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Three Weak Solutions for a Critical Non-Local Problem with Strong Singularity in High Dimension

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Abstract: In this paper, we deal with a strongly singular problem involving a non-local operator, a critical nonlinearity, and a subcritical perturbation. We apply techniques from non-smooth analysis to the energy functional, in combination with the study of the topological properties of the sublevels of its smooth part, to prove the existence of three weak solutions: two points of local minimum and a third one as a mountain pass critical point.

Keywords: strong singularity; critical nonlinearity; non-local operator; subcritical perturbation; topology of sublevels; Szulkin functional

MSC: 35J20; 35B33

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1. Introduction

In the present paper, we study a non-local two-parameter problem of the following type:

$$(\mathcal{P}) \qquad \begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \mu \dfrac{s(x)}{u^\gamma} + |u|^{2^\star-2}u + \lambda f(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ ($N \geq 4$), $s : \Omega \mapsto [0, +\infty]$ is in $L^{(2^\star)'}(\Omega)$, and $f : \Omega \times \mathbb{R} \mapsto [0, +\infty]$ is a Caratheòdory function not identically zero and satisfying $f(x, 0) = 0$ a.e. in Ω. Moreover, $a, b > 0$, $\gamma > 1$, and $\lambda, \mu > 0$ are real parameters. As usual, we denote by $2^\star := \frac{2N}{N-2}$ the Sobolev critical exponent and by $(2^\star)' = \frac{2^\star}{2^\star-1}$ its conjugate.

The peculiarity of the above problem is the combination of a Kirchhoff-type operator, which is responsible for the non-local nature of the problem, of a strong singular term, a critical nonlinearity, and a subcritical perturbation. The existence or multiplicity of solutions to Kirchhoff-type problems with critical terms is frequently studied by mountain pass arguments combined with the Lions concentration-compactness principle, both when $N < 4$ (see [1–3]) and in the higher dimensions ($N \geq 4$) (see [4–7]). Note that, in order to employ the concentration-compactness principle, a and b need to satisfy suitable constraints. Moreover, in high dimensions, the effect of a non-local operator, combined with a critical nonlinearity, forces the energy functional to be coercive, and the interplay between the Kirchhoff and the critical term allows us to establish some variational properties that will be crucial to our arguments (see [8]).

After the pioneering work of [9], the interest in singular problems has been increasing over the years; existence and multiplicity results have been obtained, both for the weak singular case (i.e., $\gamma < 1$) and for the strong singular case (i.e., $\gamma \geq 1$). Indeed, due to the presence of the singular term, when $\gamma < 1$, problem $(\mathcal{P})$ is variational, and it is possible to associate with $(\mathcal{P})$ an energy functional $E_{\lambda,\mu}$ which, although not differentiable, has useful properties in the natural Sobolev space $H_0^1(\Omega)$. On the contrary, in the case $\gamma \geq 1$, in general, the energy functional $E_{\lambda,\mu}$ is no longer defined on the whole $H_0^1(\Omega)$. In the seminal work of [10], it is proved that, even in the semilinear case, the pure singular problem is not expected to possess solutions in $H_0^1(\Omega)$ for $\gamma \geq 3$. However, if the singular term is multiplied by a suitable positive weight, one can still obtain solutions in $H_0^1(\Omega)$ via sub- and super-solution techniques (see [11]).

The presence of a non-local term, in the context of singular problems, makes the analysis more challenging, since it becomes quite difficult to establish any sort of comparison principle (which is one of the main tools to produce solutions in the presence of a strong singularity).

In the present work, we study the interaction between a non-local, a critical, and a singular term to investigate the existence of multiple solutions for $(\mathcal{P})$ in $H_0^1(\Omega)$. Note that, if a and b are big enough and $\lambda = 0$, problem $(\mathcal{P})$ has precisely one solution, which corresponds to the unique global minimizer of $E_{0,\mu}$ (see Remark 3). The presence of a perturbation breaks this uniqueness property, and the existence of three weak solutions is obtained for big λs and small μs. The study of existence of three solutions for singular problems driven by the p-Laplace operator has been developed in [12] in the low-dimensional case and in [13] for any dimension. Later, in [14], an equivalence-type result was obtained in the setting of Orlicz spaces for the non-local case.

In the present work, we extend the results of [13] to a more general problem: with respect to [13], the presence of a non-local operator and a critical term requires some extra variational properties. Thus, even if the underlying idea is the same, our multiplicity result is not straightforward. Employing the results of [8], we are still able to prove that the energy functional $E_{\lambda,\mu}$ associated with $(\mathcal{P})$ is sequentially weakly lower semicontinuous and that its derivative satisfies some form of compactness property.

We will prove the existence of two local minimizers via topological arguments. The idea is to show that the functional $E_{\lambda,0}$ for $\lambda > 0$ is large enough, has two local minimizers, and that the topology of $E_{\lambda,0}$, after being perturbed by the singular term, changes little enough so that $E_{\lambda,\mu}$ still has two local minimizers in $H_0^1(\Omega)$, as long as the parameter $\mu > 0$ is small enough. The existence of a third solution follows at once by employing a suitable version of the mountain pass theorem for Szulkin functionals.

Let us first present the definition of weak solution in our framework (see [11]).

Definition 1. *A weak solution for $(\mathcal{P})$ is a function $u \in H_0^1(\Omega)$ such that*

(i) $u > 0$ *almost everywhere in* Ω,

(ii) $su^{-\gamma}\phi \in L^1(\Omega)\ \forall\, \phi \in H_0^1(\Omega)$,

(iii) $\forall \phi \in H_0^1(\Omega)$; *there holds*

$$\left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla\phi dx - \mu\int_\Omega su^{-\gamma}\phi dx - \int_\Omega u^{2^\star-1}\phi dx - \lambda\int_\Omega f(x,u)\phi dx = 0.$$

In the subsequent work, we will assume the following:

(f_1) there exist $c_1, c_2 > 0$, $q \in (2, 2^\star)$, such that $0 \leq f(x,t) \leq c_1 + c_2 t^{q-1}$ for a.e. $x \in \Omega$, $t \geq 0$;

(f_2) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$ uniformly in Ω;

(s_1) there exists $\overline{u} \in C_0^1(\overline{\Omega})$, with $\overline{u} > 0$, such that $s\overline{u}^{-\gamma} \in L^{(2^\star)'}(\Omega)$.

Let $F : \Omega \times [0, +\infty] \to \mathbb{R}$ be a primitive of f, i.e.,

$$F(x,t) = \int_0^t f(x,s)ds.$$

Also put $f(x,t) = 0$ for all $t \leq 0$.

In order to state our main result, let us introduce some useful notation. We endow $H_0^1(\Omega)$ with the classical norm $\|\cdot\| := \left(\int_\Omega |\nabla \cdot|^2 dx\right)^{\frac{1}{2}}$ and $L^p(\Omega)$ with the standard Lebesgue norm $\|\cdot\|_p := \left(\int_\Omega |\cdot|^p\right)^{\frac{1}{p}}$, $p \leq 2^\star$. Let $u_+ := \max\{u, 0\}$ and $u_- := \max\{-u, 0\}$.

Denote by S_0 the embedding constant of $H_0^1(\Omega) \hookrightarrow L^{2^\star}(\Omega)$, i.e.,

$$S_0 = \sup_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|u\|_{2^\star}^{2^\star}}{\|u\|^{2^\star}}.$$

Now, define the constants

$$C_1(N) := \begin{cases} \dfrac{4(N-4)^{\frac{N-4}{2}}}{S_0^{-\frac{N-2}{2}} N^{\frac{N-2}{2}}} & \text{if} \quad N > 4 \\ S_0 & \text{if} \quad N = 4 \end{cases},$$

$$C_2(N) := \begin{cases} \dfrac{2(N-4)^{\frac{N-4}{2}}}{S_0^{-\frac{N-2}{2}}} \dfrac{1}{\left[\frac{(N-2)^2}{N+2}\right]^{\frac{N-2}{2}}} & \text{if} \quad N > 4 \\ 3S_0 & \text{if} \quad N = 4. \end{cases}$$

The above constants were introduced in [8], where some useful variational properties of the energy functional involving a Kirchhoff and a critical term were proved.

The following result is the main purpose of this work.

Theorem 1. *Assume conditions (f_1), (f_2), and (s_1), and put*

$$\lambda^\star = \inf\left\{ \frac{\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^\star}\|u_+\|_{2^\star}^{2^\star}}{\int_\Omega F(x, u_+(x))\, dx} \ : \ u \in H_0^1(\Omega),\ \int_\Omega F(x, u_+(x))\, dx > 0 \right\}. \tag{1}$$

Then, the following holds true.

(i) *If $a^{\frac{N-4}{2}} b > C_1(N)$ then, for each $\lambda > \lambda^\star$, there exists $\mu^\star = \mu^\star(\lambda) > 0$ such that, for all $0 < \mu < \mu^\star$, problem $(\mathcal{P})$ has at least two weak solutions.*

(ii) *If $a^{\frac{N-4}{2}} b > C_2(N)$ then, for each $\lambda > \lambda^\star$, there exists $\mu^\star = \mu^\star(\lambda) > 0$ such that, for all $0 < \mu < \mu^\star$, problem $(\mathcal{P})$ has at least three weak solutions.*

Remark 1. *Note that $C_2(N) > C_1(N)$ for $N \geq 4$.*

2. Abstract Results

We now state the preparatory results upon which the proof of Theorem 1 is based.

Define $S : \Omega \times \mathbb{R} \mapsto [-\infty, +\infty]$ by

$$S(x,t) = \begin{cases} \dfrac{1}{\gamma - 1} s(x) t^{1-\gamma}, & \text{if} \quad x \in \Omega \ \text{and} \ t > 0, \\ +\infty, & \text{if} \quad x \in \Omega \ \text{and} \ t \leq 0. \end{cases}$$

The singular operator $\Psi : H_0^1(\Omega) \to [-\infty, +\infty]$ is given by

$$\Psi(u) = \begin{cases} \displaystyle\int_\Omega S(x,u)dx, & \text{if} \quad S(x,u) \in L^1(\Omega), \\ +\infty, & \text{if} \quad S(x,u) \notin L^1(\Omega). \end{cases} \tag{2}$$

Also, we define a functional $\Phi_\lambda : H_0^1(\Omega) \mapsto \mathbb{R}$ by

$$\Phi_\lambda(u) := \frac{1}{2} a\|u\|^2 + b\frac{1}{4}\|u\|^4 - \frac{1}{2^\star}\|u_+\|_{2^\star}^{2^\star} - \lambda \int_\Omega F(x, u_+(x))dx.$$

Finally, we can introduce the extended energy functional $E_{\lambda,\mu} : H_0^1(\Omega) \mapsto [-\infty, +\infty]$ by

$$E_{\lambda,\mu}(u) := \Phi_\lambda(u) + \mu\Psi(u).$$

Lemma 1. *Assume $a^{\frac{N-4}{2}} b \geq C_1(N)$. Then, the functional Φ_0 is sequentially weakly lower semicontinuous.*

The proof is the same as in Lemma 2.1 of [8]. Here, we indicate the main steps for completeness.

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ such that $u_n \rightharpoonup u \in H_0^1(\Omega)$ (thus, $\{u_n\}$ is bounded). Let us define an auxiliary functional $\Gamma : H_0^1(\Omega) \mapsto \mathbb{R}$ by

$$\Gamma(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \frac{1}{2^\star}\|u\|_{2^\star}^{2^\star}.$$

Simple computations and the Brezis–Lieb lemma give us the relations

$$\begin{aligned} \|u_n\|^2 - \|u\|^2 &= \|u_n - u\|^2 + o_n(1), \\ \|u_n\|^4 - \|u\|^4 &\geq \|u_n - u\|^4 + o_n(1), \\ \|u_n\|_{2^\star}^{2^\star} - \|u\|_{2^\star}^{2^\star} &= \|u_n - u\|_{2^\star}^{2^\star} + o_n(1). \end{aligned}$$

Combining the relations above, we obtain

$$\Gamma(u_n) - \Gamma(u) \quad \geq \quad \|u_n - u\|^2\left(\tfrac{1}{2}a + \tfrac{1}{4}b\|u_n - u\|^2 - \tfrac{S_0}{2^\star}\|u_n - u\|^{2^\star - 2}\right) + o_n(1).$$

Define, for $N \geq 4$, the convex function $g : [0, +\infty[\mapsto \mathbb{R}$ given by

$$g(t) \equiv \left(\frac{a}{2} + \frac{b}{4}t^2 - \frac{S_0}{2^\star}t^{2^\star - 2}\right).$$

For $N > 4$, g attains its minimum at

$$t_0 = \left(\frac{2(2^\star - 2)}{2^\star}\frac{S_0}{b}\right)^{\frac{1}{4 - 2^\star}},$$

and simple computations show that

$$g(t_0) \geq 0 \quad \iff \quad a^{\frac{N-4}{2}} b \geq C_1(N).$$

For $N = 4$ (in this case, $2^\star = 4$), we rewrite $g(t) \equiv \left(\frac{a}{2} + (b - S_0)\frac{t^2}{4}\right)$, which attains its minimum value $\frac{a}{2}$ at $t_0 = 0$ for $b \geq S_0$.

Therefore, for $N \geq 4$, if $a^{\frac{N-4}{2}} b \geq C_1(N)$, it holds that

$$\liminf_{n \to +\infty}[\Gamma(u_n) - \Gamma(u)] \geq \liminf_{n \to +\infty}\left[\|u_n - u\|^2 g(t_0)\right] \geq 0.$$

To finish the proof, it is enough to note that

$$\Phi_0(u) \equiv \Gamma(u) + \frac{1}{2^\star}\|u_-\|_{2^\star}^{2^\star}.$$

□

Corollary 1. *Assume $a^{\frac{N-4}{2}} b \geq C_1(N)$ and (f_1). Then, for each $\lambda > 0$, the functional Φ_λ is sequentially weakly lower semicontinuous.*

Lemma 2. *Assume the following: either* $N = 4$ *and* $a^{\frac{N-4}{2}} b \geq C_1(N)$*, or* $N > 4$ *and* $a, b > 0$ *are arbitrary. Then, the functional* Φ_0 *is coercive.*

Proof. Indeed,

$$\begin{aligned} \Phi_0(u) &\geq \frac{1}{2}a\|u\|^2 + b\frac{1}{4}\|u\|^4 - \frac{1}{2^\star}\|u\|_{2^\star}^{2^\star} \\ &\geq \frac{1}{2}a\|u\|^2 + b\frac{1}{4}\|u\|^4 - S_0\frac{1}{2^\star}\|u\|^{2^\star}. \end{aligned}$$

Therefore, for $2^\star < 4$ (when $N > 4$), the claim is true for all $a, b > 0$; if $2^\star = 4$ (when $N = 4$), the conclusion follows only for $b \geq S_0$. □

Corollary 2. *Under the same conditions as in Lemma 2, the energy functional* $E_{\lambda,\mu}$ *is coercive for all* $\lambda, \mu \geq 0$.

We denote by $\mathcal{W}$ the class of functionals $\Phi : X \to \mathbb{R}$ having the following property:

$(\mathcal{W})$ if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $u_n \rightharpoonup u$ (weakly) and

$$\liminf_n \Phi(u_n) \leq \Phi(u);$$

then, it has a sub-sequence strongly converging to u.

Proposition 1. *Assume* $a^{\frac{N-4}{2}} b > C_1(N)$*. Then,* $\Phi_0 \in \mathcal{W}$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ and $\liminf_n \Phi_0(u_n) \leq \Phi_0(u)$. By Lemma 1, $\liminf_n \Phi_0(u_n) = \Phi_0(u)$. Thus, recalling that

$$\Gamma(u) = \Phi_0(u) - \frac{1}{2^\star}\|u_-\|_{2^\star}^{2^\star},$$

by the weak lower semicontinuity of the norm, we deduce that

$$\liminf_{n\to\infty} \Gamma(u_n) \leq \Phi_0(u) - \frac{1}{2^\star}\|u_-\|_{2^\star}^{2^\star} = \Gamma(u). \tag{3}$$

From the proof of Lemma 1, we observe that

$$\Gamma(u_n) - \Gamma(u) \geq \|u_n - u\|^2 g(t_0) + o_n(1),$$

and thus

$$\liminf_{n\to+\infty}[\Gamma(u_n) - \Gamma(u)] \geq \liminf_{n\to+\infty}\Big[\|u_n - u\|^2 g(t_0)\Big].$$

Now, if $a^{\frac{N-4}{2}} b > C_1(N)$, one has $g(t_0) > 0$, and the claim follows by (3). □

The following result will be crucial in the subsequent work.

Theorem 2 ([15])**.** *Let* X *be a reflexive and separable real Banach space, and let* $I, J : X \to \mathbb{R}$ *be two sequentially weakly lower semicontinuous functionals with* $I \in \mathcal{W}$*. Assume that*

$$\lim_{\|u\|\to\infty} I(u) + J(u) = +\infty.$$

Then, any strict local minimizer of $I + J$ *in the strong topology is the same in the weak topology.*

Corollary 3. *Assume* $a^{\frac{N-4}{2}} b > C_1(N)$ *and* $(f_1), (f_2)$*. Then, 0 is a local minimizer of* Φ_λ *in the weak topology.*

Proof. We apply Theorem 2 choosing $I = \Phi_0$, and J is defined by $J(u) = -\lambda \int_\Omega F(x, u_+(x))\,dx$. By Lemma 1 and assumption (f_1), I and J are sequentially weakly lower semicontinuous, and Proposition 1 ensures that $I \in \mathcal{W}$. The coercivity of $I + J = \Phi_\lambda$ follows by Lemma 2 (for $\mu = 0$). By (f_2), 0 is a local minimizer of Φ_λ in the strong topology, and the claim follows. □

The singular term prevents the application of the classical critical point theory for C^1 functionals. We will need the following.

Definition 2 ([16])**.** *Let X be a real Banach space, $\Phi \in C^1(X)$, and $\Psi : X \mapsto \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous. Then, $I = \Phi + \Psi$ is referred to as the Szulkin functional. Moreover, a point $u \in X$ is said to be critical for the Szulkin functional I if $u \in \mathrm{dom}\Psi(u) = \{u \in X : \Psi(u) < +\infty\}$ and*

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq 0 \quad \text{for all } v \in X.$$

It is well-known that a local minimum of I is a critical point of I.

Lemma 3 ([13], Lemma 3.1)**.** *Assume (s_1). Then,*

$$\mathrm{int}(C_0^1(\overline{\Omega})_+) \subset \mathrm{dom}(\Psi)$$

where $\mathrm{int}(C_0^1(\overline{\Omega})_+)$ denotes the interior in the ordered Banach space $C_0^1(\overline{\Omega})$ of the positive cone

$$C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \ \forall\, x \in \Omega\}.$$

Remark 2. *The energy functional $E_{\lambda,\mu}$ associated with problem $(\mathcal{P})$ is a Szulkin functional. Indeed, Φ_λ is of class $C^1(H_0^1(\Omega))$, while Ψ, defined as in (2), is a convex and lower semicontinuous (l.s.c.) functional. Moreover, it is proper from the previous result.*

Definition 3. *We say that the operator $T : X \mapsto X^\star$ satisfies the $(S)_+$ condition if the following is true: let $\{u_n\}_{n\geq 1} \subset X$ such that $u_n \rightharpoonup u \in X$ and*

$$\limsup_{n\to+\infty} \langle T(u_n), u_n - u\rangle \leq 0.$$

Then, $u_n \mapsto u \in X$.

Proposition 2. *Assume $a^{\frac{N-4}{2}} b > C_2(N)$. Then, the operator Φ_0' satisfies the $(S)_+$ condition.*

Proof. Let $n \geq 1$ and $t \in [0,1]$. A straightforward calculation shows that

$$\begin{aligned}
&\Phi_0''(tu + (1-t)u_n)(u_n - u)(u_n - u) \geq \\
&a\|u_n - u\|^2 + b\|tu + (1-t)u_n\|^2 \|u_n - u\|^2 - (2^\star - 1)S_0 \|(tu + (1-t)u_n)_+\|^{\frac{4}{N-2}} \|u_n - u\|^2 \geq \\
&a\|u_n - u\|^2 + b\|tu + (1-t)u_n\|^2 \|u_n - u\|^2 - (2^\star - 1)S_0 \|tu + (1-t)u_n\|^{\frac{4}{N-2}} \|u_n - u\|^2 = \\
&\left[a + b\|tu + (1-t)u_n\|^2 - (2^\star - 1)S_0 \|tu + (1-t)u_n\|^{\frac{4}{N-2}}\right] \|u_n - u\|^2.
\end{aligned}$$

Now, define the auxiliary function

$$g(t) \equiv a + bt^2 - \frac{N+2}{N-2} S_0 t^{\frac{4}{N-2}}.$$

If $N > 4$, g attains its minimum at

$$t_0 = \left[S_0 \frac{2(N+2)}{b(N-2)^2}\right]^{\frac{N-2}{2(N-4)}},$$

and $g(t_0) > 0$ if and only if

$$ab^{\frac{2}{N-4}} > \left[\frac{N-4}{2}\right]\left[\frac{2(N+2)}{(N-2)^2}S_0\right]^{\frac{N-2}{N-4}}.$$

If $N = 4$,

$$g(t) = a + (b - 3S_0)t^2,$$

and if $b > 3S_0$, $g(t) \geq a$. Subsequently, if $a^{\frac{N-4}{2}}b > C_2(N)$, there exists a constant $C > 0$ such that

$$\Phi_0''(tu + (1-t)u_n)(u_n - u)(u_n - u) \geq C\|u_n - u\|^2.$$

Let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ and

$$\limsup_{n\to+\infty}\langle \Phi_0{}'(u_n), u_n - u\rangle \leq 0.$$

Then,

$$\limsup_{n\to+\infty}\langle \Phi_0{}'(u), u_n - u\rangle = 0,$$

and so

$$\begin{aligned} 0 &\geq \limsup_{n\to+\infty}\langle \Phi_0{}'(u_n), u_n - u\rangle = \limsup_{n\to+\infty}\left\langle \Phi_0'(u_n) - \Phi_0'(u), u_n - u\right\rangle \\ &= \limsup_{n\to+\infty}\int_0^1 \Phi_0''(tu + (1-t)u_n)(u_n - u)(u_n - u)dt \\ &\geq \limsup_{n\to+\infty} C\|u_n - u\|^2. \end{aligned}$$

Thus, we conclude that

$$\lim_{n\to+\infty}\|u_n - u\|^2 = 0;$$

that is our claim. □

From Proposition 2, it follows that:

Corollary 4. *Assume $a^{\frac{N-4}{2}}b > C_2(N)$ and (f_1). Then, the operator Φ_λ' satisfies the $(S)_+$ condition.*

Remark 3. *Proposition 2 also shows that Φ_0' is strongly monotone; thus, in particular, Φ_0 is strictly convex.*

Definition 4. *Let $I = \Phi + \Psi$ be a Szulkin functional defined on a Banach space X. We say that I satisfies the Palais–Smale condition if, for any sequence $\{u_n\}_{n\geq1} \subset X$ and $\{\epsilon_n\}_{n\geq1} \subset \mathbb{R}_+$ such that $I(u_n) \mapsto c \in \mathbb{R}$, $\epsilon_n \mapsto 0$, and*

$$\left\langle \Phi'(u_n), v - u_n\right\rangle + \Psi(v) - \Psi(u_n) \geq -\epsilon_n\|v - u_n\|$$

for all $n \geq 1$ and for all $v \in X$, $\{u_n\}_{n\geq1}$ possesses a strongly convergent sub-sequence.

Proposition 3. *Assume $a^{\frac{N-4}{2}}b > C_2(N)$, (f_1). Then, the energy functional $E_{\lambda,\mu}$ satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}_{n\geq1}$ be a sequence such that $\{E_{\lambda,\mu}(u_n)\}$ converges to some $c \in \mathbb{R}$, and let $\{\epsilon_n\}_{n\geq1}$ be a sequence of positive real numbers such that $\epsilon_n \mapsto 0$. Since $E_{\lambda,\mu}$ is coercive by Corollary 2, $u_n \rightharpoonup u \in H_0^1(\Omega)$ up to a sub-sequence. Moreover, we note that $E_{\lambda,\mu}$

is sequentially weakly lower semicontinuous, and thus $E_{\lambda,\mu}(u) < +\infty$. Subsequently, $u \in \mathrm{dom}(\Psi)$. Setting $v = u$ in the inequality in Definition 4, we obtain

$$\left\langle \Phi'_\lambda(u_n), u_n - u \right\rangle \leq \mu\Psi(u) - \mu\Psi(u_n) + \epsilon_n \|u - u_n\|.$$

Now, since Ψ is sequentially weakly lower semicontinuous, there holds

$$\begin{aligned} \limsup_{n\to+\infty} \left\langle \Phi'_\lambda(u_n), u_n - u \right\rangle &\leq \limsup_{n\to+\infty} (\mu\Psi(u) - \mu\Psi(u_n) + \epsilon_n \|u - u_n\|) \\ &= \mu \limsup_{n\to+\infty} (\Psi(u) - \Psi(u_n)) \\ &\leq 0. \end{aligned}$$

From Corollary 4, Φ'_λ is of type $(S)_+$, and thus $u_n \mapsto u$ strongly in $H_0^1(\Omega)$. □

Proposition 4. *Any critical point $u \in H_0^1(\Omega)$ of $E_{\lambda,\mu}$ (in the sense of Szulkin) is a weak solution for problem $(\mathcal{P})$.*

Proof. By Definition 2, for all $v \in H_0^1(\Omega)$, there holds

$$\left\langle \Phi'_\lambda(u), v - u \right\rangle + \mu(\Psi(v) - \Psi(u)) \geq 0. \tag{4}$$

Since $u \in \mathrm{dom}(\Psi)$, by the definition of S, it is clear that $u > 0$ almost everywhere in Ω. Let $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$, and $v := u + \epsilon\varphi$ in (4).

Then,

$$\left(a + b\|u\|^2\right) \int_\Omega \langle \nabla u, \nabla \varphi \rangle dx - \int_\Omega u^{2^\star - 1}\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx \geq \mu \int_\Omega \frac{S(x,u) - S(x,u+\epsilon\varphi)}{\epsilon} dx.$$

Taking $\epsilon \mapsto 0$ and applying Fatou's Lemma, we obtain that

$$s(x)u^{-\gamma}\varphi \in L^1(\Omega)$$

and

$$\left(a + b\|u\|^2\right) \int_\Omega \langle \nabla u, \nabla \varphi \rangle dx - \int_\Omega u^{2^\star - 1}\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx \geq \mu \int_\Omega s(x)u^{-\gamma}\varphi dx. \tag{5}$$

Let $v := (1-\epsilon)u$, with $\epsilon \in (0,1)$, in (4).

Then,

$$\left(a + b\|u\|^2\right) \int_\Omega \langle \nabla u, \nabla u \rangle dx - \int_\Omega u^{2^\star} dx - \lambda \int_\Omega f(x,u)u dx - \mu \int_\Omega \frac{S(x,(1-\epsilon)u) - S(x,u)}{\epsilon} dx \leq 0. \tag{6}$$

Applying the mean value theorem for functions of a real variable, there exists $\tau = \tau(\epsilon) \in (0,\epsilon)$ such that (6) can be rewritten as

$$\left(a + b\|u\|^2\right) \int_\Omega \langle \nabla u, \nabla u \rangle dx - \int_\Omega u^{2^\star - 1}u dx - \lambda \int_\Omega f(x,u)u dx - \mu(1-\tau(\epsilon))^{-\gamma} \int_\Omega s(x)u^{1-\gamma} dx \leq 0.$$

Taking $\epsilon \mapsto 0$, we obtain

$$\left(a + b\|u\|^2\right) \int_\Omega \langle \nabla u, \nabla u \rangle dx - \int_\Omega u^{2^\star} dx - \lambda \int_\Omega f(x,u)u dx - \mu \int_\Omega s(x)u^{1-\gamma} dx \leq 0.$$

Therefore, by (5), we deduce that

$$\left(a + b\|u\|^2\right) \int_\Omega \langle \nabla u, \nabla u \rangle dx - \int_\Omega u^{2^\star} dx - \lambda \int_\Omega f(x,u)u dx - \mu \int_\Omega s(x)u^{1-\gamma} dx = 0.$$

Now, take any $\varphi \in H_0^1(\Omega)$ to test (5) with the function

$$v := (u + \epsilon\varphi)^+$$

and obtain

$$\Big(a+b\|u\|^2\Big)\int_\Omega\langle\nabla u,\nabla(u+\epsilon\varphi)\rangle dx-\int_\Omega u^{2^\star-1}(u+\epsilon\varphi)dx-\lambda\int_\Omega f(x,u)(u+\epsilon\varphi)dx$$
$$-\mu\int_\Omega s(x)u^{-\gamma}(u+\epsilon\varphi)dx-\Big(a+b\|u\|^2\Big)\int_{\{u+\epsilon\varphi<0\}}\langle\nabla u,\nabla(u+\epsilon\varphi)\rangle dx$$
$$+\int_{\{u+\epsilon\varphi<0\}}u^{2^\star-1}(u+\epsilon\varphi)dx+\lambda\int_{\{u+\epsilon\varphi<0\}}f(x,u)(u+\epsilon\varphi)dx+\mu\int_{\{u+\epsilon\varphi<0\}}s(x)u^{-\gamma}(u+\epsilon\varphi)dx\geq 0.$$

From the inequality above, we have

$$\overbrace{\Big(a+b\|u\|^2\Big)\int_\Omega|\nabla u|^2dx-\int_\Omega u^{2^\star}dx-\lambda\int_\Omega f(x,u)udx-\mu\int_\Omega s(x)u^{1-\gamma}dx}^{=0}+$$
$$\epsilon\Big[\Big(a+b\|u\|^2\Big)\int_\Omega\langle\nabla u,\nabla\varphi\rangle dx-\int_\Omega u^{2^\star-1}\varphi dx-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega s(x)u^{-\gamma}\varphi dx\Big]+$$
$$\overbrace{-\Big(a+b\|u\|^2\Big)\int_{\{u+\epsilon\varphi<0\}}|\nabla u|^2dx}^{\leq 0}-\epsilon\Big(a+b\|u\|^2\Big)\int_{\{u+\epsilon\varphi<0\}}\langle\nabla u,\nabla\varphi\rangle dx+$$
$$\overbrace{\int_{\{u+\epsilon\varphi<0\}}u^{2^\star-1}(u+\epsilon\varphi)dx+\lambda\int_{\{u+\epsilon\varphi<0\}}f(x,u)(u+\epsilon\varphi)dx+\mu\int_{\{u+\epsilon\varphi<0\}}s(x)u^{-\gamma}(u+\epsilon\varphi)dx}^{\leq 0}\geq 0.$$

In other words,

$$\Big(a+b\|u\|^2\Big)\int_\Omega\langle\nabla u,\nabla\varphi\rangle dx-\int_\Omega u^{2^\star-1}\varphi dx-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega s(x)u^{-\gamma}\varphi dx$$
$$-\Big(a+b\|u\|^2\Big)\int_{\{u+\epsilon\varphi<0\}}\langle\nabla u,\nabla\varphi\rangle dx\geq 0.$$

Taking the limit as $\epsilon\mapsto 0$, we conclude that

$$\Big(a+b\|u\|^2\Big)\int_\Omega\langle\nabla u,\nabla\varphi\rangle dx-\int_\Omega u^{2^\star-1}\varphi dx-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega s(x)u^{-\gamma}\varphi dx\geq 0,$$

and, by the arbitrariness of φ, we conclude that

$$\Big(a+b\|u\|^2\Big)\int_\Omega\langle\nabla u,\nabla\varphi\rangle dx-\int_\Omega u^{2^\star-1}\varphi dx-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega s(x)u^{-\gamma}\varphi dx=0;$$

that is, u is a weak solution to $(\mathcal{P})$. □

Lemma 4. *Under condition (f_1), all critical points $u\in H_0^1(\Omega)$ of Φ_λ belong to* $\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Proof. Assume u as in the statement and note first that $u\geq 0$. Put $k=\big(a+b\|u\|^2\big)$. Thus, we can write

$$\begin{cases}-\Delta u=a(x)(1+u(x)) & in\ \Omega\\ u=0 & in\ \partial\Omega,\end{cases}\tag{7}$$

where

$$a(x):=\frac{1}{k}\cdot\frac{u(x)^{2^\star-1}+\lambda f(x,(u(x))}{1+u(x)}.$$

Let us prove that $a(x)\in L^{\frac{N}{2}}(\Omega)$.

In fact, by (f_1), denoting by c a constant whose value may vary from line to line, there holds

$$\begin{aligned}\left(\frac{1}{k}\cdot\frac{u(x)^{2^\star-1}+\lambda f(x,(u(x))}{1+u(x)}\right)^{\frac{N}{2}} &\le c\left(\frac{u(x)^{2^\star-1}+c(1+u(x)^{q-1})}{1+u(x)}\right)^{\frac{N}{2}}\\ &\le c(1+u(x)^{2^\star})\in L^1(\Omega).\end{aligned}$$

Therefore, by the Brezis–Kato arguments (see, for instance, Lemma B.3 of [17]), $u\in L^p(\Omega)$ for all $p>1$. This proves that the right-hand side of (7) belongs to $L^p(\Omega)$ for all $1<p<+\infty$, which in turn allows us to use the Calderón–Zigmund inequalities to prove that $u\in W_0^{2,p}(\Omega)$ for all $1<p<+\infty$. Finally, the Sobolev immersions imply that $u\in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$. From Theorem 1 in [18], we conclude that $u>0$ a.e. in Ω. □

We conclude this section recalling the following topological result, which will be useful in the subsequent work. Its proof will be used in the proof of our main result and, for this reason, we provide it here.

Proposition 5 ([13], Proposition 2.2)**.** *Let X be a Hausdorff topological space. Let $\{K_n\}_{n\ge1}\subset X$ be a decreasing sequence of compact subsets such that $\bigcap_{n\ge1}K_n$ is the disjoint union of non-empty compact sets $\mathcal{S}_i$ $(i=1,2)$. Then, for some $\overline{n}\ge1$, the set $K_{\overline{n}}$ is the disjoint union of non-empty compact sets $\mathcal{C}_i$ $(i=1,2)$, where $\mathcal{S}_i\subset\mathcal{C}_i$ $(i=1,2)$.*

Proof. Since X is Hausdorff and $\mathcal{S}_i$ is compact, there are disjoint open sets $\mathcal{O}_i$ $(i=1,2)$ so that $\mathcal{S}_i\subset\mathcal{O}_i$ $(i=1,2)$. Moreover,

$$\begin{aligned}K_1 &\subset \left[X\backslash\bigcap_{n=1}^{\infty}K_n\right]\bigcup\left[\bigcap_{n=1}^{\infty}K_n\right]\\ &= \left[\bigcup_{n=1}^{\infty}(X\backslash K_n)\right]\bigcup\left[\bigcup_{i=1}^{2}\mathcal{S}_i\right]\\ &\subseteq \left[\bigcup_{n=1}^{\infty}(X\backslash K_n)\right]\bigcup_{i=1}^{2}\mathcal{O}_i.\end{aligned}$$

By the compactness of K_1, there is a finite index set J such that

$$K_1\subset\left[\bigcup_{n\in J}(X\backslash K_n)\right]\bigcup_{i=1}^{2}\mathcal{O}_i.$$

Since the set sequence $\{X\backslash K_n\}_{n\ge1}$ is increasing, for $\overline{n}=\max J$, there holds

$$K_{\overline{n}}\subset K_1\subset(X\backslash K_{\overline{n}})\bigcup_{i=1}^{2}\mathcal{O}_i.$$

In other words,

$$K_{\overline{n}}\subset\bigcup_{i=1}^{2}\mathcal{O}_i. \tag{8}$$

Define

$$C_i:=K_{\overline{n}}\cap\mathcal{O}_i\quad for\quad i=1,2. \tag{9}$$

Relations (8) and (9) tell us that $\mathcal{C}_i$ $(i=1,2)$ are disjoint and compact. It is also clear that $\mathcal{S}_i\subset\mathcal{C}_i$ $(i=1,2)$. □

3. Main Result

Now we are in a position to prove Theorem 1. First, we prove the existence of two local minimizers for the energy functional associated with problem $(\mathcal{P})$. The proof follows as in [13], Theorem 1.1 (see also [19]).

Lemma 5. *Assume $a^{\frac{N-4}{2}}b > C_1(N)$. Under conditions (f_1), (f_2), and (s_1), the energy functional $E_{\lambda,\mu}$ has at least two local minimizers.*

Proof. From assumption (f_2), it follows that $0 \in H_0^1(\Omega)$ is a strict local minimizer of Φ_λ in the strong topology. Consequently, 0 is also a strict local minimizer in the weak topology from Corollary 3.

In other words, there is a weak neighborhood of zero $\mathcal{O} \subset H_0^1(\Omega)$ such that $\Phi_\lambda(u) > 0 \; \forall u \in \mathcal{O}\backslash\{0\}$. Thus,

$$\Phi_\lambda^{-1}([-\infty, 0]) = \bigcap_{n\geq 1}^{\infty} \Phi_\lambda^{-1}\left(\left[-\infty, \frac{1}{n}\right]\right) = \{0\} \cup \left(\Phi_\lambda^{-1}([-\infty, 0])\backslash\mathcal{O}\right).$$

By the definition of $\lambda^\star$, for $\lambda > \lambda^\star$, $\inf_{u\in H_0^1(\Omega)} \Phi_\lambda < 0$, and thus $\Phi_\lambda^{-1}([-\infty, 0])\backslash\mathcal{O} \neq \emptyset$. Note that the sublevel sets of Φ_λ are weakly compact and metrizable (thus sequentially weakly compact) with respect to the weak topology. By Proposition 5, for some $\overline{n} \geq 1$, there exist weakly compact and sequentially weakly compact disjoint sets $\mathcal{C}_i$ $(i = 1, 2)$ with $\{0\} \subset \mathcal{C}_1$ and $\Phi_\lambda^{-1}([-\infty, 0])\backslash\mathcal{O} \subset \mathcal{C}_2$, such that

$$\Phi_\lambda^{-1}\left(\left[-\infty, \frac{1}{\overline{n}}\right]\right) = \mathcal{C}_1 \cup \mathcal{C}_2.$$

Since $\Phi_\lambda + \mu\Psi$ is sequentially weakly lower semicontinuous and $\mathcal{C}_i$ $(i = 1, 2)$ are sequentially weakly compact, we obtain the existence of $u_i \in \mathcal{C}_i$ $(i = 1, 2)$ such that

$$\inf_{\mathcal{C}_i}(\Phi_\lambda + \mu\Psi) = \Phi_\lambda(u_i) + \mu\Psi(u_i).$$

We wish to prove that, for $i = 1, 2$, there holds

$$u_i \in \mathcal{O}_i' := \{u \in \mathcal{O}_i : \; \Phi_\lambda(u) < \frac{1}{\overline{n}}\},$$

where $\mathcal{O}_i$ $(i = 1, 2)$ come from the proof of Proposition 5. Notice also that $\mathcal{O}_i' \subset \mathcal{C}_i$ $(i = 1, 2)$.

The $\mathcal{O}_i$ values are weakly open and therefore strongly open, and Φ_λ is strongly continuous. Therefore, the $\mathcal{O}_i'$ values are strongly open. Let us prove that

$$\mathrm{dom}(\Psi) \cap \mathcal{O}_i' \neq \emptyset.$$

Indeed, by Lemma 3, it is enough to prove that

$$\mathrm{int}(C_0^1(\overline{\Omega})) \cap \mathcal{O}_i' \neq \emptyset.$$

Since $0 \in \mathcal{O}_1'$, then there exists a positive number ε such that $B(0, \varepsilon) \subset \mathcal{O}_1'$. Then, $\mathrm{dom}(\Psi) \cap \mathcal{O}_1' \neq \emptyset$.

On the other hand, by the coercivity and the sequential weak lower semicontinuity of Φ_λ, there exists $u_0 \in \mathcal{O}_2'$ as a global minimizer of Φ_λ. Thus, u_0 is a critical point of Φ_λ, and by Lemma 4, $u_0 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$; thus, $\mathrm{dom}(\Psi) \cap \mathcal{O}_2' \neq \emptyset$.

Then, we may define in $\mathbb{R}$ the constant

$$m_i := \inf_{\mathrm{dom}(\Psi)\cap\mathcal{O}_i'} \frac{\Psi - \min_{\mathcal{C}_i}\Psi}{\frac{1}{n} - \Phi_\lambda} \qquad i = 1,2.$$

Take $\mu^\star > 0$ such that

$$\frac{1}{\mu^\star} > \max\{m_i : \ i = 1,2\}.$$

Therefore, for all $0 < \mu < \mu^\star$, there are $y_i \in \mathrm{dom}(\Psi) \cap \mathcal{O}_i'$ such that

$$\mu\Psi(y_i) + \Phi_\lambda(y_i) < \mu \min_{\mathcal{C}_i} \Psi + \frac{1}{n}.$$

Thus, we may find $u_1 \in \mathrm{dom}(\Psi) \cap C_1, \ u_2 \in \mathrm{dom}(\Psi) \cap C_2$ s.t.:

$$\min_{C_i}(\Phi_\lambda + \mu\Psi) = \Phi_\lambda(u_i) + \mu\Psi(u_i), \quad i = 1,2.$$

We wish to prove that $u_i \in \mathcal{O}_i'$, $i = 1,2$. Assume by contradiction that, for some $i \in \{1,2\}$, it holds that $u_i \notin \mathcal{O}_i'$. This means that $\Phi_\lambda(u_i) \geq \frac{1}{n}$.

Then,

$$\begin{aligned} \inf_{\mathcal{O}_i'}(\Phi_\lambda + \mu\Psi) &\geq \inf_{\mathcal{C}_i}(\Phi_\lambda + \mu\Psi) \\ &= \Phi_\lambda(u_i) + \mu\Psi(u_i) \\ &\geq \frac{1}{n} + \mu \min_{\mathcal{C}_i} \Psi \\ &> \mu\Psi(y_i) + \Phi_\lambda(y_i), \end{aligned}$$

which is a contradiction. □

Proof. (1) By Lemma 5, the energy functional $E_{\lambda,\mu}$ has at least two local minimizers. Proposition 1.1 of [16] guarantees that local minimizers are critical points in the sense of Szulkin, and from Proposition 4, the claim follows.

(2) From Proposition (3), the energy functional $E_{\lambda,\mu}$ satisfies the Palais–Smale condition. Since $C_2(N) > C_1(N)$, $E_{\lambda,\mu}$ has two local minimizers and given Corollary 3.3 of [16], the thesis is proved. □

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