \quad (6)$$

where R, R^F, and $R^{\mathbb{N}}$ are the Riemannian curvature tensor of $\mathbb{M}$, that of $\mathbb{B}$ with respect to the induced affine connection ∇, and that of $\mathbb{N}$ with respect to $\mathcal{D}$. Note that $\mathcal{T}_{\mathbb{U}}\mathbb{V}$ is the second fundamental form of each fiber. If $\mathcal{T}_{\mathbb{U}}\mathbb{V} = 0 = \tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{V}$ is satisfied for vertical vector fields $\mathbb{U}$ and $\mathbb{V}$, then ϑ is referred to as a statistical submersion with doubly isometric fibers. But we use isometric fibers instead of doubly isometric fibers.

Lemma 2. *The tensor fields $\mathcal{T}$, $\mathcal{A}$, $\tilde{\mathcal{T}}$, and $\tilde{\mathcal{A}}$ on $\mathbb{M}$ have the following properties:*

1. $\mathcal{T}_{\mathbb{U}}\mathbb{V} = \mathcal{T}_{\mathbb{V}}\mathbb{U}$; $\quad \tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{V} = \tilde{\mathcal{T}}_{\mathbb{V}}\mathbb{U}$,
2. $\mathcal{A}_{\mathbb{X}}\mathbb{Y} = -\tilde{\mathcal{A}}_{\mathbb{Y}}\mathbb{X}$,
3. $\mathcal{A}_{\mathbb{X}}\mathbb{Y} - \mathcal{A}_{\mathbb{Y}}\mathbb{X} = V[\mathbb{X}, \mathbb{Y}]$,
4. $\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{Y} - \tilde{\mathcal{A}}_{\mathbb{Y}}\mathbb{X} = V[\mathbb{X}, \mathbb{Y}]$,
5. $G(\mathcal{T}_{\mathbb{U}}\mathbb{V}, \mathbb{X}) = -G(\mathbb{V}, \tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X})$,
6. $G(\mathcal{A}_{\mathbb{X}}\mathbb{Y}, \mathbb{U}) = -G(\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U})$.

3. Statistical Submersions with Isometric Fibers

Consider an isometric immersion $(\mathbb{M}, D, G)$ of dimension r into an m-dimensional statistical manifold $(\overline{\mathbb{M}}, \overline{\nabla}, \overline{G})$. We denote the conjugate connection of $\overline{\nabla}$ on $\overline{\mathbb{M}}$ as $\overline{\nabla}'$. Then, we have the following:

1. $\overline{R}$ and $\overline{R}'$ are the Riemannian curvature tensors of $\overline{\mathbb{M}}$ with respect to $\overline{\nabla}$ and $\overline{\nabla}'$, respectively;
2. S and $\overline{S}$ are the sectional curvature functions of $\mathbb{M}$ and $\overline{\mathbb{M}}$;
3. $\tau(y)$ and $\overline{\tau}(T_y\mathbb{M})$ are the scalar curvatures of $\mathbb{M}$ and $\overline{\mathbb{M}}$;
4. h and $\tilde{h}$ are the symmetric bilinear forms called the embedding curvature tensors of $\mathbb{M}$ in $\overline{\mathbb{M}}$ for $\overline{\nabla}$ and $\overline{\nabla}'$, respectively;
5. $\mathcal{H} = \frac{1}{r}trace(h)$ and $\tilde{\mathcal{H}} = \frac{1}{r}trace(\tilde{h})$ are the mean curvature vector fields of $\mathbb{M}$ for $\overline{\nabla}$ and $\overline{\nabla}'$, respectively;
6. h^0 and $\mathcal{H}^0 = \frac{1}{r}trace(h^0)$ are the second fundamental form and the mean curvature vector field of $\mathbb{M}$ for the Levi–Civita connection $\overline{\nabla^0}$ on $\overline{\mathbb{M}}$.

For an orthonormal basis $\{\mathcal{E}_I | I = 1, 2, 3, \cdots, r\}$ of $T_y\mathbb{M}$, $y \in \mathbb{M}$, we define

$$\tau(y) = \sum_{I<J} S(\mathcal{E}_I, \mathcal{E}_J) \quad \text{and} \quad \overline{\tau}(T_y\mathbb{M}) = \sum_{I<J} S(\mathcal{E}_I, \mathcal{E}_J). \tag{7}$$

First, we prepare the following lemma.

Lemma 3. *Let $\vartheta : (\mathbb{M}, D, G) \to (\mathbb{N}, \mathcal{D}, g)$ be a statistical submersion. The lemma states the following:*

1. *For the horizontal vector fields $\mathbb{X}$ and $\mathbb{Y}$,*

$$\begin{aligned} R_{\mathbb{X},\mathbb{Y};\mathbb{Y},\mathbb{X}} &= R^{\mathbb{N}}_{\mathbb{X},\mathbb{Y};\mathbb{Y},\mathbb{X}} - 2||\mathcal{A}_{\mathbb{X}}\mathbb{Y}||^2 - G(\mathcal{A}_{\mathbb{Y}}\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}) \\ &\quad - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{Y}, \mathcal{A}_{\mathbb{X}}\mathbb{Y}). \end{aligned}$$

2. *For the horizontal vector field $\mathbb{X}$ and the vertical vector field $\mathbb{U}$,*

$$\begin{aligned} R_{\mathbb{X},\mathbb{U};\mathbb{U},\mathbb{X}} &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{U}) \\ &\quad + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}), \end{aligned}$$

and

$$\begin{aligned} R_{\mathbb{X},\mathbb{U};\mathbb{U},\mathbb{X}} &= -\tilde{R}_{\mathbb{X},\mathbb{U};\mathbb{X},\mathbb{U}} \\ &= -G((D'_{\mathbb{X}}\tilde{\mathcal{T}})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{U}) \\ &\quad + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}), \end{aligned}$$

where $\tilde{R}$ is the Riemannian curvature tensor of $\mathbb{M}$ with respect to D'.

Proof. From formula (4), we have

$$\begin{aligned} R_{\mathbb{X},\mathbb{Y};\mathbb{Y},\mathbb{X}} &= R^{\mathbb{N}}_{\mathbb{X},\mathbb{Y};\mathbb{Y},\mathbb{X}} - G(\mathcal{A}_{\mathbb{Y}}\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}) + G(\mathcal{A}_{\mathbb{X}}\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{Y}}\mathbb{X}) \\ &\quad + G((\mathcal{A}_{\mathbb{X}} + \tilde{\mathcal{A}}_{\mathbb{X}})\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{Y}}\mathbb{X}). \end{aligned}$$

Since we know that $\mathcal{A}_{\mathbb{X}}\mathbb{Y} = -\tilde{\mathcal{A}}_{\mathbb{Y}}\mathbb{X}$ holds for horizontal vector fields $\mathbb{X}$ and $\mathbb{Y}$, we obtain

$$\begin{aligned} R_{\mathbb{X},\mathbb{Y};\mathbb{Y},\mathbb{X}} &= R^{\mathbb{N}}_{\mathbb{X},\mathbb{Y};\mathbb{Y},\mathbb{X}} - G(\mathcal{A}_{\mathbb{Y}}\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}) - G(\mathcal{A}_{\mathbb{X}}\mathbb{Y}, \mathcal{A}_{\mathbb{X}}\mathbb{Y}) \\ &\quad - G((\mathcal{A}_{\mathbb{X}} + \tilde{\mathcal{A}}_{\mathbb{X}})\mathbb{Y}, \mathcal{A}_{\mathbb{X}}\mathbb{Y}) \\ &= R^{\mathbb{N}}_{\mathbb{X},\mathbb{Y};\mathbb{Y},\mathbb{X}} - 2||\mathcal{A}_{\mathbb{X}}\mathbb{Y}||^2 - G(\mathcal{A}_{\mathbb{Y}}\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}) \\ &\quad - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{Y}, \mathcal{A}_{\mathbb{X}}\mathbb{Y}). \end{aligned}$$

Next, we utilize (5) to derive the second equation of this lemma. Consequently, we have

$$\begin{aligned} R_{\mathbb{X},\mathbb{U};\mathbb{U},\mathbb{X}} &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) - G((D_{\mathbb{U}}\mathcal{A})_{\mathbb{X}}\mathbb{U}, \mathbb{X}) \\ &\quad + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}) \\ &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) - G(D_{\mathbb{U}}(\mathcal{A}_{\mathbb{X}}\mathbb{U}), \mathbb{X}) \\ &\quad + G(\mathcal{A}_{D_{\mathbb{U}}\mathbb{X}}\mathbb{U}, \mathbb{X}) + G(\mathcal{A}_{\mathbb{X}}(D_{\mathbb{U}}\mathbb{U}), \mathbb{X}) \\ &\quad + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}) \\ &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) - \mathbb{U}G(\mathcal{A}_{\mathbb{X}}\mathbb{U}, \mathbb{X}) + G(\mathcal{A}_{\mathbb{X}}\mathbb{U}, D'_{\mathbb{U}}\mathbb{X}) \\ &\quad - G(\mathbb{U}, \tilde{\mathcal{A}}_{D_{\mathbb{U}}\mathbb{X}}\mathbb{X}) - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}, D_{\mathbb{U}}\mathbb{U}) \\ &\quad + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}), \end{aligned}$$

where we have applied the definition of the statistical manifold and $G(\mathcal{A}_{\mathbb{X}}\mathbb{Y}, \mathbb{U}) = -G(\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U})$ (see Lemma 2).

Further, we have

$$\begin{aligned} R_{\mathbb{X},\mathbb{U};\mathbb{U},\mathbb{X}} &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) + \mathbb{U}G(\mathbb{U}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}) + G(\mathcal{A}_{\mathbb{X}}\mathbb{U}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}) \\ &\quad + G(\mathbb{U}, \mathcal{A}_{\mathbb{X}}D_{\mathbb{U}}\mathbb{X}) - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}, D_{\mathbb{U}}\mathbb{U}) \\ &\quad + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}) \\ &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) + G(\mathcal{A}_{\mathbb{X}}\mathbb{U}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, D_{\mathbb{U}}\mathbb{X}) \\ &\quad - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}, D_{\mathbb{U}}\mathbb{U}) + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) \\ &\quad - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}) \\ &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) + G(\mathcal{A}_{\mathbb{X}}\mathbb{U}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) \\ &\quad - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{U}) + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) \\ &\quad - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}) \\ &= G((D_{\mathbb{X}}\mathcal{T})_{\mathbb{U}}\mathbb{U}, \mathbb{X}) - G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{U}) \\ &\quad + G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \mathcal{A}_{\mathbb{X}}\mathbb{U}) - G(\tilde{\mathcal{T}}_{\mathbb{U}}\mathbb{X}, \mathcal{T}_{\mathbb{U}}\mathbb{X}). \end{aligned}$$

Similarly, it is worth noting that the last formula can be easily derived from Equation (6). □

Proposition 1. *Let $(\mathbb{M}, D, G)$ be a statistical manifold of non-positive sectional curvature. If a statistical submersion $\vartheta : (\mathbb{M}, D, G) \to (\mathbb{N}, \mathcal{D}, g)$ has isometric fibers, then the following results hold:*

1. *The horizontal distribution is integrable for D.*
2. *The total space $(\mathbb{M}, D, G)$ is a product space of the base space $(\mathbb{N}, \mathcal{D}, g)$ and the fiber.*
3. *$(\mathbb{N}, \mathcal{D}, g)$ has non-positive sectional curvature.*

Proof. Suppose that $\mathcal{A}_{\mathbb{X}}\mathbb{Y} \neq 0$ for the orthonormal horizontal vector fields $\mathbb{X}$ and $\mathbb{Y}$. By Lemma 2, we have

$$-G(\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}) = G(\mathcal{A}_{\mathbb{X}}\mathbb{Y}, \mathbb{U}) \neq 0,$$

for a unit vector $\mathbb{U}$ on $\mathbb{M}$. Therefore, by Lemma 3, it follows that

$$S(\mathbb{X} \wedge \mathbb{U}) = G(\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}) > 0.$$

This contradicts our assumption, implying that $\mathcal{A}$ must vanish identically. Since $\mathcal{A}$ is related to the integrability of the horizontal distribution, $\mathcal{A}$ is symmetric for horizontal vectors if and only if the horizontal distribution is integrable for D. Hence, the second part directly follows. The third part follows straightforwardly from Lemma 3 and the first part of this corollary. □

Example 1. *Consider a statistical manifold* $\mathbb{M}_r^{2r} = \{(x_1, y_1, \cdots, x_r, y_r) \in \mathbb{R}_r^{2r} | y_i \neq 0 \; for \; i = 1, 2, 3, \cdots, r)\}$ *with the metric*

$$G = \begin{pmatrix} \frac{2}{y_i^2}\delta_{ij} & 0 \\ 0 & \frac{1}{y_i^2}\delta_{ij} \end{pmatrix},$$

and the affine connection D*, defined by*

$$D_{\partial x_i}\partial x_j = -D_{\partial y_i}\partial y_j = \frac{4}{3y_i}\delta_{ij}\partial y_i,$$
$$D_{\partial x_i}\partial y_j = D_{\partial y_j}\partial x_i = \frac{-4}{3y_i}\delta_{ij}\partial x_i.$$

We define the statistical submersion $\vartheta : (\mathbb{M}_r^{2r}, D, G) \to (\mathbb{M}_l^{2l}, D, G)$ *by*

$$\vartheta(x_1, y_1, \cdots, x_r, y_r) = (x_1, y_1, \cdots, x_l, y_l) \quad (l < r).$$

Then, we find that ϑ *has isometric fibers and* $\mathcal{A} = 0$*. Consequently,* $\mathbb{M}_r^{2r}$ *is a product space of* $\mathbb{M}_l^{2l}$ *and the fiber.*

4. Sharp Relationship between Statistical Submersions and Doubly Minimal Immersions

We use the Gauss equation for r-dimensional $(\mathbb{M}, D, G)$ in $(\overline{\mathbb{M}}, \overline{\nabla}, \overline{G})$ of dimension m, and we find that

$$\begin{aligned} 2\overline{\tau}(T_y\mathbb{M}) &= 2\tau(y) - r^2\overline{G}(\mathcal{H}, \tilde{\mathcal{H}}) + \sum_{\alpha=r+1}^{m}\sum_{I,J=1}^{r}\overline{G}(h(\mathcal{E}_I, \mathcal{E}_J), \tilde{h}(\mathcal{E}_I, \mathcal{E}_J)) \\ &= 2\tau(y) - r^2\overline{G}(\mathcal{H}, \tilde{\mathcal{H}}) + \sum_{\alpha=r+1}^{m}\sum_{I,J=1}^{r} h_{IJ}^{\alpha}\tilde{h}_{IJ}^{\alpha} \end{aligned}$$

for an orthonormal basis $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \cdots, \mathcal{E}_r, \mathcal{E}_{r+1}, \mathcal{E}_{r+2}, \cdots, \mathcal{E}_m\}$.

On using $2\mathcal{H}^0 = \mathcal{H} + \tilde{\mathcal{H}}$ and $2h^0 = h + \tilde{h}$, our above equation is modified as

$$\begin{aligned} 2\overline{\tau}(T_y\mathbb{M}) &= 2\tau(y) - \frac{r^2}{4}[(\mathcal{H} + \tilde{\mathcal{H}})^2 - (\mathcal{H} - \tilde{\mathcal{H}})^2] \\ &\quad + \frac{1}{4}[(h^{\alpha}_{IJ} + \tilde{h}^{\alpha}_{IJ})^2 - (h^{\alpha}_{IJ} - \tilde{h}^{\alpha}_{IJ})^2]. \\ &= 2\tau(y) - r^2\mathcal{H}^{02} + \frac{r^2}{4}(\mathcal{H} - \tilde{\mathcal{H}})^2 \\ &\quad + ||h^0||^2 - \frac{1}{4}\sum_{\alpha=r+1}^{m}\sum_{I,J=1}^{r}(h^{\alpha}_{IJ} - \tilde{h}^{\alpha}_{IJ})^2, \end{aligned}$$

that is,

$$\begin{aligned} 2\tau(y) &= 2\overline{\tau}(T_y\mathbb{M}) + r^2\mathcal{H}^{02} - ||h^0||^2 - \frac{r^2}{4}(\mathcal{H} - \tilde{\mathcal{H}})^2 \\ &\quad + \frac{1}{4}\sum_{\alpha=r+1}^{m}\sum_{I,J=1}^{r}(h^{\alpha}_{IJ} - \tilde{h}^{\alpha}_{IJ})^2. \end{aligned} \tag{8}$$

We define

$$\begin{aligned} \lambda &= 2\tau - 2\overline{\tau}(T_y\mathbb{M}) - \frac{r^2}{2}\mathcal{H}^{02} + \frac{r^2}{4}(\mathcal{H} - \tilde{\mathcal{H}})^2 \\ &\quad - \frac{1}{4}\sum_{\alpha=r+1}^{m}\sum_{I,J=1}^{r}(h^{\alpha}_{IJ} - \tilde{h}^{\alpha}_{IJ})^2. \end{aligned} \tag{9}$$

Then, Equation (8) simplifies to

$$r^2\mathcal{H}^{02} = 2\lambda + 2||h^0||^2. \tag{10}$$

We now select a local orthonormal frame $\{\mathcal{E}_1, \mathcal{E}_s, \mathcal{E}_{s+1}, \cdots, \mathcal{E}_r, \cdots, \mathcal{E}_m\}$ such that $\{\mathcal{E}_1 \cdots, \mathcal{E}_s\}$ are horizontal vector fields of $\mathbb{M}$, $\{\mathcal{E}_{s+1}, \cdots, \mathcal{E}_r\}$ are vertical vector fields of $\mathbb{M}$, and $\mathcal{E}_{r+1}$ is a unit normal vector field parallel to the mean curvature vector field of $\mathbb{M}$. With this choice, Equation (10) becomes

$$\left(\sum_{I=1}^{r} h^{0r+1}_{II}\right)^2 = 2\left[\lambda + \sum_{I=1}^{r}(h^{0r+1}_{II})^2 + \sum_{I\neq J}(h^{0r+1}_{IJ})^2 + \sum_{\alpha=r+2}^{m}\sum_{I,J=1}^{r}(h^{0\alpha}_{IJ})^2\right]. \tag{11}$$

To use Lemma 1, we rewrite (11) as

$$\begin{aligned} &\left[h^{0r+1}_{11} + (h^{0r+1}_{22} + h^{0r+1}_{33} + \cdots + h^{0r+1}_{ss}) + (h^{0r+1}_{s+1\,s+1} + \cdots + h^{0r+1}_{rr})\right]^2 \\ = &\ 2\Bigg[\lambda + (h^{0r+1}_{11})^2 + (h^{0r+1}_{22} + h^{0r+1}_{33} + \cdots + h^{0r+1}_{ss})^2 + (h^{0r+1}_{s+1\,s+1} + \cdots + h^{0r+1}_{rr})^2 \\ &+ 2\sum_{1\leq I<J\leq r}(h^{0r+1}_{IJ})^2 + \sum_{\alpha=r+2}^{m}\sum_{I,J=1}^{r}(h^{0\alpha}_{IJ})^2 - 2\sum_{2\leq J<K\leq s} h^{0r+1}_{JJ}h^{0r+1}_{KK} \\ &- 2\sum_{s+1\leq i<j\leq r} h^{0r+1}_{ii}h^{0r+1}_{jj}\Bigg]. \end{aligned} \tag{12}$$

Then, we have

$$\sum_{1\le J<K\le s} h_{JJ}^{0r+1}h_{KK}^{0r+1}+\sum_{s+1\le i<j\le r} h_{ii}^{0r+1}h_{jj}^{0r+1} \ge \frac{\lambda}{2}+\sum_{1\le i<j\le r}(h_{ij}^{0r+1})^2 +\frac{1}{2}\sum_{\alpha=r+2}^{m}\sum_{i,j=1}^{r}(h_{ij}^{0\alpha})^2. \tag{13}$$

Additionally, the sectional curvature for the plane section defined by unit horizontal and vertical vectors is given by

$$\sum_{I=1}^{s}\sum_{i=s+1}^{r} S(\mathcal{E}_I\wedge\mathcal{E}_i)=\tau-\sum_{1\le I<J\le s} S(\mathcal{E}_I,\mathcal{E}_J)-\sum_{s+1\le i<j\le r} S(\mathcal{E}_i,\mathcal{E}_j).$$

Then, by the Gauss equation for submersion, we arrive at

$$\begin{aligned}\sum_{I=1}^{s}\sum_{i=s+1}^{r} S(\mathcal{E}_I\wedge\mathcal{E}_i) &= \tau-\sum_{1\le I<J\le s}\overline{S}(\mathcal{E}_I,\mathcal{E}_J)\\ &-2\sum_{\alpha=r+1}^{m}\sum_{1\le I<J\le s}\Big(h_{II}^{0\alpha}h_{JJ}^{0\alpha}-(h_{IJ}^{0\alpha})^2\Big)\\ &+\sum_{\alpha=r+1}^{m}\sum_{1\le I<J\le s}\frac{1}{2}\Big(h_{II}^{\alpha}h_{JJ}^{\alpha}+\tilde{h}_{II}^{\alpha}\tilde{h}_{JJ}^{\alpha}\\ &-(h_{IJ}^{\alpha})^2-(\tilde{h}_{IJ}^{\alpha})^2\Big)-\sum_{s+1\le i<j\le r}\overline{S}(\mathcal{E}_i,\mathcal{E}_j)\\ &-2\sum_{\alpha=r+1}^{m}\sum_{s+1\le i<j\le r}\Big(h_{ii}^{0\alpha}h_{jj}^{0\alpha}-(h_{ij}^{0\alpha})^2\Big)\\ &+\sum_{\alpha=r+1}^{m}\sum_{s+1\le i<j\le r}\frac{1}{2}\Big(h_{ii}^{\alpha}h_{jj}^{\alpha}+\tilde{h}_{ii}^{\alpha}\tilde{h}_{jj}^{\alpha}\\ &-(h_{ij}^{\alpha})^2-(\tilde{h}_{ij}^{\alpha})^2\Big).\end{aligned} \tag{14}$$

By using the simple algebraic inequality, we derive

$$\begin{aligned}&\sum_{\alpha=r+1}^{m}\sum_{1\le I<J\le s}\frac{1}{2}\Big(h_{II}^{\alpha}h_{JJ}^{\alpha}-(h_{IJ}^{\alpha})^2+\tilde{h}_{II}^{\alpha}\tilde{h}_{JJ}^{\alpha}-(\tilde{h}_{IJ}^{\alpha})^2\Big)\\ =\ &\sum_{\alpha=r+1}^{m}\Big[\sum_{2\le J\le s}\frac{1}{2}(h_{11}^{\alpha}h_{JJ}^{\alpha}-(h_{1J}^{\alpha})^2)+\sum_{2\le I<J\le s}\frac{1}{2}(h_{II}^{\alpha}h_{JJ}^{\alpha}-(h_{IJ}^{\alpha})^2)\Big]\\ \le\ &\sum_{\alpha=r+1}^{m}\sum_{J=2}^{s}\frac{1}{2}h_{11}^{\alpha}h_{JJ}^{\alpha}-\sum_{J=2}^{s}\frac{1}{2}(h_{1J}^{r+1})^2-\sum_{\alpha=r+2}^{m}\sum_{J=2}^{s}\frac{1}{2}(h_{1J}^{\alpha})^2\\ &+\sum_{\alpha=r+1}^{m}\sum_{J=2}^{s}\frac{1}{2}\tilde{h}_{11}^{\alpha}\tilde{h}_{JJ}^{\alpha}-\sum_{J=2}^{s}\frac{1}{2}(\tilde{h}_{1J}^{r+1})^2-\sum_{\alpha=r+2}^{m}\sum_{J=2}^{s}\frac{1}{2}(\tilde{h}_{1J}^{\alpha})^2,\end{aligned} \tag{15}$$

and

$$
\begin{aligned}
& \sum_{\alpha=r+1}^{m} \sum_{s+1\leq i<j\leq r} \frac{1}{2}\Big(h^{\alpha}_{ii}h^{\alpha}_{jj} - (h^{\alpha}_{ij})^2 + \tilde{h}^{\alpha}_{ii}\tilde{h}^{\alpha}_{jj} - (\tilde{h}^{\alpha}_{ij})^2 \Big) \\
= & \sum_{\alpha=r+1}^{m} \Big[\sum_{s+2\leq j\leq r} \frac{1}{2}(h^{\alpha}_{s+1\ s+1}h^{\alpha}_{jj} - (h^{\alpha}_{s+1\ J})^2) + \sum_{s+2\leq i<j\leq s} \frac{1}{2}(h^{\alpha}_{ii}h^{\alpha}_{jj} - (h^{\alpha}_{ij})^2) \Big] \\
\leq & \sum_{\alpha=r+1}^{m} \sum_{j=s+2}^{r} \frac{1}{2}h^{\alpha}_{s+1\ s+1}h^{\alpha}_{jj} - \sum_{j=s+2}^{r} \frac{1}{2}(h^{r+1}_{s+1\ j})^2 - \sum_{\alpha=s+2}^{m} \sum_{j=s+2}^{r} \frac{1}{2}(h^{\alpha}_{s+1\ j})^2 \\
& + \sum_{\alpha=r+1}^{m} \sum_{j=s+2}^{r} \frac{1}{2}\tilde{h}^{\alpha}_{s+1\ s+1}\tilde{h}^{\alpha}_{jj} - \sum_{j=s+2}^{r} \frac{1}{2}(\tilde{h}^{r+1}_{s+1\ j})^2 - \sum_{\alpha=r+2}^{m} \sum_{j=s+2}^{r} \frac{1}{2}(\tilde{h}^{\alpha}_{s+1\ j})^2. \qquad (16)
\end{aligned}
$$

On simplifying (14), together with the calculations from [24] and the obtained inequalities (15) and (16), we obtain

$$
\begin{aligned}
\sum_{I=1}^{s} \sum_{i=s+1}^{r} S(\mathcal{E}_I \wedge \mathcal{E}_i) \leq\ & \tau - \sum_{1\leq I<J\leq s} \overline{S}(\mathcal{E}_I, \mathcal{E}_J) - \sum_{s+1\leq i<j\leq r} \overline{S}(\mathcal{E}_i, \mathcal{E}_j) \\
& -2 \sum_{\alpha=r+1}^{m} \sum_{1\leq I<J\leq s} \Big(h^{0\alpha}_{II}h^{0\alpha}_{JJ} - (h^{0\alpha}_{IJ})^2 \Big) \\
& -2 \sum_{\alpha=r+1}^{m} \sum_{s+1\leq i<j\leq r} \Big(h^{0\alpha}_{ii}h^{0\alpha}_{jj} - (h^{0\alpha}_{ij})^2 \Big) \\
& + \frac{1}{4}\Big[\sum_{\alpha=r+1}^{m} \sum_{J=1}^{s} \Big((h^{\alpha}_{JJ})^2 + (\tilde{h}^{\alpha}_{JJ})^2 \Big) \Big] \\
& + \frac{1}{4}\Big[\sum_{\alpha=r+1}^{m} \sum_{j=s+1}^{r} \Big((h^{\alpha}_{jj})^2 + (\tilde{h}^{\alpha}_{jj})^2 \Big) \Big].
\end{aligned}
$$

The mean curvature vectors $\mathcal{H}_H$ and $\tilde{\mathcal{H}}_H$ for dual affine connections on the horizontal and vertical spaces are denoted by $\mathcal{H}_V$ and $\tilde{\mathcal{H}}_V$. Thus, we have

$$
\begin{aligned}
\sum_{I=1}^{s} \sum_{i=s+1}^{r} S(\mathcal{E}_I \wedge \mathcal{E}_i) \leq\ & \tau - \sum_{1\leq I<J\leq s} \overline{S}(\mathcal{E}_I, \mathcal{E}_J) - \sum_{s+1\leq i<j\leq r} \overline{S}(\mathcal{E}_i, \mathcal{E}_j) \\
& -2 \sum_{\alpha=r+1}^{m} \sum_{1\leq I<J\leq s} \Big(h^{0\alpha}_{II}h^{0\alpha}_{JJ} - (h^{0\alpha}_{IJ})^2 \Big) \\
& -2 \sum_{\alpha=r+1}^{m} \sum_{s+1\leq i<j\leq r} \Big(h^{0\alpha}_{ii}h^{0\alpha}_{jj} - (h^{0\alpha}_{ij})^2 \Big) \\
& + \frac{(s-1)}{4}\Big(\mathcal{H}^2_H + \tilde{\mathcal{H}}^2_H \Big) + \frac{(r-s-1)}{4}\Big(\mathcal{H}^2_V + \tilde{\mathcal{H}}^2_V \Big). \qquad (17)
\end{aligned}
$$

Assuming the statistical submersion ϑ has isometric fibers, the submersion invariant $\mathcal{A}^0(\pi)$ for the Levi–Civita connection and those for the dual affine connections are denoted by $\mathcal{A}(\pi)$ and $\tilde{\mathcal{A}}(\pi)$. For the plane section spanned by unit horizontal and vertical vectors, these are given by

$$
\mathcal{A}^0(\pi) = \sum_{I=1}^{s} \sum_{i=s+1}^{r} ||\mathcal{A}^0_{\mathcal{E}_I}\mathcal{E}_i||^2,
$$

$$
\mathcal{A}(\pi) = \sum_{I=1}^{s} \sum_{i=s+1}^{r} ||\mathcal{A}_{\mathcal{E}_I}\mathcal{E}_i||^2,
$$

and

$$\tilde{\mathcal{A}}(\pi) = \sum_{I=1}^{s} \sum_{i=s+1}^{r} ||\tilde{\mathcal{A}}_{\mathcal{E}_I} \mathcal{E}_i||^2$$

From this definition and the last formula of Lemma 3, the sectional curvature can be written as

$$\begin{aligned} \sum_{I=1}^{s} \sum_{i=s+1}^{r} S(\mathcal{E}_I \wedge \mathcal{E}_i) &= \sum_{I=1}^{s} \sum_{i=s+1}^{r} G(\tilde{\mathcal{A}}_{\mathcal{E}_I} \mathcal{E}_i, \tilde{\mathcal{A}}_{\mathcal{E}_I} \mathcal{E}_i) \\ &= \sum_{I=1}^{s} \sum_{i=s+1}^{r} ||\tilde{\mathcal{A}}_{\mathcal{E}_I} \mathcal{E}_i||^2. \end{aligned}$$

Thus, we have

$$\tilde{\mathcal{A}}(\pi) = \sum_{I=1}^{s} \sum_{i=s+1}^{r} S(\mathcal{E}_I \wedge \mathcal{E}_i). \tag{18}$$

In order to derive the desired inequality, we use (9), (13), (17) and (18), and we find that

$$\begin{aligned} \tilde{\mathcal{A}}(\pi) &\leq \tau - \overline{\tau}(T_y\mathbb{M}) - \frac{\lambda}{2} + \sum_{I=1}^{s} \sum_{i=s+1}^{r} \overline{S}(\mathcal{E}_I \wedge \mathcal{E}_i) \\ &\quad + \frac{(s-1)}{4}\left(\mathcal{H}_H^2 + \tilde{\mathcal{H}}_H^2\right) + \frac{(r-s-1)}{4}\left(\mathcal{H}_V^2 + \tilde{\mathcal{H}}_V^2\right) \\ &= \frac{r^2}{4}\mathcal{H}^{02} - \frac{r^2}{8}(\mathcal{H} - \tilde{\mathcal{H}})^2 + \frac{1}{4}\sum_{\alpha=r+1}^{m} \sum_{I,J=1}^{r} (h_{IJ}^{\alpha} - \tilde{h}_{IJ}^{\alpha})^2 \\ &\quad + \frac{(s-1)}{4}\left(\mathcal{H}_H^2 + \tilde{\mathcal{H}}_H^2\right) + \frac{(r-s-1)}{4}\left(\mathcal{H}_V^2 + \tilde{\mathcal{H}}_V^2\right) \\ &\quad + \sum_{I=1}^{s} \sum_{i=s+1}^{r} \overline{S}(\mathcal{E}_I \wedge \mathcal{E}_i). \end{aligned}$$

Hence, we have the following key inequality to prove the main relationship.

Proposition 2. *If a statistical submersion $\vartheta : (\mathbb{M}, D, G) \to (\mathbb{N}, \mathcal{D}, g)$ has isometric fibers, then for any isometric immersion of $\mathbb{M}$ into a statistical manifold $(\overline{\mathbb{M}}, \nabla, \overline{G})$, the submersion invariant on $\mathbb{M}$ satisfies*

$$\begin{aligned} \tilde{\mathcal{A}}(\pi) - \frac{r^2}{4}\mathcal{H}^{02} - s(r-s)\max \overline{S}(y) &\leq -\frac{r^2}{8}(\mathcal{H} - \tilde{\mathcal{H}})^2 \\ &\quad + \frac{1}{4}\sum_{\alpha=r+1}^{m} \sum_{I,J=1}^{r} (h_{IJ}^{\alpha} - \tilde{h}_{IJ}^{\alpha})^2 \\ &\quad + \frac{(s-1)}{4}\left(\mathcal{H}_H^2 + \tilde{\mathcal{H}}_H^2\right) \\ &\quad + \frac{(r-s-1)}{4}\left(\mathcal{H}_V^2 + \tilde{\mathcal{H}}_V^2\right). \end{aligned} \tag{19}$$

where $\max \overline{S}(y)$ *denotes the maximum value of the sectional curvature function of* $\overline{\mathbb{M}}$ *when restricted to* 2*-plane sections of the tangent space* $T_y\mathbb{M}$.

Corollary 1. *Let $\vartheta : (\mathbb{M}, D, G) \to (\mathbb{N}, \mathcal{D}, g)$ be a statistical submersion with isometric fibers. Then, for any isometric immersion of $\mathbb{M}$ into a statistical manifold $(\overline{\mathbb{M}}(c), \nabla, \overline{G})$ of constant sectional curvature c, the submersion invariant on $\mathbb{M}$ satisfies*

$$\begin{aligned}\tilde{\mathcal{A}}(\pi) - \frac{r^2}{4}\mathcal{H}^{02} - s(r-s)c \quad \leq \quad & -\frac{r^2}{8}(\mathcal{H} - \tilde{\mathcal{H}})^2 \\ & + \frac{1}{4}\sum_{\alpha=r+1}^{m}\sum_{I,J=1}^{r}(h_{IJ}^{\alpha} - \tilde{h}_{IJ}^{\alpha})^2 \\ & + \frac{(s-1)}{4}\left(\mathcal{H}_H^2 + \tilde{\mathcal{H}}_H^2\right) \\ & + \frac{(r-s-1)}{4}\left(\mathcal{H}_V^2 + \tilde{\mathcal{H}}_V^2\right).\end{aligned}$$

Definition 3. *A statistical submanifold $(\mathbb{M}, D, G)$ of $(\overline{\mathbb{M}}, \nabla, \overline{G})$ is said to be doubly totally geodesic if $h = \tilde{h} = 0$ and doubly minimal if $\mathcal{H} = \tilde{\mathcal{H}} = 0$ [25].*

It is also noteworthy that the submersion invariant $\mathcal{A}^0(\pi)$ for the Levi–Civita connection can be expressed as $\mathcal{A}^0(\pi) = \frac{1}{2}(\mathcal{A}(\pi) + \tilde{\mathcal{A}}(\pi))$. Therefore, we have the main result of this article.

Theorem 1. *Let $(\mathbb{M}, D, G)$ be a statistical manifold. If $\mathbb{M}$ has a non-trivial statistical submersion with isometric fibers, then it cannot be isometrically immersed as a doubly minimal submanifold into any statistical manifold of non-positive sectional curvature, provided $h = \tilde{h}$.*

Proof. As the given statistical immersion is doubly minimal with $h = \tilde{h}$ and $\overline{\mathbb{M}}$ has non-positive sectional curvature, we derive from Proposition 2 that $\tilde{\mathcal{A}}(\pi) = 0$. For any horizontal vector field $\mathbb{X}$ and vertical vector field $\mathbb{U}$ on $(\mathbb{M}, D, D', G)$, it follows that $\tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U} = 0$. This implies that $D'_{\mathbb{X}}\mathbb{U}$ is vertical. Therefore, $D'_{\mathbb{X}}\mathbb{Y}$ is horizontal, for horizontal vector fields $\mathbb{X}, \mathbb{Y}$.

Additionally, we observe that $G(\mathcal{A}_{\mathbb{X}}\mathbb{Y}, \mathbb{U}) = -G(\mathbb{Y}, \tilde{\mathcal{A}}_{\mathbb{X}}\mathbb{U}) = 0$, which leads to the conclusion that $D_{\mathbb{X}}\mathbb{Y}$ is horizontal. Consequently, we say that $H(\mathbb{M})$ is a doubly totally geodesic distribution, meaning that $H(\mathbb{M})$ is completely integrable and its leaves are doubly totally geodesic. This contradicts the notion that neither the horizontal nor the vertical distributions of the non-trivial submersion are totally geodesic distributions. □

The following result can be obtained directly from Proposition 2.

Theorem 2. *Let $(\mathbb{M}, D, G)$ be a statistical manifold. If $\mathbb{M}$ has a non-trivial statistical submersion with isometric fibers, then it cannot be isometrically immersed into any statistical manifold of non-positive sectional curvature as a doubly totally geodesic submanifold.*

5. Concluding Remarks

In 1987 Lauritzen defined the concept of a statistical manifold as a generalization of a statistical model equipped with the Fisher metric and the Amari–Chenstov tensor. The notion of a dual connection in statistical inference is equivalent to the concept of a conjugate connection in affine geometry. The idea of the existence of non-conjugate symmetric statistical manifolds was also given by Lauritzen.

Remark 1. *In [26,27], Chen established a new inequality for Riemannian submersions and identified a significant finding about the non-existence of certain immersions on the same total space.*

Theorem 3. *Let $\vartheta : \mathbb{M} \to \mathbb{N}$ be a Riemannian submersion with totally geodesic fibers. Then, for any isometric immersion of $\mathbb{M}$ into a Riemannian manifold of non-positive sectional curvature, the submersion invariant $\mathcal{A}^0(\pi)$ on $\mathbb{M}$ satisfies*

$$\mathcal{A}^0(\pi) - \frac{r^2}{4}\mathcal{H}^{02} - s(r-s)\max \overline{S}^0(y) \leq 0. \tag{20}$$

Building on this theorem, Chen proved that if a manifold $\mathbb{M}$ admits a non-trivial Riemannian submersion ϑ with totally geodesic fibers, then $\mathbb{M}$ cannot be isometrically immersed as a minimal submanifold in any Riemannian manifold with non-positive sectional curvature. Inspired by Chen's result, we have successfully discovered a significant relationship between statistical submersions and doubly minimal immersions. These findings are applicable to numerous extensive families of statistical manifolds, reflecting the widespread occurrence of statistical submersions with isometric fibers in geometry.

Remark 2. *Let us review a few examples that illustrate the validity of Theorem 1. To begin, we examine the following examples of non-trivial statistical submersions with isometric fibers:*

1. *In [18], a statistical manifold $(\mathbb{R}^3 = \{(y,z,w)|y,z,w \in \mathbb{R}\}, \mathcal{D}, g)$ with*

$$g = e^{-x}((dy)^2 + (dw)^2) + e^x(dz)^2$$

and a locally product-like statistical manifold $(\mathbb{M}_+ = \{(x,y,z,w)|x > 0\}, D, G, \mathbb{F})$ (respectively, $(\mathbb{M}_- = \{(x,y,z,w)|x < 0\}, D, G, \mathbb{F})$) with

$$G = (e^x - 1)(dx)^2 e^{-x}((dy)^2 + (dw)^2) + e^x(dz)^2$$

and

$$\mathbb{F} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

are proved. Then, a locally product-like statistical submersion

$$\vartheta : (\mathbb{M}_{\pm}, D, G, \mathbb{F}) \to (\mathbb{R}^3, \mathcal{D}, g)$$

is defined by

$$\vartheta(x,y,z,w) = (y,z,w).$$

It is noted that each fiber is an anti-invariant of $\mathbb{M}_{\pm}$. This submersion has isometric fibers, with $\mathcal{T} = 0$.

2. *In [28], a generalized Kähler-like statistical submersion*

$$\vartheta : (\mathbb{R}^4_1, D, G, \mathbb{J}) \to (\mathbb{R}^3, \mathcal{D}, g)$$

defined by

$$\vartheta(x,y,z,w) = (x,y,z),$$

has isometric fibers. Here, $(\mathbb{R}^3 = \{(x,y,z)|x,y,z \in \mathbb{R}\}, \mathcal{D}, g)$ with

$$g = -e^{2y}(dx)^2 + e^{2y}(dz)^2$$

is a statistical manifold, and $\mathbb{R}^4_1 = \{(x,y,z,w)|x,y,z,w \in \mathbb{R}\}, D, G, \mathbb{J})$ is a K ähler-like statistical manifold with

$$G = -e^{2y}(dx)^2 + (dy)^2 + e^{2y}(dz)^2 + (dw)^2$$

and

$$\mathbb{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{21}$$

Additionally, it is worth recalling examples of doubly minimal statistical immersions as discussed in [25]:

1. *The map* $\theta : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^5$ *given by*

$$\theta(u, v) = \frac{1}{\sqrt{3}}(\cos u, \sin u, \cos v, \sin v, \cos(u+v), -\sin(u+v)) \in \mathbb{S}^5$$

defines an immersion of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ *into the Sasakian statistical manifold* $(\mathbb{S}^5, \overline{\nabla}, \overline{G}, \phi, \xi)$. *This immersion is minimal and C-totally real, meaning that the torus is a doubly minimal submanifold of the above statistical manifold.*

Remark 3. *In [27], Chen examined isometric immersions that satisfy the equality case of inequality (20). It would be valuable to extend this analysis to the statistical setting. This involves adapting inequality (19) and constructing a statistical version of Chen's investigation to address cases where the equality in (19) is achieved.*

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