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Upper Bounds for the Remainder Term in Boole's Quadrature Rule and Applications to Numerical Analysis

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Abstract: In the current study, we compute some upper bounds for the remainder term of Boole's quadrature rule involving convex mappings. First, we build a new identity for first-order differentiable mapping, an auxiliary result to establish our required estimates. We provide several upper bounds by utilizing the identity, convexity property, and bounded property of mappings and some well-known inequalities. Moreover, based on our primary findings, we deliver applications to the means, quadrature rule, special mappings, and non-linear analysis by developing a novel iterative scheme with cubic order of convergence. To the best of our knowledge, the current study is the first attempt to derive upper bounds for Boole's scheme involving convex mappings.

Keywords: Boole's rule; convex mapping; iterative methods; inequality; quadrature rule; basins of attraction

MSC: 26A51; 26D10; 26D15; 65H05

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1. Introduction and Preliminaries

Analysis based on convex sets and mappings defined over them is a crucial branch of mathematics with a large number of applications in various domains of applied sciences. Its impact on the growth of the theory of inequalities is very significant. Several fundamental inequalities can be developed by implementing the notion of convexity and its generalized versions. Moreover, error inequalities like Ostrowski's inequality, Simpson's inequality, Newton's inequality, Milne's inequality, and Maclaurin's inequality are investigated through different approaches to establish the upper bounds for the remainder terms. It is a well-known fact that by lowering the order of derivatives and involving convex mappings, upper bounds of error inequalities of classical quadrature rules can be achieved. The tight bounds for these kinds of integral inequalities can also be obtained by increasing the order of derivatives.

First, we report the notion of convex mapping.

Definition 1. *Let $\mathcal{C} \subseteq \mathbb{R}$. Any mapping $\mathcal{S} : \mathcal{C} \to \mathbb{R}$ is said to be convex if*

$$\mathcal{S}((1-\rho_1)\varkappa_1 + \rho_1\varkappa_2) \leq (1-\rho_1)\mathcal{S}(\varkappa_1) + \rho_1\mathcal{S}(\varkappa_2), \quad \forall \varkappa_1, \varkappa_2 \in \mathcal{C},\ \rho_1 \in [0,1].$$

Now, we recall the well-known double inequality established by Hermite and Hadamard independently by making use of the convex mapping and described as follows:

Let $\mathcal{S} : [\varkappa, \varkappa_3] \to \mathbb{R}$ be a convex mapping, then

$$\mathcal{S}\left(\frac{\varkappa + \varkappa_3}{2}\right) \leq \frac{1}{\varkappa_3 - \varkappa}\int_{\varkappa}^{\varkappa_3} \mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1 \leq \frac{\mathcal{S}(\varkappa) + \mathcal{S}(\varkappa_3)}{2}.$$

This inequality is necessary and sufficient for the convexity of a mapping, and it is utilized as a criterion to check the concavity of a mapping. This computes the bounds for the convex mapping, and its left and right estimations provide us with bounds for midpoint and trapezoidal rules, respectively.

Now, we mention the error inequality of Simpson's quadrature rules and state it as follows:

Theorem 1 ([1])**.** *If $\mathcal{S} : [\varkappa, \varkappa_3] \to \mathbb{R}$ is four-times continuously differentiable on $(\varkappa, \varkappa_3)$, and $||\mathcal{S}^{(4)}||_\infty = \sup_{\varkappa_1 \in (\varkappa, \varkappa_3)} |\mathcal{S}^{(4)}| < \infty$, then*

$$\left|\frac{1}{6}\left[\mathcal{S}(\varkappa) + 4\mathcal{S}\left(\frac{\varkappa + \varkappa_3}{2}\right) + \mathcal{S}(\varkappa_3)\right] - \frac{1}{\varkappa_3 - \varkappa}\int_{\varkappa}^{\varkappa_3} \mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right| \leq \frac{1}{2880}||\mathcal{S}^{(4)}||_\infty(\varkappa_3 - \varkappa)^5.$$

In 2004, Ujevic [2] investigated the sharp inequalities of Simpson's and Ostrowski's type. In 2005, Liu et al. [3] developed the Simpson's type inequality for *nth* order differentiable mappings. In 2009, Alomari et al. [4] presented the generalization of Simpson's type inequalities using the s-convexity of the mappings. In [5], Sarikaya et al. explored the Simpson's type inequalities involving the s-convexity. For more information, see [6–10]. Ostrowski's type inequalities were comprehensively discussed in [11]. In [12], the authors studied the novel Ostrowski's inequality and its applications to Simpson rule and linear combination special means. In 2000, Hanna et al. [13] devoted their efforts to deriving the two-dimensional Ostrowski inequality through the utilization of the three-point rule. In 2002, Anastassiou [14] derived the univariate Ostrowski's type inequalities and Montgomery identities for nth-order differentiable mappings. In [15], Cortez et al. examined the new advancements in Ostrowski-type inequalities in fractal space and applications. In [16], Alomari and Dragomir impressively discussed the error estimates of various Newton–Cotes schemes together with applications. In [17], Alomari established new parametric equations and developed several upper bounds for Newton–Cotes procedures by means of function of bounded variation.

Now, we mention Boole's inequality, which is explored in the following:

Theorem 2 ([18])**.** *If $\mathcal{S} : [\varkappa, \varkappa_3] \to \mathbb{R}$ is six-times continuously differentiable on $(\varkappa, \varkappa_3)$, and $||\mathcal{S}^{(6)}||_\infty = \sup_{\varkappa_1 \in (\varkappa, \varkappa_3)} |\mathcal{S}^{(6)}| < \infty$, then*

$$\left|\frac{1}{90}\left[7\mathcal{S}(\varkappa) + 32\mathcal{S}\left(\frac{3\varkappa + \varkappa_3}{4}\right) + 12\mathcal{S}\left(\frac{\varkappa + \varkappa_3}{2}\right) + 32\mathcal{S}\left(\frac{\varkappa + 3\varkappa_3}{4}\right) + 7\mathcal{S}(\varkappa_3)\right] - \frac{1}{\varkappa_3 - \varkappa}\int_{\varkappa}^{\varkappa_3} \mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right|$$
$$\leq \frac{4(\varkappa_3 - \varkappa)^6}{945}||\mathcal{S}^{(6)}||_\infty.$$

The principal aim of the current investigation is to explore Boole's inequality involving convex mappings. It is commonly known that to derive the error terms of Boole's rule, a mapping should be six-times differentiable, and we provide the method of determining the remainder of Boole's rule for first-order differentiable. To attain the desired outcomes, we structured our article into four main parts. In the initial part of the study, we give the background and facts about the problem formulation. In the next part, we establish a new equality for first-order differentiable mappings named Boole's identity. This will play a critical role in the development of Boole's type inequalities. The identity together with the convexity of the mappings and some classical inequalities are utilized to achieve Boole's type inequalities. In the third portion of the study, we provide implications of our primary findings for the theory of means, numerical integration, special mappings, and a novel iterative method with an order of convergence of four to compute the zeros of non-linear equations. Lastly, we outline the simulations conducted to validate our outcomes.

2. Upper Bounds for Boole's Inequality

In this section, we present our primary findings. The space of integrable mappings is denoted by $L[\varkappa, \varkappa_3]$.

Lemma 1. *Suppose that $\mathcal{S} : [\varkappa, \varkappa_3] \to \mathbb{R}$ is a differentiable mapping and $\mathcal{S}' \in L[\varkappa, \varkappa_3]$, then*

$$\begin{aligned} &B(\varkappa, \varkappa_3) - \frac{1}{\varkappa_3 - \varkappa} \int_{\varkappa}^{\varkappa_3} \mathcal{S}(\varkappa_1) \mathrm{d}\varkappa_1 \\ &= (\varkappa_3 - \varkappa) \Bigg[\int_0^{\frac{1}{4}} \left(\rho_1 - \frac{7}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 + \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\rho_1 - \frac{39}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 \\ &\quad + \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\rho_1 - \frac{51}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 + \int_{\frac{3}{4}}^{1} \left(\rho_1 - \frac{83}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 \Bigg], \end{aligned} \tag{1}$$

where $\rho_1 \in [0, 1]$ and

$$B(\varkappa, \varkappa_3) = \frac{1}{90} \left[7\mathcal{S}(\varkappa) + 32\mathcal{S}\left(\frac{3\varkappa + \varkappa_3}{4} \right) + 12\mathcal{S}\left(\frac{\varkappa + \varkappa_3}{2} \right) + 32\mathcal{S}\left(\frac{\varkappa + 3\varkappa_3}{4} \right) + 7\mathcal{S}(\varkappa_3) \right].$$

Proof. Consider the right side of (1),

$$I = (\varkappa_3 - \varkappa)[I_1 + I_2 + I_3 + I_4], \tag{2}$$

where

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{4}} \left(\rho_1 - \frac{7}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 \\ &= \frac{1}{\varkappa_3 - \varkappa} \left[\frac{31}{180} \mathcal{S}\left(\frac{3\varkappa + \varkappa_3}{4} \right) + \frac{7}{90} \mathcal{S}(\varkappa) \right] - \frac{1}{(\varkappa_3 - \varkappa)^2} \int_{\varkappa}^{\frac{3\varkappa + \varkappa_3}{4}} \mathcal{S}(u) \mathrm{d}u. \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\rho_1 - \frac{39}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 \\ &= \frac{1}{\varkappa_3 - \varkappa} \left[\frac{6}{90} \mathcal{S}\left(\frac{\varkappa + \varkappa_3}{2} \right) + \frac{33}{180} \mathcal{S}\left(\frac{3\varkappa + \varkappa_3}{4} \right) \right] - \frac{1}{(\varkappa_3 - \varkappa)^2} \int_{\frac{3\varkappa + \varkappa_3}{4}}^{\frac{\varkappa + \varkappa_3}{2}} \mathcal{S}(u) \mathrm{d}u. \end{aligned}$$

$$\begin{aligned} I_3 &= \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\rho_1 - \frac{51}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 \\ &= \frac{1}{\varkappa_3 - \varkappa} \left[\frac{6}{90} \mathcal{S}\left(\frac{\varkappa + \varkappa_3}{2} \right) + \frac{33}{180} \mathcal{S}\left(\frac{\varkappa + 3\varkappa_3}{4} \right) \right] - \frac{1}{(\varkappa_3 - \varkappa)^2} \int_{\frac{\varkappa + \varkappa_3}{2}}^{\frac{\varkappa + 3\varkappa_3}{4}} \mathcal{S}(u) \mathrm{d}u. \end{aligned}$$

And

$$\begin{aligned} I_4 &= \int_{\frac{3}{4}}^{1} \left(\rho_1 - \frac{83}{90} \right) \mathcal{S}'((1-\rho_1)\varkappa + \rho_1 \varkappa_3) \mathrm{d}\rho_1 \\ &= \frac{1}{\varkappa_3 - \varkappa} \left[\frac{7}{90} \mathcal{S}(\varkappa_3) + \frac{31}{180} \mathcal{S}\left(\frac{\varkappa + 3\varkappa_3}{4} \right) \right] - \frac{1}{(\varkappa_3 - \varkappa)^2} \int_{\frac{\varkappa + 3\varkappa_3}{4}}^{\varkappa_3} \mathcal{S}(u) \mathrm{d}u. \end{aligned}$$

Now, by substituting the values of I_1, I_2, I_3, and I_4 in I, we achieve our desired result. □

Now, we conclude the first bound for Boole's rule.

Theorem 3. *Suppose all the assumptions of Lemma 1 are satisfied. If $|\mathcal{S}'|$ is a convex mapping, then*

$$\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right|\le\frac{239(\varkappa_3-\varkappa)}{6480}\left[|\mathcal{S}'(\varkappa)|+|\mathcal{S}'(\varkappa_3)|\right].$$

Proof. Using the modulus property, Lemma 1, and then taking the advantage of the convexity of $|\mathcal{S}'|$, we have

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right|\\
&\le(\varkappa_3-\varkappa)\left[\int_0^{\frac14}\left|\rho_1-\frac{7}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac14}^{\frac12}\left|\rho_1-\frac{39}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\right.\\
&\quad\left.+\int_{\frac12}^{\frac34}\left|\rho_1-\frac{51}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac34}^{1}\left|\rho_1-\frac{83}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\right]\\
&\le(\varkappa_3-\varkappa)\left[\int_0^{\frac14}\left|\rho_1-\frac{7}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac14}^{\frac12}\left|\rho_1-\frac{39}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1\right.\\
&\quad\left.+\int_{\frac12}^{\frac34}\left|\rho_1-\frac{51}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac34}^{1}\left|\rho_1-\frac{83}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1\right]\\
&=(\varkappa_3-\varkappa)\left[\int_0^{\frac{7}{90}}\left(\frac{7}{90}-\rho_1\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac{7}{90}}^{\frac14}\left(\rho_1-\frac{7}{90}\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1\right.\\
&+\int_{\frac14}^{\frac{39}{90}}\left(\frac{39}{90}-\rho_1\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac{39}{90}}^{\frac12}\left(\rho_1-\frac{39}{90}\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1\\
&\quad+\int_{\frac12}^{\frac{51}{90}}\left(\frac{51}{90}-\rho_1\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac{51}{90}}^{\frac34}\left(\rho_1-\frac{51}{90}\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1\\
&\left.+\int_{\frac34}^{\frac{83}{90}}\left(\frac{83}{90}-\rho_1\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac{83}{90}}^{1}\left(\rho_1-\frac{83}{90}\right)[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1\right].
\end{aligned}$$

After simple computations, we achieve our desired result. □

Now, we give another bound for Boole's rule by utilizing the power's mean inequality.

Theorem 4. *Suppose all the assumptions of Lemma 1 are satisfied. If $|\mathcal{S}'|^q$ is a convex mapping, then*

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right|\\
&\le(\varkappa_3-\varkappa)\left[\left(\frac{1157}{64{,}800}\right)^{1-\frac1q}\left(\frac{130{,}523}{8{,}748{,}000}|\mathcal{S}'(\varkappa)|^q+\frac{3209}{1{,}093{,}500}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac1q}\right.\\
&+\left(\frac{137}{7200}\right)^{1-\frac1q}\left(\frac{4127}{324{,}000}|\mathcal{S}'(\varkappa)|^q+\frac{1019}{162{,}000}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac1q}\\
&+\left(\frac{137}{7200}\right)^{1-\frac1q}\left(\frac{1019}{162{,}000}|\mathcal{S}'(\varkappa)|^q+\frac{4127}{324{,}000}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac1q}\\
&\left.+\left(\frac{1157}{64{,}800}\right)^{1-\frac1q}\left(\frac{3209}{1{,}093{,}500}|\mathcal{S}'(\varkappa)|^q+\frac{130{,}523}{8{,}748{,}000}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac1q}\right],
\end{aligned}$$

$q \geq 1$.

Proof. Using the modulus property, Lemma 1, and the power means inequality and then taking advantage of the convexity of $|\mathcal{S}'|$, we have

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right|\\
&\leq(\varkappa_3-\varkappa)\left[\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\right.\\
&\quad\left.+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\right]\\
&\leq(\varkappa_3-\varkappa)\left[\left(\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}}\right]\\
&\leq(\varkappa_3-\varkappa)\left[\left(\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|\mathrm{d}\rho_1\right)^{1-\frac{1}{q}}\left(\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}}\right].
\end{aligned}$$

After simple computations, we achieve our desired result. □

Next, we construct a new estimate of Boole's inequality.

Theorem 5. *Suppose all the assumptions of Lemma 1 are satisfied. If $|\mathcal{S}'|^q$ is a convex mapping, then*

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right|\\
&\leq(\varkappa_3-\varkappa)\left[\left(\frac{(14)^{1+p}+(31)^{1+p}}{(180)^{1+p}(1+p)}\right)^{\frac{1}{p}}\left(\frac{7}{32}|\mathcal{S}'(\varkappa)|^q+\frac{1}{32}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{4^{1+p}+(11)^{1+p}}{(60)^{1+p}(1+p)}\right)^{\frac{1}{p}}\left(\frac{5}{32}|\mathcal{S}'(\varkappa)|^q+\frac{3}{32}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{4^{1+p}+(11)^{1+p}}{(60)^{1+p}(1+p)}\right)^{\frac{1}{p}}\left(\frac{3}{32}|\mathcal{S}'(\varkappa)|^q+\frac{5}{32}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{(14)^{1+p}+(31)^{1+p}}{(180)^{1+p}(1+p)}\right)^{\frac{1}{p}}\left(\frac{1}{32}|\mathcal{S}'(\varkappa)|^q+\frac{7}{32}|\mathcal{S}'(\varkappa_3)|^q\right)^{\frac{1}{q}}\Bigg],$$

$\frac{1}{p}+\frac{1}{q}=1.$

Proof. Using the modulus property, Lemma 1, and Hölder's inequality and then taking advantage of the convexity of $|\mathcal{S}'|$, we have

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right| \\
&\leq(\varkappa_3-\varkappa)\Bigg[\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1 \\
&\quad+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\Bigg] \\
&\leq(\varkappa_3-\varkappa)\Bigg[\left(\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|^p\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|^p\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|^p\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|^p\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1\right)^{\frac{1}{q}}\Bigg] \\
&\leq(\varkappa_3-\varkappa)\Bigg[\left(\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|^p\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|^p\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|^p\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|^P\mathrm{d}\rho_1\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

After simple computations, we achieve our desired result. □

Now, we develop a new counterpart for the remainder of Boole's quadrature rules involving Young's inequality.

Theorem 6. *Suppose all the assumptions of Lemma 1 are satisfied. If $|\mathcal{S}'|^q$ is a convex mapping, then*

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right| \\
&\leq 2(\varkappa_3-\varkappa)\left[\frac{(14)^{1+p}+(31)^{1+p}}{(180)^{1+p}p(1+p)}+\frac{4^{1+p}+(11)^{1+p}}{(60)^{1+p}p(1+p)}+\frac{1}{4q}[|\mathcal{S}'(\varkappa)|^q+|\mathcal{S}'(\varkappa_3)|^q]\right],
\end{aligned}$$

$\frac{1}{p}+\frac{1}{q}=1.$

Proof. Using the modulus property, Lemma 1, and Young's inequality and then taking the advantage of the convexity of $|\mathcal{S}'|$, we have

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right| \\
&\le(\varkappa_3-\varkappa)\left[\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\right. \\
&\quad\left.+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\right] \\
&\le(\varkappa_3-\varkappa)\left[\frac{\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_0^{\frac{1}{4}}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1}{q}\right. \\
&\quad+\frac{\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_{\frac{1}{4}}^{\frac{1}{2}}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1}{q} \\
&\quad+\frac{\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_{\frac{1}{2}}^{\frac{3}{4}}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1}{q} \\
&\quad\left.+\frac{\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_{\frac{3}{4}}^{1}|\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|^q\mathrm{d}\rho_1}{q}\right] \\
&\le(\varkappa_3-\varkappa)\left[\frac{\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_0^{\frac{1}{4}}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1|\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1}{q}\right. \\
&\quad+\frac{\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_{\frac{1}{4}}^{\frac{1}{2}}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1|\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1}{q} \\
&\quad+\frac{\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_{\frac{1}{2}}^{\frac{3}{4}}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1|\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1}{q} \\
&\quad\left.+\frac{\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|^p\mathrm{d}\rho_1}{p}+\frac{\int_{\frac{3}{4}}^{1}[(1-\rho_1)|\mathcal{S}'(\varkappa)|^q+\rho_1|\mathcal{S}'(\varkappa_3)|^q]\mathrm{d}\rho_1}{q}\right].
\end{aligned}$$

After simple computations, we achieve our desired result. □

Theorem 7. *Suppose all the assumptions of Lemma 1 are satisfied. If $|\mathcal{S}'|$ is a convex mapping, then*

$$\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right|\le\frac{239M(\varkappa_3-\varkappa)}{3240},$$

where $|f'|\le M$, $M>0$.

Proof. Using the modulus property, Lemma 1, and then taking the advantage of the convexity of $|\mathcal{S}'|$ and bounded property of $|\mathcal{S}'|$, we have

$$\begin{aligned}
&\left|B(\varkappa,\varkappa_3)-\frac{1}{\varkappa_3-\varkappa}\int_{\varkappa}^{\varkappa_3}\mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1\right| \\
&\le(\varkappa_3-\varkappa)\left[\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\right.
\end{aligned}$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\Bigg]$$

$$\leq(\varkappa_3-\varkappa)\Bigg[\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1+\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|[(1-\rho_1)|\mathcal{S}'(\varkappa)|+\rho_1|\mathcal{S}'(\varkappa_3)|]\mathrm{d}\rho_1\Bigg]$$

$$\leq(\varkappa_3-\varkappa)\Bigg[\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1+\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right||\mathcal{S}'((1-\rho_1)\varkappa+\rho_1\varkappa_3)|\mathrm{d}\rho_1\Bigg]$$

$$=M(\varkappa_3-\varkappa)\Bigg[\int_0^{\frac{1}{4}}\left|\rho_1-\frac{7}{90}\right|\mathrm{d}\rho_1+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\rho_1-\frac{39}{90}\right|\mathrm{d}\rho_1+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\rho_1-\frac{51}{90}\right|\mathrm{d}\rho_1+\int_{\frac{3}{4}}^{1}\left|\rho_1-\frac{83}{90}\right|\mathrm{d}\rho_1\Bigg].$$

After simple computations, we achieve our desired result. □

3. Applications

3.1. Applications to Means

Here, we provide novel applications of our primary findings to linear combinations of means. Let us revisit some well-known binary means of any two non-negative numbers.

1. $A(\varkappa,\varkappa_3)=\frac{\varkappa+\varkappa_3}{2}$.
2. $A_w(w_1,w_2;\varkappa,\varkappa_3)=\frac{w_1\varkappa+w_2\varkappa_3}{w_1+w_2}$.
3. $Ł_n(\varkappa,\varkappa_3)=\left[\frac{{\varkappa_3}^{n+1}-\varkappa^{n+1}}{(\varkappa_3-\varkappa)(n+1)}\right]^{\frac{1}{n}},n\in\mathbb{Z}-\{0,-1\}$.

Proposition 1. *Suppose that all the assumptions of Theorem 3 are admitted, then we have*

$$\left|\frac{1}{90}[14A(\varkappa^n,{\varkappa_3}^n)+32A_w^n(\varkappa,\varkappa_3;3,1)+32A_w^n(\varkappa,\varkappa_3;1,3)+12A^n(\varkappa,\varkappa_3)-Ł_n^n(\varkappa,\varkappa_3)]\right|$$
$$\leq\frac{239(\varkappa_3-\varkappa)}{6480}\Big[|n\varkappa^{n-1}|+|n{\varkappa_3}^{n-1}|\Big].$$

Proof. The proof follows directly by applying $\mathcal{S}(\varkappa_1)={\varkappa_1}^n,n\geq 2$ in Theorem 3. □

Proposition 2. *Suppose that all the assumptions of Theorem 4 are admitted, then we have*

$$\left|\frac{\varkappa_3-\varkappa}{90}[14A(\varkappa^n,{\varkappa_3}^n)+32A_w^n(\varkappa,\varkappa_3;3,1)+32A_w^n(\varkappa,\varkappa_3;1,3)+12A^n(\varkappa,\varkappa_3)-Ł_n^n(a,\varkappa_3)]\right|$$
$$\leq(\varkappa_3-\varkappa)\Bigg[\left(\frac{1157}{64,800}\right)^{1-\frac{1}{q}}\left(\frac{130,523}{8,748,000}|n\varkappa^{n-1}|^q+\frac{3209}{1,093,500}|n{\varkappa_3}^{n-1}|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{137}{7200}\right)^{1-\frac{1}{q}}\left(\frac{4127}{32,400}|n\varkappa^{n-1}|^q+\frac{1019}{162,000}|n{\varkappa_3}^{n-1}|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{137}{7200}\right)^{1-\frac{1}{q}}\left(\frac{1019}{162,000}|n\varkappa^{n-1}|^q+\frac{4127}{324,000}|n{\varkappa_3}^{n-1}|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{1157}{64,800}\right)^{1-\frac{1}{q}}\left(\frac{3209}{1,093,500}|n\varkappa^{n-1}|^q+\frac{130,523}{8,748,000}|n{\varkappa_3}^{n-1}|^q\right)^{\frac{1}{q}}\Bigg],$$

where $q>0$.

Proof. The proof follows directly by applying $\mathcal{S}(\varkappa_1)={\varkappa_1}^n,n\geq 2$ in Theorem 3. □

Remark 1. *We can obtain several new error estimates between special means by considering Theorem 5 to Theorem 7.*

3.2. Error Bounds

In the current portion of the study, we establish error bounds for the composite Boole's quadrature schemes for first-order differentiable mappings.

Consider a partition $\vartheta : \varkappa = \varkappa_{10} < \varkappa_{11} < \varkappa_{12} < .... < \varkappa_{1i} < \varkappa_{1i+1} < ...\varkappa_{1n} = \varkappa_3$ of the interval $[\varkappa, \varkappa_3]$, where $[\varkappa_{1i}, \varkappa_{1i+1}]$ is any arbitrary subset of $[\varkappa, \varkappa_3]$. Let $h = \varkappa_{1i+1} - \varkappa_{1i}$.

$$\begin{aligned}
&\rho_1(\vartheta, \mathcal{S}) \\
&= \sum_{i=0}^{n-1} \frac{(\varkappa_{1i+1} - \varkappa_{1i})}{90}\left[7\mathcal{S}(\varkappa_{1i}) + 32\mathcal{S}\left(\frac{3\varkappa_{1i} + \varkappa_{1i+1}}{4}\right) + 32\mathcal{S}\left(\frac{\varkappa_{1i} + 3\varkappa_{1i+1}}{4}\right) + 12\mathcal{S}\left(\frac{\varkappa_{1i} + \varkappa_{1i+1}}{2}\right) + 7\mathcal{S}(\varkappa_{1i+1})\right]. \\
&\int_{\varkappa}^{\varkappa_3} \mathcal{S}(\varkappa_1)\mathrm{d}\varkappa_1 = \rho_1(\vartheta, \mathcal{S}) + \bar{R}(\vartheta, \mathcal{S}),
\end{aligned}$$

where $\bar{R}(\vartheta, \mathcal{S})$ is the error terms.

Proposition 3. *From Theorem 3, we have*

$$|\bar{R}(\vartheta, \mathcal{S})| \leq \sum_{i=0}^{n-1} \frac{239h^2}{6480}\left[|\mathcal{S}'(\varkappa_{1i})| + |\mathcal{S}'(\varkappa_{1i+1})|\right].$$

Proof. To acquire the desired result, we apply Theorem 3 over sub-interval $[\varkappa_{1i}, \varkappa_{1i+1}]$ and taking sum from $i = 0$ to $i = n - 1$. □

Proposition 4. *From Theorem 5, we have*

$$\begin{aligned}
|\bar{R}(\vartheta, \mathcal{S})| \leq h^2 \Bigg[&\left(\frac{(14)^{1+p} + (31)^{1+p}}{(180)^{1+p}(1+p)}\right)^{1-\frac{1}{q}} \left(\frac{7}{32}|\mathcal{S}'(\varkappa_{1i})|^q + \frac{1}{32}|\mathcal{S}'(\varkappa_{1i+1})|^q\right)^{\frac{1}{q}} \\
&+ \left(\frac{4^{1+p} + (11)^{1+p}}{(60)^{1+p}(1+p)}\right)^{1-\frac{1}{q}} \left(\frac{5}{32}|\mathcal{S}'(\varkappa_{1i})|^q + \frac{3}{32}|\mathcal{S}'(\varkappa_{1i+1})|^q\right)^{\frac{1}{q}} \\
&+ \left(\frac{4^{1+p} + (11)^{1+p}}{(60)^{1+p}(1+p)}\right)^{1-\frac{1}{q}} \left(\frac{3}{32}|\mathcal{S}'(\varkappa_{1i})|^q + \frac{5}{32}|\mathcal{S}'(\varkappa_{1i+1})|^q\right)^{\frac{1}{q}} \\
&+ \left(\frac{(14)^{1+p} + (31)^{1+p}}{(180)^{1+p}(1+p)}\right)^{1-\frac{1}{q}} \left(\frac{1}{32}|\mathcal{S}'(\varkappa_{1i})|^q + \frac{7}{32}|\mathcal{S}'(\varkappa_{1i+1})|^q\right)^{\frac{1}{q}} \Bigg],
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. To acquire the desired result, We apply Theorem 5 over sub-interval $[\varkappa_{1i}, \varkappa_{1i+1}]$ and taking sum from $i = 0$ to $i = n - 1$. □

By adopting a similar technique, several bounds can be developed from other main findings.

3.3. $\mathfrak{q}$-Digamma Function

Assume that $0 < \mathfrak{q} < 1$. The $\mathfrak{q}$-digamma mapping $\Upsilon_{\mathfrak{q}}(\mathfrak{u})$ (for further information, refer to [19]) can be expressed as

$$\Upsilon_{\mathfrak{q}}(\mathfrak{u}) = -\ln(1-\mathfrak{q}) + \ln(\mathfrak{q}) \sum_{i=0}^{\infty} \frac{\mathfrak{q}^{i+\mathfrak{u}}}{1-\mathfrak{q}^{i+\mathfrak{u}}} = -\ln(1-\mathfrak{q}) + \ln(\mathfrak{q}) \sum_{i=0}^{\infty} \frac{\mathfrak{q}^{i\mathfrak{u}}}{1-\mathfrak{q}^{i\mathfrak{u}}}.$$

If $\mathfrak{q} > 1$ and $\mathfrak{u} > 0$, the $\mathfrak{q}$-digamma mapping $\Upsilon_{\mathfrak{q}}$ can be represented as follows:

$$\begin{aligned}\Upsilon_{\mathfrak{q}}(\mathfrak{u}) &= -\ln(\mathfrak{q}-1) + \ln(\mathfrak{q})\left[\mathfrak{u} - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{\mathfrak{q}^{-(i+\mathfrak{u})}}{1-\mathfrak{q}^{-(i+\mathfrak{u})}}\right]. \\ &= -\ln(\mathfrak{q}-1) + \ln(\mathfrak{q})\left[\mathfrak{u} - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{\mathfrak{q}^{-iu}}{1-\mathfrak{q}^{-iu}}\right].\end{aligned}$$

For $\mathfrak{q} > 0$, the mapping $\Upsilon'_{\mathfrak{q}}(\mathfrak{u})$ is completely monotonic on the interval $(0,\infty)$, which implies that it is a convex mapping. These facts allow us to formulate the following significant results regarding $\mathfrak{q}$-digamma mapping.

Proposition 5. *Considering Theorem 6, we obtain*

$$\begin{aligned}&\left|\frac{\varkappa_3-\varkappa}{90}\left[7\Upsilon'_{\mathfrak{q}}(\varkappa) + 32\Upsilon'_{\mathfrak{q}}\left(\frac{3\varkappa+\varkappa_3}{4}\right) + 12\Upsilon_{\mathfrak{q}}\left(\frac{\varkappa+\varkappa_3}{2}\right) + 32\Upsilon'_{\mathfrak{q}}\left(\frac{\varkappa+3\varkappa_3}{4}\right) + 7\Upsilon'_{\mathfrak{q}}(\varkappa_3)\right] - \frac{\Upsilon_{\mathfrak{q}}(\varkappa)+\Upsilon'_{\mathfrak{q}}(\varkappa_3)}{\varkappa_3-\varkappa}\right| \\ &\leq 2(\varkappa_3-\varkappa)\left[\frac{(14)^{1+p}+(31)^{1+p}}{(180)^{1+p}p(1+p)} + \frac{4^{1+p}+(11)^{1+p}}{(60)^{1+p}p(1+p)} + \frac{1}{4q}[|\Upsilon''_{\mathfrak{q}}|^q + |\Upsilon''_{\mathfrak{q}}(\varkappa_3)|^q]\right],\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By applying $\mathcal{S}(\varkappa_1) \mapsto \Upsilon'_q(\varkappa_1)$, the desired result follows directly. □

Proposition 6. *Considering Theorem 7, we obtain*

$$\begin{aligned}&\left|\frac{\varkappa_3-\varkappa}{90}\left[7\Upsilon'_{\mathfrak{q}}(\varkappa) + 32\Upsilon'_{\mathfrak{q}}\left(\frac{3\varkappa+\varkappa_3}{4}\right) + 12\Upsilon_{\mathfrak{q}}\left(\frac{\varkappa+\varkappa_3}{2}\right) + 32\Upsilon'_{\mathfrak{q}}\left(\frac{\varkappa+3\varkappa_3}{4}\right) + 7\Upsilon'_{\mathfrak{q}}(\varkappa_3)\right] - \frac{\Upsilon_{\mathfrak{q}}(\varkappa)+\Upsilon'_{\mathfrak{q}}(\varkappa_3)}{\varkappa_3-\varkappa}\right| \\ &\leq \frac{239M(\varkappa_3-\varkappa)}{3240}.\end{aligned}$$

Proof. By applying $\mathcal{S}(\varkappa_1) \mapsto \Upsilon'_q(\varkappa_1)$, the desired result follows directly. □

3.4. Modified Bessel Functions

Suppose $\lambda_\delta : \mathbb{R} \to (0,1]$ which is defined as

$$\lambda_\delta(v) = 2^\delta \Gamma(1+\delta) v^{-\varkappa_3} I_\delta(v).$$

Now, we give an overview of modified Bessel mappings.

$$I_\delta(v) = \sum_{u \geq 0} \frac{\left(\frac{v}{2}\right)^{\delta+2u}}{u!\Gamma(\delta+u+1)}.$$

The first- and nth-order derivative formula's for $\lambda_\delta(v)$ are given as follows:

$$\lambda'_\delta(v) = \frac{v}{2(1+\delta)}\lambda_{\delta+1}(v), \qquad \frac{\partial^n \lambda_\delta}{\partial^n v} = 2^{n-2\delta}\sqrt{\pi} v^{\delta-n}\Gamma(1+\delta)_2\mathcal{S}_3\left(\frac{1+\delta}{2}, \frac{2+\delta}{2}; \frac{1+\delta-n}{2}, \frac{2+\delta-n}{2}, 1+\delta; \frac{v^2}{4}\right),$$

where ${}_2F_3(.,.,.)$ is a hypergeometric mapping which is represented as follows:

$${}_2F_3\left(\frac{1+\delta}{2}, \frac{2+\delta}{2}; \frac{1+\delta-n}{2}, \frac{2+\delta-n}{2}, (1+\delta); \frac{v^2}{4}\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1+\delta}{2}\right)_k \left(\frac{2+\delta}{2}\right)_k v^{2k}}{\left(\frac{1+\delta-n}{2}\right)_k \left(\frac{2+\delta-n}{2}\right)_k (1+\delta)_k 4^k k!}.$$

For details, see [20,21].

Proposition 7. *For* $[\varkappa, \varkappa_3] \in \mathbb{R}$, *and* $\delta > -1$ *then*

$$\left|\frac{1}{180(1+\delta)}\left[7\varkappa\lambda_{\delta+1}(\varkappa) + 32\left(\frac{3\varkappa+\varkappa_3}{4}\right)\lambda_{\delta+1}\left(\frac{3\varkappa+\varkappa_3}{4}\right) + 12\left(\frac{\varkappa+\varkappa_3}{2}\right)\lambda_{\delta+1}\left(\frac{\varkappa+\varkappa_3}{2}\right)\right.\right.$$
$$\left.\left.+32\left(\frac{\varkappa+3\varkappa_3}{4}\right)\lambda_{\delta+1}\left(\frac{\varkappa+3\varkappa_3}{4}\right) + 7\varkappa_3\lambda_{\delta+1}(\varkappa_3)\right] - \frac{\lambda_\delta(\varkappa_3) - \lambda_\delta(\varkappa)}{\varkappa_3 - \varkappa}\right|$$
$$\leq \frac{(\varkappa_3-\varkappa)2^{2-2\delta}\sqrt{\pi}\Gamma(1+\delta)}{6480}\left[329|\varkappa^{\delta-2}|\left|{}_2F_3\left(\frac{1+\delta}{2}, \frac{2+\delta}{2}; \frac{\delta-1}{2}, \frac{\delta}{2}, (1+\delta); \frac{\varkappa^2}{4}\right)\right|\right.$$
$$\left.+239|\varkappa^{\delta-2}|\left|{}_2F_3\left(\frac{1+\delta}{2}, \frac{2+\delta}{2}; \frac{\delta-1}{2}, \frac{\delta}{2}, (1+\delta); \frac{\varkappa_3{}^2}{4}\right)\right|\right].$$

Proof. To conclude our desired outcome, we apply $\mathcal{S}(\varkappa_1) = \lambda'_\delta(\varkappa_1)$ in Theorem 3. □

Proposition 8. *For* $[\varkappa, \varkappa_3] \in \mathbb{R}$, *and* $\delta > -1$, *then*

$$\left|\frac{1}{180(1+\delta)}\left[7\varkappa\lambda_{\delta+1}(\varkappa) + 32\left(\frac{3\varkappa+\varkappa_3}{4}\right)\lambda_{\delta+1}\left(\frac{3\varkappa+\varkappa_3}{4}\right) + 12\left(\frac{\varkappa+\varkappa_3}{2}\right)\lambda_{\delta+1}\left(\frac{\varkappa+\varkappa_3}{2}\right)\right.\right.$$
$$\left.\left.+32\left(\frac{\varkappa+3\varkappa_3}{4}\right)\lambda_{\delta+1}\left(\frac{\varkappa+3\varkappa_3}{4}\right) + 7\varkappa_3\lambda_{\delta+1}(\varkappa_3)\right] - \frac{\lambda_\delta(\varkappa_3) - \lambda_\delta(\varkappa)}{\varkappa_3 - \varkappa}\right|$$
$$\leq \frac{239M(\varkappa_3-\varkappa)}{3240}.$$

Proof. To conclude our desired outcome, we apply $\mathcal{S}(\varkappa_1) = \lambda'_\delta(\varkappa_1)$ in Theorem 7. □

4. Visual Demonstration

In the current part of the investigation, we present some visuals to validate our primary results, which provide the bounds for Boole's rule.

First, we demonstrate a graphical explanation of Theorem 3.

Example 1. *For* $\mathcal{S}(\varkappa_1) = \frac{m}{r+2m}\varkappa_1{}^{\frac{r}{m}+2}, r \geq 1$ *&* $m > 1$ *and* $[\varkappa, \varkappa_3] = [0, 2]$, *then Theorem 3 can be written as follows:*

$$\left|\frac{m}{90(r+2m)}\left[32\left((0.5)^{\frac{r+2m}{m}} + (1.5)^{\frac{r+2m}{m}}\right) + 7(2)^{\frac{r+2m}{m}} + 12\right] - \frac{m^2(2)^{\frac{r+2m}{m}}}{(r+2m)(r+3m)}\right|$$
$$\leq \frac{239}{1620}|2^{\frac{r+m}{m}}|.$$

For Figure 1a–c, we select $r, m \in [1, 5]$, $r \in [1, 5]$ and $m \in [1, 10]$, respectively, as unknown to construct the visual demonstration of left and right side of Theorem 3.

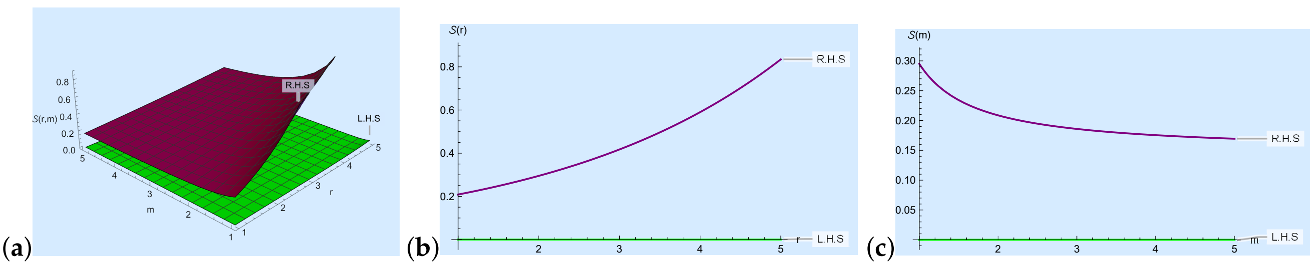


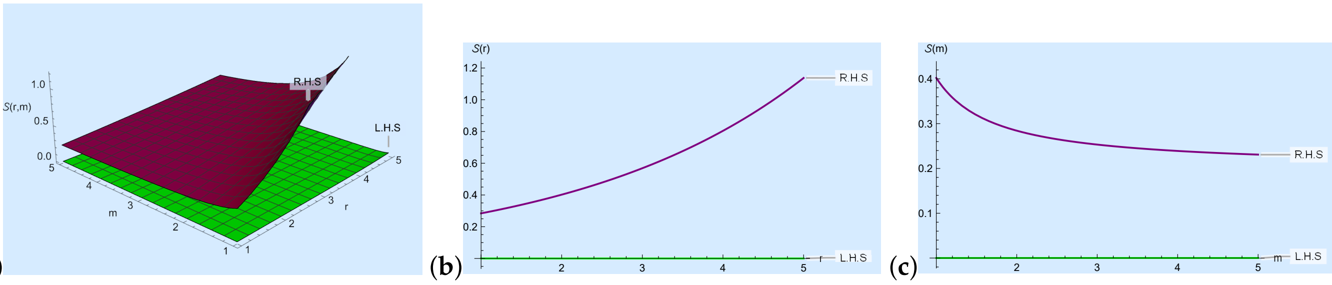
Figure 1. Graphical illustration of **left** (green) and **right** (purple) sides of Theorem 3.

Figure 1a validate **left** (green) and **right** (purple) sides of Theorem 3 for $r, m \in [1,5]$. Figure 1b validate **left** (green) and **right** (purple) sides of Theorem 3 for $r \in [1,5]$. Figure 1c validate **left** (green) and **right** (purple) sides of Theorem 3 for $m \in [1,5]$.

Example 2. *For* $\mathcal{S}(\varkappa_1) = \frac{m}{r+2m}\varkappa_1^{\frac{r}{m}+2}, r \geq 1$ *&* $m > 1, q = 2$ *and* $[\varkappa, \varkappa_3] = [0,2]$*, then Theorem 4 can be written as follows:*

$$\left|\frac{m}{90(r+2m)}\left[32\left((0.5)^{\frac{r+2m}{m}}+(1.5)^{\frac{r+2m}{m}}\right)+7(2)^{\frac{r+2m}{m}}+12\right]-\frac{m^2(2)^{\frac{r+2m}{m}}}{(r+2m)(r+3m)}\right|$$
$$\leq 2^{\frac{r+2m}{m}}\left[\sqrt{\frac{1157}{64,800}}\left(\sqrt{\frac{130,523}{8,748,000}}+\sqrt{\frac{3209}{10,393,500}}\right)+\sqrt{\frac{137}{7200}}\left(\sqrt{\frac{1019}{162,000}}+\sqrt{\frac{4127}{324,000}}\right)\right].$$

For Figure 2a–c, we select $r, m \in [1,5]$, $r \in [1,5]$ and $m \in [1,10]$, respectively, as unknown to construct the visual demonstration of left and right side of Theorem 4.



(**a**) (**b**) (**c**)

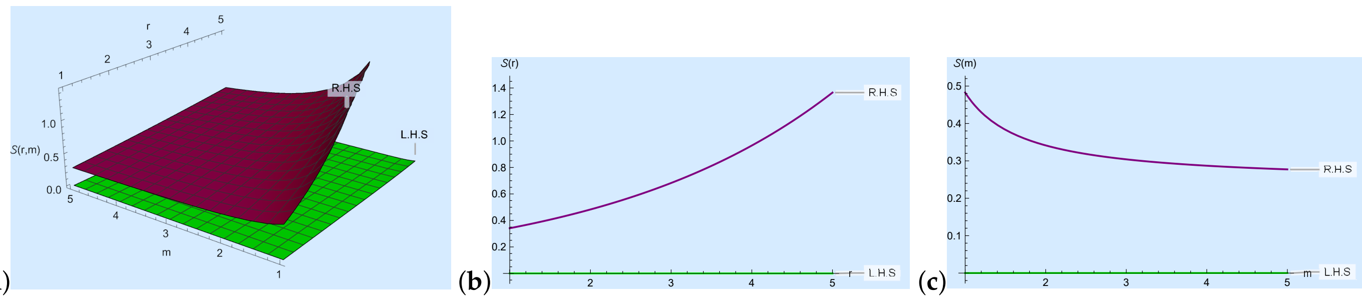
Figure 2. Graphical illustration of **left** (green) and **right** (purple) sides of Theorem 4.

Figure 2a validate **left** (green) and **right** (purple) sides of Theorem 3 for $r, m \in [1,5]$. Figure 2b validate **left** (green) and **right** (purple) sides of Theorem 3 for $r \in [1,5]$. Figure 2c validate **left** (green) and **right** (purple) sides of Theorem 3 for $m \in [1,5]$.

Example 3. *For* $\mathcal{S}(\varkappa_1) = \frac{m}{r+2m}\varkappa_1^{\frac{r}{m}+2}, r \geq 1$ *&* $m > 1, p = 2 = q$ *and* $[\varkappa, \varkappa_3] = [0,2]$*, then Theorem 5 can be written as follows:*

$$\left|\frac{m}{90(r+2m)}\left[32\left((0.5)^{\frac{r+2m}{m}}+(1.5)^{\frac{r+2m}{m}}\right)+7(2)^{\frac{r+2m}{m}}+12\right]-\frac{m^2(2)^{\frac{r+2m}{m}}}{(r+2m)(r+3m)}\right|$$
$$\leq 2^{\frac{r+3m}{m}}\left[\sqrt{\frac{241}{129,600}}\left(\sqrt{\frac{1}{32}}+\sqrt{\frac{7}{32}}\right)+\sqrt{\frac{31}{14,400}}\left(\sqrt{\frac{3}{32}}+\sqrt{\frac{5}{32}}\right)\right].$$

For Figure 3a–c, we select $r, m \in [1,5]$, $r \in [1,5]$ and $m \in [1,10]$, respectively, as unknown to construct the visual demonstration of left and right side of Theorem 5.



(**a**) (**b**) (**c**)

Figure 3. Graphical illustration of **left** (green) and **right** (purple) sides of Theorem 5.

Figure 3a validate **left** (green) and **right** (purple) sides of Theorem 3 for $r, m \in [1,5]$. Figure 3b validate **left** (green) and **right** (purple) sides of Theorem 3 for $r \in [1,5]$. Figure 3c validate **left** (green) and **right** (purple) sides of Theorem 3 for $m \in [1,5]$.

Example 4. *For* $\mathcal{S}(\varkappa_1) = \frac{m}{r+2m}\varkappa_1^{\frac{r}{m}+2}, r \geq 1$ *&* $m > 1, p = 2 = q$ *and* $[\varkappa, \varkappa_3] = [0,2]$, *then Theorem 6 can be written as follows:*

$$\left| \frac{m}{90(r+2m)}\left[32\left((0.5)^{\frac{r+2m}{m}} + (1.5)^{\frac{r+2m}{m}}\right) + 7(2)^{\frac{r+2m}{m}} + 12\right] - \frac{m^2(2)^{\frac{r+2m}{m}}}{(r+2m)(r+3m)} \right|$$
$$\leq \frac{104}{6480} + (2)^{\frac{2r+2m}{m}}.$$

For Figure 4a–c, we select $r, m \in [1,5]$, $r \in [1,5]$ and $m \in [1,10]$, respectively, as unknown to construct the visual demonstration of left and right side of Theorem 6.

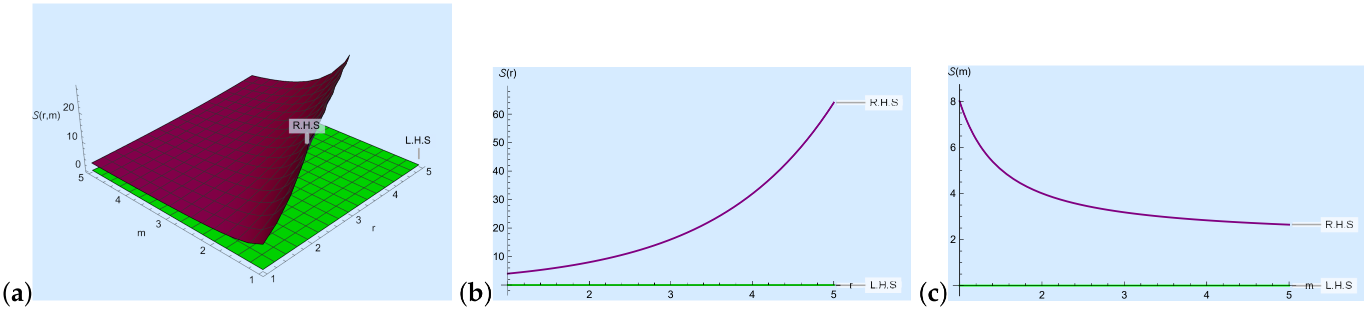


Figure 4. Graphical illustration of **left** (green) and **right** (purple) sides of Theorem 6.

Figure 4a validate **left** (green) and **right** (purple) sides of Theorem 3 for $r, m \in [1,5]$. Figure 4b validate **left** (green) and **right** (purple) sides of Theorem 3 for $r \in [1,5]$. Figure 4c validate **left** (green) and **right** (purple) sides of Theorem 3 for $m \in [1,5]$.

Example 5. *For* $\mathcal{S}(\varkappa_1) = \frac{m}{r+2m}\varkappa_1^{\frac{r}{m}+2}, r \geq 1$ *&* $m > 1$ *and* $[\varkappa, \varkappa_3] = [0,2]$, *then Theorem 7 can be written as follows:*

$$\left| \frac{m}{90(r+2m)}\left[32\left((0.5)^{\frac{r+2m}{m}} + (1.5)^{\frac{r+2m}{m}}\right) + 7(2)^{\frac{r+2m}{m}} + 12\right] - \frac{m^2(2)^{\frac{r+2m}{m}}}{(r+2m)(r+3m)} \right|$$
$$\leq \frac{239}{1620}|2^{\frac{r+m}{m}}|.$$

For Figure 5a–c, we select $r, m \in [1,5]$, $r \in [1,5]$ and $m \in [1,10]$, respectively, as unknown to construct the visual demonstration of left and right side of Theorem 7.

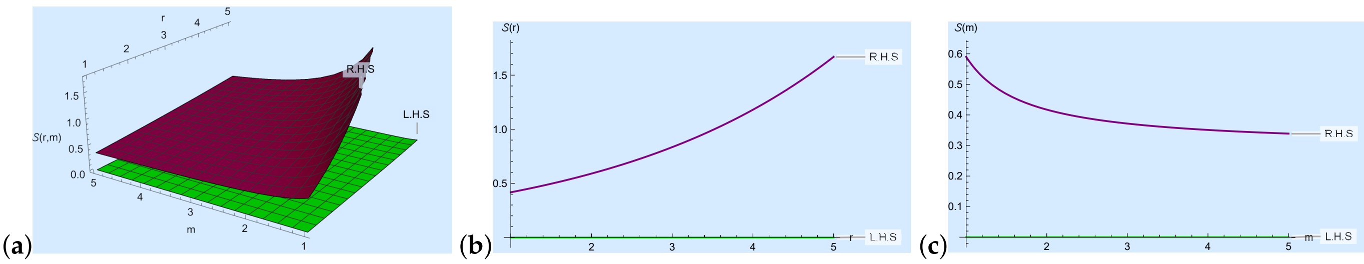


Figure 5. Graphical illustration of **left** (green) and **right** (purple) sides of Theorem 7.

Figure 5a validate **left** (green) and **right** (purple) sides of Theorem 3 for $r, m \in [1,5]$. Figure 5b validate **left** (green) and **right** (purple) sides of Theorem 3 for $r \in [1,5]$. Figure 5c validate **left** (green) and **right** (purple) sides of Theorem 3 for $m \in [1,5]$.

4.1. Application to Numerical Scheme for Finding the Solutions of Non-Linear Equations

Now, we give novel applications to non-linear analysis. We propose a new iterative scheme to evaluate the nonlinear equations.

Consider non-linear equation

$$\mathcal{S}(\varkappa_1) = 0. \tag{3}$$

Various methods and novel techniques have been developed in the literature to compute the zeros of non-linear equations. Also, it is an interesting aspect of research in the current era. Iterative schemes, such as Newton's method, have been extensively examined. Several approaches, including quadrature formulae, Taylor's series, interpolating polynomials, and decomposition procedures, have also been identified. Based on the significance of iterative methods, we propose a new efficient scheme to solve the nonlinear equations and a few examples along with graphical explanations.

In [22], Weerakoon and Fernando addressed the association between quadrature and iterative strategies for Newton's indefinite integral expression. Now, we present Newton's integral form, which is described in [23] as follows:

$$\mathcal{S}(\varkappa_1) = \mathcal{S}(\varkappa_{1n}) + \int_{\varkappa_{1n}}^{\varkappa_1} \mathcal{S}'(\mu)\mathrm{d}\mu. \tag{4}$$

Example 6. *For any $[\varkappa, \varkappa_3] \subset \mathbb{R}$ such that $\mathcal{S}(\varkappa_1) = 0$ is a non-linear equation, then*

$$\varkappa_{1n+1} = \varkappa_{1n} - \frac{90\mathcal{S}(\varkappa_{1n})}{7\mathcal{S}'(\varkappa_{1n}) + 32\mathcal{S}'\left(\frac{3\varkappa_{1n}+\gamma_n}{4}\right) + 12\mathcal{S}'\left(\frac{\gamma_n+\varkappa_{1n}}{2}\right) + 32\mathcal{S}'\left(\frac{\varkappa_{1n}+3\gamma_n}{4}\right) + 7\mathcal{S}'(\gamma_n)}, \tag{5}$$

where

$$\gamma_n = \varkappa_{1n} - \frac{\mathcal{S}(\varkappa_{1n})}{\mathcal{S}'(\varkappa_{1n})}.$$

Proof. One can easily obtain the scheme utilizing Equation (4) in Theorem 7. □

Theorem 8. *Let $r \in I$ be a simple zero of sufficiently differentiable mapping $\mathcal{S}$ on I°. If $\varkappa_{1_\circ}$ is sufficiently close to r, then Example 6 exhibits a cubic order of convergence and satisfies the following error equation:*

$$\varkappa_{1n+1} = c_2^2 e_n^3 + O(e_n^4).$$

Proof. Let r be a zero of differentiable $\mathcal{S}$, by expanding $\mathcal{S}(\varkappa_{1n})$ and $\mathcal{S}'(\varkappa_{1n})$ about r, we have

$$\mathcal{S}(\varkappa_{1n}) = \mathcal{S}'(r)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \ldots], \tag{6}$$

and

$$\mathcal{S}'(\varkappa_{1n}) = \mathcal{S}'(r)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_4 e_n^4 + \ldots]. \tag{7}$$

where $c_k = \frac{1}{k!}\frac{\mathcal{S}^k(r)}{\mathcal{S}'(r)}, k = 1, 2, 3, \ldots$, where $e_n = \varkappa_{1n} - r$. Now, from Equations (6) and (7), we have

$$\gamma_n = \varkappa_{1n} - \frac{\mathcal{S}(\varkappa_{1n})}{\mathcal{S}'(\varkappa_{1n})} = [r + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + \ldots]. \tag{8}$$

This implies that

$$\mathcal{S}(\gamma_n) = \mathcal{S}'(r)[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 5c_2^3 + 3c_4)e_n^4 + \ldots]. \tag{9}$$

This implies that

$$\mathcal{S}'(\gamma_n) = \mathcal{S}'(r)[1 + 2c_2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + (-11c_2^2c_3 + 8c_2^4 + 6c_2c_4)e_n^4 + \ldots]. \quad (10)$$

Also

$$\mathcal{S}'\left(\frac{3\varkappa_{1n} + \gamma_n}{4}\right) = \mathcal{S}'(r)[c_1 + \frac{3}{2}c_1c_2e_n + c_1\left(\frac{27}{16}c_3 + \frac{c_2^2}{2}\right)e_n^2 + c_1\left(\frac{27}{16}c_4 + \frac{9}{8}c_2c_3 + 2\left(-\frac{c_2^3}{2} + \frac{c_2c_3}{2}\right)\right)e_n^3 + \ldots]. \quad (11)$$

This implies that

$$\mathcal{S}'\left(\frac{\varkappa_{1n} + \gamma_n}{2}\right) = \mathcal{S}'(r)[c_1 + c_1c_2e_n + c_1\left(\frac{3}{4}c_3 + c_2^2\right)e_n^2 + c_1\left(\frac{c_4}{2} + \frac{3}{2}c_2c_3 + 2\left(-c_2^3 + c_2c_3\right)\right)e_n^3 + \ldots]. \quad (12)$$

Also

$$\mathcal{S}'\left(\frac{\varkappa_{1n} + 3\gamma_n}{4}\right) = \mathcal{S}'(r)[c_1 + \frac{1}{2}c_1c_2e_n + c_1\left(\frac{3}{16}c_3 + \frac{3c_2^2}{2}\right)e_n^2 + c_1\left(\frac{c_4}{16} + \frac{9}{8}c_2c_3 + 2\left(-\frac{3c_2^2c_3}{2} + \frac{3c_3^2}{2}\right)\right)e_n^3 + \ldots]. \quad (13)$$

By using (8–13) in (5), we achieve

$$\varkappa_{1n+1} = c_2^2e_n^3 + O(e_n^4).$$

Hence, the result is acquired. □

4.2. Numerical Analysis

In the following subsequent portion, we demonstrate the numerical analysis of the developed Example 6. For this, we consider some physical problems.

1. We consider the Blood Rheology and Fractional Non-Linear Equations Model ([24]). To evaluate the plug flow of Casson fluids, we utilize the following non-linear fractional equation, where a drop-in rate of flow is computed by:

$$\mathcal{S}(\varkappa_1) = 1 - \frac{16}{7}\sqrt{\varkappa_1} + \frac{4}{3}\varkappa_1 - \frac{1}{21}{\varkappa_1}^4 - G,$$

   where the fall of in flow rate is measured by $G = 0.4$. Selecting the initial guess of $\varkappa_{10} = 0.1$ then employing the developed Example 6 provides the desired solution $\varkappa_1 = 0.10469865153654822811639269775$ in three iterations.
2. Now, we consider fluid permeability in Biogels ([24]). The relation between velocity and pressure in a porous medium can be visualized by the given equation:

$$\mathcal{S}(\varkappa_1) = \Re_e{\varkappa_1}^3 - 20\kappa(1 - \varkappa_1)^2,$$

   where $\Re_e = 10 \times 10^{-9}$ and $\kappa = 0.3655$. Through initial guess of $\varkappa_{10} = 2$, the developed Example 6 provides the desired root in $\varkappa_1 = 1.0000370035782964266680525774$ in 12 iterations.
3. Lastly, we formulate the root of the non-linear equation given below ([25])

$$\mathcal{S}(\varkappa_1) = \frac{\varkappa_1}{1 - \varkappa_1} - 5\log\left[\frac{0.4(1 - \varkappa_1)}{0.4 - 0.5\varkappa_1}\right] + 4.45977, \quad (14)$$

   where $\varkappa_1$ indicates the transformation of species A in a chemical reactor and $\varkappa_1 \in [0, 1]$. Implementing Example 6, we obtain the solution $\varkappa_1 = 0.75739624625375387945964129779$ after three iterations.

Now, we present the comparative study of Example 6. For this purpose, we consider the following non-linear equations:

1. $\mathcal{S}_1(\varkappa_1) = {\varkappa_1}^3 + 4{\varkappa_1}^2 - 15$;
2. $\mathcal{S}_2(\varkappa_1) = xe^{{\varkappa_1}^2} - \sin^2 \varkappa_1 + 3\cos \varkappa_1 + 5$;
3. $\mathcal{S}_3(\varkappa_1) = 10\varkappa_1 e^{-{\varkappa_1}^2} - 1$;
4. $\mathcal{S}_4(\varkappa_1) = e^{-\varkappa_1} + \cos \varkappa_1$.

We present the comparative study of our proposed Example 6 with well-known schemes such as the Newton method (NM) [26], Abbasbandy's method (AM) [27], Halley's method (HM) [26], and Chun's method (CM) [28]. To approximate the root, we take a tolerance of $\epsilon = 10^{-15}$. The subsequent termination conditions are utilized for computer Example 6:

1. $|\varkappa_{1n+1} - \varkappa_{1n}| < \epsilon$;
2. $|\mathcal{S}(\varkappa_{1n+1})| < \epsilon$.

Numerical tests were conducted on an Intel(R) Core(TM) i5 processor with 1.60 GHz and 16 GB RAM. Maple 2020 was used for coding, while graphical analysis was carried out using MATLAB 2021.

After performing the numerical tests on the software, we obtained the present tabular as well as visual illustrations of Example 6 for the above-mentioned examples.

Table 1 provides a comparison of newly proposed Example 6 with the classical method by considering examples $\mathcal{S}_1$ to $\mathcal{S}_4$.

Table 1. Comparison analysis for different examples.

Methods	$\varkappa_{10}$	IT	$\varkappa_{1n}$	$\lvert\mathcal{S}(\varkappa_{1n})\rvert$	δ	Example
NM	2	5	1.6319808055660635175	0	4.77035×10^{-14}	$\mathcal{S}_1$
AM	2	4	1.6319808055660635175	0	0	$\mathcal{S}_1$
HM	2	4	1.6319808055660635175	0	0	$\mathcal{S}_1$
CM	2	4	1.6319808055660635175	0	0	$\mathcal{S}_1$
ALG	2	4	1.6319808055660635175	0	0	$\mathcal{S}_1$
NM	-1	6	-1.2076478271309189270	4.0×10^{-19}	7.58×10^{-17}	$\mathcal{S}_2$
AM	-1	5	-1.2076478271309189270	4.0×10^{-19}	0	$\mathcal{S}_2$
HM	-1	4	-1.2076478271309189270	4.0×10^{-19}	0	$\mathcal{S}_2$
CM	-1	5	-1.2076478271309189270	4.0×10^{-19}	0	$\mathcal{S}_2$
ALG	-1	4	-1.2076478271309189270	4.0×10^{-19}	2.62456×10^{-14}	$\mathcal{S}_2$

Table 1. *Cont.*

Methods	$\varkappa_{10}$	IT	$\varkappa_{1n}$	$\lvert\mathcal{S}(\varkappa_{1n})\rvert$	δ	Example
NM	1.8	5	1.6796306104284499407	-9×10^{-20}	4.7395×10^{-15}	$\mathcal{S}_3$
AM	1.8	4	1.6796306104284499407	-9×10^{-20}	1.0×10^{-19}	$\mathcal{S}_3$
HM	1.8	4	1.6796306104284499407	-9×10^{-20}	0	$\mathcal{S}_3$
CM	1.8	4	1.6796306104284499407	2.0×10^{-19}	0	$\mathcal{S}_3$
ALG	1.8	4	1.6796306104284499407	-9×10^{-20}	0	$\mathcal{S}_3$
NM	2	4	1.7461395304080124177	6.0×10^{-20}	$1.611907606 \times 10^{-19}$	$\mathcal{S}_4$
AM	2	4	1.7461395304080124177	-6×10^{-20}	1.0×10^{-19}	$\mathcal{S}_4$
HM	2	4	1.7461395304080124176	6.0×10^{-20}	1.0×10^{-19}	$\mathcal{S}_4$
CM	2	3	1.7461395304080124177	-6×10^{-20}	4.63×10^{-17}	$\mathcal{S}_4$
ALG	2	3	1.7461395304080124177	-6×10^{-20}	$8.0159910 \times 10^{-12}$	$\mathcal{S}_4$

Next, we give the graphical demonstration of the examples under consideration based on the root values and number of iterations.

Figure 6a–d illustrate the comparative study of our proposed Example 6 with classical method by the means of root value and number of iterations for examples $\mathcal{S}_1$ to $\mathcal{S}_4$ respectively.

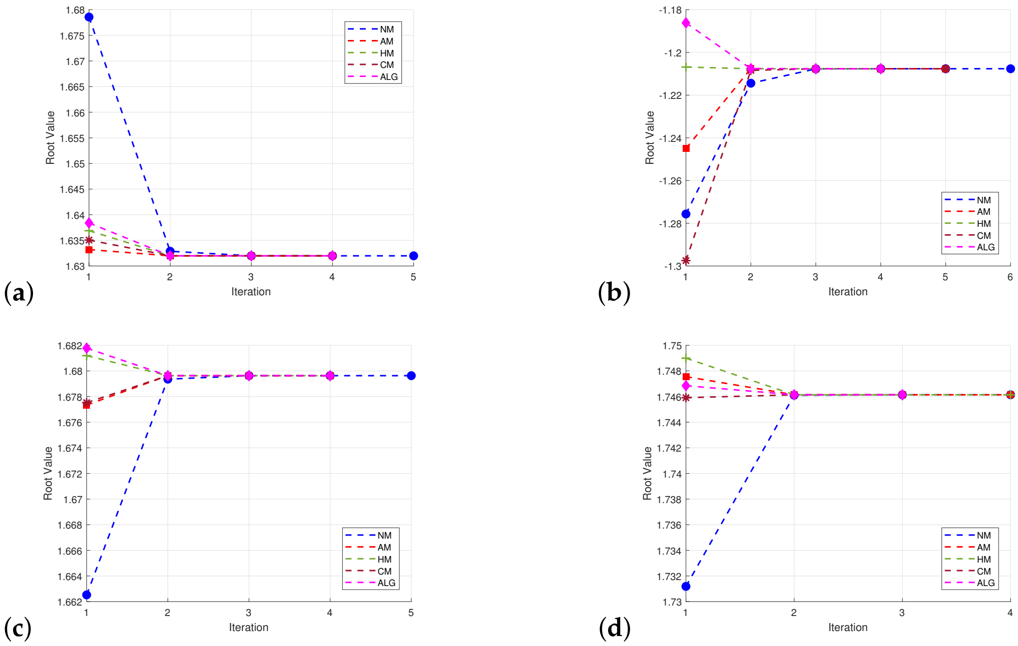


Figure 6. Comparison Analysis

4.3. Basins of Attraction

Here, we explore Example 6 by presenting the basin of attraction and visuals for CPU time to produce the basin of attraction. We implemented our developed Example 6 on $[-2,2] \times [-2,2]$ with a 500×500 points grid by selecting the tolerance $|\mathcal{S}(\varkappa_{1n})| < 1 \times 10^{-10}$ and the maximum number of iterations of 20. For this purpose, we considered the following examples.

Figure 7a presents basin of attraction for polynomial ${\varkappa_1}^2 - 1$ and Figure 7b gives the visual analysis of CPU time required to produce the basin of attraction per iteration.

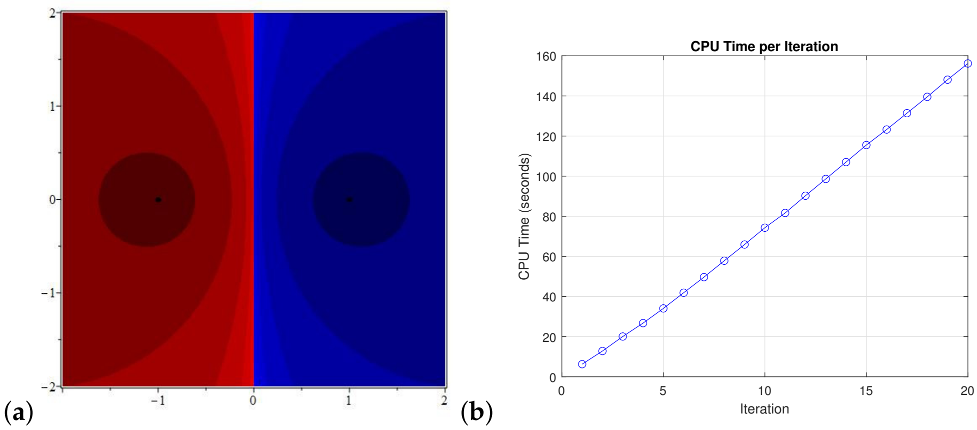


Figure 7. (**a**) Basin of attraction for ${\varkappa_1}^2 - 1$ and (**b**) the CPU time to produce the basin of attraction.

Figure 8a presents basin of attraction for polynomial ${\varkappa_1}^3 - 1$ and Figure 8b gives the visual analysis of CPU time required to produce the basin of attraction per iteration.

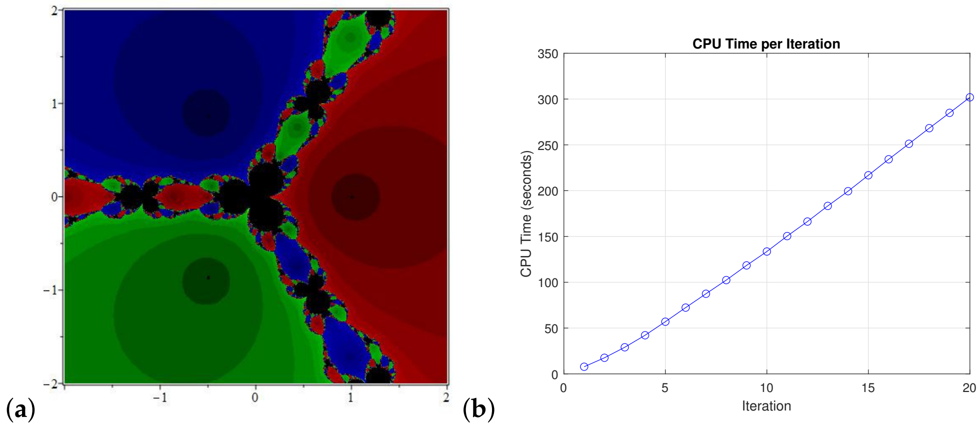


Figure 8. (**a**) Basin of attraction for ${\varkappa_1}^3 - 1$ and (**b**) the CPU time to produce the basin of attraction.

Figure 9a presents basin of attraction for polynomial ${\varkappa_1}^4 - 1$ and Figure 9b gives the visual analysis of CPU time required to produce the basin of attraction per iteration.

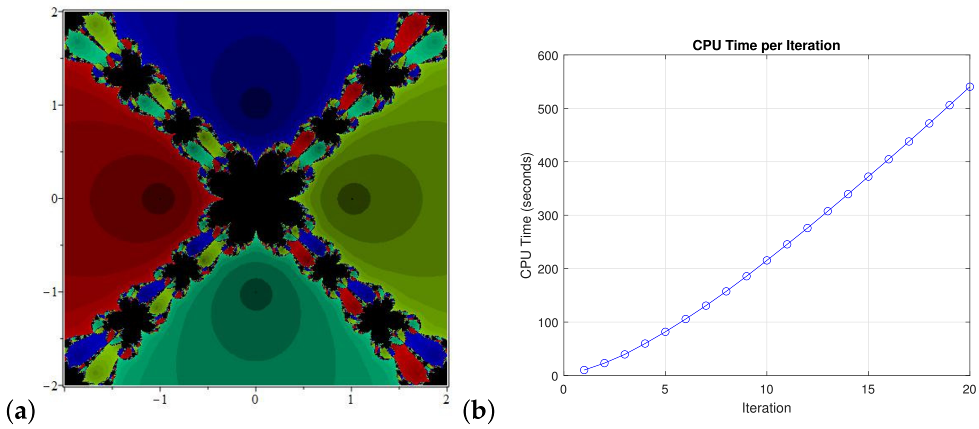


Figure 9. (**a**) Basin of attraction for ${\varkappa_1}^4 - 1$ and (**b**) the CPU time to produce the basin of attraction.

Figure 10a presents basin of attraction for polynomial ${\varkappa_1}^3 - \varkappa_1$ and Figure 10b gives the visual analysis of CPU time required to produce the basin of attraction per iteration.

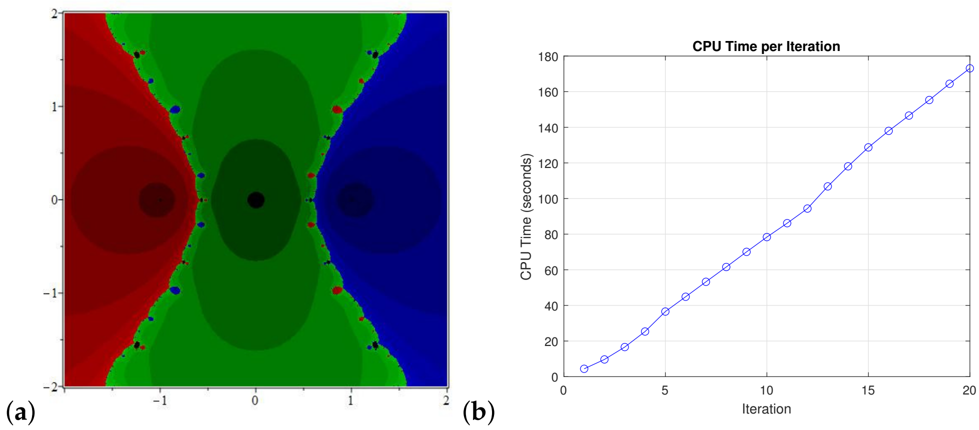


Figure 10. (**a**) Basin of attraction for ${\varkappa_1}^3 - \varkappa_1$ and (**b**) the CPU time to produce the basin of attraction.

5. Conclusions

Error inequalities of Newton–Cotes quadrature procedures have been investigated using multiple approaches to attain various upper and tight bounds. The derivation of various integral inequalities through convex functions and its generalization is a very active field of research. In the present work, we have studied Boole's type inequalities involving convex mappings. We have developed the various error estimates for Boole's inequality and their applications to the theory of means, special mappings, and error bounds to the composite Boole's rule and an iterative method to solve nonlinear equations. Also, we have provided a visual explanation of the primary findings. Moreover, we have discussed the convergence of the proposed iterative method and provided several examples in support of our proposed method. The novelty of this study is that it provides new upper bounds for Boole's error inequality for differentiable convex and bounded mappings. The results obtained in the study have a large number of applications in numerical analysis, the theory of means, etc. In the future, we will extend Boole's type inequalities for non-convex function classes in the frameworks of fractional calculus and quantum calculus.

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