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Some Fractional Integral and Derivative Formulas Revisited

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Abstract: In the most common literature about fractional calculus, we find that ${}_aD_t^\alpha f(t) = {}_aI_t^{-\alpha} f(t)$ is assumed implicitly in the tables of fractional integrals and derivatives. However, this is not straightforward from the definitions of ${}_aI_t^\alpha f(t)$ and ${}_aD_t^\alpha f(t)$. In this sense, we prove that ${}_0D_t^\alpha f(t) = {}_0I_t^{-\alpha} f(t)$ is true for $f(t) = t^{\nu-1}\log t$, and $f(t) = e^{\lambda t}$, despite the fact that these derivations are highly nontrivial. Moreover, the corresponding formulas for ${}_{-\infty}D_t^\alpha |t|^{-\delta}$ and ${}_{-\infty}I_t^\alpha |t|^{-\delta}$ found in the literature are incorrect; thus, we derive the correct ones, proving in turn that ${}_{-\infty}D_t^\alpha |t|^{-\delta} = {}_{-\infty}I_t^{-\alpha} |t|^{-\delta}$ holds true.

Keywords: Riemann–Liouville fractional integral; Riemann–Liouville fractional derivative; Weyl fractional integral; Weyl fractional derivative

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1. Introduction

From the very beginning of the invention of differential calculus, important inquiries regarding the significance of the non-integer operations of integrals and derivatives calculus were brought up. In this sense, it is well known that Leibniz initially introduced a symbolic approach and employed the notation $d^n y/dt^n = D^n y$ to represent the n-th derivative, with n being a non-negative integer. However, L'Hospital asked in a letter to Leibniz dated in 1695 [1]: "What if n is $1/2$?" Leibniz replied, "It will lead to a paradox." But he added, "From this apparent paradox, one day useful consequences will be drawn". From this initial "paradox", fractional calculus was developed through contributions from mathematicians such as Euler, Lagrange, Laplace, Fourier, and others during the 18th and early 19th centuries (a comprehensive summary of its historical progression can be found in [2] [Chap. I]). Despite the efforts of these great mathematicians, a satisfactory expression for the generalization of integration to fractional powers was not developed until the mid-19th century, through the work of Liouville [3]. However, it is worth noting that Abel set the notation that was used later by Liouville (and also used nowadays) for fractional-order integration when solving the generalization of the tautochrone problem (see [4] and the references therein). For a rigorous understanding of fractional calculus as a theory involving operators of integration and differentiation of arbitrary order, we recommend the book by Samko, Kilbas, and Marichev [5].

Definition 1 (Riemann–Liouville fractional integral)**.** *For $\alpha > 0$ [6] ([Chap. XIII])*

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{1} $$

Remark 1. *Usually, $a = 0$ in many textbooks and applications [7] [Eqn. 1.2]. However, when $a \to -\infty$ in (1), we obtain the Weyl fractional integral, i.e.,*

$$ {}_{-\infty}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{2} $$

It is worth noting that in [6] [Chap. XIII], we find other definition of the Weyl fractional integral, i.e.,

$$ \mathfrak{W}_{\alpha}\{f(\tau);t\} = \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (\tau-t)^{\alpha-1} f(\tau) d\tau. $$

It is easy to prove that

$$ {}_{-\infty}I_t^{\alpha} f(t) = \mathfrak{W}_{\alpha}\{f(-\tau);-t\}. $$

As pointed out in [7] [Sect. 1.2], one is tempted to substitute α with $-\alpha$ in (1) in order to obtain a definition for the fractional derivative ${}_aD_t^{\alpha} f(t)$. Nevertheless, some care is required in the integration for this generalization, and the theory of generalized functions has to be invoked. In order to avoid the use of generalized functions, we find in [8] [Sect. 2.3.3] the following definition:

Definition 2 (Riemann–Liouville fractional derivative)**.** *For $m \in \mathbb{N}$, and $\alpha > 0$*

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_a^t \dfrac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{3} $$

Remark 2. *Usually, $a = 0$ in many textbooks and applications [7] [Eqn. 1.13b]. However, when $a \to -\infty$ in (3), we obtain the Weyl fractional derivative [7] [Eqn. 1.108], i.e., for $m \in \mathbb{N}$, and $\alpha > 0$*

$$ {}_{-\infty}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_{-\infty}^t \dfrac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{4} $$

Nonetheless, in the existing literature, we find tables of Riemann–Liouville fractional derivatives, wherein they just substitute α with $-\alpha$ in the corresponding Riemann–Liouville fractional integral. For instance, if $E_{\mu,\nu}(z)$ denotes the Mittag–Leffler function, and $\psi(z)$ denotes the digamma function, we find in [6] [Eqn. 13.1.(24)] and [9] [Table IV.1], respectively,

$$ {}_0I_t^{\alpha}\left[t^{\nu-1}\log t\right] = \frac{t^{\alpha+\nu-1}\,\Gamma(\nu)}{\Gamma(\alpha+\nu)}[\log t + \psi(\nu) - \psi(\alpha+\nu)], \tag{5} $$

$$ {}_0I_t^{\alpha} e^{\lambda t} = t^{\alpha} E_{1,1+\alpha}(\lambda t), \tag{6} $$

Nevertheless, in [8] [Appendix], we find that α is changed by $-\alpha$ in (5) and (6) in order to obtain the corresponding fractional derivatives, i.e., ${}_0D_t^{\alpha}\left[t^{\nu-1}\log t\right]$ and ${}_0D_t^{\alpha} e^{\lambda t}$.

Also, we find in [7] [Eqn. 1.112]

$$ {}_{-\infty}D_t^{\alpha}|t|^{-\delta} = \frac{\Gamma(\delta+\alpha)}{\Gamma(\delta)}\,|t|^{-\alpha-\delta} = {}_{-\infty}I_t^{-\alpha}|t|^{-\delta}. \tag{7} $$

However, according to our numerical experiments, it seems that (7) does not hold true. Consequently, the aim of this paper is twofold. On the one hand, we want to justify that

$$ \begin{aligned} {}_0D_t^{\alpha}\left[t^{\nu-1}\log t\right] &= {}_0I_t^{-\alpha}\left[t^{\nu-1}\log t\right], \\ {}_0D_t^{\alpha} e^{\lambda t} &= {}_0I_t^{-\alpha} e^{\lambda t}, \end{aligned} $$

from the definitions given in (1) and (3). We will see that these proofs are highly non-trivial. On the other hand, we would like to calculate the Weyl fractional integral and derivative for the power function, i.e., ${}_{-\infty}I_t^{\alpha}|t|^{-\delta}$ and ${}_{-\infty}D_t^{\alpha}|t|^{-\delta}$, as well as to justify that

$$ {}_{-\infty}D_t^{\alpha}|t|^{-\delta} = {}_{-\infty}I_t^{-\alpha}|t|^{-\delta}. $$

This paper is organized as follows. Section 2 collects all the definitions of the special functions and polynomials that appear throughout the paper. Section 3 calculates the fractional integrals ${}_0I_t^{\alpha}\left[t^{\nu-1}\log t\right]$, ${}_0I_t^{\alpha}e^{\lambda t}$, and ${}_{-\infty}I_t^{\alpha}|t|^{-\delta}$. Despite the fact that ${}_0I_t^{\alpha}\left[t^{\nu-1}\log t\right]$, and ${}_0I_t^{\alpha}e^{\lambda t}$ are found in the existing literature, it is worth performing these calculations, as they will be useful in the following section. In Section 4, we calculate ${}_0D_t^{\alpha}\left[t^{\nu-1}\log t\right]$, ${}_0D_t^{\alpha}e^{\lambda t}$ and ${}_{-\infty}D_t^{\alpha}|t|^{-\delta}$. Finally, we collect our conclusions in Section 5.

2. Preliminaries

In this section, we collect all the definitions of the special functions and polynomials that appear throughout the paper.

Definition 3. *For* $\operatorname{Re}\alpha > 0$*, the gamma function is defined as [10] [Eqn. 1.1.1]*

$$ \Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt. \tag{8} $$

Definition 4. *The digamma function is defined as [10] [Eqn. 1.3.1]*

$$ \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{9} $$

Definition 5. *The Pochhammer polynomial is defined as [11] [Eqn. 18:12:1]*

$$ (x)_n = x\,(x+1)(x+2)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{10} $$

Definition 6. *For* $\operatorname{Re}a > 0$, $\operatorname{Re}b > 0$*, the beta function is defined as [10] [Eqns. 1.5.2&5]*

$$ \mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}. \tag{11} $$

Remark 3. *Note that if we are considering* $a, b \in \mathbb{R}$*, then the condition* $\operatorname{Re}a > 0$, $\operatorname{Re}b > 0$*, becomes* $a, b > 0$*.*

Definition 7. *The lower incomplete gamma function is defined as [11] [Eqn. 45:3:1]*

$$ \gamma(\alpha,z) = \int_0^z t^{\alpha-1}e^{-t}dt. \tag{12} $$

Definition 8. *The two-parameter Mittag–Leffler function is defined as [12] [Eqn. 10.46.3],*

$$ E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu\,k+\nu)}. \tag{13} $$

Definition 9. *The generalized hypergeometric function is defined as [12] [Eqn. 16.2.1]*

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}. \tag{14} $$

3. Fractional Integrals

3.1. Fractional Integral of the Power Function

Theorem 1. *For $\alpha > 0$, $\gamma > -1$, and $t > 0$, we have [6] [Eqn. 13.1.(7)]*

$$ {}_0I_t^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\gamma+\alpha}. \tag{15} $$

Proof. Apply definition (1), and perform the change in variables $t - \tau = st$, with $t > 0$ to obtain

$$ \begin{aligned} {}_0I_t^{\alpha} t^{\gamma} &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\gamma}\, d\tau \\ &= \frac{t^{\alpha+\gamma}}{\Gamma(\alpha)} \underbrace{\int_0^1 s^{\alpha-1} (1-s)^{\gamma}\, d\tau}_{\mathrm{B}(\alpha,1+\gamma)}. \end{aligned} $$

Finally, apply (11) to obtain (15) as we wanted to prove. Note that we consider $\alpha, \gamma \in \mathbb{R}$; thus, according to Remark 3, we have $\alpha > 0$, $\gamma > -1$. □

As mentioned in the Introduction, the Weyl fractional integral for the power function given in (7) does not seem to hold true. Therefore, next, we are going to calculate ${}_{-\infty}I_t^{\alpha} |t|^{-\delta}$. For this purpose, let us first prove the following lemma.

Lemma 1. *For $\alpha < -\beta - 1 < 0$, and $t > 0$, the following integral formula holds true:*

$$ \begin{aligned} \mathcal{I}_{\alpha,\beta}(t) &= \int_{-\infty}^{0} (t-\tau)^{\alpha} |\tau|^{\beta} d\tau \\ &= t^{\alpha+\beta+1} \frac{\Gamma(-1-\alpha-\beta)\,\Gamma(\beta+1)}{\Gamma(-\alpha)}. \end{aligned} \tag{16} $$

Proof. Perform the change in variables $u = -\tau$ to obtain

$$ \begin{aligned} \mathcal{I}_{\alpha,\beta}(t) &= \int_{-\infty}^{0} (t-\tau)^{\alpha} (-\tau)^{\beta} d\tau \\ &= \int_0^{\infty} (t+u)^{\alpha} u^{\beta} du. \end{aligned} $$

Perform the change in variables $u = st/(1-s)$ with $t > 0$, and use the definition of the beta function (11) to obtain

$$ \begin{aligned} \mathcal{I}_{\alpha,\beta}(t) &= \int_0^1 \left(t + \frac{st}{1-s} \right)^{\alpha} \left(\frac{st}{1-s} \right)^{\beta} t \frac{ds}{(1-s)^2} \\ &= t^{\alpha+\beta+1} \underbrace{\int_0^1 (1-s)^{-\alpha-\beta-2} s^{\beta} ds}_{\mathrm{B}(-\alpha-\beta-1,\beta+1)}. \end{aligned} $$

Applying again (11), we finally arrive at (16), as we wanted to prove. According to Remark 3, we have $\alpha < -\beta - 1 < 0$. □

Theorem 2. *For $0 < \alpha < \delta < 1$, and $t > 0$*

$$ {}_{-\infty}I_t^{\alpha} |t|^{-\delta} = \frac{\Gamma(\delta-\alpha)}{\Gamma(\delta)} \frac{\cos\left(\frac{\pi\delta}{2} - \pi\alpha\right)}{\cos\left(\frac{\pi\delta}{2}\right)} t^{\alpha-\delta}. \tag{17} $$

Proof. Note that

$$ {}_{-\infty}I_t^{\alpha}|t|^{-\delta} = {}_{-\infty}I_0^{\alpha}|t|^{-\delta} + {}_0I_t^{\alpha}|t|^{-\delta}. $$

Also, from (1) and (15), for $\alpha > 0$, $\delta < 1$ and $t > 0$, we have

$$ {}_0I_t^{\alpha}|t|^{-\delta} = {}_0I_t^{\alpha}\, t^{-\delta} = \frac{\Gamma(1-\delta)}{\Gamma(1-\delta+\alpha)} t^{\alpha-\delta}. \tag{18} $$

Moreover, from definition (1) and lemma (16), we have for $\alpha < \delta < 1$, and $t > 0$

$$ {}_{-\infty}I_0^{\alpha}|t|^{-\delta} = \frac{1}{\Gamma(\alpha)} \underbrace{\int_{-\infty}^{0} (t-\tau)^{\alpha-1}|\tau|^{-\delta}}_{\mathcal{I}_{\alpha-1,-\delta}(t)} = t^{\alpha-\delta} \frac{\Gamma(\delta-\alpha)\,\Gamma(1-\delta)}{\Gamma(1-\alpha)}, \tag{19} $$

and thus, according to (18) and (19), we obtain

$$ \begin{aligned} & {}_{-\infty}I_t^{\alpha}|t|^{-\delta} \\ = \;& t^{\alpha-\delta}\,\Gamma(1-\delta)\left[\frac{\Gamma(\delta-\alpha)}{\Gamma(\alpha)\,\Gamma(1-\alpha)} + \frac{1}{\Gamma(1+\alpha-\delta)}\right] \\ = \;& t^{\alpha-\delta}\,\frac{\Gamma(\delta-\alpha)\,\Gamma(1-\delta)\,\Gamma(\delta)}{\Gamma(\delta)} \\ & \left[\frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)} + \frac{1}{\Gamma(1+\alpha-\delta)\,\Gamma(\delta-\alpha)}\right]. \end{aligned} $$

Now, apply the property [10] [Eqn. 1.2.2]

$$ \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{20} $$

to obtain

$$ {}_{-\infty}I_t^{\alpha}|t|^{-\delta} = t^{\alpha-\delta}\,\frac{\Gamma(\delta-\alpha)}{\Gamma(\delta)\sin(\pi\delta)}[\sin(\pi\alpha) + \sin(\pi(\delta-\alpha))]. $$

Finally, apply the property [13] [Eqn. 5.61]

$$ \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \tag{21} $$

to arrive at (17) as we wanted to prove. □

3.2. Fractional Integral of the Exponential Function

Lemma 2. *The following identity holds true:*

$$ \gamma(\alpha, z) = z^{\alpha}\,\Gamma(\alpha)\,e^{-z}\,E_{1,1+\alpha}(z). \tag{22} $$

Proof. Consider the expansion [11] [Eqn. 45:6:2]

$$ e^{z}\,\gamma(\alpha, z) = \frac{z^{\alpha}}{\alpha}\sum_{k=0}^{\infty}\frac{z^k}{(\alpha+1)_k}. $$

Taking into account the factorial property of the gamma function [10] [Eqn. 1.2.1], i.e.,

$$ \Gamma(z+1) = z\,\Gamma(z), \tag{23} $$

and the definition of the Pochhammer symbol (10), we have

$$ e^{z}\,\gamma(\alpha, z) = z^{\alpha}\,\Gamma(\alpha)\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha+1+k)}. $$

Finally, apply the definition of the Mittag–Leffler function (13) to complete the proof. □

Theorem 3. *For $t > 0$, and $\alpha > 0$, the following fractional integral holds true [9] [Table IV.1]:*

$$ {}_0I_t^{\alpha} e^{\lambda t} = t^{\alpha} E_{1,1+\alpha}(\lambda t). \tag{24}$$

Proof. According to the definition of the Riemann–Liouville fractional integral (1), we have

$$\begin{aligned} {}_0I_t^{\alpha} e^{\lambda t} &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{\lambda\tau} d\tau \\ &= \frac{\lambda^{-\alpha}}{\Gamma(\alpha)} \int_0^t (\lambda t - \lambda\tau)^{\alpha-1} e^{\lambda\tau} \lambda d\tau, \end{aligned}$$

thus performing the change in variables $u = \lambda(t-\tau)$, and taking into account the definition of the lower incomplete gamma function (12), we obtain

$$\begin{aligned} {}_0I_t^{\alpha} e^{\lambda t} &= \frac{\lambda^{-\alpha} e^{\lambda t}}{\Gamma(\alpha)} \int_0^{\lambda t} u^{\alpha-1} e^{-u} du \\ &= \frac{\lambda^{-\alpha} e^{\lambda t}}{\Gamma(\alpha)} \gamma(\alpha, \lambda t). \end{aligned}$$

Finally, apply (22) to complete the proof. □

3.3. Fractional Integral Formula of the Logarithmic Function

Lemma 3. *The following integral formula holds true:*

$$\begin{aligned} &\int_0^1 t^{a-1}(1-t)^{b-1} \log t \, dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} [\psi(a) - \psi(a+b)], \\ &\mathrm{Re}\, a > 0,\ \mathrm{Re}\, b > 0. \end{aligned} \tag{25}$$

Proof. Perform the derivative with respect to the first parameter in the beta function definition (11),

$$\begin{aligned} &\frac{\partial}{\partial a} \mathrm{B}(a,b) \\ =\ &\int_0^1 t^{a-1}(1-t)^{b-1} \log t \, dt = \Gamma(b) \frac{\Gamma'(a)\,\Gamma(a+b) - \Gamma(a)\,\Gamma'(a+b)}{\Gamma^2(a+b)}, \end{aligned} \tag{26}$$

and take into account the definition of the digamma function (9) to complete the proof. □

Theorem 4. *The following fractional integral holds true [6] [Eqn. 13.1.(24)]:*

$$\begin{aligned} &{}_0I_t^{\alpha} \left[t^{\nu-1} \log t \right] = \frac{t^{\alpha+\nu-1}\,\Gamma(\nu)}{\Gamma(\alpha+\nu)} [\log t + \psi(\nu) - \psi(\alpha+\nu)], \\ &\mathrm{Re}\, \alpha > 0,\ \mathrm{Re}\, \nu > 0. \end{aligned} \tag{27}$$

Proof. Apply the definition (1) and perform the change in variables $\tau = t\,u$ with $t > 0$ to obtain

$$\begin{aligned} &{}_0I_t^{\alpha} \left[t^{\nu-1} \log t \right] \\ =\ &\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\nu-1} \log \tau \, d\tau \\ =\ &\frac{t^{\alpha+\nu-1}}{\Gamma(\alpha)} \left\{ \log t \int_0^1 (1-u)^{\alpha-1} u^{\nu-1} du + \int_0^1 (1-u)^{\alpha-1} u^{\nu-1} \log u \, du \right\}. \end{aligned} \tag{28}$$

Use the definition of the beta function (11) to calculate the first integral in (28) as

$$\int_0^1 (1-u)^{\alpha-1} u^{\nu-1} du = \frac{\Gamma(\nu)\,\Gamma(\alpha)}{\Gamma(\alpha+\nu)}, \quad (29)$$

$$\operatorname{Re}\alpha > 0,\ \operatorname{Re}\nu > 0.$$

The second integral in (28) is given in lemma (25). Therefore, substituting (25) and (29) in (28), we arrive at (27) as we wanted to prove. □

4. Fractional Derivatives

4.1. Fractional Derivative of the Power Function

Lemma 4. *The following n-th derivative formula holds:*

$$\frac{d^n}{dt^n} t^a = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} t^{a-n}. \quad (30)$$

Proof. According to the definition of the Pochhamer symbol (10), we have

$$\begin{aligned} \frac{d^n}{dt^n} t^a &= a(a-1)\cdots(a-n+1) t^{a-n} \\ &= (a-n+1)_n\, t^{a-n} \\ &= \frac{\Gamma(a+1)}{\Gamma(a-n+1)} t^{a-n}. \end{aligned}$$

□

Theorem 5. *For $\alpha > 0$, $\gamma > -1$, and $t > 0$ [9] [Table IV.1], [8] [Appendix], the following fractional derivative holds true:*

$${}_0D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}. \quad (31)$$

Proof. According to definition (3), and the results given in (15) and (30),

$$\begin{aligned} {}_0D_t^\alpha t^\gamma &= \left(D_t^m \circ {}_0I_t^{m-\alpha}\right) t^\gamma \\ &= \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+m-\alpha)} \frac{d^m}{dt^m} t^{\gamma+m-\alpha} \\ &= \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+m-\alpha)} \frac{\Gamma(\gamma+m-\alpha+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma+m-\alpha-m}, \end{aligned}$$

as we wanted to prove. □

Now, we calculate the Weyl fractional derivative corresponding to the Weyl fractional integral calculated in (17).

Theorem 6. *For $0 < \delta < 1$, $\alpha > 0$, and $t > 0$,*

$${}_{-\infty}D_t^\alpha |t|^{-\delta} = \frac{\Gamma(\delta+\alpha)}{\Gamma(\delta)} \frac{\cos\left(\frac{\pi\delta}{2} + \pi\alpha\right)}{\cos\left(\frac{\pi\delta}{2}\right)} t^{-\alpha-\delta}. \quad (32)$$

Proof. According to the definition of the Weyl fractional integral (4), we have

$${}_{-\infty}D_t^\alpha |t|^{-\delta} = {}_{-\infty}D_0^\alpha |t|^{-\delta} + {}_0D_t^\alpha |t|^{-\delta}. \quad (33)$$

On the one hand, apply definition (3), taking into account that $\alpha \notin \mathbb{N}$ and $m - \alpha > 0$,

$$ {}_{-\infty}D_0^{\alpha}|t|^{-\delta} = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \underbrace{\int_{-\infty}^{0} \frac{|\tau|^{-\delta}}{(t-\tau)^{\alpha+1-m}} d\tau}_{\mathcal{I}_{m-1-\alpha,-\delta}(t)}. $$

Now, apply lemmas (16) and (30) to obtain for $m - \alpha < \delta < 1$

$$ \begin{aligned} {}_{-\infty}D_0^{\alpha}|t|^{-\delta} &= \frac{\Gamma(\alpha+\delta-m)\,\Gamma(1-\delta)}{\Gamma(m-\alpha)\,\Gamma(1+\alpha-m)} \frac{d^m}{dt^m} t^{m-\alpha-\delta} \\ &= \frac{\Gamma(\alpha+\delta-m)\,\Gamma(1-\delta)}{\Gamma(m-\alpha)\,\Gamma(1+\alpha-m)} \frac{\Gamma(m-\alpha-\delta+1)}{\Gamma(-\alpha-\delta+1)} t^{-\alpha-\delta}. \end{aligned} \tag{34} $$

On the other hand, taking into account that $t > 0$ and $\alpha \notin \mathbb{N}$, apply (31) to obtain for $\delta < 1$,

$$ {}_0D_t^{\alpha}|t|^{-\delta} = {}_0D_t^{\alpha}\, t^{-\delta} = \frac{\Gamma(1-\delta)}{\Gamma(1-\delta-\alpha)} t^{-\delta-\alpha}. \tag{35} $$

Substitute results (34) and (35) in (33) to arrive at

$$ {}_{-\infty}D_t^{\alpha}|t|^{-\delta} = \frac{t^{-\alpha-\delta}\,\Gamma(1-\delta)}{\Gamma(1-\alpha-\delta)} \left[\frac{\Gamma(\alpha+\delta-m)\,\Gamma(1-\alpha-\delta+m)}{\Gamma(m-\alpha)\,\Gamma(1+\alpha-m)} + 1 \right]. $$

Now, apply (20)

$$ \begin{aligned} & {}_{-\infty}D_t^{\alpha}|t|^{-\delta} \\ =\;& t^{-\alpha-\delta} \frac{\Gamma(\alpha+\delta)}{\Gamma(\delta)} \frac{\Gamma(1-\delta)\,\Gamma(\delta)}{\Gamma(1-\alpha-\delta)\,\Gamma(\alpha+\delta)} \left[\frac{\sin \pi(m-\alpha)}{\sin \pi(\alpha+\delta-m)} + 1 \right] \\ =\;& t^{-\alpha-\delta} \frac{\Gamma(\alpha+\delta)}{\Gamma(\delta)} \frac{\sin \pi(\alpha+\delta)}{\sin \pi\delta} \left[1 - \frac{\sin \pi\alpha}{\sin \pi(\alpha+\delta)} \right] \\ =\;& t^{-\alpha-\delta} \frac{\Gamma(\alpha+\delta)}{\Gamma(\delta)} \left[\frac{\sin \pi(\alpha+\delta) - \sin \pi\alpha}{\sin \pi\delta} \right]. \end{aligned} $$

Finally, apply (21) and simplify to arrive at (32) as we wanted to prove. □

4.2. Fractional Derivative of the Exponential Function

Theorem 7. *For $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $t > 0$*

$$ {}_0D_t^{\alpha} e^{\lambda t} = t^{-\alpha} E_{1,1-\alpha}(\lambda t). \tag{36} $$

Proof. Apply the definition (3) and expand the exponential fraction in its Maclaurin series to obtain

$$ \begin{aligned} {}_0D_t^{\alpha} e^{\lambda t} &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} e^{\lambda\tau} d\tau \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} \sum_{k=0}^{\infty} \frac{(\lambda\tau)^k}{k!} d\tau \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t \frac{\tau^k}{(t-\tau)^{\alpha-m+1}} d\tau \right). \end{aligned} \tag{37} $$

Perform the change in variables $\tau = u\,t$ with $t > 0$ in (37), and apply the definition of the beta function (11). Thus, for $m - \alpha > 0$, we have

$$\begin{aligned} {}_0D_t^\alpha e^{\lambda t} &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \left(t^{m-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \int_0^1 u^k (1-u)^{m-\alpha-1} du \right) \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \left(t^{m-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \mathrm{B}(k+1, m-\alpha) \right) \\ &= \frac{d^m}{dt^m} \sum_{k=0}^{\infty} \frac{\lambda^k\, t^{m-\alpha+k}}{\Gamma(m-\alpha+k+1)}. \end{aligned}$$

Now, apply the differentiation formula (30) to arrive at

$$ {}_0D_t^\alpha e^{\lambda t} = t^{-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{\Gamma(k+1-\alpha)}. $$

Finally, apply the definition of the Mittag–Leffler function (13) to complete the proof. □

4.3. Fractional Derivative of the Logarithm Function

Lemma 5. *The following n-th derivative formula holds true:*

$$ \frac{d^n}{dt^n}\left(t^\beta \log t\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} t^{\beta-n} [\log t + \psi(\beta+1) - \psi(\beta-n+1)]. \tag{38} $$

Proof. According to Leibniz's differentiation formula [12] [Eqn. 1.4.12],

$$ \frac{d^n}{dx^n}[f(x)g(x)] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} f(x) \frac{d^k}{dx^k} g(x), \tag{39} $$

and applying the n-th derivative formula given in (30), as well as the following one (which can be easily proved by induction)

$$ \frac{d^n}{dt^n} \log t = \frac{(-1)^{n-1}(n-1)!}{t^n}, \quad n = 1, 2, \ldots $$

after simplification, we arrive at

$$\begin{aligned} &\frac{d^n}{dt^n}\left(t^\beta \log t\right) \\ = &\sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dt^{n-k}} t^\beta \frac{d^k}{dt^k} \log t \\ = &\log t \frac{d^n}{dt^n} t^\beta + \sum_{k=1}^{n} \binom{n}{k} \frac{d^{n-k}}{dt^{n-k}} t^\beta \frac{d^k}{dt^k} \log t \\ = &\Gamma(\beta+1)\, t^{\beta-n} \left[\frac{\log t}{\Gamma(\beta-n+1)} + n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k\,(n-k)!\,\Gamma(\beta-n+k+1)} \right]. \end{aligned} \tag{40}$$

In order to calculate the finite sum given in (40), consider the following function $f(z)$ which can be recast as a hypergeometric function [14] [Sect. 2.1]

$$ f(z) = \sum_{k=1}^{n} \frac{z^{k-1}}{(n-k)!\,\Gamma(\beta-n+k+1)} \tag{41} $$

$$ = \frac{1}{\Gamma(n)\Gamma(\beta-n+2)}\, {}_2F_1\left(\begin{array}{c} 1, 1-n \\ \beta-n+2 \end{array} \middle| -z \right). \tag{42} $$

On the one hand, integrating term by term in (41)

$$g(z) = \int_0^z f(t)\,dt = \sum_{k=1}^{n} \frac{z^k}{k\,(n-k)!\,\Gamma(\beta-n+k+1)}. \tag{43}$$

On the other hand, applying the integration formula given in [14] [Eqn. 2.2.3]

$$\begin{aligned} & {}_{p+1}F_{q+1}\left(\begin{array}{c} a_1,\ldots,a_p,a_{p+1} \\ b_1,\ldots,b_q,b_{q+1} \end{array} \middle| x \right) \\ = \; & \frac{\Gamma\left(b_{q+1}\right) x^{1-b_{q+1}}}{\Gamma\left(a_{p+1}\right)\Gamma\left(b_{q+1}-a_{p+1}\right)} \\ & \int_0^x t^{a_{p+1}-1}(x-t)^{b_{q+1}-a_{p+1}-1}\,{}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array} \middle| t \right) dt, \end{aligned}$$

taking $a_1 = 1$, $a_2 = 1-n$, $b_1 = \beta-n+2$, we obtain

$$\begin{aligned} & g(z) \\ = \; & \int_0^z f(t)\,dt = \frac{1}{\Gamma(n)\Gamma(\beta-n+2)} \int_0^z {}_2F_1\left(\begin{array}{c} 1,1-n \\ \beta-n+2 \end{array} \middle| -t \right) dt \\ = \; & \frac{z}{\Gamma(n)\Gamma(\beta-n+2)}\,{}_3F_2\left(\begin{array}{c} 1,1,1-n \\ 2,\beta-n+2 \end{array} \middle| -z \right) \end{aligned} \tag{44}$$

Therefore, from (43) and (44),

$$\begin{aligned} -g(-1) &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k\,(n-k)!\,\Gamma(\beta-n+k+1)} \\ &= \frac{1}{\Gamma(n)\Gamma(\beta-n+2)}\,{}_3F_2\left(\begin{array}{c} 1,1,1-n \\ 2,\beta-n+2 \end{array} \middle| 1 \right). \end{aligned}$$

Now, consider the reduction formula [15] [Eqn. 7.4.4(40)]

$${}_3F_2\left(\begin{array}{c} 1,1,a \\ 2,b \end{array} \middle| 1 \right) = \frac{b-1}{a-1}[\psi(b-1)-\psi(b-a)],$$

to arrive at

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k\,(n-k)!\Gamma(\beta-n+k+1)} = \frac{\psi(\beta+1)-\psi(\beta-n+1)}{n!\,\Gamma(\beta-n+1)}. \tag{45}$$

Finally, substitute (45) in (40) to arrive at the desired result. □

Theorem 8. *For $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\mathrm{Re}\,\nu > 0$, and $t > 0$, the following fractional derivative holds true:*

$${}_0D_t^{\alpha}\left[t^{\nu-1}\log t\right] = \frac{t^{\nu-\alpha-1}\,\Gamma(\nu)}{\Gamma(\nu-\alpha)}[\log t + \psi(\nu) - \psi(\nu-\alpha)]. \tag{46}$$

Proof. Apply the definition (3) and perform the change in variables $\tau = t\,u$ with $t > 0$, to obtain for $\alpha > 0$,

$$\begin{aligned} & {}_0D_t^\alpha\left[t^{\nu-1}\log t\right] \\ = & \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}\tau^{\nu-1}\log\tau\,d\tau \\ = & \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\left\{t^{m-\alpha+\nu-1}\left[\log t\int_0^1 (1-u)^{m-\alpha-1}u^{\nu-1}du \right.\right. \\ & \left.\left. +\int_0^1 (1-u)^{m-\alpha-1}u^{\nu-1}\log u\,du\right]\right\}. \end{aligned} \tag{47}$$

The first integral in (47) is just a beta function (11); thus, for $\mathrm{Re}\,\nu > 0$, we have

$$\int_0^1 (1-u)^{m-\alpha-1}u^{\nu-1}du = \frac{\Gamma(\nu)\,\Gamma(m-\alpha)}{\Gamma(m-\alpha+\nu)},$$

and for the second one, we can apply (25); thus, for $\mathrm{Re}\,\nu > 0$, we have

$$\int_0^1 (1-u)^{m-\alpha-1}u^{\nu-1}\log u\,du = \frac{\Gamma(\nu)\,\Gamma(m-\alpha)}{\Gamma(m-\alpha+\nu)}[\psi(\nu)-\psi(m-\alpha+\nu)],$$

thereby

$$\begin{aligned} & {}_0D_t^\alpha\left[t^{\nu-1}\log t\right] \\ = & \frac{\Gamma(\nu)}{\Gamma(m-\alpha+\nu)} \\ & \left\{\frac{d^m}{dt^m}\left(t^{m-\alpha+\nu-1}\log t\right)+[\psi(\nu)-\psi(m-\alpha+\nu)]\frac{d^m}{dt^m}t^{m-\alpha+\nu-1}\right\}. \end{aligned} \tag{48}$$

Apply (38) to obtain

$$\begin{aligned} & \frac{d^m}{dt^m}\left(t^{m-\alpha+\nu-1}\log t\right) \\ = & \frac{\Gamma(m-\alpha+\nu)}{\Gamma(\nu-\alpha)}t^{-\alpha+\nu-1}[\log t+\psi(m-\alpha+\nu)-\psi(\nu-\alpha)], \end{aligned} \tag{49}$$

and apply (30) to obtain

$$\frac{d^m}{dt^m}t^{m-\alpha+\nu-1} = \frac{\Gamma(m-\alpha+\nu)}{\Gamma(\nu-\alpha)}t^{-\alpha+\nu-1}. \tag{50}$$

Insert (49) and (50) in (48), and simplify the result to complete the proof. □

5. Conclusions

On the one hand, according to (27), (46), (24) and (36), we have analytically justified that

$$\begin{aligned} {}_0D_t^\alpha\left[t^{\nu-1}\log t\right] &= {}_0I_t^{-\alpha}\left[t^{\nu-1}\log t\right] \\ &= \frac{t^{\nu-\alpha-1}\,\Gamma(\nu)}{\Gamma(\nu-\alpha)}[\log t+\psi(\nu)-\psi(\nu-\alpha)], \end{aligned} \tag{51}$$

$${}_0D_t^\alpha e^{\lambda t} = {}_0I_t^{-\alpha}e^{\lambda t} = t^{-\alpha}E_{1,1-\alpha}(\lambda t)\,, \tag{52}$$

applying the corresponding definitions of the Riemann–Liouville fractional integral (1) and the Riemann–Liouville fractional derivative (3). Note that the fractional derivatives calculated in (51) and (52) can be obtained from the corresponding fractional integrals, substituting α by $-\alpha$. However, the corresponding derivations from the Riemann–Liouville definitions of the fractional integral and the fractional derivative in order to arrive to this conclusion are highly non-trivial.

On the other hand, from the definitions of the Weyl fractional integral (2), and the Weyl fractional derivative (4), we have calculated the novel formulas (17), and (32), i.e.,

$$ {}_{-\infty}D_t^{\alpha}|t|^{-\delta} = \frac{\Gamma(\delta+\alpha)}{\Gamma(\delta)} \frac{\cos\left(\frac{\pi\delta}{2}+\pi\alpha\right)}{\cos\left(\frac{\pi\delta}{2}\right)} t^{-\alpha-\delta} = {}_{-\infty}I_t^{-\alpha}|t|^{-\delta}. $$

Again, we can obtain the derivative formula ${}_{-\infty}D_t^{\alpha}|t|^{-\delta}$ substituting α by $-\alpha$ in the corresponding formula for ${}_{-\infty}I_t^{\alpha}|t|^{-\delta}$. Nevertheless, according to the corresponding derivations, this property is not straightforward as in the case of (51) and (52). In general, this occurs because the definition of the Riemann–Liouville fractional derivative (3) involves an m-th derivative. Meanwhile, this is not the case for the definition of the Riemann–Liouville fractional integral (1). It would be interesting to investigate the conditions under which it is satisfied that

$$ {}_aD_t^{\alpha} f(t) = {}_aI_t^{-\alpha} f(t), \tag{53} $$

from the definitions given in (1) and (3) since (53) is implicitly taken for granted in the most common literature about fractional calculus, to the knowledge of the authors.

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