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Mathematical Model for Optimal Agri-Food Industry Residual Streams Flow Management: A Valorization Decision Support Tool

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Abstract: We present a mathematical model for agri-food industry residual streams flow management, which serves as a decision support tool for optimizing their valorization. The aim is to determine, under a cost-benefit analysis approach, the best strategy at a global level. The proposed mathematical model provides the optimal valorization scenario, namely the set of routes followed by agri-food industry residual streams that maximizes the total profit obtained. The model takes into account the complete stoichiometry of the residual stream at each step of the valorization route. Furthermore, the model allows for the calculations of different scenarios to support decision-making. The proposed approach is illustrated through a case study using a real-case network of a region. The case study bears evidence that the use of the model can lead to significant profit increases compared to those obtained with current practices. Moreover, notable profit improvements are obtained in the case study if the selling price of all the value-added products considered increases or if the processing cost of the animal feed producer decreases. Therefore, our model enables the detection of key factors that influence the optimal strategy, making it a powerful decision-support tool for optimizing the valorization of agri-food industry residual streams.

Keywords: agri-food residual streams; valorization; optimization; mathematical modeling; value chain

MSC: 90-10; 90B06; 90B50; 90B90

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1. Introduction

Optimal management of agri-food industry residual streams is linked to sustainable food processing, and the reutilization (valorization) of such organic waste is a relevant issue of global concern under a circular economy approach.

Agri-food industry organic waste deposited in landfills represents a cost to the agri-food processor and is responsible for greenhouse emissions and air pollution as well as groundwater contamination. The social and environmental negative impact would be mitigated if the organic waste of one food processing sector is used as a resource (feedstock) for another. Hence, the use of agri-food industry residual streams as raw material for obtaining new value-added products is of great interest and makes valorization a revenue source.

In the literature there exist many references regarding agri-food waste valorization, mainly under technical and economic aspects, but they are focused on one valorization process or on one particular agri-food waste. However, valorization requires collaboration across the entire industry chain, and improving the logistics system is a challenge to be addressed [1]. Therefore, a broader approach is needed in order to determine the best strategy at a global level. Such an approach must consider the logistics of multi-step valorization routes in big networks. It should also include environmental aspects for management prioritization. Moreover, for the optimization of the valorization processes considered in those routes, the complete stoichiometry of the residual streams should be taken into account.

To address these challenges, in this paper we present a mathematical model for optimal agri-food industry residual streams flow management. This model allows to find the optimal valorization scenario handling together different agri-food industries, with their corresponding residual streams, multiple bio-based industries, and all the available routes that residual streams can follow through bio-based industry processes (i.e., valorization processes). Our model includes the complete stoichiometry of the residual streams for accurate process modeling and supports the inclusion of economic, social, and environmental factors. Furthermore, variation of the model's parameters enables the detection of key factors that influence the optimal strategy, making the mathematical model a powerful decision support tool for optimizing the valorization of agri-food industry residual streams.

The rest of the paper is structured as follows: In Section 2 we present a literature review. In Section 3, we present the description of the mathematical model and the optimization problem to be solved. In Section 4, the use of the proposed model is illustrated through a case study. Finally, in Section 5, we present the conclusions of this work.

2. Literature Review

Although transport and treatment of agri-food industry residual streams can sometimes be very expensive, those costs may be covered and eventually overcome by higher benefits due to valorization's economic, social, and environmental impact. In fact, bio-based industries have gained special attention under the European Circular Economy strategy since they allow to recover valuable compounds and to obtain new value-added products from residual streams, reducing both waste and environmental footprint. However, according to [2], literature evidencing the application of waste valorization models is still limited.

Commonly adopted strategies for the management and valorization of agri-food industry residual streams are to process them in order to obtain biofuels, bioproducts, animal feed, or high value-added compounds (see, e.g., [3–6]). Therefore, there will be multiple possible valorization scenarios available for agri-food industry residual streams flow management, and the aim of this work is to find the optimal one. In this study, a valorization scenario is understood as a set of agri-food industries, with their corresponding residual streams, a set of bio-based industries, and a set of available routes that residual streams can follow from agri-food industries through different bio-based industries processes (i.e., valorization processes). It should be mentioned that valorization processes, in addition to obtaining value-added products, may generate new residual streams, which in turn can be re-introduced in another valorization process as feedstock.

In the literature, different methodological approaches have been employed to evaluate several potential waste valorization strategies: life cycle assessment (LCA), life cycle sustainability assessment (LCSA), life cycle costing (LCC), cost-benefit analysis (CBA), full cost accounting (FCA), and variations on multi-criteria decision analysis (MCDA). Those methodologies assess different valorization scenarios according to their economic viability, environmental and social sustainability, or technological maturity [7]. Following one of these approaches, a company can determine what is likely to be its optimal valorization option (see, e.g., [8,9]). In this work, we follow a CBA approach where profits of multiple potential strategies are compared. However, the aim is to determine the best strategy at a global level and not necessarily help a specific company in its decision-making.

For the optimization of agri-food industry residual streams flow management, it is necessary to consider logistics but also the mass and energy flows that allow the optimization of industrial processes.

Currently, there are commercial programs for the simulation of industrial processes such as Aspen Plus®, Chemcad®, Pro/II®, Prosim®, SuperPro®, or COCO® among many others. These programs, based on mass and energy balances, allow predicting the response of the system to scenarios or disturbances in a wide range of sectors (chemical, petroleum, pharmaceutical, bio-technological, wastewater treatment, etc.). However, the model library for food sector processes has not followed the same development, and there are still missing models of numerous main processes (see [10]). The existing commercial products focus on providing tools for the description of the bio-industry and allow the residual stream generated in the different sectors to be taken as raw material. The biggest limitation of those commercial products is that they are mainly based on the analysis of specific facilities and not on broader analyses of big networks with various industries, in which logistics plays a key role.

Parallel to the development and updating of these commercial programs, the scientific community is making considerable progress in the development of models to simulate and predict the value-added products that can be obtained through fermentation (see, e.g., [11–14]) and extraction [15,16].

Furthermore, the aforementioned commercial programs base optimizations or decisions on technical and economic factors, although there are more and more attempts to include environmental aspects. An example of this is the Aspen Plus "Environmental & Safety Analysis" module. Numerous works are collected in the literature that combine LCA and industrial process modeling (see, e.g., [17–19]).

In the literature, there also exist many references focused on the productivity and sustainability of just one valorization process with one input and one output (see [20–24] among others). Their aim is to model and optimize the performance of a single operation. Our mathematical model has a more global approach, and as many valorization processes as wanted can be considered. It should be mentioned that in order to find the optimal valorization scenario, the complete stoichiometry of the residual stream may be essential.

For the logistic analysis, there are specific programs such as Arena®, Flexsim®, or TRUX Haul-it, among others, that allow the analysis of commercial or logistic operations. There are programs for optimising the time of decision-making but not for the optimization of mass flows such as industrial simulators.

In the literature, waste exploitation is not a very common issue. There exist references dealing with the design of biomass supply chains for the production of bio-based products (see [25–30] and references therein). The supply chain involves raw material suppliers, processing plants, and demand. Unlike here, in many of those references, biomass is the raw material considered instead of agri-food industry residual streams, and it usually comes from dedicated crops. Moreover, in many of those references, only biorefineries are regarded as bio-based industries, while in this work as many management options as wanted can be considered. Furthermore, the design of a supply chain is linked to meeting customers' demands, while in our case such an approach is not entirely appropriate. Our work is focused on the management and valorization of residual streams, and hence, talking about meeting customers' demands makes no sense.

In the literature, there exist also many references dealing with the problem of where to locate a new biorefinery or the problem of establishing the number, size, and location of all the potential biorefineries in the network (see, e.g., [31–37]). In this work, we assume that the number and location of the bio-based industries are known. Nevertheless, our model allows us to study different layouts of the network in order to determine the optimal one.

3. Methodology

In this section, we first present the description of the model considered and the optimization problem to be solved. Secondly, we introduce the mathematical formulation of the optimization problem.

3.1. Model Description and Assumptions

The model considered represents a valorization scenario and consists of a set of agri-food industries (AFI), a set of bio-based industries (BBI), a set of available valorization routes, a set of residual streams generated, and a residual stream distribution. The mathematical model is developed according to the following criteria:

- A residual stream from a given AFI (by-products) is reintroduced as raw material in a valorization route. There may be different routes available. Without loss of generality, we assume that each AFI generates only one by-product (if more than one by-product were generated, as many instances of the AFI as needed could be used).
- Available valorization routes for a given residual stream are made up of different steps. Each step corresponds to the set of BBIs that can use such a residual stream as raw material.
- The relations between the input and the outputs of a BBI are modeled as linear applications and therefore can be modeled by using a matrix. Mathematical models used for describing mass transformations in the bio-industries processes are out of the scope of this paper. However, since many of those transformations are usually modeled by using differential equations, it is not unusual to assume that some valorization processes conform to a linear transformation and therefore can be modeled by using a transformation matrix.
- BBIs use a residual stream as raw material to produce value-added products and can also generate a new residual stream. The relation between the residual stream at the input and at the output of a BBI and the relation between the residual stream at the input and the value-added products at the output of a BBI are modeled separately.
- The residual stream generated in one BBI must be sent to the next step of the valorization route.
- If a BBI does not produce a residual stream (e.g., landfills, incineration plants, waste water treatment plants, etc.) or the residual stream generated cannot be used as feedstock for another valorization process, then the valorization route ends.
- The residual stream might be modeled considering the by-products that it contains as well as considering its chemical composition.
- It is assumed that the composition of the residual stream does not change with transport or storage. This assumption may not be true in real conditions, but it could be easily overcome by including in the model a transformation matrix that represents those changes.
- A residual stream from a given BBI is never reintroduced in the same BBI.
- The profit obtained in a BBI at each step is the difference between the revenue from the sale of the value-added products obtained and the cost derived from the valorization of the residual stream used as feedstock in such step.
- The model supports the inclusion of economic, social, and environmental costs/benefits.
- BBIs are assumed to have enough capacity for processing all the residual streams that they receive.

The aim of this work is to determine the optimal valorization scenario, that is, the residual streams distribution that maximizes the sum of profits generated at BBIs throughout the different available steps of the valorization routes from the AFIs through one or more BBIs.

3.2. Mathematical Formulation of the Optimization Problem

In this subsection, we present the mathematical formulation of the optimization problem to be solved. First we introduce some notation.

Assume, without loss of generality, a valorization scenario with N AFIs (AFI_1, AFI_2, ..., AFI_N) and N BBIs (BBI_1, BBI_2, ..., BBI_N). Figure 1 shows an S-step model of the valorization scenario considered, with $s \in \{1, \dots, S\}$ being the step number of the valorization route. The S-th step is the final step where all the valorization routes of the residual streams have ended. That is, either the residual stream has been deposited in a landfill (or in an incineration plant, in a waste water treatment plant, etc.) or the residual stream generated cannot be used as feedstock for another valorization process. A connection between BBIs (or between AFIs and BBIs) in Figure 1 represents a possible path of the valorization route.

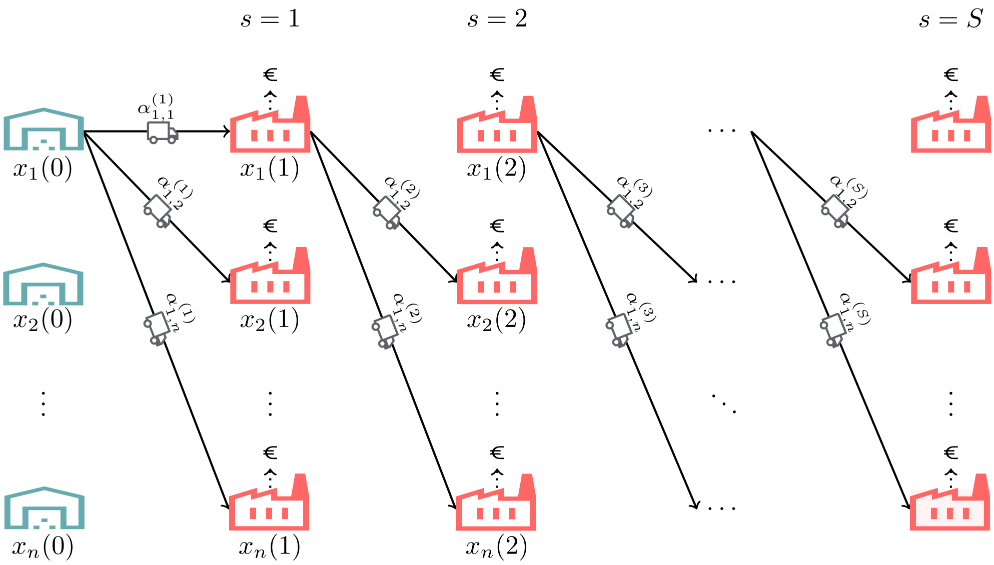

Figure 1. Simplified diagram of an S-step model.

Let Q be the number of by-products and chemical compounds that the residual streams considered in the scenario might contain. Then, a residual stream is modeled as a $Q \times 1$ real column vector. The residual stream at the output of the BBI_i at step s, $s \in \{1, \dots, S-1\}$, is given by

$$\mathbf{x}_i(s) = \mathbf{A}_i \sum_{j=1}^{N} \alpha_{j,i}^{(s)} \mathbf{x}_j(s-1), \qquad i \in \{1, \dots, N\}, \tag{1}$$

with $\mathbf{x}_i(0)$ being a $Q \times 1$ real column vector that models the residual stream generated at AFI_i, $\mathbf{A}_i$ being the $Q \times Q$ real matrix that models the relation between the residual stream at the input and the residual stream at the output of BBI_i, and $\alpha_{j,i}^{(s)} \geq 0$ being the proportion of residual stream that is received at BBI_i from BBI_j at step s (or from AFI_j if $s = 1$).

According to the assumptions, $\alpha_{j,i}^{(s)} = 0$ if $i = j$, for all $s \in \{2, \dots, S\}$. Moreover, $\sum_{i=1}^{N} \alpha_{j,i}^{(s)} = 1$ for all $j \in \{1, \dots, N\}$ and $s \in \{1, \dots, S\}$.

Let M_i be the number of different value-added products at the output of BBI_i. The revenue from the sale of those value-added products obtained at step s, $s \in \{1, \dots, S\}$, is given by

$$r_i(s) = \mathbf{p}_i^{\top} \mathbf{B}_i \sum_{j=1}^{N} \alpha_{j,i}^{(s)} \mathbf{x}_j(s-1), \qquad i \in \{1, \dots, N\}, \tag{2}$$

with $\top$ denoting transpose, $\mathbf{p}_i$ being a $M_i \times 1$ real column vector with the selling prices of the value-added products obtained at BBI_i, $\mathbf{B}_i$ being the $M_i \times Q$ real matrix that models the relation between the residual stream at the input and the value-added products at the output of BBI_i, and $\mathbf{x}_j(s)$ given in (1).

The cost derived from the valorization of the residual stream received at BBI_i at step s, $s \in \{1, \ldots, S\}$, is given by

$$c_i(s) = \sum_{j=1}^{N} \left(\mathbf{c}_{j,i}(s)\right)^\top \alpha_{j,i}^{(s)} \mathbf{x}_j(s-1), \qquad i \in \{1, \ldots, N\}, \tag{3}$$

with $\mathbf{c}_{j,i}(s)$ being a $Q \times 1$ real column vector that models the costs at BBI_i derived from the valorization of the residual stream received from BBI_j (or from AFI_j if $s = 1$) and $\mathbf{x}_j(s)$ given in (1).

Therefore, from (2) and (3), the profit obtained at BBI_i at step s, $s \in \{1, \ldots, S\}$, is given by

$$b_i(s) = \sum_{j=1}^{N} \left(\mathbf{b}_{j,i}(s)\right)^\top \alpha_{j,i}^{(s)} \mathbf{x}_j(s-1), \qquad i \in \{1, \ldots, N\},$$

with $\mathbf{b}_{j,i}(s) = \mathbf{B}_i^\top \mathbf{p}_i - \mathbf{c}_{j,i}(s)$. Observe that $\mathbf{b}_{j,i}(s)$ models the profit obtained at BBI_i when the residual stream received from BBI_j (or from AFI_j if $s = 1$) is processed. It should be mentioned that the profit vector, $\mathbf{b}_{j,i}(s)$, supports the inclusion of revenues from the sales of value-added products obtained, costs derived from the valorization of the residual stream (transport, energy, labour, etc.), as well as social and environmental costs/benefits.

Table 1 shows a summary of the different model parameters and decision variables.

Table 1. Summary of the different parameters and decision variables in the proposed mathematical model.

Parameters	**Description**
$\mathbf{x}_i(s)$	Residual stream at the output of BBI_i at step s
$\mathbf{A}_i$	Relation between the residual stream at the input and the output of BBI_i
$r_i(s)$	Revenue from the sale of value-added products obtained at BBI_i at step s
$\mathbf{p}_i$	Selling prices of the value-added products obtained at BBI_i
$\mathbf{B}_i$	Relation between the residual stream at the input and the value-added products at the output of BBI_i
$c_i(s)$	Cost derived from the valorization of the residual stream received at BBI_i at step s
$\mathbf{c}_{j,i}(s)$	Costs at BBI_i derived from the valorization of the residual stream received from BBI_j
$b_i(s)$	Profit obtained at BBI_i at step s
$\mathbf{b}_{j,i}(s)$	Profit obtained at BBI_i at step s from the sale of value-added products obtained from the residual stream coming from BBI_j
Variables	**Description**
$\alpha_{j,i}^{(s)}$	Proportion of residual stream received at BBI_i from BBI_j at step s

Now we present the mathematical formulation of the optimization problem to be solved, which is expressed as the maximization of an objective function subject to a number of constraints. We aim to obtain the residual stream distribution that maximizes the sum of profits generated throughout the different available steps of the valorization routes. Hence, we aim to solve the following maximization problem:

$$\begin{aligned} &\text{Maximize} \quad \sum_{s=1}^{S} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\mathbf{b}_{j,i}(s)\right)^\top \alpha_{j,i}^{(s)} \mathbf{x}_j(s-1) \\ &\text{Subject to} \\ &\qquad \alpha_{i,j}^{(s)} \geq 0 \quad i, j \in \{1, \ldots, N\}, s \in \{1, \ldots, S\} \\ &\qquad \sum_{j=1}^{n} \alpha_{i,j}^{(s)} = 1 \quad i \in \{1, \ldots, N\}, s \in \{1, \ldots, S\} \\ &\qquad \alpha_{i,j}^{(s)} = 0 \quad i = j, s \in \{2, \ldots, S\} \end{aligned} \tag{4}$$

The maximization is with respect to $\alpha_{i,j}^{(s)}$, with $i, j \in \{1, \ldots, N\}$ and $s \in \{1, \ldots, S\}$, and the objective function of the optimization problem (4) is a non-linear function. There are numerical methods that are able to solve the aforementioned optimization problem in a reasonable amount of time.

4. Results of a Case Study

In this section, the proposed approach is illustrated through a case study. Using a real-case network of a region, we compare, under a CBA approach, the current agri-food industry residual streams management strategy with the optimal management strategy obtained by solving the optimization problem presented in Section 3.2. Our mathematical model allows for the calculations of different scenarios to support decision-making. Different factors affect the revenue and/or the cost of the case under study. A sensitivity analysis that takes these factors into account has been carried out to show the potential of the mathematical model.

4.1. Case Study Description

Consider the region in Figure 2a. The set of agri-food industries and the set of BBIs that can be found in the considered region are shown in Figure 2b,c, respectively.

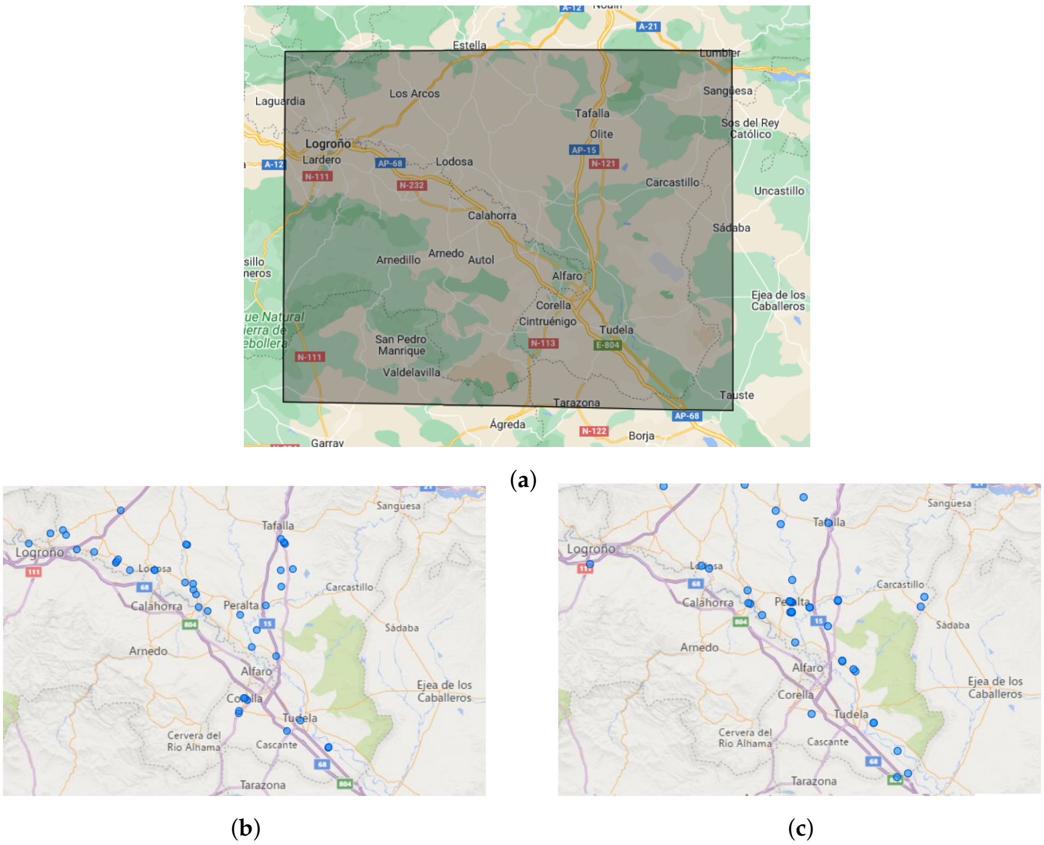

Figure 2. Considered region, agri-food industries and BBIs. (**a**) A 100 × 100 kilometer region of northern Spain considered for optimization. (**b**) Plot of the different agri-food industries found in the considered region. Each point corresponds to one agri-food industry. (**c**) Plot of the different BBIs found in the considered region. Each point corresponds to a different BBI.

Although the mathematical model allows us to consider as many agri-food industries and BBIs as desired, for the sake of simplicity, we will select for the example a valorization scenario with two agri-food industries (AFI_1 and AFI_2) and five BBIs: two anaerobic digesters operating at mesophilic temperature (BBI_1 and BBI_2), two composting plants (BBI_3 and BBI_4), and one animal feed producer (BBI_5). Detailed descriptions of the treatments carried out in the anaerobic digester, the composting plant, and the animal feed producer are depicted in Figures 3–5 respectively.

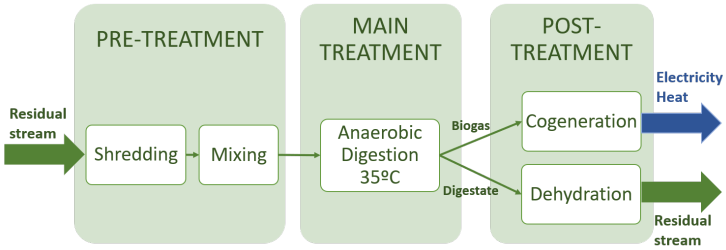

Figure 3. Diagram representing the treatments carried out in the anaerobic digester.

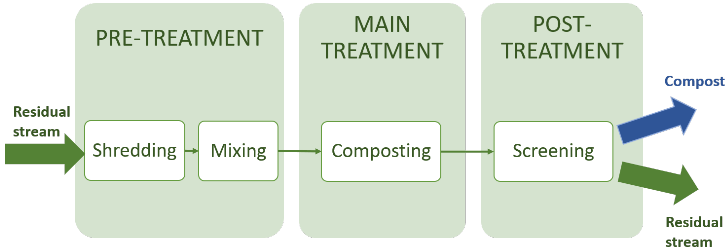

Figure 4. Diagram representing the treatments carried out in the composting plant.

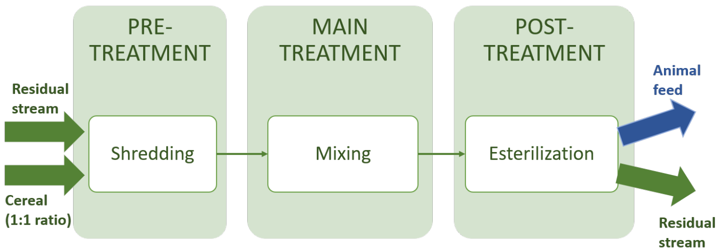

Figure 5. Diagram representing the treatments carried out in the animal feed producer.

As it can be seen in Figure 6, the residual streams may go through different valorization routes. At first, both residual streams may be sent to any of the anaerobic digesters, to any of the composting plants (ending the valorization route), or to the animal feed producer (ending the valorization route as well). In the case that the agri-food industry residual stream is sent to an anaerobic digester, the resulting residual stream is then sent to any of the composting plants, ending the valorization route.

The residual streams in the scenario are modeled considering two different by-products (potato peels, which is the processing waste from AFI_1, and artichoke bracts, which is the processing waste from AFI_2) and 68 chemical compounds. Hence, a residual stream is modeled as a 70×1 real column vector where each entry reflects the amount in tons of by-product or chemical compound that the residual stream contains. The complete list of the considered chemical compounds can be found in [dataset] [38]. For this example, the considered amount of potato peels is 2 tons and the considered amount of artichoke bracts is 5 tons.

Residual streams at the output of AFI_1 and AFI_2, $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$, can be found in [dataset] [38]. According to (1) residual streams at the output of BBI_1 and BBI_2 at $s = 1$ are given by

$$\mathbf{x}_1(1) = \mathbf{A}_1\left(\alpha_{1,1}^{(1)}\mathbf{x}_1(0) + \alpha_{2,1}^{(1)}\mathbf{x}_2(0)\right),$$
$$\mathbf{x}_2(1) = \mathbf{A}_2\left(\alpha_{1,2}^{(1)}\mathbf{x}_1(0) + \alpha_{2,2}^{(1)}\mathbf{x}_2(0)\right),$$

with $\mathbf{A}_1$ and $\mathbf{A}_2$ being the 70×70 real matrices that model the relation between the residual stream at the input and the residual stream at the output of BBI_1 and BBI_2, respectively. Matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ can be found in [dataset] [38].

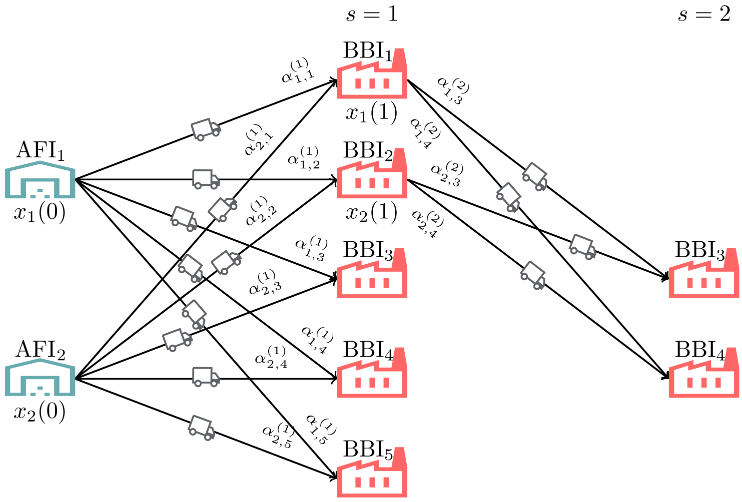

Figure 6. Diagram representing the possible valorization routes of the proposed example.

Three value-added products can be obtained: biogas, which is used to produce electricity and heat (from BBI_1 and BBI_2), compost (from BBI_3 and BBI_4), and animal feed (from BBI_5). Hence, the matrices that model the relation between the residual stream at the input and the value-added products at the output of BBI_i, $\mathbf{B}_i$ with $i \in \{1, \ldots, 5\}$, turn out to be 1×70 real row vectors (which can be found in [dataset] [38]). Moreover, $\mathbf{p}_i$ with $i \in \{1, \ldots, 5\}$, have only one entry, p_i, which corresponds to the selling price of the value-added product obtained at BBI_i, $i \in \{1, \ldots, 5\}$. Revenue from the sale depends on the amount of value-added products obtained and the selling price of such products.

In this example, we classify the costs derived from the valorization of a residual stream into two categories: processing cost (c_p) and transport cost (c_t). We assume that processing cost depends on the total mass processed at the BBI and that transport cost additionally depends on the BBI from which the waste stream processed comes (distance).

We now compute the profit vector at each BBI in this example, which depends on the tons of residual stream processed. A summary of the information provided by the companies for computing the profit vector can be found in Table 2. Distances between each industry ($d_{i,j}$), computed using openrouteservice [39], are presented in Table 3.

Table 2. Value of the different model parameters.

Parameters		Value	Unit	Notation
Processing cost	BBI_1, BBI_2	9.3	€/ton	c_{p1}, c_{p2}
	BBI_3, BBI_4	18.77	€/ton	c_{p3}, c_{p4}
	BBI_5	436.51	€/ton	c_{p5}
Transport cost		0.0468	€/km ton	c_t
Selling price	biogas	490	€/ton	p_1, p_2
	compost	34	€/ton	p_3, p_4
	animal feed	316.83	€/ton	p_5

Table 3. Distance matrix of the considered industries, in kilometres.

	$\mathbf{AFI_1}$	$\mathbf{AFI_2}$	$\mathbf{BBI_1}$	$\mathbf{BBI_2}$	$\mathbf{BBI_3}$	$\mathbf{BBI_4}$	$\mathbf{BBI_5}$
AFI_1	0	13.01	34.74	39.53	54.8	35.17	17.59
AFI_2		0	16.88	50.92	41.89	31.37	16.57
BBI_1			0	34.61	38.75	55.93	38.91
BBI_2				0	50.62	78.45	60.1
BBI_3					0	72.13	57.33
BBI_4						0	18.41
BBI_5							0

Using the information in Tables 2 and 3, we compute the profit obtained at BBI_i at step s, with $s \in \{1,2\}$, when residual stream from BBI_j (or from AFI_j if $s = 1$) is processed as

$$\mathbf{b}_{j,i}(s) = p_i \cdot \mathbf{B}_i^\top - (c_{pi} + d_{j,i}c_t)\mathbf{1}_{70} \quad i,j \in \{1,\ldots,5\},$$

where $\mathbf{1}_H$ is the $H \times 1$ vector of ones.

Hence, according to (4), we aim to solve the following maximization problem:

$$\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{5}\sum_{j=1}^{2}\left(\mathbf{b}_{j,i}(1)\right)^\top \alpha_{j,i}^{(1)}\mathbf{x}_j(0) + \sum_{i=3}^{4}\sum_{j=1}^{2}\left(\mathbf{b}_{j,i}(2)\right)^\top \alpha_{j,i}^{(2)}\mathbf{x}_j(1), \\ &\text{Subject to} \\ &\qquad \alpha_{i,j}^{(1)} \geq 0 \qquad i \in \{1,2\}, j \in \{1,\ldots,5\}, \\ &\qquad \alpha_{i,j}^{(2)} \geq 0 \qquad i \in \{1,2\}, j \in \{3,4\}, \\ &\qquad \textstyle\sum_{j=1}^{5} \alpha_{i,j}^{(1)} = 1 \quad i \in \{1,2\}, \\ &\qquad \textstyle\sum_{j=3}^{4} \alpha_{i,j}^{(2)} = 1 \quad i \in \{1,2\} \end{aligned} \tag{5}$$

The objective function of the optimization problem is a non-linear function. Non-linear optimization problems are considered to be harder than linear problems, and optimizing the non-linear objective function analytically is not an easy task. There are numerical methods that are able to solve the aforementioned optimization problem in a reasonable amount of time and, among them, we use the well-known gradient descent or steepest descent method (see, e.g., [40]).

4.2. Optimization Results and Sensitivity Analysis

In the considered example, the current agri-food industry strategy for the management of processing waste is to send all the residual streams to produce animal feed. Therefore, $\alpha_{1,5}^{(1)} = 1$, $\alpha_{2,5}^{(1)} = 1$, and the profit obtained for the considered amount of by-products generated by AFI_1 and AFI_2 would be -161€. Solving the maximization problem in (5), we obtain that the optimal strategy for the management of the processing waste is to use all the residual streams to generate biogas in the first step and to produce compost in the second step. That is, $\alpha_{1,1}^{(1)} = 1$, $\alpha_{2,1}^{(1)} = 1$, $\alpha_{1,3}^{(2)} = 1$, and the profit obtained for the considered amount of by-products generated by AFI_1 and AFI_2 would be 592€. This result bears evidence that the use of the model can lead to significant profit increases compared to those obtained with current industry practices. Figure 7 compares the current and the optimal residual stream distribution.

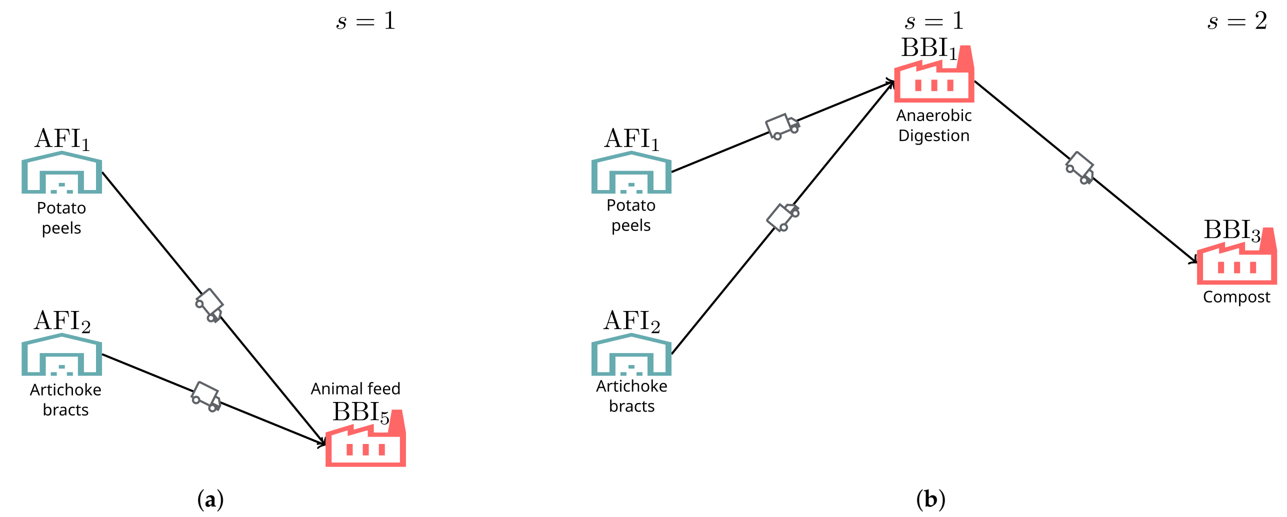

Figure 7. Diagram representing residual stream distribution. (**a**) Current distribution. (**b**) Optimal distribution obtained by solving the maximization problem in (5).

As it has been mentioned, different factors may affect the profit and the optimal residual stream distribution of the case under study. Just to show the potential of the model, a brief sensitivity analysis has been carried out to study the impact of variations in parameters related with the revenues or with the costs.

As an example, we have proposed two groups of scenarios. In the first group of scenarios, we have changed the value of parameters related to the revenue obtained from the sales. Specifically, the selling prices of the biogas, the compost, and the animal feed have been changed by ±35%. These adjustments were made separately as well as simultaneously, resulting in eight different scenarios. Table 4 shows the variation in the solution of (5) after adjusting the prices, and Figure 8 also depicts the optimal residual stream distributions in this group of scenarios. As it is shown, the optimal residual stream distribution changes only in two scenarios: when the selling price of animal feed increases or when the selling prices of all the value-added products increase. If the selling price of animal feed increases, the optimal management strategy would be to send all the residual streams to produce animal feed. If the selling prices of all the value-added products increase, the optimal management strategy would be to use the artichoke bracts to produce animal feed and to use the potato peels to produce first biogas and secondly compost. Interestingly, a decrease in the selling price of the animal feed does not affect either the residual stream distribution or the profit obtained.

Table 4. Summary of the impact variations in selling prices (an up arrow represents an increase and a down arrow represents a decrease).

Value-Added Product	Selling Price	Profit	Optimal Strategy
biogas	↑35%	↑35%	Figure 8c
	↓35%	↓35%	Figure 8c
compost	↑35%	↑11.5%	Figure 8c
	↓35%	↓11.5%	Figure 8c
animal feed	↑35%	↑44.1%	Figure 8a
	↓35%	0%	Figure 8c
all	↑35%	↑48.6%	Figure 8b
	↓35%	↓46.4%	Figure 8c

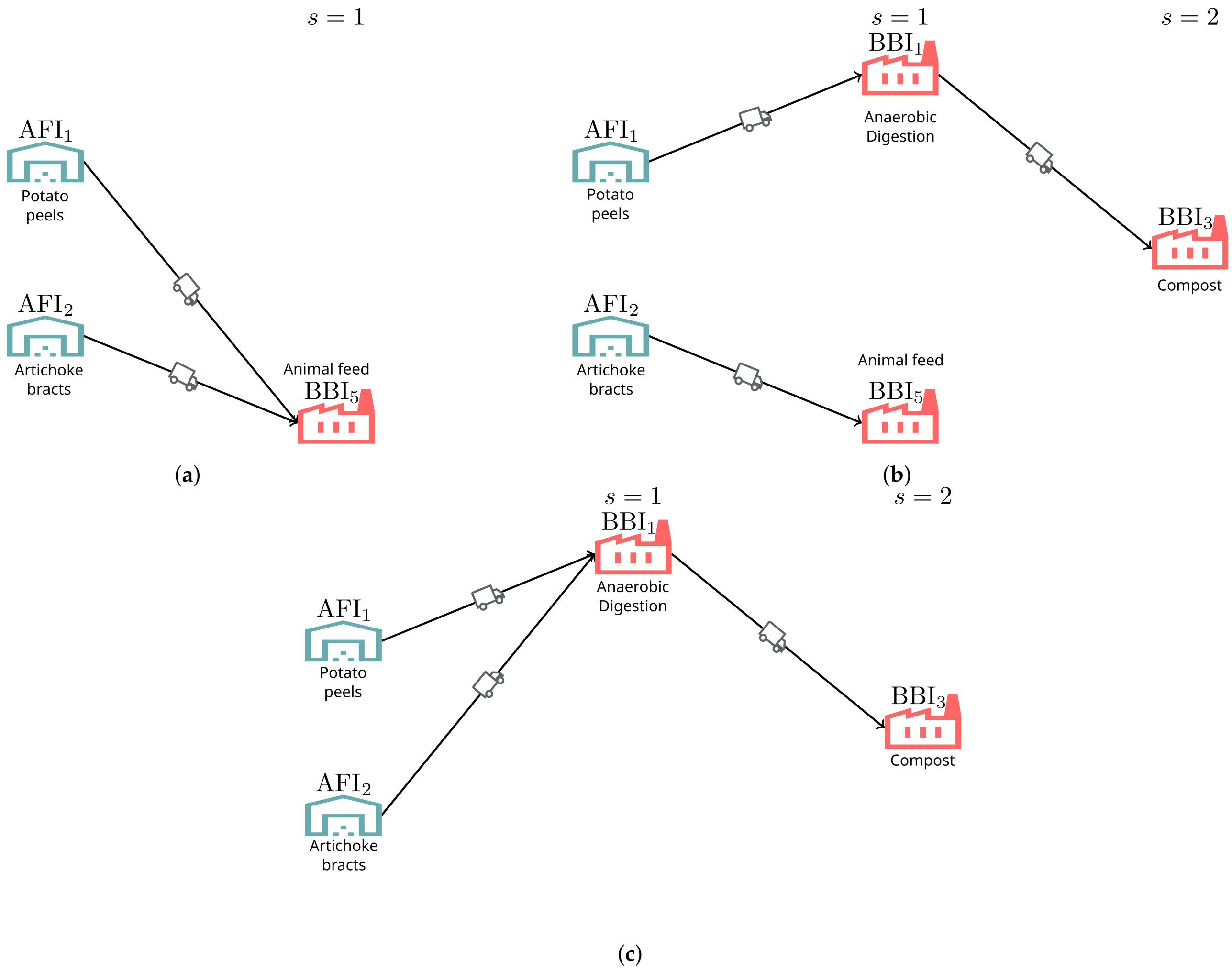

Figure 8. Results obtained when modifying selling prices. (**a**) Optimal distribution when the animal feed price increases 35%. (**b**) Optimal distribution when all selling prices increase 35%. (**c**) Optimal distribution for the rest of resulting scenarios

In the second group of scenarios, we have changed the value of parameters related to the processing costs of the valorization processes. Specifically, the processing cost of the anaerobic digestion plant, the composting plant, and the animal feed producer have changed by $\pm 35\%$. As in the first group, these adjustments were made both separately and simultaneously, resulting in eight different scenarios. Table 5 shows the variation in the solution of (5) after adjusting the processing costs, and Figure 9 depicts the optimal residual stream distributions in the second group of scenarios. As it is shown, the optimal residual stream distribution remains unchanged in every scenario, except when the processing cost of BBI_5 (that is, the animal feed producer) decreases. In that case, the optimal management strategy would be to send all the residual streams to produce animal feed.

The result bears evidence that our model enables us to study the impact of different factors on the profit and on the optimal residual stream distribution. This brief sensitivity analysis shows the potential of the mathematical model as a powerful decision-support tool for optimizing the valorization of agri-food industry residual streams.

Table 5. Summary of the impact of variations in costs derived from the valorization of the residual streams (an up arrow represents an increase and a down arrow represents a decrease).

BBI	Processing Cost	Profit	Optimal Strategy
BBI_1, BBI_2	↑35%	↓3.8%	Figure 9b
	↓35%	↑3.8%	Figure 9b
BBI_3, BBI_4	↑35%	↓6.4%	Figure 9b
	↓35%	↑6.4%	Figure 9b
BBI_5	↑35%	0%	Figure 9b
	↓35%	↑53.2%	Figure 9a
all	↑35%	↓10.4%	Figure 9b
	↓35%	↑10.4%	Figure 9b

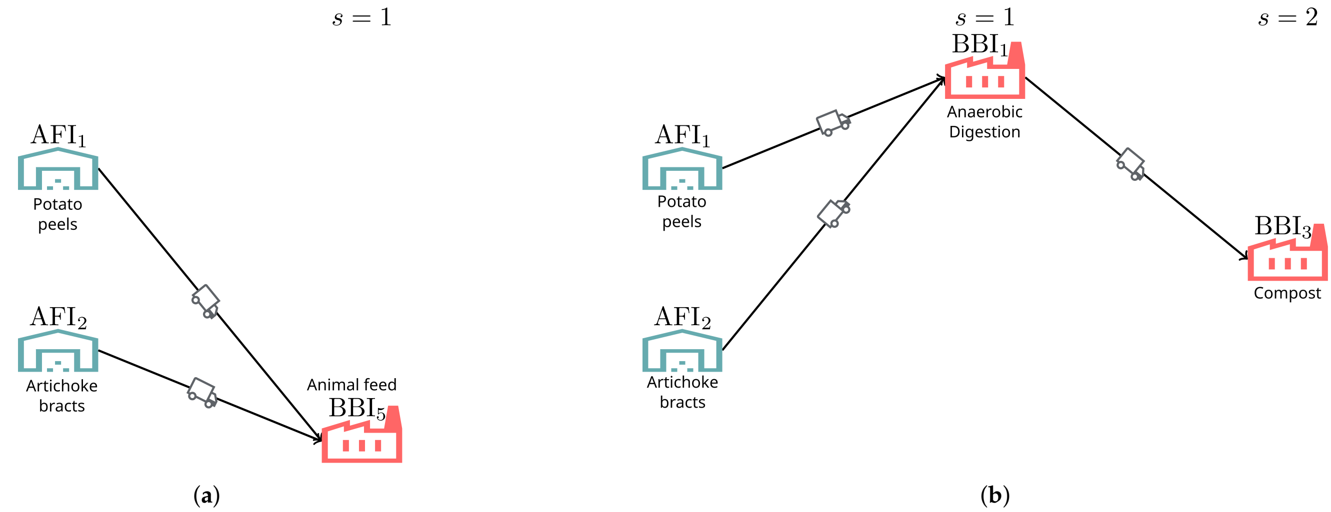

Figure 9. Results obtained when modifying the production costs. (**a**) Optimal distribution when the processing cost of animal feed production decreases 35%. (**b**) Optimal distribution for the rest of resulting scenarios.

5. Conclusions

In this paper, we have presented a mathematical model for optimal agri-food industry residual streams flow management. Our model includes the complete stoichiometry of the residual streams for accurate process modeling and supports the inclusion of economic, social, and environmental factors. Furthermore, variation of the model's parameters enables the detection of key factors that influence the best recommended way for the valorization of agri-food industry residual streams from a holistic perspective. This makes the mathematical model a powerful decision-support tool.

To show the potential of our model, the proposed approach is illustrated through a case study using a real-case network of a region. Although it is a simplified example, the results obtained bear evidence that the use of our model can lead to significant profit increases compared to those obtained with current practices. Moreover, the brief sensitive analysis conducted uncovers several useful management insights and shows that the model has a wide range of possible applications, such as:

- Given a valorization network of industries in a region, the model allows to improve the profits compared to those achieved through existing industry practices.
- Enables to test different scenarios to make informed decisions. For instance, it enables us to study the impact of variations in the selling prizes and/or the valorization costs (individually and simultaneously) in the profit and in the optimal residual stream distribution. This knowledge helps to recognize potential challenges or opportunities in advance.

- Helps to detect key factors that influence the optimal strategy and to identify optimal conditions for achieving the best strategy at a global level.
- Allows us to deal with the problem of establishing the number, size, and location of all the potential BBIs of a valorization network and to study different layouts of the network in order to determine the optimal one.
- Given a complex valorization network of industries, with many different multi-step valorization routes available, the model enables to avoid unprofitable valorization options.
- Helps to anticipate the impact of potential risks by simulating different scenarios. This will improve the resilience of the valorization network, leading to more strategic decision-making.

Interesting future lines would be to explore strategies for addressing possible data limitations in real-world applications, to study how to evaluate in the long run the possible effects of the optimal strategy on the environment and the economy, and to extend the model in order to relax the linearity assumption of the valorization processes.

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