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Article

On LP-Kenmotsu Manifold with Regard to Generalized Symmetric Metric Connection of Type (α, β)

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Abstract: In the current article, we examine Lorentzian para-Kenmotsu (shortly, LP-Kenmotsu) manifolds with regard to the generalized symmetric metric connection $\nabla^{\mathcal{G}}$ of type (α, β). First, we obtain the expressions for curvature tensor, Ricci tensor and scalar curvature of an LP-Kenmotsu manifold with regard to the connection $\nabla^{\mathcal{G}}$. Next, we analyze LP-Kenmotsu manifolds equipped with the connection $\nabla^{\mathcal{G}}$ that are locally symmetric, Ricci semi-symmetric, and φ-Ricci symmetric and also demonstrated that in all these situations the manifold is an Einstein one with regard to the connection $\nabla^{\mathcal{G}}$. Moreover, we obtain some conclusions about projectively flat, projectively semi-symmetric and φ-projectively flat LP-Kenmotsu manifolds concerning the connection $\nabla^{\mathcal{G}}$ along with several consequences through corollaries. Ultimately, we provide a 5-dimensional LP-Kenmotsu manifold example to validate the derived expressions.

Keywords: Lorentzian para-Kenmotsu manifolds; generalized symmetric metric connection of type (α, β); Einstein manifold; projective curvature tensor

MSC: 53C05; 53C15; 53C25; 53C50

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1. Introduction

In 1924, Friedman and Schouten first proposed the notion of a semi-symmetric linear connection on a differentiable manifold [1]. The geometric significance associated with such a connection was provided by Bartolotti [2] in 1930. A metric connection known as a semi-symmetric metric connection with a non-zero torsion on a Riemannian manifold was first introduced and investigated in 1932 by Hayden [3]. Yano has conducted a thorough investigation of a semi-symmetric metric connection upon a Riemannian manifold [4]. A quarter-symmetric linear connection on a differentiable manifold was first proposed by Golab [5] in 1975 as a more generalized form of a semi-symmetric linear connection. Rastogi [6] carried out a subsequent systematic investigation into the quarter-symmetric metric connection on a Riemannian manifold. The study on these connections was further studied by various authors. At this moment we refer to the papers [7–10] and references therein for the extensive study on these connections.

If the torsion tensor $\mathcal{T}$ of a linear connection on a (semi-)Riemannian manifold M is said to be a generalized symmetric connection, then $\mathcal{T}$ is defined as

$$\mathcal{T}(U_1, U_2) = \alpha\{\pi(U_2)U_1 - \pi(U_1)U_2\} + \beta\{\pi(U_2)\varphi U_1 - \pi(U_1)\varphi U_2\}, \quad (1)$$

for U_1, U_2 vector fields on M, where smooth functions are α and β on M. Here, φ denotes tensor of type $(1,1)$ and π is regarded as a 1-form and satisfies $\pi(U_1) = g(U_1, \nu)$

for a vector field ν on M. In addition, if there is a Riemannian metric g in M such that $\nabla^{\mathcal{G}} g = 0$, then the connection is considered to be a generalized symmetric metric connection (shortly, GSM-connection) of type (α, β); if not, it is non-metric. Many authors have examined the properties of Riemannian and semi-Riemannian manifolds with this connection (see [11–13]). The connection in Equation (1) is referred to as a β-quarter-symmetric connection (resp. α-semi-symmetric connection) if $\alpha = 0$ (resp. $\beta = 0$). Furthermore, the GSM-connection of type (α, β) simplifies to a semi-symmetric connection and quarter-symmetric connection, respectively, if we put $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$. As a result, semi-symmetric and quarter-symmetric connections can be described as generalizations of generalized symmetric connections. These two connections play an important role in various geometrical and physical aspects.

On the other side, the analysis of differentiable manifolds using the Lorentzian metric is a natural and interesting topic in differential geometry. In 1989, the idea of nearly para-contact metric manifolds with the Lorentzian metric, in particular, Lorentzian para-Sasakian (shortly, LP-Sasakian) manifolds were introduced by Matsumoto [14]. Later, in 1995, Sinha and Sai Prasad [15] defined a class of almost paracontact metric manifolds namely para-Kenmotsu manifolds similar to para-Sasakian manifolds. Also, they obtained important characterizations of para-Kenmotsu manifolds. In 2018, Haseeb and Prasad [16] defined a class of Lorentzian almost paracontact metric manifolds, namely, Lorentzian para-Kenmotsu (shortly, LP- Kenmotsu) manifolds. Submanifolds in LP-Kenmotsu manifolds have been studied by several authors in [17–19] . LP-Kenmotsu manifolds admitting Ricci solitons have been studied in [20–23]. Many interesting results on LP-Kenmotsu manifolds have been studied by many geometers (see, [24–26]).

2. Preliminaries

Suppose that M is a n-dimensional differentiable manifold that possesses a contravariant vector field ρ, (1, 1)-tensor field φ, a 1-form θ and Lorentzian metric g that fulfills the following

$$\varphi^2 U_1 = U_1 + \theta(U_1)\rho, \quad \theta(\rho) = -1, \tag{2}$$

$$g(\varphi U_1, \varphi U_2) = g(U_1, U_2) + \theta(U_1)\theta(U_2), \quad g(U_1, \rho) = \theta(U_1), \tag{3}$$

for certain U_1, U_2 vector fields on M, then such a manifold $M(\varphi, \rho, \theta, g)$ is referred to as a Lorentzian almost paracontact metric manifold [14]. In this manifold, the following conditions are satisfied:

$$\varphi\rho = 0, \quad \theta(\varphi U_1) = 0, \quad \Phi(U_1, U_2) = g(\varphi U_1, U_2) = \Phi(U_2, U_1), \tag{4}$$

where Φ is the fundamental two-form.

A Lorentzian almost paracontact metric manifold M is recognized as an LP-Kenmotsu manifold [16,26], if

$$(\nabla_{U_1}\varphi)U_2 = -g(\varphi U_1, U_2)\rho - \theta(U_2)\varphi U_1 \tag{5}$$

for U_1, U_2 vector fields on M. The following are satisfied by the LP-Kenmotsu manifold:

$$\nabla_{U_1}\rho = -U_1 - \theta(U_1)\rho, \tag{6}$$

$$(\nabla_{U_1}\theta)U_2 = -g(U_1, U_2) - \theta(U_1)\theta(U_2), \tag{7}$$

where ∇ indicates the Levi–Civita connection with regard to the Lorentzian metric g.

Further, an n-dimensional LP-Kenmotsu manifold follows the relations [16,26]:

$$g(\mathcal{R}(U_1, U_2)U_3, \rho) = \theta(\mathcal{R}(U_1, U_2)U_3) = g(U_2, U_3)\theta(U_1) - g(U_1, U_3)\theta(U_2), \quad (8)$$

$$\mathcal{R}(\rho, U_1)U_2 = g(U_1, U_2)\rho - \theta(U_2)U_1, \quad (9)$$

$$\mathcal{R}(U_1, U_2)\rho = \theta(U_2)U_1 - \theta(U_1)U_2, \quad (10)$$

$$\mathcal{S}(U_1, \rho) = (n-1)\theta(U_1), \mathcal{S}(\rho, \rho) = -(n-1), \quad (11)$$

$$\mathcal{S}(\varphi U_1, \varphi U_2) = \mathcal{S}(U_1, U_2) + (n-1)\theta(U_1)\theta(U_2) \quad (12)$$

for U_1, U_2, U_3 vector fields on M, in which $\mathcal{S}$ and $\mathcal{R}$ can be viewed as the Ricci tensor and the curvature tensor of M, respectively.

If the non-vanishing Ricci tensor S of an LP-Kenmotsu manifold M meets the following relation, then M is a generalized θ-Einstein manifold. The relation is as follows

$$\mathcal{S}(U_1, U_2) = ag(U_1, U_2) + b\theta(U_1)\theta(U_2) + cg(\varphi U_1, U_2)$$

for any U_1, U_2 vector fields on M and the scalar functions on M are a, b and c. When $c = 0$, then M is regarded as an θ-Einstein manifold. Furthermore, M is an Einstein manifold if $b = 0$ and $c = 0$.

3. Relation between the Levi–Civita Connection and GSM-Connection of Type (α, β)

In an LP-Kenmotsu manifold M, assuming that $\nabla^{\mathcal{G}}$ is a linear connection and ∇ is the Levi–Civita connection such that

$$\nabla^{\mathcal{G}}_{U_1} U_2 = \nabla_{U_1} U_2 + \mathcal{U}(U_1, U_2), \quad (13)$$

for any U_1 and U_2 vector fields on M. In this instance, $\mathcal{U}$ represents a tensor of type $(1,2)$, which is acquired in such a way that $\nabla^{\mathcal{G}}$ indicates a generalized-symmetric metric connection of ∇ in M as:

$$\mathcal{U}(U_1, U_2) = \frac{1}{2}[\mathcal{T}(U_1, U_2) + \mathcal{T}'(U_1, U_2) + \mathcal{T}'(U_2, U_1)], \quad (14)$$

where $\mathcal{T}$ indicated as the torsion tensor of $\nabla^{\mathcal{G}}$ and

$$g(\mathcal{T}'(U_1, U_2), U_3) = g(\mathcal{T}(U_3, U_1), U_2). \quad (15)$$

Plugging (1) in (15), we arrive at the following:

$$\mathcal{T}'(U_1, U_2) = \alpha\{\theta(U_1)U_2 - g(U_1, U_2)\rho\} + \beta\{\theta(U_1)\varphi U_2 - g(\varphi U_1, U_2)\rho\}. \quad (16)$$

Substituting (1) and (16) in (14), we obtain

$$\mathcal{U}(U_1, U_2) = \alpha\{\theta(U_2)U_1 - g(U_1, U_2)\rho\} + \beta\{\theta(U_2)\varphi U_1 - g(\varphi U_1, U_2)\rho\}. \quad (17)$$

Hence, a generalized symmetric metric connection $\nabla^{\mathcal{G}}$ of type (α, β) in an LP-Kenmotsu manifold is defined as

$$\nabla^{\mathcal{G}}_{U_1} U_2 = \nabla_{U_1} U_2 + \alpha\{\theta(U_2)U_1 - g(U_1, U_2)\rho\} + \beta\{\theta(U_2)\varphi U_1 - g(\varphi U_1, U_2)\rho\}. \quad (18)$$

Conversely, with the help of (18), the torsion tensor with respect to the connection $\nabla^{\mathcal{G}}$ is defined as follows

$$\begin{aligned} \mathcal{T}(U_1, U_2) &= \nabla^{\mathcal{G}}_{U_1} U_2 - \nabla^{\mathcal{G}}_{U_2} U_1 - [U_1, U_2] \\ &= \alpha\{\theta(U_2)U_1 - \theta(U_1)U_2\} + \beta\{\theta(U_2)\varphi U_1 - \theta(U_1)\varphi U_2\}. \quad (19) \end{aligned}$$

This shows that the connection $\nabla^{\mathcal{G}}$ in an LP-Kenmotsu manifold is a generalized symmetric connection. Also, we have

$$\begin{aligned}(\nabla^{\mathcal{G}}_{U_1}g)(U_2,U_3) &= U_1g(U_2,U_3)-g(\nabla^{\mathcal{G}}_{U_1}U_2,U_3)-g(U_2,\nabla^{\mathcal{G}}_{U_1}U_3)\\ &= 0. \end{aligned} \tag{20}$$

From (19) and (20), we determine that $\nabla^{\mathcal{G}}$ is a GSM-connection of type (α,β). This is recorded as follows:

Corollary 1. *Let M be an LP-Kenmotsu manifold, then the relation between Levi–Civita connection ∇ and a GSM-connection $\nabla^{\mathcal{G}}$ of type (α,β) on M is defined as (18).*

The GSM-connection is scaled down to a semi-symmetric metric connection and a quarter-symmetric metric connection, respectively, if we take $(\alpha,\beta)=(1,0)$ and $(\alpha,\beta)=(0,1)$, as shown in the following:

$$\nabla^{\mathcal{G}}_{U_1}U_2=\nabla_{U_1}U_2+\theta(U_2)U_1-g(U_1,U_2)\rho \tag{21}$$

and

$$\nabla^{\mathcal{G}}_{U_1}U_2=\nabla_{U_1}U_2+\theta(U_2)\varphi U_1-g(\varphi U_1,U_2)\rho. \tag{22}$$

Next, from (5), (6) and (18) we have the following:

Lemma 1. *In an LP-Kenmotsu manifold M with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type (α,β), we have the following relations:*

$$\begin{aligned}(\nabla^{\mathcal{G}}_{U_1}\varphi)U_2 &= (\alpha+1)\{-g(\varphi U_1,U_2)\rho-\theta(U_2)\varphi U_1\}\\ &- \beta\{g(U_1,U_2)\rho+\theta(U_2)U_1+2\theta(U_1)\theta(U_2)\rho\}, \end{aligned} \tag{23}$$

$$\nabla^{\mathcal{G}}_{U_1}\rho = -(\alpha+1)\{U_1+\theta(U_1)\rho\}-\beta\varphi U_1, \tag{24}$$

for any U_1,U_2 vector fields on M.

4. Curvature Tensor with Regard to GSM-Connection $\nabla^{\mathcal{G}}$ of Type (α,β)

For an LP-Kenmotsu manifold M, we define its curvature tensor with respect to the connection $\nabla^{\mathcal{G}}$ of type (α,β) by

$$\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3=\nabla^{\mathcal{G}}_{U_1}\nabla^{\mathcal{G}}_{U_2}U_3-\nabla^{\mathcal{G}}_{U_2}\nabla^{\mathcal{G}}_{U_1}U_3-\nabla^{\mathcal{G}}_{[U_1,U_2]}U_3. \tag{25}$$

From (18) it follows that

$$\begin{aligned}\nabla^{\mathcal{G}}_{U_1}\nabla^{\mathcal{G}}_{U_2}U_3 &= \nabla^{\mathcal{G}}_{U_1}\nabla_{U_2}U_3+\alpha[\nabla^{\mathcal{G}}_{U_1}(\theta(U_3)U_2)-\nabla^{\mathcal{G}}_{U_1}(g(U_2,U_3)\rho)]\\ &+ \beta[\nabla^{\mathcal{G}}_{U_1}(\theta(U_3)\varphi U_2)-\nabla^{\mathcal{G}}_{U_1}(g(\varphi U_2,U_3)\rho)]. \end{aligned} \tag{26}$$

In view of Equations (18), (25) and (26) we obtain the formula for the curvature tensor $\mathcal{R}^{\mathcal{G}}$ of the connection $\nabla^{\mathcal{G}}$ as

$$\begin{aligned}\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3 &= \mathcal{R}(U_1,U_2)U_3+(\alpha^2+2\alpha)[g(U_2,U_3)U_1-g(U_1,U_3)U_2]\\ &+ (\alpha^2+\alpha)[g(U_2,U_3)\theta(U_1)\rho-g(U_1,U_3)\theta(U_2)\rho+\theta(U_2)\theta(U_3)U_1\\ &- \theta(U_1)\theta(U_3)U_2]+(\alpha+1)\beta[g(\varphi U_2,U_3)U_1-g(\varphi U_1,U_3)U_2\\ &+ g(U_2,U_3)\varphi U_1-g(U_1,U_3)\varphi U_2]+(\alpha\beta)[g(\varphi U_2,U_3)\theta(U_1)\rho\\ &- g(\varphi U_1,U_3)\theta(U_2)\rho+\theta(U_2)\theta(U_3)\varphi U_1-\theta(U_1)\theta(U_3)\varphi U_2]\\ &+ \beta^2[g(\varphi U_2,U_3)\varphi U_1-g(\varphi U_1,U_3)\varphi U_2], \end{aligned} \tag{27}$$

where

$$\mathcal{R}(U_1, U_2)U_3 = \nabla_{U_1}\nabla_{U_2}U_3 - \nabla_{U_2}\nabla_{U_1}U_3 - \nabla_{[U_1,\, U_2]}U_3.$$

Therefore, Equation (27) represents the relationship between the curvature tensor of M with regard to GSM-connection $\nabla^{\mathcal{G}}$ and the Levi–Civita connection ∇ on an LP-Kenmotsu manifold. Further, U_6 is treated as the inner product of (27), we obtain

$$\begin{aligned} \mathcal{R}^{\mathcal{G}}(U_1, U_2, U_3, U_6) &= \mathcal{R}(U_1, U_2, U_3, U_6) \\ &+ (\alpha^2 + 2\alpha)[g(U_2, U_3)g(U_1, U_6) - g(U_1, U_3)g(U_2, U_6)] \\ &+ (\alpha^2 + \alpha)[g(U_2, U_3)\theta(U_1)\theta(U_6) - g(U_1, U_3)\theta(U_2)\theta(U_6) \\ &+ \theta(U_2)\theta(U_3)g(U_1, U_6) - \theta(U_1)\theta(U_3)g(U_2, U_6)] \\ &+ (\alpha + 1)\beta[g(\varphi U_2, U_3)g(U_1, U_6) - g(\varphi U_1, U_3)g(U_2, U_6) \\ &+ g(U_2, U_3)g(\varphi U_1, U_6) - g(U_1, U_3)g(\varphi U_2, U_6)] \\ &+ (\alpha\beta)[g(\varphi U_2, U_3)\theta(U_1)\theta(U_6) - g(\varphi U_1, U_3)\theta(U_2)\theta(U_6) \\ &+ \theta(U_2)\theta(U_3)g(\varphi U_1, U_6) - \theta(U_1)\theta(U_3)g(\varphi U_2, U_6)], \\ &+ \beta^2[g(\varphi U_2, U_3)g(\varphi U_1, U_6) - g(\varphi U_1, U_3)g(\varphi U_2, U_6)], \end{aligned} \tag{28}$$

where $\mathcal{R}^{\mathcal{G}}(U_1, U_2, U_3, U_6) = g(\mathcal{R}^{\mathcal{G}}(U_1, U_2)U_3, U_6)$ and $\mathcal{R}(U_1, U_2, U_3, U_6) = g(\mathcal{R}(U_1, U_2)U_3, U_6)$. Contracting (28) upon U_1 and U_6, we obtain

$$\begin{aligned} \mathcal{S}^{\mathcal{G}}(U_2, U_3) &= \mathcal{S}(U_2, U_3) + [(n-2)\alpha^2 + (2n-3)\alpha + (\alpha+1)\psi\beta - \beta^2]g(U_2, U_3) \\ &+ [(n-3)\alpha\beta + (n-2)\beta + \psi\beta^2]g(\varphi U_2, U_3) \\ &+ [(n-2)(\alpha^2 + \alpha) + \alpha\beta\psi - \beta^2]\theta(U_2)\theta(U_3), \end{aligned} \tag{29}$$

where S and $\mathcal{S}^{\mathcal{G}}$ are the Ricci tensors of the connection ∇ and $\nabla^{\mathcal{G}}$, respectively, on M and $\psi = trace\ \varphi$. Since in an LP-Kenmotsu manifold, the $(1,1)$-tensor field φ is symmetric and the Ricci tensor S with regard to the connection $\nabla^{\mathcal{G}}$ is symmetric and also $\nabla^{\mathcal{G}}$ satisfies the relation $\mathcal{S}^{\mathcal{G}}(U_2, U_3) = \mathcal{S}^{\mathcal{G}}(U_3, U_2)$.

From (12) and (29) it follows that

$$\mathcal{S}^{\mathcal{G}}(\varphi U_2, \varphi U_3) = \mathcal{S}^{\mathcal{G}}(U_2, U_3) + [(n-1)(\alpha+1) + \psi\beta]\theta(U_2)\theta(U_3). \tag{30}$$

Again, from (29), we have

$$\begin{aligned} \mathcal{Q}^{\mathcal{G}}U_2 &= \mathcal{Q}U_2 + [(n-2)\alpha^2 + (2n-3)\alpha + (\alpha+1)\psi\beta - \beta^2]U_2 \\ &+ [(n-3)\alpha\beta + (n-2)\beta + \psi\beta^2]\varphi U_2 \\ &+ [(n-2)(\alpha^2 + \alpha) + \alpha\beta\psi - \beta^2]\theta(U_2)\rho, \end{aligned} \tag{31}$$

where $\mathcal{Q}^{\mathcal{G}}$ and $\mathcal{Q}$ are denoted as the Ricci operators on M with regard to the connections $\nabla^{\mathcal{G}}$ and ∇, respectively. Contracting (29) upon U_2 and U_3, we have the following

$$\begin{aligned} r^{\mathcal{G}} &= r + (n-1)(n-2)\alpha^2 + 2(n^2 - 2n + 1)\alpha \\ &+ 2(n-1)(\alpha+1)\beta\psi + (\psi^2 - (n-1))\beta^2, \end{aligned} \tag{32}$$

where r^G and r are denoted as the scalar curvatures on M with regard to the connections $\nabla^{\mathcal{G}}$ and ∇, respectively.

So, we define the theorem:

Theorem 1. *For an n-dimensional LP-Kenmotsu manifold M with regard to GSM-connection* $\nabla^{\mathcal{G}}$ *of type* (α, β)*:*
1. *The curvature tensor* $\nabla^{\mathcal{G}}$ *is given by (27),*
2. $\mathcal{R}^{\mathcal{G}}(U_1, U_2, U_3, U_6) = -\mathcal{R}^{\mathcal{G}}(U_2, U_1, U_3, U_6)$,
3. $\mathcal{R}^{\mathcal{G}}(U_1, U_2, U_3, U_6) = -\mathcal{R}^{\mathcal{G}}(U_1, U_2, U_6, U_3)$,

4. $\mathcal{R}^{\mathcal{G}}(U_1,U_2,U_3,U_6)=\mathcal{R}^{\mathcal{G}}(U_2,U_1,U_6,U_3)=\mathcal{R}^{\mathcal{G}}(U_6,U_3,U_2,U_1)$,
5. $\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3+\mathcal{R}^{\mathcal{G}}(U_2,U_3)U_1+\mathcal{R}^{\mathcal{G}}(U_3,U_1)U_2=0$,
6. The Ricci tensor $\mathcal{S}^{\mathcal{G}}$ *is given by (29),*
7. The scalar curvature $r^{\mathcal{G}}$ *is given by (32).*

Particularly, when $(\alpha,\beta)=(1,0)$ and $(\alpha,\beta)=(0,1)$ are taken into consideration, we have the following:

Corollary 2. *The Ricci tensor* $\mathcal{S}^{\mathcal{G}}$ *and curvature tensor* $\mathcal{R}^{\mathcal{G}}$ *with regard to the semi-symmetric metric connection on an LP-Kenmotsu manifold M are displayed as in the following way:*

$$\begin{aligned}\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3 &= \mathcal{R}(U_1,U_2)U_3+3\{g(U_2,U_3)U_1-g(U_1,U_3)U_2\}\\ &+ 2\{g(U_2,U_3)\theta(U_1)\rho-g(U_1,U_3)\theta(U_2)\rho\\ &+ \theta(U_2)\theta(U_3)U_1-\theta(U_1)\theta(U_3)U_2\}\end{aligned}$$

and

$$\mathcal{S}^{\mathcal{G}}(U_2,U_3)=\mathcal{S}(U_2,U_3)+(n-5)g(U_2,U_3)+2(n-2)\theta(U_2)\theta(U_3).$$

Corollary 3. *The Ricci tensor* $\mathcal{S}^{\mathcal{G}}$ *and the curvature tensor* $\mathcal{R}^{\mathcal{G}}$ *with regard to the quarter-symmetric metric connection on an LP-Kenmotsu manifold M is displayed as in the following way:*

$$\begin{aligned}\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3 &= \mathcal{R}(U_1,U_2)U_3+g(\varphi U_2,U_3)U_1-g(\varphi U_1,U_3)U_2\\ &+ g(\varphi U_2,U_3)\varphi U_1-g(\varphi U_1,U_3)\varphi U_2+g(U_2,U_3)\varphi U_1-g(U_1,U_3)\varphi U_2\end{aligned}$$

and

$$\mathcal{S}^{\mathcal{G}}(U_2,U_3)=\mathcal{S}(U_2,U_3)+(\psi-1)g(U_2,U_3)+(n-2+\psi)g(\varphi U_2,U_3)-\theta(U_2)\theta(U_3).$$

The following Lemma is presented using (27) and (29).

Lemma 2. *Suppose that M is an n-dimensional LP-Kenmotsu manifold with regard to GSM-connection of type* (α,β)*. Then*

$$\begin{aligned}\theta(\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3) &= (\alpha+1)[g(U_2,U_3)\theta(U_1)-g(U_1,U_3)\theta(U_2)]\\ &+ \beta[g(\varphi U_2,U_3)\theta(U_1)-g(\varphi U_1,U_3)\theta(U_2)], \quad (33)\\ \mathcal{R}^{\mathcal{G}}(U_1,U_2)\rho &= (\alpha+1)[\theta(U_2)U_1-\theta(U_1)U_2]\\ &+ \beta[\theta(U_2)\varphi U_1-\theta(U_1)\varphi U_2], \quad (34)\\ \mathcal{R}^{\mathcal{G}}(\rho,U_2)U_3 &= (\alpha+1)[g(U_2,U_3)\rho-\theta(U_3)U_2]\\ &+ \beta[g(\varphi U_2,U_3)\rho-\theta(U_3)\varphi U_2], \quad (35)\\ \mathcal{S}^{\mathcal{G}}(U_2,\rho) &= ((n-1)(\alpha+1)+\psi\beta)\theta(U_2), \quad (36)\\ \mathcal{S}^{\mathcal{G}}(\rho,\rho) &= -((n-1)(\alpha+1)+\psi\beta),\\ \mathcal{Q}^{\mathcal{G}}\rho &= ((n-1)(\alpha+1)+\psi\beta)\rho \quad (37)\end{aligned}$$

for any U_1,U_2,U_3 *vector fields on M.*

5. Some Results on LP-Kenmotsu Manifolds with Regard to GSM-Connection $\nabla^{\mathcal{G}}$ of Type (α,β)

In this section, we characterize locally symmetric, Ricci semi-symmetric and φ-Ricci symmetric LP-Kenmotsu manifolds with regard to GSM-connection.

5.1. Locally Symmetric LP-Kenmotsu Manifold with Regard to $\nabla^{\mathcal{G}}$

The study of Riemann symmetric manifolds began with the work of Cartan [27]. According to Cartan [27], a Riemannian manifold M is said to be locally symmetric if the curvature tensor $\mathcal{R}$ satisfies the relation $\nabla\mathcal{R} = 0$, where ∇ denotes the Levi–Civita connection on M. If the curvature tensor $\mathcal{R}$ of an LP-Kenmotsu manifold M fulfills the condition $\nabla\mathcal{R} = 0$, where ∇ is the Levi–Civita connection of M, then M is said to be locally symmetric.

Assuming that M is a locally symmetric LP-Kenmotsu manifold with regard to the connection $\nabla^{\mathcal{G}}$, then

$$(\nabla^{\mathcal{G}}_{U_1}\mathcal{R}^{\mathcal{G}})(U_2, U_3)U_6 = 0, \tag{38}$$

for any U_1, U_2, U_3, U_6 vector fields on M. With a suitable contraction of this equation, we have

$$(\nabla^{\mathcal{G}}_{U_1}\mathcal{S}^{\mathcal{G}})(U_3, U_6) = \nabla^{\mathcal{G}}_{U_1}\mathcal{S}^{\mathcal{G}}(U_3, U_6) - \mathcal{S}^{\mathcal{G}}(\nabla^{\mathcal{G}}_{U_1}U_3, U_6) - \mathcal{S}^{\mathcal{G}}(U_3, \nabla^{\mathcal{G}}_{U_1}U_6) = 0. \tag{39}$$

Taking $U_6 = \rho$ in (39), we have

$$\nabla^{\mathcal{G}}_{U_1}\mathcal{S}^{\mathcal{G}}(U_3, \rho) - \mathcal{S}^{\mathcal{G}}(\nabla^{\mathcal{G}}_{U_1}U_3, \rho) - \mathcal{S}^{\mathcal{G}}(U_3, \nabla^{\mathcal{G}}_{U_1}\rho) = 0. \tag{40}$$

Now using (6), (18) and (36), we obtain from (40) that

$$\begin{aligned} & (\alpha+1)\mathcal{S}^{\mathcal{G}}(U_1, U_3) + \beta\mathcal{S}^{\mathcal{G}}(\varphi U_1, U_3) \\ = \; & \{(n-1)(\alpha+1) + \psi\beta\}[(\alpha+1)g(U_1, U_3) + \beta g(\varphi U_1, U_3)]. \end{aligned} \tag{41}$$

Substituting U_1 by φU_1 in the above equation and using (2) and (36) we obtain

$$\begin{aligned} \mathcal{S}^{\mathcal{G}}(\varphi U_1, U_3) \; = \; & \frac{\beta}{(\alpha+1)}[-\mathcal{S}^{\mathcal{G}}(U_1, U_3) + ((n-1)(\alpha+1) + \psi\beta)g(U_1, U_3)] \\ + \; & ((n-1)(\alpha+1) + \psi\beta)g(\varphi U_1, U_3). \end{aligned} \tag{42}$$

Taking account of the above equation in (41) we obtain

$$\mathcal{S}^{\mathcal{G}}(U_1, U_3) = ((n-1)(\alpha+1) + \psi\beta)g(U_1, U_3), \tag{43}$$

provided $\alpha \neq -1$ and $\beta^2 \neq (\alpha+1)^2$. Thus, M is an Einstein manifold with regard to the connection $\nabla^{\mathcal{G}}$. Hence, we obtain the theorem:

Theorem 2. *If M is an n-dimensional locally symmetric LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type (α, β), where $\alpha \neq -1$ and $\beta^2 \neq (\alpha+1)^2$, then M is regarded as an Einstein one with regard to the connection $\nabla^{\mathcal{G}}$.*

5.2. Ricci Semi-Symmetric LP-Kenmotsu Manifold with Regard to $\nabla^{\mathcal{G}}$

As a generalization of locally symmetric manifolds, many geometers have examined semi-symmetric manifold and their generalizations. If the curvature tensor $\mathcal{R}$ satisfies the below condition, a (semi-)Riemannian manifold is said to be semi-symmetric

$$\mathcal{R}(U_1, U_2) \cdot \mathcal{R} = 0,$$

for any U_1, U_2 vector fields on M. These conditions are found in the works of E. Cartan and also Shirokov, who were the first to study spaces with a condition $\nabla R = 0$. N.S. Sinyukov, in 1954, introduced the term semi-symmetric space in his study of geodesic mappings of semi-symmetric spaces, see [28] and, for example, [29]. Mikeš continued these investigations, notably in [30]. In this paper, symmetric and semi-symmetric projective flat spaces are also examined. Among other things, the results indicate the existence of semi-symmetric spaces that are not symmetric, as demonstrated explicitly in Tagaki's work [31]. For instance, the example of a semi-symmetric not locally symmetric Riemannian manifold was given by Takagi [31]. While in the Riemmanina case, every homogeneous semi-symmetric manifold is actually locally symmetric, in the Lorentzian case they are homogeneous semi-symmetric Lorentzian manifolds which are not locally symmetric. It is mentioned that Szabo [32] provided a fully intrinsic classification of these spaces.

Also, a (semi-)Riemannian manifold M is referred to as Ricci semi-symmetric, if its curvature tensor $\mathcal{R}$ satisfies the condition

$$\mathcal{R}(U_1, U_2) \cdot \mathcal{S} = 0,$$

for any U_1, U_2 vector fields on M, where the Ricci tensor S of type (0, 2) is regarded as a field of the linear operator on $\mathcal{R}(U_1, U_2)$.

Suppose that M is a Ricci semi-symmetric LP-Kenmotsu manifold with regard to the connection $\nabla^{\mathcal{G}}$. So M^{2n+1} satisfies the condition

$$(\mathcal{R}^{\mathcal{G}}(U_1, U_2) \cdot \mathcal{S}^{\mathcal{G}})(U_3, U_6) = 0$$

for any U_1, U_2, U_3, U_6 vector fields on M. We obtain

$$\mathcal{S}^{\mathcal{G}}(\mathcal{R}^{\mathcal{G}}(U_1, U_2)U_3, U_6) + \mathcal{S}^{\mathcal{G}}(U_3, \mathcal{R}^{\mathcal{G}}(U_1, U_2)U_6) = 0. \tag{44}$$

Putting $U_1 = U_3 = \rho$ in (44), then we arrive at the following

$$\mathcal{S}^{\mathcal{G}}(\mathcal{R}^{\mathcal{G}}(\rho, U_2)\rho, U_6) + \mathcal{S}^{\mathcal{G}}(\rho, \mathcal{R}^{\mathcal{G}}(\rho, U_2)U_6) = 0. \tag{45}$$

Using (35) and (36) in (45), we arrive at (41). Further, continuing the proceeding according to the previous Section 5.1, we obtained at (43). Thus, we obtain the following:

Theorem 3. *Suppose that M is an n-dimensional Ricci semi-symmetric LP-Kenmotsu manifold with regard to GSM-connection $\bar{\nabla}$ of type (α, β) with $\alpha \neq -1$ and $\beta^2 \neq (\alpha+1)^2$, then the manifold is an Einstein one with regard to GSM-connection $\nabla^{\mathcal{G}}$.*

5.3. φ-Ricci Symmetric LP-Kenmotsu Manifold with Regard to $\nabla^{\mathcal{G}}$

The concept of local symmetry of Riemannian manifolds has been diminished by many authors in a variety of ways to a different extent. Takahashi [33] developed the idea of local φ-symmetry on Sasakian manifolds as a weaker version of local symmetry. If the following condition is true

$$\varphi^2(\nabla_{U_1} Q)U_2 = 0$$

for any vector fields U_1, U_2 on M, then an LP-Kenmotsu manifold is φ-Ricci symmetric. Here, Q is treated as the Ricci operator, i.e., $g(QU_1, U_2) = \mathcal{S}(U_1, U_2)$ for all U_1, U_2 vector fields. If U_1, U_2 are horizontal vector fields, then the manifold is known as locally φ-Ricci symmetric.

Suppose that M is a φ-Ricci symmetric LP-Kenmotsu manifold with regard to the connection $\nabla^{\mathcal{G}}$. Then, the Ricci operator $\mathcal{Q}^{\mathcal{G}}$ fulfills the condition

$$\varphi^2(\nabla^{\mathcal{G}}_{U_1} \mathcal{Q}^{\mathcal{G}})U_2 = 0, \tag{46}$$

for any U_1, U_2 vector fields on M, which by using (2) we obtain the following

$$(\nabla^{\mathcal{G}}_{U_1} \mathcal{Q}^{\mathcal{G}})U_2 + \theta((\nabla^{\mathcal{G}}_{U_1} \mathcal{Q}^{\mathcal{G}})(U_2))\rho = 0. \tag{47}$$

The inner product of (47) with U_3 is given by

$$g((\nabla^{\mathcal{G}}_{U_1} \mathcal{Q}^{\mathcal{G}})U_2, U_3) + \theta((\nabla^{\mathcal{G}}_{U_1} \mathcal{Q}^{\mathcal{G}})(U_2))\theta(U_3) = 0, \tag{48}$$

which after simplification takes the form

$$g(\nabla^{\mathcal{G}}_{U_1}(\mathcal{Q}^{\mathcal{G}}U_2),U_3)-\mathcal{S}^{\mathcal{G}}(\nabla^{\mathcal{G}}_{U_1}U_2,U_3)+\theta((\nabla^{\mathcal{G}}_{U_1}\mathcal{Q}^{\mathcal{G}})(U_2))\theta(U_3)=0. \quad (49)$$

Taking $U_2=\rho$ in the above equation and using (24), (36) and (37) we arrive at

$$\begin{aligned}&(\alpha+1)[\mathcal{S}^{\mathcal{G}}(U_1,U_3)-((n-1)(\alpha+1)+\psi\beta)g(U_1,U_3)]\\ =\;&\beta[\mathcal{S}^{\mathcal{G}}(\varphi U_1,U_3)-((n-1)(\alpha+1)+\psi\beta)g(\varphi U_1,U_3)]. \quad (50)\end{aligned}$$

Replacing U_1 by φU_1 in (50), we obtain

$$\begin{aligned}\mathcal{S}^{\mathcal{G}}(\varphi U_1,U_3) &= \frac{\beta}{(\alpha+1)}[\mathcal{S}^{\mathcal{G}}(U_1,U_3)-((n-1)(\alpha+1)+\psi\beta)g(U_1,U_3)]\\ &+ ((n-1)(\alpha+1)+\psi\beta)g(\varphi U_1,U_3). \quad (51)\end{aligned}$$

Using (51) in (50) we obtain

$$\mathcal{S}^{\mathcal{G}}(U_1,U_3)=((n-1)(\alpha+1)+\psi\beta)g(U_1,U_3), \quad (52)$$

provided $\alpha\neq-1$ and $\beta^2\neq(\alpha+1)^2$. That is, the Einstein manifold M with regard to $\nabla^{\mathcal{G}}$. Therefore, we conclude the following theorem:

Theorem 4. *Suppose that M is an n-dimensional φ-Ricci symmetric LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type (α,β) with $\alpha\neq-1$ and $\beta^2\neq(\alpha+1)^2$, then M is treated as Einstein one with regard to GSM-connection $\nabla^{\mathcal{G}}$.*

It is observed from the Theorems 2, 3 and 4 that if the manifold M is locally symmetric (or, Ricci semi-symmetric, or φ-Ricci symmetric) with regard to the connection $\nabla^{\mathcal{G}}$ then the manifold is an Einstein manifold with regard to the connection ∇.

On comparing (43) with (29) we obtain

$$\begin{aligned}\mathcal{S}(U_1,U_3) &= [(n-1)-(n-2)(\alpha^2+\alpha)-\alpha\psi\beta+\beta^2]g(U_1,U_3)\\ &- [(n-3)\alpha\beta+(n-2)\beta+\psi\beta^2]g(\varphi U_1,U_3)\\ &- [(n-2)(\alpha^2+\alpha)+\alpha\psi\beta-\beta^2]\theta(U_1)\theta(U_3). \quad (53)\end{aligned}$$

This helps us to state the succeeding corollary:

Corollary 4. *Suppose that M is an n-dimensional locally symmetric (or Ricci semi-symmetric, or φ-Ricci symmetric) LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type (α,β), then the expressions are obtained as follows:*
(i) M is an θ-Einstein manifold defined as $\mathcal{S}(U_1,U_3)=-(n-3)g(U_1,U_3)-2(n-2)\theta(U_1)\theta(U_3)$ in regard to the connection $\nabla^{\mathcal{G}}$ of type $(1,0)$.
(ii) M is a generalized θ-Einstein manifold given by $\mathcal{S}(U_1,U_3)=ng(U_1,U_3)-(n-2+\psi)g(\varphi U_1,U_3)+\theta(U_1)\theta(U_3)$ in regard to the connection $\nabla^{\mathcal{G}}$ of type $(0,1)$.

6. Projective Curvature Tensor on LP-Kenmotsu Manifold with Regard to GSM-Connection $\nabla^{\mathcal{G}}$ of Type (α,β)

Suppose that M is an n-dimensional LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$. We define the projective curvature tensor $\mathcal{P}^{\mathcal{G}}$ of type (1, 3) with regard to the connection $\nabla^{\mathcal{G}}$ of M as

$$\mathcal{P}^{\mathcal{G}}(U_1,U_2)U_3=\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3-\frac{1}{n-1}[\mathcal{S}^{\mathcal{G}}(U_2,U_3)U_1-\mathcal{S}^{\mathcal{G}}(U_1,U_3)U_2]. \quad (54)$$

for any U_1,U_2,U_3 vector fields on M, where $\mathcal{R}^{\mathcal{G}}$ and $\mathcal{S}^{\mathcal{G}}$ are the curvature tensor and the Ricci tensor with regard to the connection $\nabla^{\mathcal{G}}$, respectively.

Here, we begin with the following:

Let us assume that M is a projectively flat LP-Kenmotsu manifold with regard to the connection $\nabla^{\mathcal{G}}$. Then the condition

$$\mathcal{P}^{\mathcal{G}}(U_1,U_2)U_3=0,$$

holds for any U_1,U_2,U_3 vector fields on M. Then, using the above equation in (54), we arrive at the following

$$\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3=\frac{1}{n-1}[\mathcal{S}^{\mathcal{G}}(U_2,U_3)U_1-\mathcal{S}^{\mathcal{G}}(U_1,U_3)U_2]. \tag{55}$$

Putting $U_1=\rho$ and on applying the inner product with ρ in (55), we obtain

$$\mathcal{S}^{\mathcal{G}}(U_2,U_3)=-(n-1)\theta(\mathcal{R}^{\mathcal{G}}(\rho,U_2)U_3)-((n-1)(\alpha+1)+\psi\beta)\theta(U_2)\theta(U_3). \tag{56}$$

Taking the help of (33) in the above equation, we obtain

$$\mathcal{S}^{\mathcal{G}}(U_2,U_3)=(n-1)(\alpha+1)g(U_2,U_3)-\psi\beta\theta(U_2)\theta(U_3)+(n-1)\beta g(U_2,\varphi U_3). \tag{57}$$

Hence, it leads to the following:

Theorem 5. *Suppose that M is an n-dimensional projectively flat LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type (α,β), then M is a generalized θ-Einstein manifold with regard to GSM-connconnection $\nabla^{\mathcal{G}}$.*

In particular, if we choose $(\alpha,\beta)=(\alpha,0)$ then from (57) we have

$$\mathcal{S}^{\mathcal{G}}(U_2,U_3)=(\alpha+1)(n-1)g(U_2,U_3). \tag{58}$$

Thus, M is defined as the Einstein manifold with regard to the connection $\nabla^{\mathcal{G}}$ of type $(\alpha,0)$. Therefore, we infer the following:

Corollary 5. *Suppose that M is an n-dimensional projectively flat LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type $(\alpha,0)$, then M is treated as an Einstein one with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type $(\alpha,0)$.*

From Lemma 2 and the relation (54), we obtain the following:

$$\begin{aligned}\theta(\mathcal{P}^{\mathcal{G}}(U_1,U_2)U_3)= &\quad (\alpha+1)[g(U_2,U_3)\theta(U_1)-g(U_1,U_3)\theta(U_2)]\\ &+\beta[g(\varphi U_2,U_3)\theta(U_1)-g(\varphi U_1,U_3)\theta(U_2)]\\ &-\frac{1}{n-1}[\mathcal{S}^{\mathcal{G}}(U_2,U_3)\theta(U_1)-\mathcal{S}^{\mathcal{G}}(U_1,U_3)\theta(U_2)], \qquad (59)\\ \theta(\mathcal{P}^{\mathcal{G}}(U_1,U_2)\rho) &= 0, \qquad (60)\\ \theta(\mathcal{P}^{\mathcal{G}}(\rho,U_2)U_3) &= \frac{1}{n-1}\mathcal{S}^{\mathcal{G}}(U_2,U_3)\\ &-(\alpha+1)g(U_2,U_3)-\beta g(\varphi U_2,U_3)+\frac{\psi\beta}{n-1}\theta(U_2)\theta(U_3) \qquad (61)\end{aligned}$$

for any U_1,U_2,U_3 vector fields on M.

Consider a projectively semi-symmetric LP-Kenmotsu manifold M that admits a GSM-connection $\nabla^{\mathcal{G}}$. Then, the following the condition holds

$$\mathcal{R}^{\mathcal{G}}(U_1,U_2)\cdot\mathcal{P}^{\mathcal{G}}=0, \tag{62}$$

for any U_1, U_2 vector fields on M. In the virtue of (62), we obtain

$$\begin{aligned} &\mathcal{R}^{\mathcal{G}}(U_1,U_2)\mathcal{P}^{\mathcal{G}}(U_4,U_5)U_3 - \mathcal{P}^{\mathcal{G}}(\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_4,U_5)U_3 \\ &-\mathcal{P}^{\mathcal{G}}(U_4,\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_5)U_3 - \mathcal{P}^{\mathcal{G}}(U_4,U_5)\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3 = 0, \end{aligned}$$

for any U_1, U_2, U_3, U_4, U_5 vector fields on M. Therefore,

$$\begin{aligned} &g(\mathcal{R}^{\mathcal{G}}(U_1,U_2)\mathcal{P}^{\mathcal{G}}(U_4,U_5)U_3,\rho) - g(\mathcal{P}^{\mathcal{G}}(\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_4,U_5)U_3,\rho) \\ - \; &g(\mathcal{P}^{\mathcal{G}}(U_4,\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_5)U_3,\rho) - g(\mathcal{P}^{\mathcal{G}}(U_4,U_5)\mathcal{R}^{\mathcal{G}}(U_1,U_2)U_3,\rho) = 0. \end{aligned}$$

Then by taking $U_1 = \rho$, it follows that

$$\begin{aligned} &(\alpha+1)[-\mathcal{P}^{\mathcal{G}}(U_4,U_5,U_3,U_2) - \theta(\mathcal{P}^{\mathcal{G}}(U_4,U_5)U_3)\theta(U_2) \\ - \; &g(U_2,U_4)\theta(\mathcal{P}^{\mathcal{G}}(\rho,U_5)U_3) + \theta(U_4)\theta(\mathcal{P}^{\mathcal{G}}(U_2,U_5)U_3) - g(U_2,U_5)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)U_3) \\ + \; &\theta(U_5)\theta(\mathcal{P}^{\mathcal{G}}(U_4,U_2)U_3) + \theta(U_3)\theta(\mathcal{P}^{\mathcal{G}}(U_4,U_5)U_2)] + \beta[-\mathcal{P}^{\mathcal{G}}(U_4,U_5,U_3,\varphi U_2) \\ - \; &g(\varphi U_2,U_4)\theta(\mathcal{P}^{\mathcal{G}}(\rho,U_5)U_3) + \theta(U_4)\theta(\mathcal{P}^{\mathcal{G}}(\varphi U_2,U_5)U_3) \\ - \; &g(\varphi U_2,U_5)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)U_3) + \theta(U_5)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\varphi U_2)U_3) \\ + \; &\theta(U_3)\theta(\mathcal{P}^{\mathcal{G}}(U_4,U_5)\varphi U_2)] = 0. \end{aligned} \tag{63}$$

where $\mathcal{P}^{\mathcal{G}}(U_4,U_5,U_3,U_2) = g(\mathcal{P}^{\mathcal{G}}(U_4,U_5)U_3,U_2)$. Suppose that $\{e_1, e_2, \ldots\ldots, e_{n-1}, \rho\}$ is a local orthonormal basis of vector fields in M. With the help of this, we define $\{\varphi e_1, \varphi e_2, \ldots\ldots., \varphi e_{n-1}, \rho\}$ as a local orthonormal basis in M. We put $U_5 = U_3 = e_i$ in (63) and adding with regard to i, we arrive at

$$\begin{aligned} &(\alpha+1)\sum_{i=1}^{n-1}[-\mathcal{P}^{\mathcal{G}}(U_4,e_i,e_i,U_2) - \theta(\mathcal{P}^{\mathcal{G}}(U_4,e_i)e_i)\theta(U_2) - g(U_2,U_4)\theta(\mathcal{P}^{\mathcal{G}}(\rho,e_i)e_i) \\ + \; &\theta(U_4)\theta(\mathcal{P}^{\mathcal{G}}(U_2,e_i)e_i) - g(U_2,e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)e_i) + \theta(e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,U_2)e_i) \\ + \; &\theta(e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,e_i)U_2)] + \beta\sum_{i=1}^{n-1}[-\mathcal{P}^{\mathcal{G}}(U_4,e_i,e_i,\varphi U_2) - g(\varphi U_2,U_4)\theta(\mathcal{P}^{\mathcal{G}}(\rho,e_i)e_i) \\ + \; &\theta(U_4)\theta(\mathcal{P}^{\mathcal{G}}(\varphi U_2,e_i)e_i) - g(\varphi U_2,e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)e_i) + \theta(e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\varphi U_2)e_i) \\ + \; &\theta(e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,e_i)\varphi U_2)] = 0. \end{aligned} \tag{64}$$

Using (59) and (60), it can be easily verified that

$$\sum_{i=1}^{n-1}\theta(\mathcal{P}^{\mathcal{G}}(U_4,e_i)e_i) = \frac{1}{n-1}[n((n-1)(\alpha+1)+\psi\beta) - r^{\mathcal{G}}]\theta(U_4), \tag{65}$$

$$\sum_{i=1}^{n-1}g(U_2,e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)e_i) = \theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)U_2), \tag{66}$$

$$\sum_{i=1}^{n-1}\theta(e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,U_2)e_i) = \theta(\mathcal{P}^{\mathcal{G}}(U_4,U_5)\rho) = 0, \tag{67}$$

$$\sum_{i=1}^{n-1}\theta(e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,e_i)U_2) = 2\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)U_2), \tag{68}$$

$$\sum_{i=1}^{n-1}g(\varphi U_2,e_i)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)e_i) = \theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)\varphi U_2), \tag{69}$$

$$\sum_{i=1}^{n-1}\theta(e_i)\theta\Big(\mathcal{P}^{\mathcal{G}}(U_4,e_i)\varphi U_2\Big) = 2\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)\varphi U_2). \tag{70}$$

Using (65) to (70), it follows from (64) that

$$
\begin{aligned}
&(\alpha+1)\sum_{i=1}^{n-1}[-\mathcal{P}^{\mathcal{G}}(U_4,e_i,e_i,U_2)-g(U_2,U_4)\theta(\mathcal{P}^{\mathcal{G}}(\rho,e_i)e_i)]+(\alpha+1)\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)U_2)\\
+\;&\beta\sum_{i=1}^{n-1}[-\mathcal{P}^{\mathcal{G}}(U_4,e_i,e_i,\varphi U_2)-g(\varphi U_2,U_4)\theta(\mathcal{P}^{\mathcal{G}}(\rho,e_i)e_i)]+\beta\theta(\mathcal{P}^{\mathcal{G}}(U_4,\rho)\varphi U_2)]=0.
\end{aligned}
\tag{71}
$$

Again, with the help of (54), we obtain

$$
\begin{aligned}
\sum_{i=1}^{n-1}\mathcal{P}^{\mathcal{G}}(U_4,e_i,e_i,U_2) &= \frac{1}{n-1}[n\mathcal{S}^{\mathcal{G}}(U_4,U_2)-r^{\mathcal{G}}g(U_4,U_2)]\\
&- \frac{\psi\beta}{n-1}[g(U_4,U_2)+\theta(U_4)\theta(U_2)\}+\beta g(\varphi U_2,U_4),
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
\sum_{i=1}^{n-1}\mathcal{P}^{\mathcal{G}}(U_4,e_i,e_i,\varphi U_2) &= \frac{1}{n-1}[n\mathcal{S}^{\mathcal{G}}(U_4,\varphi U_2)-r^{\mathcal{G}}g(U_4,\varphi U_2)]\\
&- \frac{\psi\beta}{n-1}g(\varphi U_2,U_4)+\beta[g(\theta U_2,\theta U_4)].
\end{aligned}
\tag{73}
$$

Taking account of (72) and (73), the Equation (71) reduces to

$$
\begin{aligned}
&(\alpha+1)[-\mathcal{S}^{\mathcal{G}}(U_4,U_2)+\{(n-1)(\alpha+1)+\psi\beta\}g(U_4,U_2)\\
+\;&\beta[-\mathcal{S}^{\mathcal{G}}(U_4,\varphi U_2)+\{(n-1)(\alpha+1)+\psi\beta\}g(U_4,\varphi U_2)]=0.
\end{aligned}
\tag{74}
$$

Replacing U_2 by φU_2 in (74), we obtain

$$
\begin{aligned}
\mathcal{S}^{\mathcal{G}}(U_4,\varphi U_2) &= -\frac{\beta}{(\alpha+1)}\mathcal{S}^{\mathcal{G}}(U_4,U_2)+\{(n-1)(\alpha+1)+\psi\beta\}g(U_4,\varphi U_2)\\
&+ \frac{\beta}{(\alpha+1)}\{(n-1)(\alpha+1)+\psi\beta\}g(U_4,U_2).
\end{aligned}
\tag{75}
$$

By taking account of the above in (74), we obtain

$$
\mathcal{S}^{\mathcal{G}}(U_4,U_2) = \{(n-1)(\alpha+1)+\psi\beta\}g(U_4,U_2),
\tag{76}
$$

provided that $\alpha\neq -1$ and $(\alpha+1)^2\neq\beta^2$. So, we have the following result:

Theorem 6. *Suppose that M is an n-dimensional projectively semi-symmetric LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type (α,β) with $\alpha\neq -1$ and $\beta^2\neq(\alpha+1)^2$, then M is regarded as an Einstein manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$.*

In particular, if we choose $(\alpha,\beta)=(\alpha,0)$ then from (76) we have

$$
\mathcal{S}^{\mathcal{G}}(U_4,U_2)=(n-1)(\alpha+1)g(U_4,U_2).
\tag{77}
$$

Now, with the help of (77), the Equations (59) and (61) gives that $\theta(\mathcal{P}^{\mathcal{G}}(U_4,U_5)U_3)=0$ and $\theta(\mathcal{P}^{\mathcal{G}}(\rho,U_4)U_5)=0$, respectively. By taking these in the Equation (63), we have

$$
\mathcal{P}^{\mathcal{G}}(U_4,U_5,U_3,U_2)=0.
\tag{78}
$$

Therefore, M is projectively flat with regard to the connection $\nabla^{\mathcal{G}}$. Conversely, (78) trivially implies (62). Hence, we state the following:

Corollary 6. *Suppose that M is an n-dimensional LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type $(\alpha,0)$ with $\alpha\neq 1$. Then, it is projectively semi-symmetric if and only if it is projectively flat.*

Next, let us consider a φ-projectively flat LP-Kenmotsu manifold M that admits a connection $\nabla^{\mathcal{G}}$. Then, the condition

$$\varphi^2\mathcal{P}^{\mathcal{G}}(\varphi U_1, \varphi U_2)\varphi U_3 = 0, \tag{79}$$

holds for any U_1, U_2, U_3 vector fields on M. Then, $\varphi^2\mathcal{P}^{\mathcal{G}}(\varphi U_1, \varphi U_2)\varphi U_3 = 0$ holds if and only if

$$g(\mathcal{P}^{\mathcal{G}}(\varphi U_1, \varphi U_2)\varphi U_3, \varphi U_6) = 0, \tag{80}$$

holds for any U_1, U_2, U_3, U_6 vector fields on M.
The φ-projectively flat can be defined using (54) and (80) as

$$\begin{aligned} & g(\mathcal{R}^{\mathcal{G}}(\varphi U_1, \varphi U_2)\varphi U_3, \varphi U_6) \\ = & \frac{1}{n-1}\Big[\mathcal{S}^{\mathcal{G}}(\varphi U_2, \varphi U_3)g(\varphi U_1, \varphi U_6) - \mathcal{S}^{\mathcal{G}}(\varphi U_1, \varphi U_3)g(\varphi U_2, \varphi U_6)\Big] \end{aligned} \tag{81}$$

for any vector fields U_1, U_2, U_3, U_6 on M. For the local orthonormal basis $\{\varphi e_1, \varphi e_2, \ldots, \varphi e_{n-1}, \rho\}$ of vector fields in M, choosing $U_1 = U_6 = e_i$ in (81) and adding with regard to $i = 1, 2, \ldots n$, we have

$$\begin{aligned} & \sum_{i=1}^{n-1} g(\mathcal{R}^{\mathcal{G}}(\varphi e_i, \varphi U_2)\varphi U_3, \varphi e_i) \\ = & \frac{1}{n-1}\sum_{i=1}^{n-1}\Big[\mathcal{S}^{\mathcal{G}}(\varphi U_2, \varphi U_3)g(\varphi e_i, \varphi e_i) - \mathcal{S}^{\mathcal{G}}(\varphi e_i, \varphi U_3)g(\varphi U_2, \varphi e_i)\Big] \end{aligned} \tag{82}$$

for any U_2, U_3 vector fields on M. So, it can be easily verified using (28) and (30)

$$\begin{aligned} \sum_{i=1}^{n-1} g(\mathcal{R}^{\mathcal{G}}(\varphi e_i, \varphi U_2)\varphi U_3, \varphi e_i) &= \mathcal{S}^{\mathcal{G}}(\varphi U_2, \varphi U_3) \\ &+ (\alpha+1)g(\varphi U_2, \varphi U_3) + \beta g(U_2, \varphi U_3), \end{aligned} \tag{83}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n+1, \tag{84}$$

$$\sum_{i=1}^{n-1} \mathcal{S}^{\mathcal{G}}(\varphi U_2, \varphi e_i)g(\varphi e_i, \varphi U_3) = \mathcal{S}^{\mathcal{G}}(\varphi U_2, \varphi U_3). \tag{85}$$

In view of (83)–(85), (82) becomes

$$\mathcal{S}^{\mathcal{G}}(\varphi U_2, \varphi U_3) = (n-1)[(\alpha+1)g(\varphi U_2, \varphi U_3) + \beta g(U_2, \varphi U_3)]. \tag{86}$$

for any U_2, U_3 vector fields on M. With the assistance of (30), Equation (86) reduces to (57). Thus, M is a generalized θ-Einstein manifold. Hence, we state the following:

Theorem 7. *Suppose that M is an n-dimensional φ-projectively flat LP-Kenmotsu manifold with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type (α, β) with $\alpha \neq -1$ and $\beta^2 \neq (\alpha+1)^2$, then M is a generalized θ-Einstein manifold with regard to GSM-connconnection $\nabla^{\mathcal{G}}$.*

In particular, if we choose $(\alpha, \beta) = (\alpha, 0)$ then using (86) in (81), we obtain

$$\begin{aligned} g(\mathcal{R}^{\mathcal{G}}(\varphi U_1, \varphi U_2)\varphi U_3, \varphi U_6) &= (\alpha+1)\Big[g(\varphi U_2, \varphi U_3)g(\varphi U_1, \varphi U_6) \\ &- g(\varphi U_1, \varphi U_3)g(\varphi U_2, \varphi U_6)\Big], \end{aligned} \tag{87}$$

for any U_1, U_2, U_3, U_6 vector fields on M. The converse part is also true. So, we define the following

Theorem 8. *Suppose that M is an n-dimensional LP-Kenmotsu manifold. The φ-projectively flat M with regard to GSM-connection of type $(\alpha, 0)$ if and only if M fulfills (87).*

Finally, we give the following statements:

Corollary 7. *Suppose that M is an n-dimensional LP-Kenmotsu manifold. Then, the underlying statements are equivalent:*
(1) M is projectively flat with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type $(\alpha, 0)$ with $\alpha \neq 1$,
(2) M is projectively semi-symmetric with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type $(\alpha, 0)$ with $\alpha \neq 1$,
(3) M is φ-projectively flat with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type $(\alpha, 0)$ with $\alpha \neq 1$,
(4) The curvature tensor with regard to GSM-connection $\nabla^{\mathcal{G}}$ of type $(\alpha, 0)$ with $\alpha \neq 1$ of M is given by

$$\mathcal{R}^{\mathcal{G}}(U_1, U_2)U_3 = (\alpha+1)[g(U_2, U_3)U_1 - g(U_1, U_3)U_2] \tag{88}$$

for any U_1, U_2, U_3 vector fields on M.

Proof. Assume that M is an n-dimensional LP-Kenmotsu manifold. From, Corollary 6, it is stated that (1) and (2) are equivalent, and also (2) implies (3) obviously. Now, we have to assume that (3) is true. In an LP-Kenmotsu manifold, using (35), we can verify

$$\begin{aligned}
& \mathcal{R}^{\mathcal{G}}(\varphi^2 U_1, \varphi^2 U_2, \varphi^2 U_3, \varphi^2 U_6) \\
= \; & \mathcal{R}^{\mathcal{G}}(U_1, U_2, U_3, U_6) \\
+ \; & (\alpha+1)[g(U_2, U_3)\theta(U_1)\theta(U_6) - g(U_1, U_3)\theta(U_2)\theta(U_6) \\
- \; & g(U_2, U_6)\theta(U_1)\theta(U_3) + g(U_1, U_6)\theta(U_2)\theta(U_3)]
\end{aligned} \tag{89}$$

for any vector fields U_1, U_2, U_3, U_6 on M. By interchanging U_1, U_2, U_3, U_6 to $\varphi^2 U_1$, $\varphi^2 U_2$, $\varphi^2 U_3$, $\varphi^2 U_6$, respectively, in (87) and using (89), we have (88). Hence, the statement (3) implies (4) satisfies. Next, assuming that the statement (4) is true. On contracting (88), it follows (77). Using (77) and (88) in (54), we arrive at the statement (1). This ends the proof. □

7. Example of an LP-Kenmotsu Manifold with Regard to the Connection $\nabla^{\mathcal{G}}$

Consider a five-dimensional manifold $M = \{(x_1, x_2, x_3, x_4, z) \in \mathbb{R}^5, z \neq 0\}$, where (x_1, x_2, x_3, x_4, z) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields,

$$e_1 = z\frac{\partial}{\partial x_1}, e_2 = z\frac{\partial}{\partial x_2}, e_3 = z\frac{\partial}{\partial x_3}, e_4 = z\frac{\partial}{\partial x_4}, e_5 = z\frac{\partial}{\partial z} = \rho$$

and which are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$\begin{aligned}
g(e_i, e_i) &= 1 \text{ for } 1 \leq i \leq 4 \text{ and } g(e_5, e_5) = -1; \\
g(e_i, e_j) &= 0 \text{ for } i \neq j, \; 1 \leq i, j \leq 5.
\end{aligned} \tag{90}$$

We define θ, a 1-form as $\theta(U_1) = g(U_1, e_5)$ for any vector field U_1 on M and let φ be the $(1,1)$-tensor field defined by

$$\varphi e_1 = -e_2, \; \varphi e_2 = -e_1, \; \varphi e_3 = -e_4, \; \varphi e_4 = -e_3, \; \varphi e_5 = 0.$$

The linearity property of φ and g yields that

$$\theta(\rho) = g(\rho, \rho) = -1, \ \varphi^2 U_1 = U_1 + \theta(U_1)\rho, \ g(\varphi U_1, \varphi U_2) = g(U_1, U_2) + \theta(U_1)\theta(U_2),$$

for any vector fields U_1, U_2 on M. Thus, for $e_5 = \rho$, the structure $(\varphi, \rho, \theta, g)$ defines a Lorentzian almost para-contact metric structure on M. Then, we have

$$\begin{aligned} &[e_i, e_j] = 0, \text{ if } i \neq j, \ 1 \leq i, \ j \leq 4, \\ &[e_i, e_5] = -e_i, \text{ for } 1 \leq i \leq 4. \end{aligned}$$

By using the well-known Koszul's formula, we obtain

$$\begin{aligned} \nabla_{e_i} e_i &= -e_5, \text{ for } 1 \leq i \leq 4, \\ \nabla_{e_i} e_5 &= -e_i, \text{ for } 1 \leq i \leq 4, \\ \nabla_{e_i} e_j &= 0, \text{ for } i \neq j, 1 \leq i, j \leq 4. \end{aligned} \tag{91}$$

Also, one can easily verify that $\nabla_{U_1}\rho = -U_1 - \theta(U_1)\rho$ and $(\nabla_{U_1}\varphi)U_2 = -g(\varphi U_1, U_2)\rho - \theta(U_2)\varphi U_1$, for any arbitrary vector field $U_1 = \sum U_{1_i} e_i$ and $U_2 = \sum U_{2_i} e_i$ on M. Therefore, $M(\varphi, \rho, \theta, g)$ is a five-dimensional LP-Kenmotsu manifold.

Now, we can make similar calculations for the connection. Using (18) in the above equations, we obtain

$$\nabla^{\mathcal{G}}_{e_1} e_1 = -(\alpha+1)e_5, \ \nabla^{\mathcal{G}}_{e_1} e_2 = \beta e_5, \ \nabla^{\mathcal{G}}_{e_1} e_3 = 0, \ \nabla^{\mathcal{G}}_{e_1} e_4 = 0, \nabla^{\mathcal{G}}_{e_1} e_5 = -(\alpha+1)e_1 + \beta e_2,$$

$$\nabla^{\mathcal{G}}_{e_2} e_1 = \beta e_5, \ \nabla^{\mathcal{G}}_{e_2} e_2 = -(\alpha+1)e_5, \ \nabla^{\mathcal{G}}_{e_2} e_3 = 0, \ \nabla^{\mathcal{G}}_{e_2} e_4 = 0, \ \nabla^{\mathcal{G}}_{e_2} e_5 = -(\alpha+1)e_2 + \beta e_1,$$

$$\nabla^{\mathcal{G}}_{e_3} e_1 = 0, \ \nabla^{\mathcal{G}}_{e_3} e_2 = 0, \ \nabla^{\mathcal{G}}_{e_3} e_3 = -(\alpha+1)e_5, \ \nabla^{\mathcal{G}}_{e_3} e_4 = \beta e_5, \ \nabla^{\mathcal{G}}_{e_3} e_5 = -(\alpha+1)e_3 + \beta e_4,$$

$$\nabla^{\mathcal{G}}_{e_4} e_1 = 0, \ \nabla^{\mathcal{G}}_{e_4} e_2 = 0, \ \nabla^{\mathcal{G}}_{e_4} e_3 = \beta e_5, \ \nabla^{\mathcal{G}}_{e_4} e_4 = -(\alpha+1)e_5, \ \nabla^{\mathcal{G}}_{e_4} e_5 = -(\alpha+1)e_4 + \beta e_3,$$

$$\nabla^{\mathcal{G}}_{e_5} e_1 = 0, \ \nabla^{\mathcal{G}}_{e_5} e_2 = 0, \ \nabla^{\mathcal{G}}_{e_5} e_3 = 0, \ \nabla^{\mathcal{G}}_{e_5} e_4 = 0, \ \nabla^{\mathcal{G}}_{e_5} e_5 = 0.$$

The relations presented above remark that $(\nabla^{\mathcal{G}}_{U_1}\varphi)U_2 = (\alpha+1)\{-g(\varphi U_1, U_2)\rho - \theta(U_2)\varphi U_1\} - \beta\{g(U_1, U_2)\rho + \theta(U_2)U_1 + 2\theta(U_1)\theta(U_2)\rho\}$ and $\nabla^{\mathcal{G}}_{U_1}\rho = (\alpha+1)\{U_1 + \theta(U_1)\rho\} - \beta\{\varphi U_1\}$, for all $e_5 = \rho$. Thus, $\nabla^{\mathcal{G}}$ is a GSM-connection on M.

We can make calculations of the components of the curvature tensor regarding the connection $\nabla^{\mathcal{G}}$ as follows:

$$\mathcal{R}^{\mathcal{G}}(e_1, e_2)e_1 = -(\alpha^2 + 2\alpha + 1 - \beta^2)e_2, \ \mathcal{R}^{\mathcal{G}}(e_1, e_2)e_2 = (\alpha^2 + 2\alpha + 1 - \beta^2)e_1,$$

$$\mathcal{R}^{\mathcal{G}}(e_1, e_3)e_1 = -(\alpha^2+2\alpha+1)e_3 + (\alpha+1)\beta e_4, \ \mathcal{R}^{\mathcal{G}}(e_1, e_3)e_3 = (\alpha^2+2\alpha+1)e_1 - (\alpha+1)\beta e_2,$$

$$\mathcal{R}^{\mathcal{G}}(e_1, e_4)e_1 = -(\alpha^2+2\alpha+1)e_4 + (\alpha+1)\beta e_3, \ \mathcal{R}^{\mathcal{G}}(e_1, e_4)e_4 = (\alpha^2+2\alpha+1)e_1 - (\alpha+1)\beta e_2,$$

$$\mathcal{R}^{\mathcal{G}}(e_1, e_5)e_1 = -(\alpha+1)e_5, \ \mathcal{R}^{\mathcal{G}}(e_1, e_5)e_5 = -(\alpha+1)e_1 + \beta e_2,$$

$$\mathcal{R}^{\mathcal{G}}(e_2, e_3)e_2 = -(\alpha^2+2\alpha+1)e_3 + (\alpha+1)\beta e_4, \ \mathcal{R}^{\mathcal{G}}(e_2, e_3)e_3 = (\alpha^2+2\alpha+1)e_2 - (\alpha+1)\beta e_1,$$

$$\mathcal{R}^{\mathcal{G}}(e_2, e_4)e_2 = -(\alpha^2+2\alpha+1)e_4 + (\alpha+1)\beta e_3, \ \mathcal{R}^{\mathcal{G}}(e_2, e_4)e_4 = (\alpha^2+2\alpha+1)e_2 - (\alpha+1)\beta e_1,$$

$$\mathcal{R}^{\mathcal{G}}(e_2, e_5)e_2 = -(\alpha+1)e_5, \ \mathcal{R}^{\mathcal{G}}(e_2, e_5)e_5 = -(\alpha+1)e_2 + \beta e_1,$$

$$\mathcal{R}^{\mathcal{G}}(e_3, e_4)e_3 = -(\alpha^2 + 2\alpha + 1 - \beta^2)e_4, \ \mathcal{R}^{\mathcal{G}}(e_3, e_4)e_4 = (\alpha^2 + 2\alpha + 1 - \beta^2)e_3,$$

$$\mathcal{R}^{\mathcal{G}}(e_3, e_5)e_3 = -(\alpha+1)e_5, \ \mathcal{R}^{\mathcal{G}}(e_3, e_5)e_5 = -(\alpha+1)e_3 + \beta e_4,$$

$$\mathcal{R}^{\mathcal{G}}(e_4, e_5)e_4 = -(\alpha+1)e_5, \ \mathcal{R}^{\mathcal{G}}(e_4, e_5)e_5 = -(\alpha+1)e_4 + \beta e_3. \tag{92}$$

We calculate the Ricci tensor with regard to the connection $\nabla^{\mathcal{G}}$ as follows:

$$\begin{aligned} \mathcal{S}^{\mathcal{G}}(e_i, e_i) &= 3\alpha^2 + 7\alpha + 4 - \beta^2, \quad \text{for } 1 \leq i \leq 4, \\ \mathcal{S}^{\mathcal{G}}(e_5, e_5) &= -4(\alpha + 1). \end{aligned} \tag{93}$$

Hence, (92) and (93) are verified through the Equations (27) and (29), respectively. Moreover, the scalar regarding the Levi–Civita connection and generalized symmetric metric connection are $r = 20$ and $r^{\mathcal{G}} = 20 + 12\alpha^2 + 32\alpha - 4\beta^2$, which also verified (32).

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