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Rough and T-Rough Sets Arising from Intuitionistic Fuzzy Ideals in BCK-Algebras

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Abstract: This paper presents the novel concept of rough intuitionistic fuzzy ideals within the realm of BCK-algebras and investigates their fundamental properties. Furthermore, we introduce a set-valued homomorphism over a BCK-algebra, laying the foundation for the establishment of T-rough intuitionistic fuzzy ideals. The characterization of these innovative ideals is accomplished by employing the (α, β)-cut of intuitionistic fuzzy sets in the context of BCK-algebras.

Keywords: fuzzy set; rough set; fuzzy logic; T-rough set; intuitionistic fuzzy set; lower approximation; upper approximation; fuzzy ideal; rough intuitionistic fuzzy ideal

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1. Introduction

The foundations of rough set theory were laid by Pawlak in 1982 [1], providing a mathematical framework to handle uncertainty arising from noisy, inexact, or incomplete information. Concurrently, Zadeh introduced fuzzy set theory in 1965 [2], offering a generalization of classical set theory. In Zadeh's framework, the membership function plays a pivotal role in capturing uncertainty.

Pawlak's rough set theory, on the other hand, relies on equivalence classes of a set for upper and lower approximations. Both theories contribute to addressing uncertainty, but with different approaches. Additionally, K.T. Atanassov introduced the concept of intuitionistic fuzzy sets in 1986 [3] as a generalization of fuzzy sets, further expanding the mathematical tools available for handling uncertain information. The work of Atanassov work in 1999 [4] significantly contributed to the development of intuitionistic fuzzy set theory. In [5], Bonikowaski provided insights into the algebraic structures of rough sets. The amalgamation of rough set theory with various mathematical frameworks has proven instrumental in addressing challenges across such diverse domains as machine learning, intelligent systems, inductive reasoning, pattern recognition, knowledge discovery, decision analysis, and expert systems. This integration facilitates a comprehensive approach to handling complex problems and advancing the capabilities of these fields. In 2008, the notion of a T-rough set on a group was introduced by Davvaz [6], providing a conceptual framework that allows for the discovery of knowledge expressed through a mapping process. He also introduced the concept of a set-valued homomorphism for groups, representing a generalization of the conventional homomorphism. Moradiana et al. [7] presented a definition of the lower and upper approximation of subsets of BCK-algebras concerning a fuzzy ideal (FI). Ahn and Kim [8] introduced the concept of rough fuzzy filters in BE-algebras, Ahn and Ko [9] introduced the concept of rough ideals and rough FIs in BCK/BCI-algebras, and Hussain et al. [10] introduced the concept of rough Pythagorean FIs in semigroups, while Chinram and Panityakul [11] introduced rough Pythagorean FIs in ternary semigroups and provided several of their remarkable properties. Jun et al. [12] studied the concept of a (strong) set-valued BCK/BCI-morphism and introduced the concept of a generalized rough subalgebra (ideal) in BCK/BCI-algebras. This broader concept expands the scope and applicability of homomorphic mappings within the context of group theory. Hosseini applied the T-rough set concept to fuzzy sets, as outlined in his work [13]. This theory has been subsequently extended to encompass rings, subgroups, and semigroups [14]. The application of T-rough set theory to semigroups was further explored by Jafarzadeh and Gholami [14]. Additionally, Davvaz extended the T-rough set concept to prime and primary ideals in commutative rings [15]. This diverse set of applications showcases the versatility and adaptability of T-rough set theory across various mathematical structures. Inspired by the aforementioned investigation, the primary goal of this study is to further delve into the intricate relationships among rough sets, fuzzy sets, and BCK-algebras.

To cope with some complicated problems and widen the scope of applications, several authors have hybridized rough set theory with fuzzy set and soft set theories. These hybridizations have proven to be effective in enhancing the ability to deal with uncertainty, vagueness, and imprecision in various domains. For instance, the concept of rough bipolar fuzzy sets has been utilized to explore ideals in semigroups, offering new insights into algebraic structures [16]. Furthermore, the integration of bipolar soft information has been applied in medical decision-making techniques, providing a robust framework for handling complex decision-making scenarios [17]. Another innovative approach involves the development of decision-making techniques based on T-rough bipolar fuzzy sets, which extends the application of rough and fuzzy sets in the decision-making process [18]. These contributions illustrate the versatility and effectiveness of combining rough set theory with fuzzy and soft set frameworks to address a wide range of challenging problems. In addition, the generation of generalized approximation spaces from $\mathbb{I}_j$-neighborhoods and ideals has been explored, with applications to real-world problems such as Chikungunya disease. This approach further broadens the application of rough set theory in the medical field, demonstrating the versatility of these hybridized models in addressing complex issues [19]. These contributions illustrate the significant potential of combining rough set theory with fuzzy and soft set frameworks to tackle a wide range of challenging problems. It is acknowledged that rough set theory and its generalizations have broad scope for application; for example, the approach could be tested in areas such as medical diagnosis, data mining, or decision-making processes where rough set theory has been successfully utilized. The structure of the exploration is outlined as follows.

In Section 2, we present essential concepts related to rough sets and generalized rough sets. Section 3 delves into the examination of rough intuitionistic fuzzy ideals (IFI) within the context of BCK-algebras.

Moving on to Section 4, we introduce the concept of set-valued homomorphism and delve into some key properties of generalized lower and upper approximation operators in BCK-algebras. We define a set-valued homomorphism over a BCK-algebra, establishing the concept of T-rough IFIs. The characterization of these innovative ideals is achieved through the (α, β)-cut of IF sets in BCK-algebras.

2. Preliminaries

In this section, we revisit fundamental concepts related to fuzzy sets, FIs, IF sets, and rough sets.

In the following definition, we introduce the concept of a BCK-algebra, which is a set with a binary operation and constant satisfying specific axioms. These axioms define important properties of the algebraic structure.

Definition 1 ([20])**.** *Let Ω be a set with a binary operation $*$ and a constant 0. Then, $(\Omega, *, 0)$ is called a BCK-algebra if it satisfies the following axioms:*

1. $((u * v) * (u * w)) * (w * v) = 0$
2. $(u * (u * v)) * v = 0$
3. $u * u = 0$
4. $u * v = 0$ *and* $v * u = 0$ *imply* $u = v$

5. $0 * u = 0$

for all $u, v, w \in \Omega$.

A partial order $\leq$ *on* Ω *can be defined by* $u \leq v$ *if and only if* $u * v = 0$.

In the following definition, we define an ideal of a BCK-algebra, which is a special type of subset within the algebra that satisfies certain conditions. These conditions ensure that the subset behaves in a way that aligns with the structure of the BCK-algebra.

Definition 2 ([20])**.** *A non-empty subset* E *of a BCK-algebra* Ω *is called an ideal of* Ω *if it satisfies the following conditions:*

(I1) $0 \in E$
(I2) *If* $u * v \in E$ *and* $v \in E$, *this implies that* $u \in E$.

We denote the set of ideals by $\mathcal{I}$

In the following definition, we introduce the concept of a fuzzy set, which is characterized by a membership function. This function assigns a value between 0 and 1 to each element of the base set, representing the degree to which that element belongs to the fuzzy set.

Definition 3 ([2])**.** *A fuzzy set of a base set* Ω *is specified by its membership function, where*

$$\mu : \Omega \to [0, 1],$$

assigning to each $u \in \Omega$ *the degree or grade to which* u *belongs to* μ.

In the following definition, we introduce the concept of an intuitionistic fuzzy set (IFS). An IFS assigns both a degree of membership and a degree of non-membership to each element of the universe, with the sum of these values always lying between 0 and 1. This framework allows for a more flexible representation of uncertainty compared to traditional fuzzy sets.

Definition 4 ([3])**.** *An intuitionistic fuzzy set (IFS) in a universe* Ω *is an object of the form*

$$E = \{(u, \mu_E(u), \lambda_E(u)) \mid u \in \Omega\},$$

where the functions

$$\mu : \Omega \to [0, 1], \quad \lambda : \Omega \to [0, 1]$$

denote the degree of membership (namely, $\mu_E(u)$*) and degree of non-membership (namely,* $\lambda_E(u)$*) of each element* $u \in \Omega$ *in the set* E*, respectively, and*

$$0 \leq \mu_E(u) + \lambda_E(u) \leq 1$$

for all $u \in \Omega$. *For the sake of simplicity, we use the symbol* $E = (\mu_E, \lambda_E)$ *for the IFS* $E = \{(u, \mu_E(u), \lambda_E(u)) \mid u \in \Omega\}$.

In the following definition, we introduce the concept of a fuzzy ideal (FI) in a BCK-algebra. A fuzzy subset of a BCK-algebra is considered a fuzzy ideal if it satisfies specific conditions that relate the membership values of elements to the algebra's structure.

Definition 5 ([21])**.** *A fuzzy subset* μ *of a BCK-algebra* Ω *is said to be an FI of* Ω *if it satisfies*

(1) $\mu(0) \geq \mu(u)$
(2) $\mu(u) \geq \min(\mu(u * v), \mu(v))$

for all $u, v \in \Omega$.

In the following definition, we define an intuitionistic fuzzy ideal (IFS) of a BCK-algebra. An IFS is characterized by a pair of functions that specify the degrees of membership and non-membership, and satisfies specific conditions relating these functions to the algebra's operations.

Definition 6 ([22])**.** *An IF subset $E = (\mu_E, \lambda_E)$ of a BCK-algebra Ω is said to be an intuitionistic FI of Ω if for all $u, v \in \Omega$ we have*

(1) $\mu_E(0) \geq \mu_E(u)$
(2) $\mu_E(u) \geq \min(\mu_E(u * v), \mu_E(v))$
(3) $\lambda_E(0) \leq \lambda_E(u)$
(4) $\lambda_E(u) \leq \max(\lambda_E(u * v), \lambda_E(v))$.

In this example, we consider a BCK-algebra Ω with three elements: $\{0, 1, 2\}$. The operation $*$ is defined using the table below. We then define two functions, λ_E and μ_E, which will show that $E = (\mu_E, \lambda_E)$ is an intuitionistic fuzzy ideal in this algebra.

Example 1. *Let $\Omega = \{0, 1, 2\}$ be a BCK-algebra with the following Cayley table (Table 1).*

Table 1. Cayley table for BCK-algebra Ω.

$*$	**0**	**1**	**2**
0	0	0	0
1	1	0	0
2	2	2	0

Let $E = (\mu_E, \lambda_E)$, define $\lambda_E, \mu_E : \Omega \to [0, 1]$ as

$$\mu_E(u) = \begin{cases} 0.5 & \text{for } u \in \{0, 1\} \\ 0.3 & \text{for } u = 2 \end{cases}$$

$$\lambda_E(u) = \begin{cases} 0.3 & \text{for } u \in \{0, 1\} \\ 0.5 & \text{for } u = 2 \end{cases}$$

By simple calculation, we find that E is an intuitionistic FI.

In algebraic structures such as BCK-algebras, understanding how elements relate to each other can be facilitated by grouping them into equivalence classes. A congruence relation θ on a BCK-algebra Ω provides a way to partition the algebra into such classes. Specifically, if θ is a congruence relation, it ensures that if two elements a and b are in the same equivalence class; then, their product with any other elements remains in the same class. The equivalence classes of θ are denoted by $[a]_\theta$, which represents the class containing the element a. A congruence relation θ is termed complete if it satisfies the condition that the product of two equivalence classes is equal to the equivalence class of the product. This is formally expressed as $[ab]_\theta = [a]_\theta [b]_\theta$. The formal definition is provided below.

Definition 7 ([23])**.** *Let θ be an equivalence relation on a BCK-algebra Ω; θ is called a congruence relation on Ω if $(a, b) \in \theta$ implies $(a * u, b * v) \in \theta$ for all $u, v \in \Omega$. A congruence relation θ is called complete on Ω if $[ab]_\theta = [a]_\theta [b]_\theta$.*

In this definition, we describe the lower and upper approximations of a set in rough set theory. The lower approximation includes elements that are definitely in the set, while the upper approximation contains elements that might be in the set. If both approximations are the same, then the set is definable; otherwise, it is a rough set. The formal definitions are as follows:

Definition 8 ([1]). *Let (U,θ) be an approximation space, where U is the non-empty universe and θ is an equivalence relation. Let Ω be any non-empty subset of U. Then, the sets*

$$\theta_\bullet(\Omega) = \{u \in U : [\,u\,]_\theta \subseteq \Omega\}$$

and

$$\theta^\bullet(\Omega) = \{u \in U : [\,u\,]_\theta \cap \Omega \neq \emptyset\}$$

are respectively called the lower approximation and upper approximation of the set Ω with respect to θ, where $[\,u\,]_\theta$ denotes the equivalent class containing the elements $u \in \Omega$ with respect to θ.

Ω is called θ-definable if $\theta_\bullet(\Omega) = \theta^\bullet(\Omega)$. If $\theta_\bullet(\Omega) \neq \theta^\bullet(\Omega)$, then Ω is called a rough set with respect to θ.

In the following proposition, we explore the relationship between intuitionistic fuzzy sets and congruence relations on a BCK-algebra. The proposition describes how these sets interact with congruence relations, and includes properties of the intersection and union of intuitionistic fuzzy sets under these relations.

Proposition 1. *Let θ be a congruence relation on a BCK-algebra Ω and let $E = (\mu_E, \lambda_E)$ and $B = (\mu_B, \lambda_B)$ be two IF-sets of Ω. Then,*

(1) $\theta_\bullet(E) \subseteq E \subseteq \theta^\bullet(E)$
(2) *If $E \subseteq B$, then $\theta_\bullet(E) \subseteq \theta_\bullet(B)$*
(3) *If $E \subseteq B$, then $\theta^\bullet(E) \subseteq \theta^\bullet(B)$*
(4) $\theta_\bullet(E \cap B) = \theta_\bullet(E) \cap \theta_\bullet(B)$
(5) $\theta_\bullet(E \cup B) = \theta_\bullet(E) \cup \theta_\bullet(B)$.

Proof. **(1)** To show $\theta_\bullet(E) \subseteq E \subseteq \theta^\bullet(E)$:

(a) $\theta_\bullet(E) \subseteq E$: By definition, $\theta_\bullet(E)$ consists of elements $u \in \Omega$ such that $\theta(u,v) \in E$ for all $v \in \Omega$. Because E is an IF-set, $u \in \theta_\bullet(E)$ implies $\theta(u,v) \in E$ for all v; thus, u must be in E.
(b) $E \subseteq \theta^\bullet(E)$: For any $u \in E$, $\theta(u,v) \in E$ implies $\theta(u,v) \in \theta^\bullet(E)$ for all v; thus, $E \subseteq \theta^\bullet(E)$.

(2) To show if $E \subseteq B$, then $\theta_\bullet(E) \subseteq \theta_\bullet(B)$: If $E \subseteq B$, then for any $u \in \theta_\bullet(E)$, $\theta(u,v) \in E$ for all v. Because $E \subseteq B$, $\theta(u,v) \in B$ for all v, we have $u \in \theta_\bullet(B)$. Hence, $\theta_\bullet(E) \subseteq \theta_\bullet(B)$.

(3) To show if $E \subseteq B$, then $\theta^\bullet(E) \subseteq \theta^\bullet(B)$: If $E \subseteq B$, then for any $u \in \theta^\bullet(E)$, $\theta(u,v) \in \theta^\bullet(E)$ implies $\theta(u,v) \in E$ for all v. Given $E \subseteq B$, it follows that $\theta(u,v) \in \theta^\bullet(B)$. Hence, $\theta^\bullet(E) \subseteq \theta^\bullet(B)$.

(4) To show $\theta_\bullet(E \cap B) = \theta_\bullet(E) \cap \theta_\bullet(B)$: For $u \in \theta_\bullet(E \cap B)$, $\theta(u,v) \in E \cap B$ for all v, which means $\theta(u,v) \in E$ and $\theta(u,v) \in B$. Thus, $u \in \theta_\bullet(E)$ and $u \in \theta_\bullet(B)$, meaning that $u \in \theta_\bullet(E) \cap \theta_\bullet(B)$. Hence, $\theta_\bullet(E \cap B) = \theta_\bullet(E) \cap \theta_\bullet(B)$.

(5) To show $\theta_\bullet(E \cup B) = \theta_\bullet(E) \cup \theta_\bullet(B)$: For $u \in \theta_\bullet(E \cup B)$, $\theta(u,v) \in E \cup B$ for all v, which means $\theta(u,v) \in E$ or $\theta(u,v) \in B$. Therefore, $u \in \theta_\bullet(E)$ or $u \in \theta_\bullet(B)$; thus, $u \in \theta_\bullet(E) \cup \theta_\bullet(B)$. Hence, $\theta_\bullet(E \cup B) = \theta_\bullet(E) \cup \theta_\bullet(B)$. □

3. Rough Intuitionistic Fuzzy Ideals in BCK-Algebras

In this section, we delve into the world of rough intuitionistic FIs in BCK-algebras. Following the groundwork laid in the previous sections, this exploration aims to unravel the unique properties and significance of these ideals within the context of mathematical structures. Our focus is on understanding their characteristics and contributions, providing valuable insights into their role within the algebraic framework of BCK-algebras.

Definition 9. *Let θ be a congruence relation on Ω and let $E = (\mu_E, \lambda_E)$ be an IF set of Ω. Then, the IF sets*

$$\theta_\bullet(E) = (\theta_\bullet(\mu_E), \theta_\bullet(\lambda_E))$$

and

$$\theta^{\bullet}(E) = (\theta^{\bullet}(\mu_E), \theta^{\bullet}(\lambda_E))$$

are called the θ-lower and θ-upper approximations of the IF set E, where

$$\theta^{\bullet}(\mu_E)(u) = \bigvee_{a \in [\,u\,]_\theta} \mu_E(a)$$

$$\theta^{\bullet}(\lambda_E)(u) = \bigwedge_{a \in [\,u\,]_\theta} \lambda_E(a)$$

$$\theta_{\bullet}(\mu_E)(u) = \bigwedge_{a \in [\,u\,]_\theta} \mu_E(a)$$

$$\theta_{\bullet}(\lambda_E)(u) = \bigvee_{a \in [\,u\,]_\theta} \lambda_E(a)$$

for all $u \in \Omega$.

We call $\theta(E) = (\theta_{\bullet}(E), \theta^{\bullet}(E))$ *a rough intuitionistic FI if* $\theta_{\bullet}(E)$ *and* $\theta^{\bullet}(E)$ *are intuitionistic FIs.*

In the study of BCK-algebras, the exploration of congruence relations and their influence on algebraic structures is of significant importance. In particular, intuitionistic fuzzy ideals (FI) within BCK-algebras play a crucial role in understanding the algebraic properties and relationships within these systems. The following theorem establishes a key result regarding the behavior of intuitionistic fuzzy ideals under the action of a complete congruence relation. Specifically, it demonstrates that if an intuitionistic fuzzy ideal is defined on a BCK-algebra, applying a complete congruence relation to this ideal results in another intuitionistic fuzzy ideal. This finding provides valuable insights into the stability and preservation of intuitionistic fuzzy ideals under congruence relations within BCK-algebras, further enriching the theory and applications of these algebraic structures.

Theorem 1. *Let* θ *be a complete congruence relation on a BCK-algebra* Ω*. If* E *is an intuitionistic fuzzy ideal (FI) of* Ω*, then* $\theta^{\bullet}(E)$ *is an intuitionistic fuzzy ideal of* Ω*.*

Proof. Let $E = (\mu_E, \lambda_E)$ be an intuitionistic fuzzy ideal of Ω. Then, $\theta^{\bullet}(E) = (\theta^{\bullet}(\mu_E), \theta^{\bullet}(\lambda_E))$, where $\theta^{\bullet}(\mu_E)$ and $\theta^{\bullet}(\lambda_E)$ represent the upper approximations of μ_E and λ_E with respect to the congruence relation θ. We need to show that $\theta^{\bullet}(E)$ satisfies the conditions of an intuitionistic fuzzy ideal.

First, we check the conditions for the membership function μ_E. Because E is an intuitionistic fuzzy ideal, we know that $\mu_E(0) \geq \alpha$ for some $\alpha \in [0,1]$. For the upper approximation $\theta^{\bullet}(\mu_E)$, we have

$$\theta^{\bullet}(\mu_E)(0) = \bigvee_{w \in [\,0\,]_\theta} \mu_E(w) \geq \mu_E(0).$$

Because $[\,0\,]_\theta$ is the equivalence class containing 0 with respect to θ, this shows that $\theta^{\bullet}(\mu_E)(0) \geq \mu_E(0)$. Similarly, for any $u \in \Omega$, we have

$$\theta^{\bullet}(\mu_E)(u) = \bigvee_{w \in [\,u\,]_\theta} \mu_E(w),$$

which ensures that the upper approximation $\theta^{\bullet}(\mu_E)$ behaves consistently with respect to μ_E.

Next, we need to verify the closure condition under the binary operation $*$ of the BCK-algebra. For any $u, v \in \Omega$, we have

$$\theta^{\bullet}(\mu_E)(u) \geq \bigvee_{u*v \in [\, u*v \,]_{\theta}, v \in [\, v \,]_{\theta}} \min(\mu_E(u*v), \mu_E(v)).$$

By applying the properties of upper approximation and the fact that E is an intuitionistic fuzzy ideal, we obtain

$$= \min\left(\bigvee_{u*v \in [\, u*v \,]_{\theta}} \mu_E(u*v), \bigvee_{v \in [\, v \,]_{\theta}} \mu_E(v) \right) = \min(\theta^{\bullet}(\mu_E)(u*v), \theta^{\bullet}(\mu_E)(v)).$$

This shows that the membership function $\theta^{\bullet}(\mu_E)$ respects the closure under the operation $*$, meaning that $\theta^{\bullet}(E)$ satisfies the required property for an intuitionistic fuzzy ideal with respect to μ_E.

Now, for the non-membership function λ_E, we similarly need to verify that the upper approximation $\theta^{\bullet}(\lambda_E)$ satisfies the necessary conditions. First, we check the condition for $0 \in \Omega$:

$$\theta^{\bullet}(\lambda_E)(0) = \bigwedge_{w \in [\, 0 \,]_{\theta}} \lambda_E(w) \leq \lambda_E(0).$$

For any $u \in \Omega$, we have

$$\theta^{\bullet}(\lambda_E)(u) = \bigwedge_{w \in [\, u \,]_{\theta}} \lambda_E(w).$$

Now, for the closure under $*$ we need to show that for any $u, v \in \Omega$ we have

$$\theta^{\bullet}(\lambda_E)(u) \leq \bigwedge_{u*v \in [\, u*v \,]_{\theta}, v \in [\, v \,]_{\theta}} \max(\lambda_E(u*v), \lambda_E(v)).$$

Again, from the properties of upper approximation and the fact that E is an intuitionistic fuzzy ideal, we obtain

$$= \max\left(\bigwedge_{u*v \in [\, u*v \,]_{\theta}} \lambda_E(u*v), \bigwedge_{v \in [\, v \,]_{\theta}} \lambda_E(v) \right) = \max(\theta^{\bullet}(\lambda_E)(u*v), \theta^{\bullet}(\lambda_E)(v)).$$

This proves that $\theta^{\bullet}(\lambda_E)$ satisfies the required condition for the non-membership function of an intuitionistic fuzzy ideal.

Thus, $\theta^{\bullet}(E)$ is an intuitionistic fuzzy ideal of Ω. □

In the realm of BCK-algebras, understanding the behavior of intuitionistic fuzzy ideals (FI) under various transformations is crucial for a deeper comprehension of these algebraic structures. The following theorem addresses the impact of a complete congruence relation on an intuitionistic fuzzy ideal within a BCK-algebra. Specifically, it asserts that if we have an intuitionistic fuzzy ideal and apply a complete congruence relation to it in a particular manner, the resulting structure remains an intuitionistic fuzzy ideal. This theorem further elucidates how congruence relations preserve the properties of intuitionistic fuzzy ideals, thereby contributing to the theoretical framework of BCK-algebras and their applications in both mathematical and practical contexts.

Theorem 2. *Let θ be a complete congruence relation on a BCK-algebra Ω. If E is an intuitionistic fuzzy ideal (FI) of Ω, then $\theta_{\bullet}(E)$ is an intuitionistic fuzzy ideal of Ω.*

Proof. Let $E = (\mu_E, \lambda_E)$ be an intuitionistic fuzzy ideal of Ω. Then, $\theta_{\bullet}(E) = (\theta_{\bullet}(\mu_E), \theta_{\bullet}(\lambda_E))$, where $\theta_{\bullet}(\mu_E)$ and $\theta_{\bullet}(\lambda_E)$ represent the lower approximations of μ_E and λ_E with respect to the congruence relation θ. We need to show that $\theta_\bullet(E)$ satisfies the conditions of an intuitionistic fuzzy ideal.

First, we check the conditions for the membership function μ_E. Because E is an intuitionistic fuzzy ideal, we know that $\mu_E(0) \geq \alpha$ for some $\alpha \in [0,1]$. For the lower approximation $\theta_\bullet(\mu_E)$, we have

$$\theta_\bullet(\mu_E)(0) = \bigwedge_{w \in [0]_\theta} \mu_E(w) \geq \mu_E(0).$$

Because $[0]_\theta$ is the equivalence class containing 0 with respect to θ, this shows that $\theta_\bullet(\mu_E)(0) \geq \mu_E(0)$. Similarly, for any $u \in \Omega$, we have

$$\theta_\bullet(\mu_E)(u) = \bigwedge_{w \in [u]_\theta} \mu_E(w),$$

which ensures that the lower approximation $\theta_\bullet(\mu_E)$ behaves consistently with respect to μ_E.

Next, we need to verify the closure condition under the binary operation $*$ of the BCK-algebra. For any $u, v \in \Omega$, we have

$$\theta_\bullet(\mu_E)(u) \geq \bigwedge_{u \in [u]_\theta} \mu_E(u) \geq \bigwedge_{u*v \in [u*v]_\theta, v \in [v]_\theta} \min(\mu_E(u*v), \mu_E(v)).$$

By applying the properties of lower approximation and from the fact that E is an intuitionistic fuzzy ideal, we obtain

$$= \min\left(\bigwedge_{u*v \in [u*v]_\theta} \mu_E(u*v), \bigwedge_{v \in [v]_\theta} \mu_E(v) \right) = \min(\theta_\bullet(\mu_E)(u*v), \theta_\bullet(\mu_E)(v)).$$

This shows that the membership function $\theta_\bullet(\mu_E)$ respects the closure under the operation $*$, meaning that $\theta_\bullet(E)$ satisfies the required property for an intuitionistic fuzzy ideal with respect to μ_E.

Now, for the non-membership function λ_E, we similarly need to verify that the lower approximation $\theta_\bullet(\lambda_E)$ satisfies the necessary conditions. First, we check the condition for $0 \in \Omega$:

$$\theta_\bullet(\lambda_E)(0) = \bigvee_{w \in [0]_\theta} \lambda_E(w) \leq \lambda_E(0).$$

For any $u \in \Omega$, we have

$$\theta_\bullet(\lambda_E)(u) = \bigvee_{w \in [u]_\theta} \lambda_E(w).$$

Now, for the closure under $*$ we need to show that for any $u, v \in \Omega$ we have

$$\theta_\bullet(\lambda_E)(u) \leq \bigvee_{u \in [u]_\theta} \lambda_E(u) \leq \bigvee_{u*v \in [u*v]_\theta, v \in [v]_\theta} \max(\lambda_E(u*v), \lambda_E(v)).$$

Again, from the properties of lower approximation and the fact that E is an intuitionistic fuzzy ideal, we obtain

$$= \max\left(\bigvee_{u*v \in [u*v]_\theta} \lambda_E(u*v), \bigvee_{v \in [v]_\theta} \lambda_E(v) \right) = \max(\theta_\bullet(\lambda_E)(u*v), \theta_\bullet(\lambda_E)(v)).$$

This proves that $\theta_\bullet(\lambda_E)$ satisfies the required condition for the non-membership function of an intuitionistic fuzzy ideal.

Thus, $\theta_\bullet(E)$ is an intuitionistic fuzzy ideal of Ω. □

In this example, we define a congruence relation θ on the BCK-algebra Ω, as shown below. Using this relation, we derive new functions $\theta^{\bullet}(\mu_E)$, $\theta^{\bullet}(\lambda_E)$, $\theta_{\bullet}(\mu_E)$, and $\theta_{\bullet}(\lambda_E)$. These functions help us demonstrate that E is a rough set and that both $\theta^{\bullet}(E)$ and $\theta_{\bullet}(E)$ are intuitionistic fuzzy ideals. Finally, we observe that the converse of Theorem 1 does not hold in general.

Example 2. *Let E be the intuitionistic FI defined in Example 1. We define a congruence relation θ on Ω as $\theta = \{(0,0),(1,1),(2,2),(0,1),(1,0),(1,2),(2,1),(0,2)\}$:*

$$\theta^{\bullet}(\mu_E)(u) = \{\ 0.5 \quad for \quad u \in \{0,1,2\}$$

$$\theta^{\bullet}(\lambda_E)(u) = \{\ 0.3 \quad for \quad u \in \{0,1,2\}$$

$$\theta_{\bullet}(\mu_E)(u) = \begin{cases} 0.5 & for \quad u = 0 \\ 0.3 & for \quad u \in \{1,2\} \end{cases}$$

$$\theta_{\bullet}(\lambda_E)(u) = \begin{cases} 0.3 & for \quad u = 0 \\ 0.5 & for \quad u \in \{1,2\}. \end{cases}$$

Clearly, E is a rough set and $\theta^{\bullet}(E)$ is an intuitionistic FI.

$$\theta_{\bullet}(\mu_E)(u) \geq \min(\theta_{\bullet}(\mu_E)(u * v), \theta_{\bullet}(\mu_E)(v))$$

$$\theta_{\bullet}(\lambda_E)(u) \leq \max(\theta_{\bullet}(\lambda_E)(u * v), \theta_{\bullet}(\lambda_E)(v))$$

Therefore, $\theta_{\bullet}(E)$ is an intuitionistic FI.
The converse of Theorem 1 does not hold in general.

Example 3. *Let $\Omega = \{0,1,2\}$ be a BCK-algebra and define a congruence relation θ on Ω as $\theta = \{(0,0),(1,1),(2,2),(0,1),(1,0),(1,2),(2,1),(0,2)\}$ (Table 2).*

Table 2. Cayley table for the BCK-algebra Ω.

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Let $E = (\mu_E, \lambda_E)$ and define $\lambda_E, \mu_E : \Omega \to [0,1]$ as

$$\mu_E(u) = \begin{cases} 0.3 & for \quad u \in \{0,2\} \\ 0.5 & for \quad u = 1. \end{cases}$$

$$\lambda_E(u) = \begin{cases} 0.5 & for \quad u \in \{0,2\} \\ 0.3 & for \quad u = 1 \end{cases}$$

Clearly, E is not an intuitionistic FI.

$$\theta^{\bullet}(\mu_E)(u) = \{\ 0.5 \quad for \quad u \in \{0,1,2\}$$

$$\theta^{\bullet}(\lambda_E)(u) = \{\ 0.3 \quad for \quad u \in \{0,1,2\}$$

Obviously, $\theta^{\bullet}(E)$ is an intuitionistic FI.

Definition 10 introduces the concept of (α,β)-cuts (or levels) of an intuitionistic fuzzy subset $E = (\mu_E, \lambda_E)$ within a BCK-algebra Ω. For any $\alpha, \beta \in [0,1]$, the (α,β)-cut of E is defined as the subset of Ω where $\mu_E(u) \geq \alpha$ and $\lambda_E(u) \leq \beta$. This subset is denoted by $E^{\alpha,\beta}$. Additionally, the strong (α, β)-cut of E, denoted by $E_s^{\alpha,\beta}$, consists of elements $u \in \Omega$ where $\mu_E(u) > \alpha$ and $\lambda_E(u) < \beta$. These definitions are essential for analyzing and understanding the structure of intuitionistic fuzzy subsets in BCK-algebras.

Definition 10. *For any IF subset $E = (\mu_E, \lambda_E)$ of a BCK-algebra Ω, we define the (α, β)-cut (level) of E as the subset $\{u \in \Omega : \mu_E(u) \geq \alpha, \lambda_E(u) \leq \beta\}$ of Ω denoted by $E^{\alpha,\beta}$; the strong (α, β)-cut (level) of E is denoted by $E_s^{\alpha,\beta}$ and defined as*

$$E_s^{\alpha,\beta} = \{u \in \Omega : \mu_E(u) > \alpha, \lambda_E(u) < \beta\}$$

for all $\alpha, \beta \in [0, 1]$.

In the study of BCK-algebras, understanding when an intuitionistic fuzzy subset qualifies as an intuitionistic fuzzy ideal (FI) is essential for analyzing the algebraic structure and its properties. The following theorem provides a criterion for such classification. Specifically, it states that an intuitionistic fuzzy subset $E = (\mu_E, \lambda_E)$ of a BCK-algebra Ω is an intuitionistic fuzzy ideal if and only if the sets $E_s^{\alpha,\beta}$ and $E^{\alpha,\beta}$ are non-empty ideals of Ω for all $\alpha, \beta \in [0, 1]$, given that the membership and non-membership functions satisfy $\mu_E(0) \geq \alpha$ and $\lambda_E(0) \leq \beta$. This result not only provides a precise condition for E to be considered an intuitionistic fuzzy ideal but also underscores the importance of these conditions in the framework of BCK-algebras.

Theorem 3. *Let $E = (\mu_E, \lambda_E)$ be an IF subset. Then, E is an intuitionistic fuzzy ideal (FI) of a BCK-algebra Ω if and only if $E_s^{\alpha,\beta}$ and $E^{\alpha,\beta}$ are non-empty ideals of Ω for all $\alpha, \beta \in [0, 1]$, where $\mu_E(0) \geq \alpha$ and $\lambda_E(0) \leq \beta$.*

Proof. Suppose that $E = (\mu_E, \lambda_E)$ is an intuitionistic fuzzy ideal (FI) of Ω and that $\mu_E(0) \geq \alpha$ and $\lambda_E(0) \leq \beta$ for all $\alpha, \beta \in [0, 1]$. If $E^{\alpha,\beta}$ is non-empty, then $E^{\alpha,\beta} \in \mathcal{I}$ (the set of ideals of Ω).

Clearly, $0 \in E^{\alpha,\beta}$, as $\mu_E(0) \geq \alpha$ and $\lambda_E(0) \leq \beta$. Let $u, v \in \Omega$ such that $u * v \in E^{\alpha,\beta}$ and $v \in E^{\alpha,\beta}$. Then, we have

$$\mu_E(u) \geq \min(\mu_E(u * v), \mu_E(v)) \geq \alpha, \quad \lambda_E(u) \leq \max(\lambda_E(u * v), \lambda_E(v)) \leq \beta.$$

Thus, $u \in E^{\alpha,\beta}$, proving that $E^{\alpha,\beta}$ is an ideal of Ω, i.e., $E^{\alpha,\beta} \in \mathcal{I}$.

Conversely, suppose that $E^{\alpha,\beta}$ is a non-empty ideal of Ω for all $\alpha, \beta \in [0, 1]$. For any $u \in \Omega$, let $\mu_E(u) = \alpha$ and $\lambda_E(u) = \beta$. Because $E^{\alpha,\beta} \neq \emptyset$ and $E^{\alpha,\beta} \in \mathcal{I}$, we have $0 \in E^{\alpha,\beta}$, which implies that

$$\mu_E(0) \geq \alpha = \mu_E(u), \quad \lambda_E(0) \leq \beta = \lambda_E(u).$$

Now, we show that

$$\mu_E(u) \geq \min(\mu_E(u * v), \mu_E(v)), \quad \lambda_E(u) \leq \max(\lambda_E(u * v), \lambda_E(v)).$$

Assuming the contrary, there exist $u^*, v^* \in \Omega$ such that

$$\mu_E(u) < \min(\mu_E(u^* * v^*), \mu_E(v^*)), \quad \lambda_E(u) > \max(\lambda_E(u^* * v^*), \lambda_E(v^*)).$$

Let

$$\alpha = \frac{1}{2}(\mu_E(v^*) + \min(\mu_E(u^* * v^*), \mu_E(v^*))), \quad \beta = \frac{1}{2}(\lambda_E(v^*) + \max(\lambda_E(u^* * v^*), \lambda_E(v^*))).$$

Then, we have

$$\mu_E(v^*) < \alpha < \min(\mu_E(u^* * v^*), \mu_E(v^*)), \quad \lambda_E(v^*) > \beta > \max(\lambda_E(u^* * v^*), \lambda_E(v^*)).$$

This implies that $v^* \notin E^{\alpha,\beta}$ and $u^* * v^* \notin E^{\alpha,\beta}$, contradicting the assumption that $E^{\alpha,\beta} \in \mathcal{I}$.

Thus, E is an intuitionistic fuzzy ideal of Ω. □

In the study of BCK-algebras, understanding the interaction between congruence relations and intuitionistic fuzzy sets is crucial. The following theorem examines how a congruence relation θ applied to an intuitionistic fuzzy set E affects certain operations. Specifically, it shows that for any $\alpha, \beta \in [0,1]$ the congruence relation preserves the structure of the intuitionistic fuzzy set in two important ways: first, $(\theta_\bullet(E))^{\alpha,\beta} = \theta_\bullet(E^{\alpha,\beta})$, and second, $(\theta^\bullet(E))^{\alpha,\beta} = \theta^\bullet(E^{\alpha,\beta})$. These results provide insight into how congruence relations interact with intuitionistic fuzzy sets, maintaining their essential properties under specific operations.

Theorem 4. *Let θ be a congruence relation on a BCK-algebra Ω. If E is an intuitionistic fuzzy (IF) set of Ω, then the following hold for any $\alpha, \beta \in [0,1]$:*

(1) $(\theta_\bullet(E))^{\alpha,\beta} = \theta_\bullet(E^{\alpha,\beta})$

(2) $(\theta^\bullet(E))^{\alpha,\beta} = \theta^\bullet(E^{\alpha,\beta})$.

Proof. 1. For $u \in (\theta_\bullet(E))^{\alpha,\beta}$, this means

$$\theta_\bullet(\mu_E)(u) \geq \alpha \quad \text{and} \quad \theta_\bullet(\lambda_E)(u) \leq \beta.$$

From the definition of the lower approximation $\theta_\bullet$, we have

$$\bigwedge_{a \in [\, u\,]_\theta} \mu_E(a) \geq \alpha \quad \text{and} \quad \bigvee_{a \in [\, u\,]_\theta} \lambda_E(a) \leq \beta.$$

This implies that for all $a \in [\, u\,]_\theta$ the membership and non-membership functions satisfy

$$\mu_E(a) \geq \alpha \quad \text{and} \quad \lambda_E(a) \leq \beta.$$

Therefore, $[\, u\,]_\theta \subseteq E^{\alpha,\beta}$; hence, $u \in \theta_\bullet(E^{\alpha,\beta})$.

Thus, $(\theta_\bullet(E))^{\alpha,\beta} = \theta_\bullet(E^{\alpha,\beta})$.

2. For $u \in (\theta^\bullet(E))^{\alpha,\beta}$, this means

$$\theta^\bullet(\mu_E)(u) \geq \alpha \quad \text{and} \quad \theta^\bullet(\lambda_E)(u) \leq \beta.$$

From the definition of the upper approximation $\theta^\bullet$, we have

$$\bigvee_{a \in [\, u\,]_\theta} \mu_E(a) \geq \alpha \quad \text{and} \quad \bigwedge_{a \in [\, u\,]_\theta} \lambda_E(a) \leq \beta.$$

This implies that there exists some $a \in [\, u\,]_\theta$ such that

$$\mu_E(a) \geq \alpha \quad \text{and} \quad \lambda_E(a) \leq \beta.$$

Therefore, $[\, u\,]_\theta \cap E^{\alpha,\beta} \neq \emptyset$; hence, $u \in \theta^\bullet(E^{\alpha,\beta})$.

Thus, $(\theta^\bullet(E))^{\alpha,\beta} = \theta^\bullet(E^{\alpha,\beta})$.

□

In the following definition, we define different types of mappings between BCK-algebras. A homomorphism preserves the algebraic operation, an epimorphism is a surjective homomorphism, and an isomorphism is both a homomorphism and a bijection, indicating that the two algebras are structurally identical.

Definition 11 ([20])**.** *Suppose that* $(\Omega, *, 0)$ *and* $(Y, *, 0)$ *are two BCK-algebras. A mapping* $f : \Omega \to Y$ *is called a homomorphism from* Ω *into* Y *if, for any* $u, v \in \Omega$,

$$f(u * v) = f(u) * f(v).$$

If in addition the mapping f *is onto, i.e.,* $f(\Omega) = Y$, *where* $f(\Omega) = \{f(u) : u \in \Omega\}$, *then* f *is called an epimorphism, and* Y *is said to be a homomorphic image of* Ω. *The mapping is called an isomorphism if it is both an epimorphism and one-to-one.*

Theorem 5 investigates how congruence relations are affected by an epimorphism between BCK-algebras. It shows that if φ is an epimorphism from a BCK-algebra Ω_1 to Ω_2 and if θ_2 is a congruence relation on Ω_2, then θ_1 defined on Ω_1 is also a congruence relation. Additionally, if θ_2 is complete and φ is injective, then θ_1 is complete as well. The theorem also establishes that $\varphi(\theta_1^{\bullet}(E))$ equals $\theta_2^{\bullet}(\varphi(E))$.

Theorem 5. *Let* φ *be an epimorphism from a BCK-algebra* Ω_1 *to a BCK-algebra* Ω_2 *and let* θ_2 *be a congruence relation on* Ω_2. *Let* E *be a subset of* Ω_1. *Then,*

1. $\theta_1 = \{(a, b) \in \Omega_1 \times \Omega_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$ *is a congruence relation on* Ω_1.
2. *If* θ_2 *is complete and* φ *is injective (one-to-one), then* θ_1 *is complete.*
3. $\varphi(\theta_1^{\bullet}(E)) = \theta_2^{\bullet}(\varphi(E))$.

Proof. (1) It is clear that θ_1 is a congruence relation on Ω_1, as it is derived from the congruence relation θ_2 on Ω_2 through the epimorphism φ.

(2) To show that θ_1 is complete, we need to prove that $[u_1 * u_2]_{\theta_1} = [u_1]_{\theta_1} * [u_2]_{\theta_1}$. Suppose that w is an element of $[u_1 * u_2]_{\theta_1}$. Because θ_2 is complete, from the definition of θ_1 we know that

$$\varphi(w) \in [\varphi(u_1 * u_2)]_{\theta_2} = [\varphi(u_1)]_{\theta_2} * [\varphi(u_2)]_{\theta_2}.$$

Because φ is an epimorphism, there exist $a, b \in \Omega_1$ such that $\varphi(a) \in [\varphi(u_1)]_{\theta_2}$, $\varphi(b) \in [\varphi(u_2)]_{\theta_2}$, and $\varphi(w) = \varphi(a * b)$.
As φ is injective, from the definition of θ_1 we have $a \in [u_1]_{\theta_1}$, $b \in [u_2]_{\theta_1}$, and $w = a * b$. Thus, $w \in [u_1]_{\theta_1} * [u_2]_{\theta_1}$. Therefore, $[u_1 * u_2]_{\theta_1} \subseteq [u_1]_{\theta_1} * [u_2]_{\theta_1}$.
Conversely, we also have $[u_1]_{\theta_1} * [u_2]_{\theta_1} \subseteq [u_1 * u_2]_{\theta_1}$. Thus, θ_1 is complete.

(3) Let v be any element of $\varphi(\theta_1^{\bullet}(E))$. Then, there exists $u \in \theta_1^{\bullet}(E)$ such that $\varphi(u) = v$. Hence,

$$[u]_{\theta_1} \cap E \neq \emptyset.$$

Thus, there exists $a \in [u]_{\theta_1} \cap E$. Consequently, $\varphi(a) \in \varphi(E)$, and from the definition of θ_1 we have $\varphi(a) \in [\varphi(u)]_{\theta_2}$. Therefore, $[\varphi(u)]_{\theta_2} \cap \varphi(E) \neq \emptyset$, which implies $v = \varphi(u) \in \theta_2^{\bullet}(\varphi(E))$. Thus, $\varphi(\theta_1^{\bullet}(E)) \subseteq \theta_2^{\bullet}(\varphi(E))$.
Conversely, let $v \in \theta_2^{\bullet}(\varphi(E))$. Then, there exists $u \in \Omega_1$ such that $\varphi(u) = v$. Hence, $[\varphi(u)]_{\theta_2} \cap \varphi(E) \neq \emptyset$, and there exists $a \in E$ such that $\varphi(a) \in \varphi(E)$ and $\varphi(a) \in [\varphi(u)]_{\theta_2}$. From the definition of θ_1, we have $a \in [u]_{\theta_1}$.
Thus, $[u]_{\theta_1} \cap E \neq \emptyset$, which implies that $u \in \theta_1^{\bullet}(E)$; Thus, $v = \varphi(u) \in \varphi(\theta_1^{\bullet}(E))$. Therefore, $\theta_2^{\bullet}(\varphi(E)) \subseteq \varphi(\theta_1^{\bullet}(E))$.
From the above, we can conclude that $\varphi(\theta_1^{\bullet}(E)) = \theta_2^{\bullet}(\varphi(E))$.
□

Theorem 6 examines the relationship between fuzzy ideals in BCK-algebras under an epimorphism. It states that if φ is an epimorphism from a BCK-algebra Ω_1 to Ω_2 and if θ_2 is a congruence relation on Ω_2, then the fuzzy ideal $\theta_1^{\bullet}(E)$ in Ω_1 is an element of the set of ideals $\mathcal{I}$ if and only if $\theta_2^{\bullet}(\varphi(E))$ is an element of $\mathcal{I}$ in Ω_2. This result connects the ideals of Ω_1 and Ω_2 through the epimorphism φ.

Theorem 6. *Let φ be an epimorphism from a BCK-algebra Ω_1 to a BCK-algebra Ω_2, and let θ_2 be a congruence relation on Ω_2. Let E be a subset of Ω_1. If*

$$\theta_1 = \{(a,b) \in \Omega_1 \times \Omega_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\},$$

then $\theta_1^{\bullet}(E) \in \mathcal{I}$ of Ω_1 if and only if $\theta_2^{\bullet}(\varphi(E)) \in \mathcal{I}$ of Ω_2.

Proof. Suppose that $\theta_1^{\bullet}(E) \in \mathcal{I}$ of Ω_1. Let $u * v \in \theta_2^{\bullet}(\varphi(E))$ and $v \in \theta_2^{\bullet}(\varphi(E))$. Because φ is an epimorphism (by Theorem 5), it follows that $u * v \in \varphi(\theta_1^{\bullet}(E))$. Therefore, there exist $a, b \in \theta_1^{\bullet}(E)$ such that

$$u = \varphi(a), \quad v = \varphi(b), \quad u * v = \varphi(a * b).$$

Because $u = \varphi(a)$ and $\theta_1^{\bullet}(E) \in \mathcal{I}$, it follows that $a * b \in \theta_1^{\bullet}(E)$. Thus,

$$u \in \varphi(\theta_1^{\bullet}(E)) \Rightarrow u \in \theta_2^{\bullet}(\varphi(E)) \Rightarrow \theta_2^{\bullet}(\varphi(E)) \in \mathcal{I}.$$

Conversely, if $\theta_2^{\bullet}(\varphi(E)) \in \mathcal{I}$ of Ω_2 and both $u * v \in \theta_1^{\bullet}(E)$ and $v \in \theta_1^{\bullet}(E)$, because φ is an epimorphism, we have:

$$\varphi(u * v) \in \varphi(\theta_1^{\bullet}(E)) \quad \text{and} \quad \varphi(v) \in \varphi(\theta_1^{\bullet}(E)).$$

Per Theorem 5, this implies that

$$\varphi(u * v) \in \theta_2^{\bullet}(\varphi(E)) \quad \text{and} \quad \varphi(v) \in \theta_2^{\bullet}(\varphi(E)).$$

Because $\theta_2^{\bullet}(\varphi(E)) \in \mathcal{I}$, we have $\varphi(u) \in \theta_2^{\bullet}(\varphi(E))$, which implies that $\varphi(u) \in \varphi(\theta_1^{\bullet}(E))$. Therefore, there exists $a \in \theta_1^{\bullet}(E)$ such that $\varphi(u) = \varphi(a)$. Consequently,

$$u \in [a]_{\theta_1} \cap E \neq \emptyset \Rightarrow u \in \theta_1^{\bullet}(E),$$

establishing that $\theta_1^{\bullet}(E) \in \mathcal{I}$. □

Theorem 7 explores the relationship between intuitionistic fuzzy ideals under an epimorphism between BCK-algebras. It states that if φ is an epimorphism from a BCK-algebra Ω_1 to Ω_2 and if θ_2 is a complete congruence relation on Ω_2, then the fuzzy ideal $\theta_1^{\bullet}(E)$ in Ω_1 is an intuitionistic fuzzy ideal if and only if $\theta_2^{\bullet}(\varphi(E))$ is an intuitionistic fuzzy ideal in Ω_2. This result shows how the properties of intuitionistic fuzzy ideals are preserved under the epimorphism φ.

Theorem 7. *Let φ be an epimorphism from a BCK-algebra Ω_1 to a BCK-algebra Ω_2 and let θ_2 be a complete congruence relation on Ω_2. Let E be a fuzzy subset of Ω_1. If*

$$\theta_1 = \{(a,b) \in \Omega_1 \times \Omega_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\},$$

then $(\theta_1^{\bullet}(E))$ is an intuitionistic fuzzy ideal (FI) of Ω_1 if and only if $(\theta_2^{\bullet}(\varphi(E)))$ is an intuitionistic fuzzy ideal (FI) of Ω_2.

Proof. By Theorem 3, we have that $(\theta_1^{\bullet}(E))$ is an intuitionistic fuzzy ideal of Ω_1 if and only if $(\theta_1^{\bullet}(E))_s^{\alpha,\beta}$ is an intuitionistic fuzzy ideal, and if it is non-empty, it is an ideal of Ω_1. By Theorem 4, we have

$$(\theta_1^{\bullet}(E))_s^{\alpha,\beta} = (\theta_1^{\bullet}(E_s^{\alpha,\beta})).$$

By Theorem 6, $\theta_1^{\bullet}(E_s^{\alpha,\beta}) \in \mathcal{I}$ of Ω_1 if and only if

$$\theta_2^{\bullet}(\varphi(E_s^{\alpha,\beta}))$$

is an ideal of Ω_2.

It is clear that $\varphi(E_s^{\alpha,\beta}) = (\varphi(E))_s^{\alpha,\beta}$. From this and Theorem 3, we have

$$(\theta_2^{\bullet}(\varphi(E_s^{\alpha,\beta}))) = (\theta_2^{\bullet}(\varphi(E)))_s^{\alpha,\beta}.$$

By Theorem 3, $(\theta_2^{\bullet}(\varphi(E)))_s^{\alpha,\beta} \in \mathcal{I}$ of Ω_2 if and only if $(\theta_2^{\bullet}(\varphi(E)))$ is an intuitionistic fuzzy ideal of Ω_2 for all $\alpha, \beta \in [0,1]$. □

4. T-Rough Intuitionistic Fuzzy Ideals in BCK-Algebras

In this section, we introduce the concept of set-valued homomorphisms and explore some fundamental properties of generalized lower and upper approximation operators in BCK-algebras. To begin, we provide key definitions that will be employed throughout this section.

Definition 12 ([6,24])**.** *Let U and W be two non-empty universes. Let T be a set-valued mapping provided by $T : U \to \mathcal{P}(W)$. Then, the triple (U, W, T) is referred to as a generalized approximation space or generalized rough set.*

Any set-valued function from U to $\mathcal{P}(W)$ defines a binary relation from U to W by setting $R_T = \{(u,v) \mid v \in T(u)\}$. Obviously, if R is an arbitrary relation from U to W, then a set-valued mapping $T_R : U \to \mathcal{P}(W)$ can be defined by $T_R(u) = \{v \in W \mid (u,v) \in R\}$, where $u \in U$.

For any set $E \subseteq W$, the lower and upper approximations $T^+(E)$ and $T^{-1}(E)$ are defined by

$$T^+(E) = \{u \in U \mid T(u) \subseteq E\}$$

$$T^{-1}(E) = \{u \in U \mid T(u) \cap E \neq \varnothing\}.$$

The pair $(T^+(E), T^{-1}(E))$ is referred to as a generalized rough set and $T^+(E)$ and $T^{-1}(E)$ are referred to as lower and upper generalized approximation operators, respectively.

Definition 13 introduces two fundamental types of inverse operations related to a set-valued mapping T between BCK-algebras Ω_1 and Ω_2. Given a subset $B \subseteq \Omega_2$, the lower inverse $T^+(B)$ is defined as the set of elements $u \in \Omega_1$ such that $T(u)$ is a subset of B. Conversely, the upper inverse $T^{-1}(B)$ consists of elements $u \in \Omega_1$ where $T(u)$ intersects B non-trivially. These definitions are crucial for understanding how elements in Ω_1 are mapped with respect to subsets of Ω_2.

Definition 13. *Let Ω_1 and Ω_2 be two BCK-algebras and $B \subseteq \Omega_2$. Let $T : \Omega_1 \to \mathcal{P}^*(\Omega_2)$ be a set-valued mapping, where $\mathcal{P}^*(\Omega_2)$ denotes the set of all non-empty subsets of Ω_2. The lower inverse and upper inverse of B under T are defined by*

$$T^+(B) = \{u \in \Omega_1 \mid T(u) \subseteq B\},$$

$$T^{-1}(B) = \{u \in \Omega_1 \mid T(u) \cap B \neq \varnothing\}.$$

Definition 14 defines the concept of a set-valued homomorphism and a strong set-valued homomorphism between BCK-algebras. A set-valued mapping T is called a set-valued homomorphism if it preserves the algebraic operations in the sense that $T(a) * T(b) \subseteq T(a * b)$ and $-T(a) \subseteq T(-a)$ for all $a, b \in \Omega_1$. On the other hand, T is called a strong set-valued homomorphism if it satisfies the equalities $T(a) * T(b) = T(a * b)$ and $-T(a) = T(-a)$. These definitions characterize the extent to which the set-valued mapping preserves the structure of the BCK-algebras.

Definition 14. *Let Ω_1 and Ω_2 be two BCK-algebras and let $T : \Omega_1 \to \mathcal{P}^*(\Omega_2)$ be a set-valued mapping, where $\mathcal{P}^*(\Omega_2)$ denotes the set of all non-empty subsets of Ω_2.*

(1) *T is called a **set-valued homomorphism** if:*

(a) *$T(a) * T(b) \subseteq T(a * b)$ for all $a, b \in \Omega_1$. Here, $*$ denotes the binary operation in Ω_2, and $T(a) * T(b)$ represents the set of all possible products of elements from $T(a)$ and $T(b)$.*

(b) *$-T(a) \subseteq T(-a)$ for all $a \in \Omega_1$. As the symbol $-$ denotes the complement in Ω_2, $-T(a)$ represents the set of all complements of elements in $T(a)$ and this set is contained within $T(-a)$.*

(2) *A set-valued mapping T is called a **strong set-valued homomorphism** if:*

(a) *$T(a) * T(b) = T(a * b)$ for all $a, b \in \Omega_1$. This means that T preserves the exact structure of the binary operation, not just an inclusion.*

(b) *$-T(a) = T(-a)$ for all $a \in \Omega_1$. This implies that T also preserves the complement operation exactly.*

Explanation of Notation

In the context of BCK-algebras and set-valued mappings:

- **Minus Operation**: In this setting, $-a$ denotes the complement of the element a in the algebra. For a set-valued mapping T, $-T(a)$ denotes the set of all complements of the elements in $T(a)$. Thus, $T(-a)$ is the image of the complement under the mapping T.

- **Equality $u = vw$**: In definitions where $u = vw$, this denotes that the element u is expressed as the product of elements v and w in the algebra. For set-valued mappings, if T is a set-valued homomorphism, this means that $T(u)$ is related to the product of $T(v)$ and $T(w)$ in some way, specifically that $T(u)$ should be contained in or be equal to the product of $T(v)$ and $T(w)$ depending on the type of homomorphism.

Theorem 8 examines the behavior of fuzzy ideals under a strong set-valued homomorphism between two BCK-algebras. It establishes two important results: first, if $T^+(E)$ is nonempty, where T is a strong set-valued homomorphism from Ω_1 to $\mathcal{P}^*(\Omega_2)$, then $T^+(E)$ is a fuzzy ideal in Ω_1. Second, if $T^{-1}(E)$ is nonempty, then $T^{-1}(E)$ is also a fuzzy ideal in Ω_1. These results demonstrate how fuzzy ideals in Ω_2 are preserved and transformed under the strong set-valued homomorphism T.

Theorem 8. *Consider two BCK-algebras Ω_1 and Ω_2, and let $T : \Omega_1 \to \mathcal{P}^*(\Omega_2)$ be a strong set-valued homomorphism. If $E \in \mathcal{I}$ of Ω_2, then the following statements hold:*

1. *If $T^+(E)$ is nonempty, then $T^+(E) \in \mathcal{I}$ of Ω_1.*
2. *If $T^{-1}(E)$ is nonempty, then $T^{-1}(E) \in \mathcal{I}$ of Ω_1.*

Proof. (1) Suppose that $E \in \mathcal{I}$ and $T^+(E)$ is nonempty. We need to show that $T^+(E)$ is an ideal of Ω_1.

To do this, consider arbitrary elements $u, v \in \Omega_1$ such that $u * v \in T^+(E)$ and $v \in T^+(E)$. From the definition of $T^+(E)$, this means that

$$T(u * v) \cap E \neq \varnothing \quad \text{and} \quad T(v) \cap E \neq \varnothing.$$

Because T is a strong set-valued homomorphism, we have

$$T(u * v) = T(u) * T(v).$$

Therefore, there exist elements $a \in T(u * v)$ and $b \in T(v)$ such that

$$a * b \in E.$$

Because $a \in T(u * v)$, it follows that $a \in T(u) * T(v)$. Given that $E \in \mathcal{I}$, if $a * b \in E$ and $b \in E$, then $a \in E$. Thus, $a \in T(u) \cap E$.

Because $a \in T(u) \cap E$, we can conclude that $u \in T^+(E)$. Therefore, $T^+(E)$ is closed under the operation $*$ and contains $T^+(E)$, which makes it an ideal of Ω_1.

(2) Now, suppose that $T^{-1}(E)$ is nonempty. We need to show that $T^{-1}(E)$ is an ideal of Ω_1.
Take arbitrary elements $u, v \in \Omega_1$ such that $u * v \in T^{-1}(E)$ and $v \in T^{-1}(E)$. From the definition of $T^{-1}(E)$, this means that

$$T(u * v) \cap E \neq \varnothing \quad \text{and} \quad T(v) \cap E \neq \varnothing.$$

Because T is a strong set-valued homomorphism, we have

$$T(u * v) = T(u) * T(v).$$

Therefore, there exist elements $a \in T(u * v)$ and $b \in T(v)$ such that

$$a * b \in E.$$

Because $a \in T(u * v)$, it follows that $a \in T(u) * T(v)$. Given that $E \in \mathcal{I}$, if $a * b \in E$ and $b \in E$, then $a \in E$. Thus, $a \in T(u) \cap E$.
Because $a \in T(u) \cap E$, we can conclude that $u \in T^{-1}(E)$. Therefore, $T^{-1}(E)$ is closed under the operation $*$ and contains $T^{-1}(E)$, which makes it an ideal of Ω_1.
□

In the following definition, we introduce the concept of T-rough intuitionistic fuzzy subsets in the context of BCK-algebras. We define how to compute the T-rough lower and upper fuzzy sets using set-valued homomorphisms and specify the conditions under which these sets form a T-rough intuitionistic fuzzy subset.

Definition 15. *Let Ω_1 and Ω_2 be two bck-algebras and $T : \Omega_1 \to P^*(\Omega_2)$ be a set-valued homomorphism. Let $E = (\mu_E, \lambda_E)$ be an IF subset of Ω_2. For every $u \in \Omega$, we define the following:*

$$T^+(\mu_E)(u) = \bigwedge_{a \in T(u)} (\mu_E)(a)$$

$$T^+(\lambda_E)(u) = \bigvee_{a \in T(u)} (\lambda_E)(a)$$

$$T^{-1}(\mu_E)(u) = \bigvee_{a \in T(u)} (\mu_E)(a)$$

$$T^{-1}(\lambda_E)(u) = \bigwedge_{a \in T(u)} (\lambda_E)(a)$$

where $T^+(E)$ and $T^{-1}(E)$ are called, respectively, the T-rough lower and T-rough upper IF subsets of Ω_1. If $T^+(E)$ and $T^{-1}(E)$ are intuitionistic FIs, then $(T^+(E), T^{-1}(E))$ is said to be a T-rough intuitionistic FI of Ω_1.

Theorem 9 investigates how intuitionistic fuzzy ideals are preserved under a set-valued homomorphism between BCK-algebras. Specifically, it states that if T is a set-valued homomorphism from a BCK-algebra Ω_1 to $\mathcal{P}^*(\Omega_2)$ and E is an intuitionistic fuzzy ideal in Ω_2, then the pre-image $T^{-1}(E)$ is also an intuitionistic fuzzy ideal in Ω_1. This result highlights the preservation of the structure of intuitionistic fuzzy ideals through the set-valued homomorphism.

Theorem 9. *Let Ω_1 and Ω_2 be two BCK-algebras and let $T : \Omega_1 \to \mathcal{P}^*(\Omega_2)$ be a set-valued homomorphism. If E is an intuitionistic FI of Ω_2, then $T^{-1}(E)$ is an intuitionistic FI of Ω_1.*

Proof. Let $E = (\mu_E, \lambda_E)$, $T^{-1}(E) = (T^{-1}(\mu_E), T^{-1}(\lambda_E))$.

$$T^{-1}(\mu_E)(0) = \bigvee_{w\in T(0)} (\mu_E)(w) \geq \bigvee_{u\in T(u)} (\mu_E)(u) = T^{-1}(\mu_E)(u)$$

For any $u, v \in \Omega_1$,

$$T^{-1}(\mu_E)(u) = \bigvee_{u\in T(u)} (\mu_E)(u) \geq \bigvee_{u*v\in T(u)*T(v),\, v\in T(v)} \min(\mu_E(u*v), (\mu_E(v)))$$

$$= \min\left(\bigvee_{u*v\in T(u*v)} \mu_E(u*v), \bigvee_{v\in T(v)} \mu_E(v)\right) = \min(T^{-1}(\mu_E)(u*v), T^{-1}(\mu_E)(v))$$

$$T^{-1}(\lambda_E)(0) = \bigwedge_{w\in T(0)} (\lambda_E)(w) \leq \bigwedge_{u\in T(u)} (\lambda_E)(u) = T^{-1}(\lambda_E)(u).$$

For any $u, v \in \Omega_1$,

$$T^{-1}(\lambda_E)(u) = \bigwedge_{u\in T(u)} (\lambda_E)(u) \leq \bigwedge_{u*v\in T(u)*T(v),\, v\in T(v)} \max(\lambda_E(u*v), (\lambda_E(v)))$$

$$= \max\left(\bigwedge_{u*v\in T(u*v)} \lambda_E(u*v), \bigwedge_{v\in T(v)} \lambda_E(v)\right) = \max(T^{-1}(\lambda_E)(u*v), T^{-1}(\lambda_E)(v)).$$

Then, $T^{-1}(E)$ is an intuitionistic FI. □

Theorem 10 addresses the behavior of intuitionistic fuzzy ideals under a set-valued homomorphism between BCK-algebras. It asserts that if T is a set-valued homomorphism from a BCK-algebra Ω_1 to $\mathcal{P}^*(\Omega_2)$ and E is an intuitionistic fuzzy ideal in Ω_2, then the image $T^+(E)$ is an intuitionistic fuzzy ideal in Ω_1. This theorem demonstrates how intuitionistic fuzzy ideals are preserved when mapped through the set-valued homomorphism.

Theorem 10. *Let Ω_1 and Ω_2 be two BCK-algebras and let $T : \Omega_1 \to \mathcal{P}^*(\Omega_2)$ be a set-valued homomorphism. If E is an intuitionistic FI of Ω_2, then $T^+(E)$ is an intuitionistic FI of Ω_1.*

Proof. Let $E = (\mu_E, \lambda_E)$, $T^+(E) = (T^+(\mu_E), T^+(\lambda_E))$.

$$T^+(\mu_E)(0) = \bigwedge_{w\in T(0)} (\mu_E)(w) \geq \bigwedge_{u\in T(u)} (\mu_E)(u) = T^+(\mu_E)(u)$$

For any $u, v \in \Omega_1$,

$$T^+(\mu_E)(u) = \bigwedge_{u\in T(u)} (\mu_E)(u) \geq \bigwedge_{u*v\in T(u)*T(v),\, v\in T(v)} \min(\mu_E(u*v), (\mu_E(v)))$$

$$= \min\left(\bigwedge_{u*v\in T(u*v)} \mu_E(u*v), \bigwedge_{v\in T(v)} \mu_E(v)\right) = \min(T^+(\mu_E)(u*v), T^+(\mu_E)(v))$$

$$T^+(\lambda_E)(0) = \bigvee_{w\in T(0)} (\lambda_E)(w) \leq \bigvee_{u\in T(u)} (\lambda_E)(u) = T^+(\lambda_E)(u).$$

For any $u, v \in \Omega_1$,

$$T^{+}(\lambda_E)(u) = \bigvee_{u \in T(u)} (\lambda_E)(u) \leq \bigvee_{u*v \in T(u)*T(v),\, v \in T(v)} \max(\lambda_E(u*v), (\lambda_E(v)))$$

$$= \max\left(\bigvee_{u*v \in T(u*v)} \lambda_E(u*v), \bigvee_{v \in T(v)} \lambda_E(v) \right) = \max(T^{+}(\lambda_E)(u*v), T^{+}(\lambda_E)(v)).$$

Then, $T^{+}(E)$ is an intuitionistic FI. □

In the following definition, we define the composition of two intuitionistic fuzzy sets (IFS) in a BCK-algebra. This composition combines the membership and non-membership functions of the two IFSs using specific operations to form a new IFS.

Definition 16. *Let $E = (\mu_E, \lambda_E)$ and $B = (\mu_B, \lambda_B)$ be any two IFSs of a BCK-algebra Ω. The composition $E \circ B$ is defined by*

$$E \circ B = (\mu_E \circ \mu_B,\ \ \lambda_E \circ \lambda_B),$$

where, for all $u \in \Omega$:

$$(\mu_E \circ \mu_B)(u) = \bigvee_{u=vw} [\mu_E(v) \wedge \mu_B(w)],$$

$$(\lambda_E \circ \lambda_B)(u) = \bigwedge_{u=vw} [\lambda_E(v) \vee \lambda_B(w)].$$

Theorem 11 explores the interaction of intuitionistic fuzzy sets under a set-valued homomorphism between BCK-algebras. It establishes that if T is a set-valued homomorphism from Ω_1 to $\mathcal{P}^*(\Omega_2)$ and if E and B are two intuitionistic fuzzy sets in Ω_2, then the composition $T^{-1}(E) \circ T^{-1}(B)$ is contained within $T^{-1}(E \circ B)$. This result shows how the composition of intuitionistic fuzzy sets is preserved under the inverse of the set-valued homomorphism.

Theorem 11. *Let Ω_1 and Ω_2 be two BCK-algebras and $T : \Omega_1 \to \mathcal{P}^*(\Omega_2)$ be a set-valued homomorphism. If E and B are two IFSs of Ω_2, then*

$$T^{-1}(E) \circ T^{-1}(B) \subseteq T^{-1}(E \circ B).$$

Proof. Let $E = (\mu_E, \lambda_E)$ and $B = (\mu_B, \lambda_B)$ be any two IFSs of a BCK-algebra Ω_2. Then,

$$T^{-1}(E) \circ T^{-1}(B) = (T^{-1}(\mu_E) \circ T^{-1}(\mu_B), T^{-1}(\lambda_E) \circ T^{-1}(\lambda_B))$$

and

$$T^{-1}(E \circ B) = (T^{-1}(\mu_E \circ \mu_B), T^{-1}(\lambda_E \circ \lambda_B)).$$

To show that $T^{-1}(E) \circ T^{-1}(B) \subseteq T^{-1}(E \circ B)$, we need to prove that for all $u \in \Omega_1$ we have

$$(T^{-1}(\mu_E) \circ T^{-1}(\mu_B))(u) \leq T^{-1}(\mu_E \circ \mu_B)(u)$$

and

$$(T^{-1}(\lambda_E) \circ T^{-1}(\lambda_B))(u) \geq T^{-1}(\lambda_E \circ \lambda_B)(u).$$

Now, for all $u \in \Omega_1$,

$$(T^{-1}(\mu_E) \circ T^{-1}(\mu_B))(u) = \bigvee_{u=vw} [\bigvee_{a \in T(v)} \mu_E(a) \wedge \bigvee_{b \in T(w)} \mu_B(b)]$$

$$\leq \bigvee_{u=vw} [\bigvee_{ab \in T(yw)} \mu_E(a) \wedge \mu_B(b)] = \bigvee_{ab \in T(u)} (\mu_E(a) \circ \mu_B(b))$$
$$= T^{-1}(\mu_E \circ \mu_B)(u).$$

Similarly,

$$(T^{-1}(\lambda_E) \circ T^{-1}(\lambda_B))(u) = \bigwedge_{u=vw} [\bigwedge_{a \in T(v)} \lambda_E(a) \vee \bigwedge_{b \in T(w)} \lambda_B(b)]$$
$$\geq \bigwedge_{u=vw} [\bigwedge_{ab \in T(yw)} \lambda_E(a) \vee \lambda_B(b)] = \bigwedge_{ab \in T(u)} (\lambda_E(a) \circ \lambda_B(b))$$
$$= T^{-1}(\lambda_E \circ \lambda_B)(u).$$

Therefore, $T^{-1}(E) \circ T^{-1}(B) \subseteq T^{-1}(E \circ B)$. □

Theorem 12 examines the behavior of intuitionistic fuzzy sets under a strong set-valued homomorphism between BCK-algebras. It states that if T is a strong set-valued homomorphism from Ω_1 to $\mathcal{P}^*(\Omega_2)$ and if E and B are two intuitionistic fuzzy sets in Ω_2, then the composition $T^+(E) \circ T^+(B)$ is contained within $T^+(E \circ B)$. This theorem highlights how the composition of intuitionistic fuzzy sets is preserved and transferred under the strong set-valued homomorphism.

Theorem 12. *Let Ω_1 and Ω_2 be two BCK-algebras and let $T : \Omega_1 \to \mathcal{P}^*(\Omega_2)$ be a strong set-valued homomorphism. If E and B are two IFSs of Ω_2, then*

$$T^+(E) \circ T^+(B) \subseteq T^+(E \circ B).$$

Proof. Let $E = (\mu_E, \lambda_E)$ and $B = (\mu_B, \lambda_B)$ be any two IFSs of a BCK-algebra Ω_2. Then,

$$T^+(E) \circ T^+(B) = (T^+(\mu_E) \circ T^+(\mu_B), T^+(\lambda_E) \circ T^+(\lambda_B)),$$
$$T^+(E \circ B) = (T^+(\mu_E \circ \mu_B), T^+(\lambda_E \circ \lambda_B)).$$

To show that $T^+(E) \circ T^+(B) \subseteq T^+(E \circ B)$, we need to prove that for all $u \in \Omega_1$ we have

$$(T^+(\mu_E) \circ T^+(\mu_B))(u) \leq T^+(\mu_E \circ \mu_B)(u),$$
$$(T^+(\lambda_E) \circ T^+(\lambda_B))(u) \geq T^+(\lambda_E \circ \lambda_B)(u).$$

Now, for all $u \in \Omega_1$:

$$\begin{aligned}
&(T^+(\mu_E) \circ T^+(\mu_B))(u) \\
&= \bigvee_{u=vw} [T^+(\mu_E)(v) \wedge T^+(\mu_B)(w)] \\
&= \bigvee_{u=vw} \left[\bigwedge_{a \in T(v)} \mu_E(a) \wedge \bigwedge_{b \in T(w)} \mu_B(b) \right] \\
&\leq \bigvee_{u=vw} \left[\bigwedge_{a \in T(v), b \in T(w)} \bigvee_{ab = \alpha\beta} (\mu_E(\alpha) \wedge \mu_B(\beta)) \right],
\end{aligned}$$

where $\alpha, \beta \in \Omega_1$.

Continuing the proof in a similar manner, we obtain the desired result. □

Theorem 13 addresses the relationship between intuitionistic fuzzy sets and set-valued homomorphisms in BCK-algebras. It establishes two key results. First, for any intuitionistic fuzzy set E in Ω_2 and for all $\alpha, \beta \in [0,1]$, the fuzzy set $(T^+(E))^{\alpha,\beta}$ is equal to $T^+(E^{\alpha,\beta})$.

Second, $(T^{-1}(E))^{\alpha,\beta}$ is equal to $T^{-1}(E^{\alpha,\beta})$. These results demonstrate how intuitionistic fuzzy sets and their parameters are transformed under the set-valued homomorphism T.

Theorem 13. *Let Ω_1 and Ω_2 be two BCK-algebras and let $T : \Omega_1 \to P^*(\Omega_2)$ be a set-valued homomorphism. If E is an IFS of Ω_2 and for any $\alpha, \beta \in [0,1]$, then*

1. $(T^+(E))^{\alpha,\beta} = T^+(E^{\alpha,\beta})$
2. $(T^{-1}(E))^{\alpha,\beta} = T^{-1}(E^{\alpha,\beta})$.

Proof. 1. If $u \in (T^+(E))^{\alpha,\beta}$, then

$$T^+(\mu_E)(u) \geq \alpha \quad \text{and} \quad T^+(\lambda_E)(u) \leq \beta,$$
$$\bigwedge_{a \in T(u)} (\mu_E)(a) \geq \alpha \quad \text{and} \quad \bigvee_{a \in T(u)} (\lambda_E)(a) \leq \beta,$$

which implies

$$a \in T(u),\ (\mu_E)(a) \geq \alpha \quad \text{and} \quad (\lambda_E)(a) \leq \beta,$$
$$\text{hence} \quad T(u) \subseteq E^{\alpha,\beta} \quad \text{so} \quad u \in T^+(E^{\alpha,\beta}).$$

2. $u \in (T^{-1}(E))^{\alpha,\beta}$ if and only if

$$T^{-1}(\mu_E)(u) \geq \alpha \quad \text{and} \quad T^{-1}(\lambda_E)(u) \leq \beta,$$
$$\bigvee_{a \in T(u)} (\mu_E)(a) \geq \alpha \quad \text{and} \quad \bigwedge_{a \in T(u)} (\lambda_E)(a) \leq \beta,$$

which implies

$$\text{There exists } a \in T(u),\ \mu_E(a) \geq \alpha \quad \text{and} \quad \lambda_E(a) \leq \beta,$$
$$\text{Hence} \quad T(u) \cap E^{\alpha,\beta} \neq \emptyset \quad \text{so,} \quad u \in T^{-1}(E^{\alpha,\beta}).$$

□

5. Application of T-Rough Intuitionistic Fuzzy Ideals in BCK-Algebras for Decision-Making

In decision-making processes, managing uncertainty and imprecision is crucial. Traditional methods often fall short when dealing with complex or vague information. Intuitionistic fuzzy sets (IFS), introduced by Atanassov, extend classical fuzzy sets by incorporating both membership and non-membership functions, providing a more comprehensive framework for handling uncertainty. T-rough intuitionistic fuzzy sets (T-IFS) refine this further by including a tolerance level T to capture degrees of uncertainty or approximation within the decision-making context.

T-rough Intuitionistic Fuzzy Sets (T-IFS): T-IFSs are used to handle situations where precise information is not available, allowing criteria to be evaluated under varying levels of tolerance.

BCK-Algebras: BCK-algebras offer a valuable algebraic structure for managing logical operations and approximations, with the binary operation $\to$ used to model decision-making scenarios with inherent uncertainty.

In this section, we apply T-rough intuitionistic fuzzy ideals in BCK-algebras to a practical decision-making problem, demonstrating how these tools can handle uncertainty and imprecision effectively.

5.1. Example Problem

We need to choose between two projects based on three criteria: cost, potential revenue, and risk. We can use T-rough intuitionistic fuzzy ideals in BCK-algebras to evaluate and rank the projects.

5.2. Define Criteria

The decision criteria are defined as follows:

- **Cost**: c
- **Revenue**: r
- **Risk**: k

5.3. Parameters

The following parameters are used to normalize the criteria:

- $C_{\max}$ is the maximum possible cost
- $R_{\max}$ is the maximum possible revenue
- $K_{\max}$ is the maximum possible risk

5.4. Membership and Non-Membership Functions

For each criterion, the membership function μ and non-membership function ν using T-rough intuitionistic fuzzy sets are defined as follows:

$$\mu_{\text{cost}}(x) = 1 - \frac{x}{C_{\max}} \tag{1}$$

$$\nu_{\text{cost}}(x) = \frac{x}{C_{\max}} \tag{2}$$

$$\mu_{\text{revenue}}(x) = \frac{x}{R_{\max}} \tag{3}$$

$$\nu_{\text{revenue}}(x) = 1 - \frac{x}{R_{\max}} \tag{4}$$

$$\mu_{\text{risk}}(x) = 1 - \frac{x}{K_{\max}} \tag{5}$$

$$\nu_{\text{risk}}(x) = \frac{x}{K_{\max}} \tag{6}$$

5.5. Example Data

We evaluate two projects with the following data:

- **Project 1:**

$$\text{Cost} = \$50000 \qquad \mu_{\text{cost}}(50000) = 1 - \frac{50000}{C_{\max}} \tag{7}$$

$$\text{Revenue} = \$200000 \qquad \mu_{\text{revenue}}(200000) = \frac{200000}{R_{\max}} \tag{8}$$

$$\text{Risk} = \$10000 \qquad \mu_{\text{risk}}(10000) = 1 - \frac{10000}{K_{\max}} \tag{9}$$

- **Project 2:**

$$\text{Cost} = \$70000 \qquad \mu_{\text{cost}}(70000) = 1 - \frac{70000}{C_{\max}} \tag{10}$$

$$\text{Revenue} = \$150000 \qquad \mu_{\text{revenue}}(150000) = \frac{150000}{R_{\max}} \tag{11}$$

$$\text{Risk} = \$15000 \qquad \mu_{\text{risk}}(15000) = 1 - \frac{15000}{K_{\max}} \tag{12}$$

5.6. Calculations and Decision Rule

Calculate the intuitionistic index $\pi(x)$ for each criterion:

$$\pi_{\text{cost}}(x) = \mu_{\text{cost}}(x) - \nu_{\text{cost}}(x) \tag{13}$$
$$\pi_{\text{revenue}}(x) = \mu_{\text{revenue}}(x) - \nu_{\text{revenue}}(x) \tag{14}$$
$$\pi_{\text{risk}}(x) = \mu_{\text{risk}}(x) - \nu_{\text{risk}}(x) \tag{15}$$

Calculate the overall decision index $\Pi(x)$ as the weighted sum of these indices, considering a tolerance level T:

$$\Pi(x) = w_c \cdot \pi_{\text{cost}}(x) + w_r \cdot \pi_{\text{revenue}}(x) + w_k \cdot \pi_{\text{risk}}(x) \tag{16}$$

Assume equal weights: $w_c = w_r = w_k = \frac{1}{3}$.

Calculation for Project 1:

$$\pi_{\text{cost}}(50000) = \left(1 - \frac{50000}{C_{\text{max}}}\right) - \frac{50000}{C_{\text{max}}} \tag{17}$$
$$\pi_{\text{revenue}}(200000) = \frac{200000}{R_{\text{max}}} - \left(1 - \frac{200000}{R_{\text{max}}}\right) \tag{18}$$
$$\pi_{\text{risk}}(10000) = \left(1 - \frac{10000}{K_{\text{max}}}\right) - \frac{10000}{K_{\text{max}}} \tag{19}$$
$$\Pi(\text{Project 1}) = \frac{1}{3} \times \pi_{\text{cost}}(50000) + \frac{1}{3} \times \pi_{\text{revenue}}(200000) + \frac{1}{3} \times \pi_{\text{risk}}(10000) \tag{20}$$

Calculation for Project 2:

$$\pi_{\text{cost}}(70000) = \left(1 - \frac{70000}{C_{\text{max}}}\right) - \frac{70000}{C_{\text{max}}} \tag{21}$$
$$\pi_{\text{revenue}}(150000) = \frac{150000}{R_{\text{max}}} - \left(1 - \frac{150000}{R_{\text{max}}}\right) \tag{22}$$
$$\pi_{\text{risk}}(15000) = \left(1 - \frac{15000}{K_{\text{max}}}\right) - \frac{15000}{K_{\text{max}}} \tag{23}$$
$$\Pi(\text{Project 2}) = \frac{1}{3} \times \pi_{\text{cost}}(70000) + \frac{1}{3} \times \pi_{\text{revenue}}(150000) + \frac{1}{3} \times \pi_{\text{risk}}(15000) \tag{24}$$

5.7. Decision

Based on the decision indices $\Pi(x)$, Project 1 (0.267) is preferred over Project 2 (-0.133).

Importance of the Contribution

This example highlights how T-rough intuitionistic fuzzy ideals in BCK-algebras can effectively manage uncertainty and imprecision in decision-making processes. By incorporating a tolerance level and using algebraic structures, this approach provides a robust framework for evaluating complex scenarios where precise data may not be available.

6. Conclusions and Perspectives

In conclusion, this paper introduces the novel concept of T-rough intuitionistic Fuzzy Ideals (FIs) within the framework of BCK-algebras, laying the groundwork for a new avenue of research at the intersection of fuzzy logic and algebraic structures. By exploring the fundamental properties of these ideals, we have provided a robust mathematical foundation that extends traditional approaches to fuzzy sets and ideals, particularly in the context of BCK-algebras.

The introduction of a set-valued homomorphism over BCK-algebras has been a key innovation, facilitating the development of T-rough intuitionistic FIs. This homomorphism enables a more refined and precise characterization of these fuzzy ideals, allowing for the capture of nuanced relationships within the algebraic structure that would otherwise remain obscured in classical treatments.

One of the most significant contributions of this work is the characterization of T-rough intuitionistic FIs through the (α, β)-cut of Intuitionistic Fuzzy (IF) sets in BCK-algebras. This approach not only solidifies the theoretical underpinnings of T-rough intuitionistic FIs but also opens up new possibilities for their application in various mathematical and decision-making contexts. The (α, β)-cut method provides a powerful tool for dissecting and analyzing the behavior of fuzzy ideals under different conditions, thereby offering a more comprehensive understanding of their structure and properties.

The distinctive nature of T-rough intuitionistic FIs, as characterized by these (α, β)-cuts, provides a solid basis for their application in complex decision-making scenarios, particularly where uncertainty and imprecision are inherent. By incorporating a tolerance level T, these ideals offer a more flexible and adaptive framework for decision-making processes, making them capable of accommodating varying degrees of uncertainty.

Perspectives

Looking forward, the concepts and methods introduced in this paper open up several promising avenues for future research and application. One potential area of exploration is the integration of T-rough intuitionistic FIs with other algebraic structures beyond BCK-algebras. Such integration could further expand their applicability and provide new insights into the complex interplay between fuzzy logic and algebraic theory.

Additionally, there is significant potential for applying T-rough intuitionistic FIs in advanced decision-making algorithms, particularly in fields where the handling of fuzzy, uncertain, or imprecise information is critical. For instance, they could be utilized in the development of more sophisticated models for risk assessment, financial forecasting, or multi-criteria decision analysis, where traditional methods may not adequately capture the complexity of the data.

Furthermore, the exploration of T-rough intuitionistic FIs in optimization problems presents another exciting research direction. By leveraging the flexibility and adaptability of these fuzzy ideals, it may be possible to devise new optimization techniques that are better suited to real-world problems characterized by high degrees of uncertainty and variability.

The potential applications of T-rough intuitionistic FIs in artificial intelligence and machine learning also warrant further investigation. Their ability to manage and process imprecise information could be particularly valuable in the development of AI systems that require robust decision-making capabilities under uncertain conditions.

In summary, this work not only advances the theoretical landscape of fuzzy logic and BCK-algebras but also lays the foundation for a wide range of practical applications. The development of T-rough intuitionistic fuzzy ideals represents a significant step forward in our understanding and utilization of fuzzy algebraic structures, with far-reaching implications for both mathematics and decision sciences. Future research in this area holds the promise of even greater insights and innovations, particularly in the context of complex decision-making and optimization challenges in the real world.

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