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A Unified Version of Weighted Weak-Type Inequalities for the One-Sided Hardy–Littlewood Maximal Function in Orlicz Classes

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Abstract: Let $M_g^+ f$ be the one-sided Hardy–Littlewood maximal function, φ_1 be a nonnegative and nondecreasing function on $[0,\infty)$, γ be a positive and nondecreasing function defined on $[0,\infty)$; let φ_2 be a quasi-convex function and u, v, w be three weight functions. In this paper, we present necessary and sufficient conditions on weight functions (u, v, w) such that the inequality $\varphi_1(\lambda) \int_{\{M_g^+ f > \lambda\}} u(x)g(x)dx \le C \int_{-\infty}^{+\infty} \varphi_2(C\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx$ holds. Then, we unify the weak and extra-weak-type one-sided Hardy–Littlewood maximal inequalities in the above inequality.

Keywords: weight; weak-type inequality; one-sided maximal function; Orlicz classes

MSC: 42B25; 26D15

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1. Introduction

As is well known, the theory of weight for one-sided maximal functions is an important part of harmonic analysis and is widely applied in PDEs, integral transforms, singular integrals and ergodic theory. Specifically, the research on weighted inequalities for one-sided maximal functions can help establish regularity results and maximum principles for solutions in the theory of PDEs, see [1,2], and can be employed to analyze the long-term average behavior of dynamical systems in ergodic theory, see [3–6], and study singular integrals associated to Calderón–Zygmund kernels, see [7–9]. Since weight theory for one-sided maximal functions has such a wide range of applications, it has attracted a lot of scholars' attention [10–16]. For instance, E. Sawyer [17] began to study the one-sided Hardy–Littlewood maximal function M^+ and proved that M^+ was bounded on $L^p(w)$ iff w satisfied the one-sided A_p condition. F. J. Martín–Reyes et al. [18] characterized pairs of weight functions (u, v) such that M_g^+ maps $L^p(v)$ to $L^p(u)$ or to weak-$L^p(u)$. Then, F. J. Martín-Reyes and A. de la Torre [19] presented some weighted inequalities for general one-sided maximal operators. In particular, if φ is a convex nondecreasing function on $(0,\infty)$, P. Ortega Salvador and L. Pick [20] considered a couple of weight functions (σ, ϱ) and gave the following weak and extra-weak-type inequalities:

(1) The weak-type inequality:

$$\varphi(\lambda)\varrho(\{x : M_g^+ f > \lambda\}) \le C \int_{-\infty}^{+\infty} \varphi(C|f(x)|)\sigma(x)dx, \tag{1}$$

where the constant C is independent of f and $\lambda > 0$.

(2) The extra-weak-type inequality:

$$\varrho(\{x : M_g^+ f > \lambda\}) \le C \int_{-\infty}^{+\infty} \varphi(\frac{C|f(x)|}{\lambda})\sigma(x)dx, \tag{2}$$

where the constant C is independent of f and $\lambda > 0$.

Subsequently, some scholars extended the study of one-sided maximal functions to Orlicz spaces (see [21–25]). For example, in 1998, P. Ortega Salvador [26] studied the two-weight weak-type inequality for the one-sided maximal function in Orlicz spaces. Recently, Wang et al. [27] characterized four-weight weak-type inequalities for the one-sided maximal operator in Orlicz classes. In this paper, we continue to explore one-sided maximal functions in Orlicz classes and obtain an equivalent condition for a three-weight (u, v, w) weak-type one-sided Hardy–Littlewood maximal inequality of the form

$$\varphi_1(\lambda)\int_{\{M_g^+ f>\lambda\}} u(x)g(x)dx \le C\int_{-\infty}^{+\infty} \varphi_2(C\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx.$$

We find that the forms of inequalities (1) and (2) can be unified in the inequality (11) (see Corollary 1 and Remark 1).

The remainder of this paper is organized as follows: In Section 2, we briefly recall some basic notions and lemmas. Section 3 is devoted to the proof of the main result and ends with an important remark. Section 4 presents the conclusion of this paper.

2. Preliminaries

In order to enable readers to clearly understand the proof of Theorem 1, let us first recall some known terminology and lemmas.

Definition 1 ([23]). *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be nonnegative, even, increasing on $[0, +\infty)$ and such that $\varphi(0+) = 0, \lim_{t\to\infty} \varphi(t) = \infty$. Additionally, if φ is also convex and satisfies $\lim_{t\to 0^+} \frac{\omega(t)}{t} = \lim_{t\to\infty} \frac{t}{\omega(t)} = 0$, we call φ is a Young function.*

Given a Young function φ, the Orlicz class is defined as the set of measurable functions f such that

$$\int_X \varphi(f(x))d\mu(x) < \infty.$$

It is a convex space of random variables. The Orlicz space is defined by

$$L^\varphi := \{f : X \to \mathbb{R}, measurable, \int_X \varphi(\lambda f(x))d\mu(x) < \infty,\ for\ some\ \lambda > 0\}.$$

It is a vector space of random variables and is the span of the Orlicz class.

Definition 2 ([23]). *A function $\varphi : [0, \infty) \to \mathbb{R}$ is said to be quasi-convex if there is a convex function ω and a constant $c > 0$ such that $\omega(t) \le \varphi(t) \le c\omega(ct)$ for any $t \ge 0$.*

Lemma 1 ([23,28]). *Let φ be the same as in Definition 1. Then, the following statements are equivalent:*

(i) *φ is a quasi-convex function on $[0, \infty)$;*

(ii) *the inequality*

$$\varphi(tx_1 + (1-t)x_2) \le C(t\varphi(Cx_1) + (1-t)\varphi(Cx_2))$$

holds for any $x_1, x_2 \in [0, +\infty)$ and all $t \in (0, 1)$ with a constant $C > 0$ independent of x_1, x_2, t;

(iii) *$\frac{\varphi(x)}{x}$ is quasi-increasing, i.e., there is a constant $C > 0$ such that*

$$\frac{\varphi(x_1)}{x_1} \le C\frac{\varphi(Cx_2)}{x_2}$$

is fulfilled for any $0 < x_1 < x_2$.

For a quasi-convex function φ, its complementary function $\widetilde{\varphi}$ is defined by

$$\widetilde{\varphi}(t) = \sup_{s\geq 0}(st - \varphi(s)).$$

The subadditivity of the supremum readily implies that $\widetilde{\varphi}$ is a Young function, and from the definition of the complementary function $\widetilde{\varphi}$, we obtain the Young inequality

$$st \leq \varphi(s) + \widetilde{\varphi}(t). \tag{3}$$

Lemma 2 ([17])**.** *For a quasi-convex function φ, we have*

$$\varepsilon\varphi(t) \leq \varphi(C\varepsilon t),\ t > 0,\ \varepsilon > 1$$

and

$$\varphi(\gamma t) \leq \gamma\varphi(Ct),\ t > 0,\ \gamma < 1, \tag{4}$$

where the constants C do not depend on t, ε, and γ.

Lemma 3 ([17])**.** *Let φ be the same as in Definition 1 which is a quasi-convex function; then, there is a constant $\delta > 0$ such that for an arbitrary $t > 0$, we have*

$$\widetilde{\varphi}(\delta\frac{\varphi(t)}{t}) \leq \varphi(t) \leq \widetilde{\varphi}(2\frac{\varphi(t)}{t})$$

and

$$\varphi(\delta\frac{\widetilde{\varphi}(t)}{t}) \leq \widetilde{\varphi}(t) \leq \varphi(2\frac{\widetilde{\varphi}(t)}{t}). \tag{5}$$

Let g be a positive locally integrable function on the real line. Then, the one-sided maximal operator M_g^+ is defined on $L_{1,loc}(\mathbb{R})$ by

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{g(x,x+h)} \int_x^{x+h} |f(y)|g(y)dy,$$

where $g(x,x+h) = \int_x^{x+h} g(y)dy$.

For $a < b < c$, the one-sided g–mean of f is defined as

$$\mu_g^+(f) = \mu_g^+(f,a,b,c) = \frac{1}{g(a,c)} \int_b^c |f(x)|g(x)dx.$$

Let (a,b) be an open interval, h be a measurable function; we denote $h(a,b) = \int_a^b h(x)dx$. As usual, $\{M_g^+ f > \lambda\}$ stands for $\{x \in \mathbb{R} : M_g^+ f > \lambda\}$.

An almost everywhere positive local integrable function $\omega : X \to \mathbb{R}$ is called a weight function.

Throughout this paper, we use C and C_i to denote positive constants, and they may denote different constants at different occurrences.

3. Main Result and Its Proof

The main result of this paper is stated as follows:

Theorem 1. *Let $f \in L_{1,loc}(\mathbb{R})$, g be a positive locally integrable function on the real line, φ_1 be a nonnegative and nondecreasing function on $[0,\infty)$, γ be a positive and nondecreasing function defined on $[0,\infty)$; let φ_2 be a quasi-convex function with its complementary function $\widetilde{\varphi}_2$ and u, v, w be three weight functions. Then, the following statements are equivalent:*

(i) *there exists a constant* $C_1 > 0$ *such that*

$$\varphi_1(\lambda)\int_{\{M_g^+ f>\lambda\}} u(x)g(x)dx \le C_1\int_{-\infty}^{+\infty}\varphi_2(C_1\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx \tag{6}$$

holds for all f *and* $\lambda > 0$*;*

(ii) *there exists a constant* $C_2 > 0$ *such that*

$$\varphi_1(\mu_g^+(f))\int_a^b u(x)g(x)dx \le C_2\int_b^c \varphi_2(C_2\frac{|f(x)|v(x)}{\gamma(\mu_g^+(f))})w(x)g(x)dx \tag{7}$$

holds for all f *and* $a < b < c$*;*

(iii) *there exist constants* $C_3 > 0$ *and* $\varepsilon > 0$ *such that*

$$\int_b^c \widetilde{\varphi}_2(\varepsilon\frac{\varphi_1(\lambda)\gamma(\lambda)\int_a^b u(y)g(y)dy}{\lambda g(a,c)v(x)w(x)})w(x)g(x)dx \le C_3\varphi_1(\lambda)\int_a^b u(x)g(x)dx \tag{8}$$

holds for all $\lambda > 0$ *and* $a < b < c$.

Proof. We now complete the proof by showing that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

The implication $(i) \Rightarrow (ii)$ is an easy consequence of the estimate

$$M_g^+ f(x) \ge \mu_g^+(f)\chi_{(a,b)}(x),$$

which is valid for all f, x and $a < b < c$. I.e., putting $\lambda = \mu_g^+(f)\chi_{(a,b)}(x)$ in (6), we obtain the inequality (7).

$(ii) \Rightarrow (iii)$. For $k \in \mathbb{N}$, we put $B_k = \{x \in (b,c) \ : \ v(x)w(x) > \frac{1}{k}\}$ and

$$A(x) = (\frac{\varphi_1(\lambda)\int_a^b u(y)g(y)dy}{\lambda g(a,c)v(x)w(x)})^{-1}\widetilde{\varphi}_2(\varepsilon\frac{\varphi_1(\lambda)\gamma(\lambda)\int_a^b u(y)g(y)dy}{\lambda g(a,c)v(x)w(x)})\chi_{B_k},$$

where χ_{B_k} denotes the characteristic function of the set B_k, and ε will be specified later. Then,

$$\begin{aligned} I &= \int_{B_k}\widetilde{\varphi}_2(\varepsilon\frac{\varphi_1(\lambda)\gamma(\lambda)\int_a^b u(y)g(y)dy}{\lambda g(a,c)v(x)w(x)})w(x)g(x)dx \\ &= \frac{\varphi_1(\lambda)}{\lambda g(a,c)}\int_b^c \frac{A(x)g(x)}{v(x)}dx \cdot \int_a^b u(x)g(x)dx. \end{aligned}$$

If $\frac{1}{g(a,c)}\int_b^c \frac{A(x)g(x)}{v(x)}dx < \lambda$, then, we have

$$I < \varphi_1(\lambda)\int_a^b u(x)g(x)dx; \tag{9}$$

If $\frac{1}{g(a,c)}\int_b^c \frac{A(x)g(x)}{v(x)}dx \ge \lambda$, it follows from (4) and (ii) for the function $f(x) = C\frac{A(x)}{v(x)}$ that

$$\begin{aligned} I &\le \varphi_1(\mu_g^+(f))\int_a^b u(x)g(x)dx \\ &\le C_2\int_b^c \varphi_2(\frac{C_2 C A(x)}{\gamma(\lambda)})w(x)g(x)dx. \end{aligned}$$

According to (9) and the above inequality, we have

$$I \leq \varphi_1(\lambda)\int_a^b u(x)g(x)dx + C_2\int_b^c \varphi_2(\frac{C_2CA(x)}{\gamma(\lambda)})w(x)g(x)dx.$$

Choosing ε so small that $\frac{C_2C^2\varepsilon}{\delta} < 1$, where the constant δ is from (5). By the definition of $A(x)$ (4) and (5), we have

$$I \leq \varphi_1(\lambda)\int_a^b u(x)g(x)dx + \frac{C_2C^2\varepsilon}{\delta}I. \tag{10}$$

Next, we will show that I is finite for a sufficiently small ε. If $\lim_{t\to\infty}\frac{\varphi_2(t)}{t} = \infty$, then, $\widetilde{\varphi}_2$ is finite everywhere, and thus,

$$\begin{aligned} I &= \int_{B_k}\widetilde{\varphi}_2(\varepsilon\frac{\varphi_1(\lambda)\gamma(\lambda)\int_a^b u(y)g(y)dy}{\lambda g(a,c)v(x)w(x)})w(x)g(x)dx \\ &= \widetilde{\varphi}_2(\varepsilon k\frac{\varphi_1(\lambda)\int_a^b u(y)g(y)dy}{\lambda g(a,c)})\int_b^c w(x)g(x)dx < \infty. \end{aligned}$$

If $\frac{\varphi_2(t)}{t}$ is bounded, then, by (7), we have

$$\varphi_1(\mu_g^+(f))\int_a^b u(x)g(x)dx \leq C\int_b^c \frac{|f(x)|v(x)}{\gamma(\mu_g^+(f))}w(x)g(x)dx.$$

Now, take $f(x) = \frac{\lambda g(a,c)}{g(b,c)}\chi_{(b,c)}$. Then,

$$\frac{\varphi_1(\lambda)}{g(a,c)}\int_a^b u(x)g(x)dx \leq \frac{C\lambda}{g(b,c)}\int_b^c \frac{v(x)w(x)g(x)}{\gamma(\lambda)}dx,$$

which yields the estimate

$$\frac{\varphi_1(\lambda)\gamma(\lambda)\int_a^b u(x)g(x)dy}{\lambda g(a,c)v(x)w(x)} \leq C$$

almost everywhere on the set (b,c), where C is independent of B_k and λ. Therefore,

$$I \leq \widetilde{\varphi}_2(\varepsilon C)\int_b^c w(x)g(x)dx.$$

Then, we can choose ε so small that $\widetilde{\varphi}_2(\varepsilon C) < \infty$, so I is finite.

Since I is finite, it follows from (10) that

$$\int_{B_k}\widetilde{\varphi}_2(\varepsilon\frac{\varphi_1(\lambda)\gamma(\lambda)\int_a^b u(y)g(y)dy}{\lambda g(a,c)v(x)w(x)})w(x)g(x)dx \leq \frac{\delta}{\delta - C_2^2C^2\varepsilon}\varphi_1(\lambda)\int_a^b u(x)g(x)dx.$$

Now, let $k \to \infty$; we then obtain (8).

$(iii) \Rightarrow (i)$. For a fixed λ, it is known that $\{M_g^+(f) > \lambda\} = \bigcup_{i=1}^\infty (a_i, b_i)$, where

$$\lambda \leq \frac{1}{g(x,b_i)}\int_x^{b_i}|f(t)|g(t)dt, \quad \forall x \in (a_i, b_i).$$

Now using the "cutting method" introduced by F. J. Martín–Reyes [13], we assume that (a,b) is one of the intervals (a_i,b_i), and wet $x_0=a$ and x_k with $x_k\nearrow b$ such that

$$\int_{x_k}^{b}|f(x)|g(x)dx=2\int_{x_{k+1}}^{b}|f(x)|g(x)dx$$

holds for any $k\in\mathbb{N}$. Notice that

$$\lambda\le\frac{4}{g(x_{k-1},x_{k+1})}\int_{x_k}^{x_{k+1}}|f(x)|g(x)dx,$$

so by (8) and Young inequality (3), we have

$$\begin{aligned}\varphi_1(\lambda)\int_{x_{k-1}}^{x_k}u(x)g(x)dx &\le \frac{4}{g(x_{k-1},x_{k+1})}\int_{x_k}^{x_{k+1}}\frac{|f(x)|v(x)}{\gamma(\lambda)}\\ &\quad\cdot\frac{\varphi_1(\lambda)\gamma(\lambda)\int_{x_{k-1}}^{x_k}u(y)g(y)dy}{\lambda v(x)w(x)}w(x)g(x)dx\\ &\le \frac{1}{2C_4}\int_{x_k}^{x_{k+1}}\varphi_2(\frac{8C_4}{\varepsilon}\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx\\ &+ \frac{1}{2C_4}\int_{x_k}^{x_{k+1}}\widetilde{\varphi}_2(\varepsilon\frac{\varphi_1(\lambda)\gamma(\lambda)\int_{x_{k-1}}^{x_k}u(y)g(y)dy}{\lambda g(x_{k-1},x_{k+1})v(x)w(x)})w(x)g(x)dx\\ &\le \frac{1}{2C_4}\int_{x_k}^{x_{k+1}}\varphi_2(\frac{8C_4}{\varepsilon}\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx+\frac{\varphi_1(\lambda)}{2}\int_{x_{k-1}}^{x_k}u(x)g(x)dx.\end{aligned}$$

It follows that

$$\varphi_1(\lambda)\int_{x_{k-1}}^{x_k}u(x)g(x)dx\le C_1\int_{x_k}^{x_{k+1}}\varphi_2(C_1\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx,$$

where $C_1=\max\{\frac{1}{C_4},\frac{8C_4}{\varepsilon}\}$.

Summing over k, we have

$$\varphi_1(\lambda)\int_a^b u(x)g(x)dx\le C_1\int_a^b\varphi_2(C_1\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx.$$

Consequently,

$$\begin{aligned}\varphi_1(\lambda)\int_{\{M_g^+(f)>\lambda\}}u(x)g(x)dx &= \sum_{i=1}^{\infty}\varphi_1(\lambda)\int_{a_i}^{b_i}u(x)g(x)dx\\ &\le C_1\sum_{i=1}^{\infty}\int_{a_i}^{b_i}\varphi_2(C_1\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx\\ &\le C_1\int_{-\infty}^{+\infty}\varphi_2(C_1\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)g(x)dx.\end{aligned}$$

The proof is complete. □

If $u(x)g(x)$ and $w(x)g(x)$ in Theorem 1 are, respectively, replaced with $u(x)$ and $w(x)$, then, we have

Corollary 1. *Let φ_1 be a nonnegative and nondecreasing function on $[0,\infty)$, let γ be a positive and nondecreasing function defined on $[0,\infty)$, and let φ_2 be a quasi-convex function with its complementary function $\widetilde{\varphi}_2$. Let u,v,w be three weight functions. Then, the following statements are equivalent:*

(i) *there exists a constant* $C_1 > 0$ *such that*

$$\varphi_1(\lambda)\int_{\{M_g^+ f>\lambda\}} u(x)dx \le C_1 \int_{-\infty}^{+\infty} \varphi_2(C_1\frac{|f(x)|v(x)}{\gamma(\lambda)})w(x)dx \tag{11}$$

holds for all f *and* $\lambda > 0$;

(ii) *there exists a constant* $C_2 > 0$ *such that*

$$\varphi_1(\mu_g^+(f))\int_a^b u(x)dx \le C_2 \int_b^c \varphi_2(C_2\frac{|f(x)|v(x)}{\gamma(\mu_g^+(f))})w(x)dx$$

holds for all f *and* $a < b < c$;

(iii) *there exist constants* $C_3 > 0$ *and* $\varepsilon > 0$ *such that*

$$\int_b^c \widetilde{\varphi}_2(\varepsilon\frac{\varphi_1(\lambda)\gamma(\lambda)\int_a^b u(y)dyg(x)}{\lambda g(a,c)v(x)w(x)})w(x)dx \le C_3\varphi_1(\lambda)\int_a^b u(x)dx$$

holds for all $\lambda > 0$ *and* $a < b < c$.

Remark 1. *In 1993, P. Ortega Salvador and L. Pick [20] gave weak and extra-weak-type inequalities for the one-sided maximal function* M^+. *At the moment, in inequality (11), if we put* $\varphi_2 = \varphi_1 = \varphi, v(x) = 1, \gamma(\lambda) = 1, u = \varrho, w = \sigma$, *then inequality (11) will become inequality (1), i.e., the weak-type characterization for the one-sided maximal function* M^+. *Additionally, in inequality (11), if we put* $\varphi_1(\lambda) = 1, \gamma(\lambda) = \lambda, \varphi_2 = \varphi, v(x) = 1, u = \varrho, w = \sigma$, *then inequality (11) will become inequality (2), i.e., the extra-weak-type inequality for the one-sided maximal function* M^+. *That is, inequalities (1) and (2) can be unified in inequality (11).*

4. Conclusions

In 2023, we obtained a three-weight weak-type one-sided Hardy–Littlewood maximal inequality on $\mathbb{R}^2$ in [25], which unified the weak and extra-weak-type inequalities for the one-sided maximal function M^+ on $\mathbb{R}^2$ in [21]. Inspired by the work [25], in this paper, we obtain a equivalent characterization for a three-weight (u, v, w) weak-type one-sided Hardy–Littlewood maximal inequality on $\mathbb{R}$ and integrate inequalities (1) and (2) into a unified inequality (11). Since the three-weight equality (11) unifies the two-weight weak and extra-weak-type inequalities for the one-sided Hardy–Littlewood maximal function M^+ in [20], it will be more convenient to study their application in various fields (there is no need to consider the weak and extra-weak inequalities separately).

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