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On the Decay in the Energy Space of Solutions to the Damped Magnetic Radial Schrödinger Equation with Non-Local Nonlinearities

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Abstract: We will explore, in any space dimension $d \geq 4$, the decay in the energy space for the damped magnetic Schrödinger equation with non-local nonlinearity and radial initial data in $H^1(\mathbb{R}^d)$. We will also display new Morawetz identities and corresponding localized Morawetz estimates.

Keywords: nonlinear Schrödinger equations; Schrödinger operators; scattering theory; non-local nonlinearity; damping

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1. Introduction

We will analyze the following Cauchy problem for the nonlinear defocusing magnetic Schrödinger equation with non-local nonlinearity in the high dimensional frame $d \geq 4$:

$$\begin{cases} i\partial_t u + \Delta_x^A u + ib(t)u - k[|\cdot|^{-(d-\gamma)} * |u|^2]u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u(0,x) = f(x) \in H^1(\mathbb{R}^d), \end{cases} \tag{1}$$

with $k \in \mathbb{R}$, where $u = u(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{C}$, $\nabla_x^A = \nabla - iA$, $A = \left(A^1, \ldots, A^d\right) \in \mathbb{C}^1_{loc}(\mathbb{R}^d \setminus \{0\}; \mathbb{R})$, so that $\operatorname{div} A = 0$, $-\Delta_x^A = -\nabla_x^A \cdot \nabla_x^A$ is self-adjoint on $L^2(\mathbb{R}^d)$ and $b : [0,\infty) \to \mathbb{C}$ is a measurable function that contains dissipative and oscillatory terms. We shall also assume that

$$|A|^2 - 2iA \cdot \nabla \in L^{\frac{d}{2},\infty}(\mathbb{R}^d), \quad A \in L^{d,\infty}(\mathbb{R}^d). \tag{2}$$

Moreover,

$$\||x|xB\|^2_{L^\infty(\mathbb{R}^d)} \leq (d-1)(d-3), \tag{3}$$

where the magnetic field $B : \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{R})$ is defined by

$$B := DA - (DA)^t,$$

with

$$(DA)_{ij} = \partial_i A^j, \quad (DA)^t_{ij} = (DA)_{ji}.$$

We will impose further the conditions on the nonlinear terms:

$$d - 2 \leq \gamma < d, \tag{4}$$

$\Re b(t), \Im b(t) \in \mathcal{C}([0,\infty))$ with $\Re b(t) \geq 0$ and

$$\mathbf{B}(t) = \int_0^t b(s)ds, \quad \inf_{t>0}\left(\frac{\Re \mathbf{B}(t)}{t}\right) \geq 0. \tag{5}$$

The last of the two conditions above, roughly speaking, means that every global solution of (1) behaves like the solution of the associated free equation (that is, $b(t) = k = 0$) as $t \to +\infty$. The main goal of this paper is to show the decay of the solutions to (1) in the energy space. More explicitly, we will prove the following theorem.

Theorem 1. *Let $d \geq 4$ and $k = 1$, and let $u \in \mathcal{C}([0,\infty); H^1(\mathbb{R}^d))$ be a global solution to (1) with radial initial data $f \in H^1(\mathbb{R}^d)$ such that (2) and the strict inequality in (3)–(5) are satisfied. Then, for $2 < r < \frac{2d}{d-2}$, one achieves*

$$\lim_{t\to\infty} e^{\Re \mathbf{B}(t)} \|u(t,x)\|_{L^r(\mathbb{R}^d)} = 0. \tag{6}$$

Equation (1) is significant in many mathematical physics models. For instance, it was introduced in quantum mechanics to analyze the behavior of Bose–Einstein condensates by considering the self-interactions of charged particles, as discussed in [1–3], and the references therein. This has spurred numerous studies on the Schrödinger–Hartree equation. For example, Ref. [4] demonstrates the asymptotic completeness and the existence of wave operators for both the nonlinear Schrödinger equation with $L^2 - H^1$ intercritical nonlinearity and the Schrödinger–Hartree equation. Subsequent improvements on these results for the Schrödinger–Hartree equation are found in [5]. Additionally, Refs. [6,7] employed the pseudo-conformal transform to study scattering solutions of the Schrödinger–Hartree equation in spaces with higher regularity than H^1. In the critical case, Ref. [8] established scattering for general data with $d \geq 5$. Scattering in the focusing case was achieved in [9,10] for small and radial data. Further references for the NLS in a general setting include [11,12]. A principal tool in studying the dynamics of solutions to (1) is the Morawetz multiplier technique and its associated estimates. In our recent work [13], we developed a method combining Morawetz inequalities, a localization step, and interpolation with a contradiction argument to achieve the decay of solutions for the Schrödinger–Hartree equation. This robust property is crucial in scattering theory, as highlighted in [12–14]. Motivated by these developments, we present a generalization of this method for the damped magnetic Schrödinger equation with Hartree-type nonlinearity. The linearly damped nonlinear Schrödinger equation plays a significant role across multiple scientific disciplines, including nonlinear optics, plasma physics, and fluid mechanics. This equation is fundamental for understanding various complex phenomena, such as the propagation of optical pulses in nonlinear media, the behavior of plasma waves in magnetized environments, and the dynamics of fluid flows under certain conditions. We quote here, for example, [15,16]. Our result is novel in the literature, and we make minimal assumptions on the magnetic function $A(x)$. Furthermore, our strategy simplifies and extends the damped magnetic Schrödinger equation to the approach used in [17–20]. We emphasize also that the approaches previously proposed, for instance, in [21–23] (see also references therein), are outperformed since we coped with the complex-valued function $b(t)$ in (1).

2. Preliminaries

Before outlining our main achievements, we will unveil some necessary notations and several useful results. For any two positive real numbers a, b, we write $a \lesssim b$ (resp. $a \gtrsim b$) to denote $a \leq Cb$ (resp. $Ca \geq b$), with $C > 0$, and we unravel the constant only when

it is necessary. We introduce the Banach space $L^r(\mathbb{R}^d) = L^r_x$ for $1 \leq r \leq \infty$. In addition, we introduce
$$H^{1,r}(\mathbb{R}^d) = (1-\Delta_x)^{-\frac{1}{2}} L^r(\mathbb{R}^d), \quad H^{1,r}(\mathbb{R}^d) = H^{1,r}_x,$$
and denote it with $H^{1,2}(\mathbb{R}^d) = H^1(\mathbb{R}^d) = H^1_x$. Given any Banach space X, we define

$$\|f\|_{L^\infty_t X} = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \|f(x)\|_X.$$

We adopt the notation $L^\infty_T X$ when one restricts $t \in [0,T)$, for $T > 0$. The following results are also useful (see [9,17,19], respectively).

Lemma 1. *Let f be a radial function in H^1_x. Then,*

$$\left\| |x|^{\frac{d-1}{2}} f \right\|^2_{L^\infty_x} \lesssim \|f\|_{L^2_x} \|\nabla_x f\|_{L^2_x}. \tag{7}$$

Proposition 1. *Let A be as in (2) and (3). For any $1 < r < d$, one obtains*

$$\left\| (-\Delta^A_x)^{\frac{1}{2}} f \right\|_{L^r_x} \lesssim \left\| (-\Delta_x)^{\frac{1}{2}} f \right\|_{L^r_x} \tag{8}$$

and

$$\left\| (-\Delta_x)^{\frac{1}{2}} f \right\|_{L^2_x} \lesssim \left\| (-\Delta^A_x)^{\frac{1}{2}} f \right\|_{L^2_x}. \tag{9}$$

We also have the following maximal estimate (see, for example, [24]), as a straightforward consequence of the Hardy inequality.

Proposition 2. *Let $0 < \gamma < d$. We have*

$$\left\| [|\cdot|^{d-\gamma} * |u|^2] \right\|_{L^\infty_x} \leq C(d,\gamma) \|u\|^2_{\dot{H}_x^{\frac{d-\gamma}{2}}}. \tag{10}$$

We recall also that the solutions to (1) satisfy the conservation laws. We summarize them in the following.

Proposition 3. *Let $d \geq 1$. Then, a sufficiently smooth solution to (1) satisfies the following identities:*

$$\|u(t)\|_{L^2_x} = e^{-\Re \mathbf{B}(t)} \|f\|_{L^2_x}, \quad H(u(t)) = H(f), \tag{11}$$

where

$$\begin{aligned} H(u(t)) = e^{2\Re\mathbf{B}(t)} \int_{\mathbb{R}^d} |\nabla^A_x u(t,x)|^2\, dx + k e^{2\Re\mathbf{B}(t)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^{d-\gamma}} dxdy \\ + 2k \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Re b(s) e^{2\Re\mathbf{B}(t)} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^{d-\gamma}} dxdy. \end{aligned} \tag{12}$$

Proof. We utilize the change in variable

$$v(t,x) := e^{\mathbf{B}(t)} u(t,x) \tag{13}$$

and see that u satisfies (1) if v solves

$$\begin{cases} i\partial_t v + \Delta^A_x v = k e^{-2\Re\mathbf{B}(t)} [|\cdot|^{-d+\gamma} * |v|^2] v, & (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ v(0,x) = u_0(x). \end{cases} \tag{14}$$

We multiply the above equation by $\bar{u}(t,x)$, integrate with respect to the x-variable, and take the imaginary part, obtaining the following, since div $A = 0$:

$$i\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|v(t,x)|^2dx + \int_{\mathbb{R}^d}\bar{v}(t,x)\Big(-\Delta u(t,x) + |A|^2u(t,x) - 2iA\cdot\nabla v(t,x)\Big)dx$$
$$+\int_{\mathbb{R}^d}ke^{-2\Re\mathbf{B}(t)}[|x|^{-d+\gamma} * |v(t,x)|^2]|v(t,x)|^2dx$$
$$= i\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|v(t,x)|^2dx - 2i\int_{\mathbb{R}^2}A\cdot\nabla\Big(|v(t,x)|^2\Big)dx = 0.$$

Thus, solutions local in time satisfy the conservation of mass

$$\|v(t)\|_{L^2}^2 = \|f\|_{L^2}^2.$$

that is, the first identity in (11). We multiply now Equation (14) by $\bar{u}(t,x)$, integrate with respect to the x-variable, and take the imaginary equation part. We have

$$\Re\int_{\mathbb{R}^d}\Big(\nabla_{\mathcal{A}}v(t,x)\nabla_{\mathcal{A}}\partial_t\bar{v}(t,x) + ke^{-2\Re\mathbf{B}(t)}[|x|^{-d+\gamma} * |v(t,x)|^2]v(t,x)\partial_t\bar{v}(t,x)\Big)dx = 0.$$

The previous identity is enhanced to

$$\int_{\mathbb{R}^d}\left(\frac{1}{2}\partial_t|\nabla_{\mathcal{A}}v(t,x)|^2 + \frac{1}{2}ke^{-2\Re\mathbf{B}(t)}[|x|^{-d+\gamma} * |v(t,x)|^2]\partial_t|v(t,x)|^2\right)dx = 0$$

and then to

$$\partial_t\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla_{\mathcal{A}}v(t,x)|^2 + \frac{1}{2}ke^{-2\Re\mathbf{B}(t)}[|x|^{-d+\gamma} * |v(t,x)|^2]|v(t,x)|^2\right)dx$$
$$= -k\int_{\mathbb{R}^d}\Re b(t)e^{-2\Re\mathbf{B}(t)}[|x|^{-d+\gamma} * |v(t,x)|^2]|v(t,x)|^2dx. \tag{15}$$

Integrating with respect to the t-variable identity (15), we obtain

$$\int_{\mathbb{R}^d}\Big(|\nabla_{\mathcal{A}}v(t,x)|^2 + ke^{-2\Re\mathbf{B}(t)}[|x|^{-d+\gamma} * |v(t,x)|^2]|v(t,x)|^2\Big)dx$$
$$+2k\int_0^t\int_{\mathbb{R}^d}\Re b(s)e^{-2\Re\mathbf{B}(s)}[|x|^{-d+\gamma} * |v(s,x)|^2]|v(s,x)|^2dxds$$
$$= \int_{\mathbb{R}^d}\Big(|\nabla_{\mathcal{A}}v(0,x)|^2 + k[|x|^{-d+\gamma} * |v(0,x)|^2]|v(0,x)|^2\Big)dx.$$

The above relation suggests that the quantity

$$\widetilde{H}(v(t)) = \int_{\mathbb{R}^d}\Big(|\nabla_{\mathcal{A}}v(t,x)|^2 + ke^{-2\Re\mathbf{B}(t)}[|x|^{-d+\gamma} * |v(t,x)|^2]|v(t,x)|^2\Big)dx$$
$$+2k\int_0^t\int_{\mathbb{R}^d}\Re b(s)e^{-2\Re\mathbf{B}(s)}[|x|^{-d+\gamma} * |v(s,x)|^2]|v(s,x)|^2dxds$$

is conserved. Hence, this implies the local conservation of the Hamiltonian in (11) with $H(u(t))$ as in (12). □

3. Well-Posedness

Here, we present the following existence and uniqueness result, which is crucial for the proof of (6). Specifically, we prove the following proposition.

Proposition 4. *Let $d \geq 3$. Assume that (2)–(5) are satisfied. Then, for all $f \in H_x^1$, there exists $T > 0$ such that problem (1) has a unique local solution $u \in \mathcal{C}([0,T);H_x^1)$ with*

$$\Big\|e^{\mathbf{B}(t)}u(t,x)u\Big\|_{L_T^\infty H_x^1} \lesssim \|f\|_{H_x^1}.$$

Moreover, the solution can be extended globally in time if $k > 0$.

Proof. We shall accomplish a fixed-point argument. Namely, consider the integral operator associated with (14) to be defined for all $f \in H^1_x$ as

$$\mathcal{T}_f(e^{\mathbf{B}(t)}u) = e^{it\Delta^A_x + \mathbf{B}(t)}f + k\int_0^t e^{-2\Re\mathbf{B}(t)}e^{i(t-\tau)\Delta^A_x}\Big([|\cdot|^{-(d-\gamma)} * |e^{\mathbf{B}(t)}u|^2]e^{\mathbf{B}(t)u}\Big)(\tau)\, d\tau.$$

We need to show that it is possible to find a $T = T\big(\|f\|_{H^1_x}\big) > 0$ and a unique

$$e^{\mathbf{B}(t)}u(t,x) \in L^\infty_T H^1_x$$

satisfying the property

$$\mathcal{T}_f(e^{\mathbf{B}(t)}u(t)) = e^{\mathbf{B}(t)}u(t), \tag{16}$$

for $t \in [0,T)$. For the sake of simplicity, we will divide the proof into different steps.

Step One: For any $e^{\mathbf{B}(t)}u \in H^1_x$, there exist $T = T\big(\|f\|_{H^1_x}\big) > 0$ and $R = R\big(\|f\|_{H^1_x}\big) > 0$ such that

$$\mathcal{T}_f(B_{L^\infty_{T'}H^1_x}(0,R)) \subset B_{L^\infty_{T'}H^1_x}(0,R),$$

for any $T' < T$.

By the classical Hardy–Littlewood–Sobolev inequality combined with (8) and (9), we have

$$\begin{aligned}\|\mathcal{T}_f(e^{\mathbf{B}(t)}u)\|_{L^\infty_T L^2_x} + \|\nabla_x \mathcal{T}_f(e^{\mathbf{B}(t)}u)\|_{L^\infty_T L^2_x} &\lesssim \|\mathcal{T}_f(e^{\mathbf{B}(t)}u)\|_{L^\infty_T L^2_x} + \|\nabla^A_x \mathcal{T}_f(e^{\mathbf{B}(t)}u)\|_{L^\infty_T L^2_x}\\ &\lesssim \|f\|_{H^1_x} + \int_0^T \|e^{-2\Re\mathbf{B}(\cdot)}[|\cdot|^{-d+\gamma} * |e^{\mathbf{B}(\cdot)}u|^2]e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x} d\tau.\end{aligned}$$

At this point, by condition (5), the last term in the above chain of inequalities can be controlled as follows:

$$\begin{aligned}&\|f\|_{H^1_x} + \int_0^T \|e^{-2\Re\mathbf{B}(\cdot)}[|\cdot|^{d-\gamma} * |e^{\mathbf{B}(\cdot)}u|^2]e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x} d\tau\\ &\lesssim \|f\|_{H^1_x} + T\Big\|e^{-2\Re\mathbf{B}(\cdot)}\Big\|_{L^\infty_t}\|[|\cdot|^{-d+\gamma} * |e^{\mathbf{B}(\cdot)}u|^2]e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x}\\ &\lesssim \|f\|_{H^1_x} + T\Big\|[|\cdot|^{-d+\gamma} * |e^{\mathbf{B}(\cdot)}u|^2]\Big\|_{L^\infty_T L^\infty_x}\|e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x}\\ &\quad + T\Big\|[|\cdot|^{-d+\gamma} * |e^{\mathbf{B}(\cdot)}u|^2]\Big\|_{L^\infty_T H_x^{1,\frac{2d}{d-\gamma}}}\|e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}}\\ &\lesssim \|f\|_{H^1_x} + T\|e^{\mathbf{B}(\cdot)}u\|^2_{L^\infty_T \dot{H}_x^{\frac{d-\gamma}{2}}}\|e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x} + T\Big\||e^{\mathbf{B}(\cdot)}u|^2\Big\|_{L^\infty_T H_x^{1,\frac{2d}{d+\gamma}}}\|e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}}\\ &\lesssim \|f\|_{H^1_x} + T\|e^{\mathbf{B}(\cdot)}u\|^2_{L^\infty_T \dot{H}_x^{\frac{d-\gamma}{2}}}\|e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x} + T\|e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x}\|e^{\mathbf{B}(\cdot)}u\|^2_{L^\infty_T L_x^{\frac{2d}{\gamma}}}\\ &\lesssim \|f\|_{H^1_x} + T\|e^{\mathbf{B}(\cdot)}u\|^2_{L^\infty_T \dot{H}_x^{\frac{d-\gamma}{2}}}\|e^{\mathbf{B}(\cdot)}u\|_{L^\infty_T H^1_x} \lesssim \|f\|_{H^1_x} + TR^3.\end{aligned}$$

By selecting R and T so that

$$2\|f\|_{H^1_x} = R, \quad 2CTR^2 \le 1,$$

we finish the proof of this step.

Step Two: Let $T, R > 0$ be as in the above step. Then, there exists $\overline{T} = \overline{T}\big(\|f\|_{H^1_x}\big) < T$ such that $\mathcal{T}_f$ is a contraction on $B_{L^\infty_{\overline{T}}H^1_x}(0,R)$, equipped with the norm $\|.\|_{L^\infty_{\overline{T}}L^2_x}$.

Consider $e^{\mathbf{B}(\cdot)}v_1, e^{\mathbf{B}(\cdot)}v_2 \in B_{L^\infty_T H^1_x}(0,R)$. We obtain, by arguing as in the previous lines, the following chain of inequalities:

$$\begin{aligned}
&\|\mathcal{T}_f e^{\mathbf{B}(\cdot)}v_1 - \mathcal{T}_f e^{\mathbf{B}(\cdot)}v_2\|_{L^\infty_T L^2_x} \\
&\lesssim T\left\|e^{-2\Re\mathbf{B}(\cdot)}\right\|_{L^\infty_t}\left\|[|\cdot|^{-d+\gamma} * |e^{\mathbf{B}(\cdot)}v_1|^2]e^{\mathbf{B}(\cdot)}v_1 - [|\cdot|^{-d+\gamma} * |e^{\mathbf{B}(\cdot)}v_2|^2]e^{\mathbf{B}(\cdot)}v_2\right\|_{L^\infty_T L^2_x} \\
&\lesssim T\left\|[|\cdot|^{-d+\gamma} * |e^{\mathbf{B}(\cdot)}v_1|^2](e^{\mathbf{B}(\cdot)}v_1 - e^{\mathbf{B}(\cdot)}v_2)\right\|_{L^\infty_T L^2_x} \\
&+T\left\|[|\cdot|^{-d+\gamma} * (|e^{\mathbf{B}(\cdot)}v_1|^2 - |e^{\mathbf{B}(\cdot)}v_2|^2)]e^{\mathbf{B}(\cdot)}v_2\right\|_{L^\infty_T L^2_x} \\
&\lesssim T\|e^{\mathbf{B}(\cdot)}v_1\|^2_{L^\infty_T H_x^{\frac{-d+\gamma}{2}}}\|e^{\mathbf{B}(\cdot)}v_1 - e^{\mathbf{B}(\cdot)}v_2\|_{L^\infty_T L^2_x} \\
&+T\left\|[|\cdot|^{-d+\gamma} * (|e^{\mathbf{B}(\cdot)}v_1|^2 - |e^{\mathbf{B}(\cdot)}v_2|^2)]\right\|_{L^\infty_T L_x^{\frac{2d}{d-\gamma}}}\|e^{\mathbf{B}(\cdot)}v_2\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}} \\
&\lesssim T\left(R^2\|e^{\mathbf{B}(\cdot)}v_1 - e^{\mathbf{B}(\cdot)}v_2\|_{L^\infty_T L^2_x} + R\|v_1+v_2\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}}\|e^{\mathbf{B}(\cdot)}v_1 - e^{\mathbf{B}(\cdot)}v_2\|_{L^\infty_T L^2_x}\right) \\
&\lesssim TR^2\|v_1 - v_2\|_{L^\infty_T L^2_x}.
\end{aligned}$$

Then, we arrive at

$$\|\mathcal{T}_f e^{\mathbf{B}(\cdot)}v_1 - \mathcal{T}_f e^{\mathbf{B}(\cdot)}v_2\|_{L^\infty_T L^2_x} \lesssim TR^2\|e^{\mathbf{B}(\cdot)}v_1 - e^{\mathbf{B}(\cdot)}v_2\|_{L^\infty_T L^2_x}.$$

This inequality allows us to say that $\mathcal{T}_f$ is a contraction on $B_{L^\infty_T H^1_x}(0,R)$ if T is chosen in a suitable manner.

Step Three: The solution exists and is unique in $L^\infty_{\overline{T}} H^1_x$, where $\overline{T}$ is as in the above step.

We can exhibit the existence and uniqueness of the solution using the contraction principle for the map $\mathcal{T}_f$ defined on the complete metric space $B_{L^\infty_{\overline{T}} H^1_x}(0,R)$, endowed with the topology induced by $\|.\|_{L^\infty_{\overline{T}} L^2_x}$.

Step Four: The solution can be extended globally.

We obtain, by conservation laws (11), (8), and (9), that

$$\left\|e^{\mathbf{B}(t)}u\right\|_{H^1_x} \lesssim \left\|e^{\mathbf{B}(t)}u\right\|_{L^2_x} + \left\|\nabla^A_x e^{\mathbf{B}(t)}u\right\|_{L^2_x} \lesssim H(u(0)) + \|f\|_{L^2_x}. \tag{17}$$

The previous bound leads to the global well-posedness for (1). □

4. Morawetz Identities and Inequalities

Our first contribution is the Morawetz equalities associated to (1). They are presented in the following.

Lemma 2. *Let $d \geq 1$ and $u \in \mathcal{C}([0,\infty); H^1_x)$ denote a global solution to (1) with radial initial data $f \in H^1_x$ such that (2) and the strict inequality in (3)–(5) are satisfied. Moreover, let $\psi = \psi(x) : \mathbb{R}^d \to \mathbb{R}$ be a sufficiently regular and decaying function, denoted by*

$$\mathcal{V}(t) := \int_{\mathbb{R}^d} \psi(x)|e^{\mathbf{B}(t)}u(t,x)|^2\,dx.$$

Then, the following identities hold:

$$\dot{\mathcal{V}}(t) = 2\Im\int_{\mathbb{R}^d} e^{2\Re\mathbf{B}(t)}\bar{u}(t,x)\nabla_x\psi(x)\cdot\nabla^A_x u(t,x)\,dx \tag{18}$$

and

$$
\begin{aligned}
\ddot{\mathcal{V}}(t) = &-\int_{\mathbb{R}^d} \Delta_x^2\psi(x)|e^{\mathbf{B}(t)}u(t,x)|^2\,dx \\
&+4\int_{\mathbb{R}^d} e^{2\Re\mathbf{B}(t)}\nabla_x^A u(t,x) D_x^2\psi(x)\cdot\overline{\nabla_x^A u(t,x)}\,dx \\
&-4\Im\int_{\mathbb{R}^d} e^{2\Re\mathbf{B}(t)} u(t,x)\nabla_x\psi(x)\cdot B(x)\overline{\nabla_x^A u(t,x)}\,dx \\
&-2k\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}\nabla_x\psi(x)\cdot\nabla_x\Big[|x|^{-(d-\gamma)} * |e^{\mathbf{B}(t)}u(t,x)|^2\Big]|e^{\mathbf{B}(t)}u(t,x)|^2\,dx,
\end{aligned} \tag{19}
$$

where $D_x^2\psi \in \mathcal{M}_{d\times d}(\mathbb{R})$ is the Hessian matrix of ψ, and $\Delta_x^2\psi = \Delta_x(\Delta_x\psi)$, the bi-Laplacian operator.

Proof. We will prove the identities for a smooth, rapidly decreasing solution $u = u(t,x)$, recovering the general case $e^{\mathbf{B}(t)}u \in \mathcal{C}(\mathbb{R}; H_x^1)$ via a density argument. The proof of (18) is similar to the one given in [20], since we can use transformation (13) and then Equation (14). We present details for obtaining (19). We have the following identity for the linear terms, using Theorem 1.2 in [20] and the $v(t,x)$ defined in (13) and (14) again. We obtain

$$
\begin{aligned}
&2\Re\int_{\mathbb{R}^d}\Big(-\Delta_x^A v(t,x)\Big)\Big(\Delta_x\psi(x)\bar{v}(t,x) + 2\nabla_x\psi(x)\cdot\overline{\nabla_x^A v(t,x)}\Big)\,dx \\
&= -\int_{\mathbb{R}^d}\Delta_x^2\psi(x)|v(t,x)|^2\,dx - 4\Im\int_{\mathbb{R}^d} v(t,x)\nabla_x\psi(x)\cdot B(x)\overline{\nabla_x^A v(t,x)}\,dx \\
&\quad +4\int_{\mathbb{R}^d}\nabla_x^A v(t,x) D_x^2\psi(x)\overline{\nabla_x^A v(t,x)}\,dx.
\end{aligned} \tag{20}
$$

In addition, for the nonlinear terms, one has

$$
\begin{aligned}
&2\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * ||^2]|v(t,x)|^2\Delta_x\psi(x)\,dx \\
&+4\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * |v(t,x)|^2]v(t,x)\nabla_x\psi(x)\cdot\overline{\nabla_x^A v(t,x)}\,dx \\
&= 2\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * |v(t,x)|^2]|v(t,x)|^2\Delta_x\psi(x)\,dx \\
&+4\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * |v(t,x)|^2]v(t,x)\nabla_x\psi(x)\cdot\nabla_x\bar{v}(t,x)\,dx.
\end{aligned}
$$

The last term of the above identity is equal to

$$
\begin{aligned}
&2\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * |v(t,x)|^2]|v(t,x)|^2\Delta_x\psi(x)\,dx \\
&+2\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * |v(t,x)|^2]\nabla_x\psi(x)\cdot\nabla_x|v(t,x)|^2\,dx.
\end{aligned}
$$

Then, through integration by parts of the second term in the last line above, one arrives at

$$
\begin{aligned}
&2\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * |v(t,x)|^2]|v(t,x)|^2\Delta_x\psi(x)\,dx \\
&+4\Re\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}[|x|^{-(d-\gamma)} * |v(t,x)|^2]v(t,x)\nabla_x\psi(x)\cdot\overline{\nabla_x^A v(t,x)}\,dx \\
&= -2\int_{\mathbb{R}^d} e^{-2\Re\mathbf{B}(t)}\nabla_x\psi(x)\cdot\nabla_x\Big[|x|^{-(d-\gamma)} * |v(t,x)|^2\Big]|v(t,x))|^2\,dx.
\end{aligned} \tag{21}
$$

Combining now identities (20) and (21) and turning back to $e^{\mathbf{B}(t)}u(t,x)$, we obtain (19). □

A Localized Morawetz Inequality

We start this section with a result that is a consequence of Lemma 2. More precisely, we have the following lemma

Lemma 3. *Assume $d \geq 4$ and let $u \in \mathcal{C}([0,\infty); H^1_x)$ be a global solution to (1) with radial initial data $f \in H^1_x$ such that (2) and the strict inequality in (3)–(5) are satisfied. Then, it holds that*

$$\int_{\mathbb{R}^d} \frac{1}{|x|^3} |e^{\mathbf{B}(t)} u(t,x)|^2 \, dx \lesssim \Im \int_{\mathbb{R}^d} e^{2\Re \mathbf{B}(t)} \bar{u}(t,x) \nabla_x \psi(x) \cdot \nabla^A_x u(t,x) \, dx. \tag{22}$$

Proof. We pick $\psi = \psi(x) = |x|$. This gives

$$\nabla_x \psi = \frac{x}{|x|}, \quad \Delta_x \psi = \frac{d-1}{|x|}, \quad \Delta^2_x \psi = -\frac{(d-1)(d-3)}{|x|^3}, \tag{23}$$

if $d \geq 4$. A change in variable (13), Equation (14), and an application of identity (19) allow us to write the following:

$$\begin{aligned} 2\partial_t \Im \int_{\mathbb{R}^d} \bar{v}(t,x) \nabla_x \psi(x) \cdot \nabla^A_x v(t,x) \, dx &= -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta^2_x \psi(x) |v(t,x)|^2 \, dx \\ &\quad -4\Im \int_{\mathbb{R}^d} v(t,x) \nabla_x \psi(x) \cdot B(x) \overline{\nabla^A_x v(t,x)} \, dx \\ &\quad +4 \int_{\mathbb{R}^d} \nabla^A_x v(t,x) D^2_x \psi(x) \overline{\nabla^A_x v(t,x)} \, dx \\ +k(d-\gamma) e^{-2\Re \mathbf{B}(t)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & \frac{1}{|x-z|^{d-\gamma+2}} |v(t,x)|^2 |v(t,z)|^2 K(x,z) \, dx dz, \end{aligned} \tag{24}$$

with

$$K(x,z) = (x-z) \cdot \left(\frac{x}{|x|} - \frac{z}{|z|} \right).$$

By the elementary inequality

$$(x-z) \cdot \left(\frac{x}{|x|} - \frac{z}{|z|} \right) = (|x||z| - (x) \cdot (z)) \left(\frac{|x| + |z|}{|x||z|} \right) \geq 0,$$

we have that $K(x,z) \geq 0$. Therefore, one can drop the last term on the right-hand side of (24). We shall focus now on the linear terms in (24), following the method utilized in [20]. Observe that the relations (23) relate to

$$\nabla^A_x v(t,x) D^2 \psi(x) \overline{\nabla^A_x v(t,x)} = \frac{\left| \nabla^\tau_A v(t,x) \right|^2}{|x|}, \tag{25}$$

(see identity (3.9) in [20]) where the operator ∇^τ_A is defined as

$$\nabla^\tau_A v(t,x) = \nabla^A_x v(t,x) - \left(\nabla^A_x v(t,x) \cdot \frac{x}{|x|} \right) \frac{x}{|x|}.$$

Therefore, utilizing (23), we have the following identity:

$$\begin{aligned} & -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta^2_x \psi(x) |v(t,x)|^2 \, dx \\ & -4\Im \int_{\mathbb{R}^d} v(t,x) \nabla_x \psi(x) \cdot B(x) \overline{\nabla^A_x v(t,x)} \, dx \\ & +4 \int_{\mathbb{R}^d} \nabla^A_x v(t,x) D^2_x \psi(x) \overline{\nabla^A_x v(t,x)} \, dx \\ = 4 \int_{\mathbb{R}^d} \frac{\left| \nabla^\tau_A v(t,x) \right|^2}{|x|} \, dx &+ (d-1)(d-3) \int_{\mathbb{R}^d} \frac{|v(t,x)|^2}{|x|^3} \, dx \\ & +4\Im \int_{\mathbb{R}^d} v(t,x) \frac{x}{|x|} B(x) \cdot \overline{\nabla^A_x v(t,x)} \, dx. \end{aligned} \tag{26}$$

The last term of the identity above can be estimated as

$$
\begin{aligned}
&-\left|\Im \int_{\mathbb{R}^d} v(t,x)\frac{x}{|x|}B(x)\cdot\overline{\nabla_x^A v(t,x)}\,dx\right| \\
&\geq -\left(\int_{\mathbb{R}^d}\frac{|v(t,x)|^2}{|x|^3}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|x|^2|xB(x)|^2|\nabla_A^\tau v(t,x)|^2\,dx\right)^{\frac{1}{2}} \\
&\geq -C_*\left(\int_{\mathbb{R}^d}\frac{|v(t,x)|^2}{|x|^3}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\frac{\left|\nabla_A^\tau v(t,x)\right|^2}{|x|}\,dx\right)^{\frac{1}{2}},
\end{aligned}
\tag{27}
$$

where

$$
C_*^2 = (d-1)(d-3).
$$

As a result, the right-hand side of (26) can be bounded as

$$
\begin{aligned}
&4\int_{\mathbb{R}^d}\frac{\left|\nabla_A^\tau v(t,x)\right|^2}{|x|}\,dx + (d-1)(d-3)\int_{\mathbb{R}^d}\frac{|v(t,x)|^2}{|x|^3}\,dx \\
&\quad +4\Im\int_{\mathbb{R}^d} v(t,x)\frac{x}{|x|}B(x)\cdot\overline{\nabla_x^A v(t,x)}\,dx \\
&\geq 4\int_{\mathbb{R}^d}\frac{\left|\nabla_A^\tau v(t,x)\right|^2}{|x|}\,dx + (d-1)(d-3)\int_{\mathbb{R}^d}\frac{|v(t,x)|^2}{|x|^3}\,dx \\
&\quad -4\widetilde{C}\left(\int_{\mathbb{R}^d}\frac{|v(t,x)|^2}{|x|^3}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\frac{\left|\nabla_A^\tau v(t,x)\right|^2}{|x|}\,dx\right)^{\frac{1}{2}} > 0.
\end{aligned}
\tag{28}
$$

Notice also that the previous inequality and a continuity argument guarantee that

$$
\begin{aligned}
&4\int_{\mathbb{R}^d}\frac{\left|\nabla_A^\tau v(t,x)\right|^2}{|x|}\,dx + (d-1)(d-3)\int_{\mathbb{R}^d}\frac{|v(t,x)|^2}{|x|^3}\,dx \\
&\quad +4\Im\int_{\mathbb{R}^d} v(t,x)\frac{x}{|x|}B(x)\cdot\overline{\nabla_x^A v(t,x)}\,dx \\
&\quad > \widetilde{\eta}(d-1)(d-3)\int_{\mathbb{R}^d}\frac{|v(t,x)|^2}{|x|^3}\,dx,
\end{aligned}
\tag{29}
$$

for a $\widetilde{\eta} > 0$. The above bound in combination with (24), the fact that $K(x,z) \geq 0$, and (26) give the proof of (22). □

We have the following corollary, which is a consequence of (22).

Corollary 1. *Let $u \in \mathcal{C}([0,\infty); H_x^1)$ be a global solution to (1) with radial initial data $f \in H_x^1$ such that (2) and the strict inequality in (3)–(5) are satisfied. Moreover, let $\mathcal{Q}_{\tilde{x}}^d(r) = \tilde{x} + [-r,r]^d$, with $r > 0$ and $\tilde{x} \in \mathbb{R}^d$. Hence, one obtains*

$$
\int_0^\infty \int_{\mathcal{Q}_{\tilde{x}}^d(r)} \frac{1}{|x|^3}|e^{\mathbf{B}(t)}u(t,x)|^2\,dxdt < \infty. \tag{30}
$$

Proof. By integrating (22) with $\psi(x)$ as in (23) with respect to the time variable on the interval $J = [t_1, t_2]$, with $t_1, t_2 \in [0,\infty)$, one arrives at

$$
\begin{aligned}
&\left[\Im\int_{\mathbb{R}^d} e^{2\Re\mathbf{B}(t)}\bar{u}(t,x)\nabla_x\psi(x)\cdot\nabla_x^A u(t,x)\,dx\right]_{t=t_1}^{t=t_2} \\
&\gtrsim \int_{t_1}^{t_2}\int_{\mathbb{R}^d}\frac{1}{|x|^3}|e^{\mathbf{B}(t)}u(t,x)|^2\,dxdt \gtrsim \int_{t_1}^{t_2}\int_{\mathcal{Q}_{\tilde{x}}^d(r)}\frac{1}{|x|^3}|e^{\mathbf{B}(t)}u(t,x)|^2\,dxdydt.
\end{aligned}
$$

Applying the Cauchy–Schwartz inequality and Proposition 1, we also infer that

$$2\left[\Im\int_{\mathbb{R}^d} e^{2\Re\mathbf{B}(t)}\bar{u}(t,x)\nabla_x\psi(x)\cdot\nabla_x^A u(t,x)\,dx\right]_{t=t_1}^{t=t_2} \lesssim \|f\|_{H_x^1}^2 < \infty, \tag{31}$$

since the H_x^1-norm is a quantity conserved by (17). Finally, we obtain (30) when $t_1 = 0$, $t_2 \to \infty$. $\square$

5. The Decay of Solutions

This section is devoted to demonstrating the main Theorem 1.

Proof. It is sufficient to prove property (6) for a suitable $2 < q < \frac{2d}{d-2}$ because the thesis for the general case follows conservation law (11) and interpolation. More precisely, it is enough to show that

$$\lim_{t\to\pm\infty} \|e^{\mathbf{B}(t)}u(t,x)\|_{L_x^{2+\frac{4}{d}}} = 0. \tag{32}$$

Then, property (6) follows for all $2 < q < \frac{2d}{d-2}$ by combining (32) with

$$\sup_{t\in\mathbb{R}} \|e^{\mathbf{B}(t)}u(t,x)\|_{H_x^1} < \infty. \tag{33}$$

We recall the following localized Gagliardo–Nirenberg inequality (see [13]):

$$\|\zeta\|_{L_x^{\frac{2d+4}{d}}}^{\frac{2d+4}{d}} \le C\left(\sup_{x\in\mathbb{R}^d}\|\zeta\|_{L^2(\mathcal{Q}_x(1))}\right)^{\frac{4}{d}}\|\zeta\|_{H_x^1}^2, \tag{34}$$

where $\mathcal{Q}_x^d(r) = x + [-1,1]^d$. Next, assume by contradiction that (32) is not true; then, by (33) and by (34), we deduce the existence of a sequence $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^d$ with $t_n \to \infty$ and $\epsilon_0 > 0$ such that

$$\inf_n \|e^{\mathbf{B}(t_n)}u(t_n,x)\|_{L^2(\mathcal{Q}_{x_n}(1))}^2 = \epsilon_0^2. \tag{35}$$

Notice that by (18) in conjunction with (33), we obtain

$$\sup_{n,t}\left|\frac{d}{dt}\int\phi(x-x_n)|e^{\mathbf{B}(t)}u(t,x)|^2\,dx\right| < \infty,$$

where $\chi(x)$ is a smooth and non-negative cut-off function, such that $\phi(x) = 1$ for $x \in \mathcal{Q}_0(1) = [-1,1]^d$ and $\phi(x) = 0$ for $x \notin \mathcal{Q}_0(2) = [-2,2]^d$. Consequently, by the Fundamental Theorem of calculus, we deduce the inequality

$$\left|\int_{\mathbb{R}^d}\phi(x-x_n)|e^{\mathbf{B}(\sigma)}u(\sigma,x)|^2dx - \int_{\mathbb{R}^d}\phi(x-x_n)|e^{\mathbf{B}(t)}u(t,x)|^2dx\right| \le \widetilde{C}|t-\sigma|, \tag{36}$$

for a $\widetilde{C} > 0$ that does not depend on n. By choosing $t = t_n$, we have

$$\int_{\mathbb{R}^d}\phi(x-x_n)|e^{\mathbf{B}(\sigma)}u(\sigma,x)|^2dx \ge \int_{\mathbb{R}^d}\phi(x-x_n)|e^{\mathbf{B}(t_n)}u(t_n,x)|^2dx - \widetilde{C}|t_n-\sigma|, \tag{37}$$

which implies the following, considering the support property of function ϕ:

$$\int_{\mathcal{Q}_{x_n}^d(2)}|e^{\mathbf{B}(\sigma)}u(\sigma,x)|^2dx \ge \int_{\mathcal{Q}_{x_n}^d(1)}|e^{\mathbf{B}(t_n)}u(t_n,x)|^2dx - \widetilde{C}|t_n-\sigma|, \tag{38}$$

for a $\widetilde{C} > 0$ independent form n. By combining this fact with (35), we have the existence of $T > 0$ such that

$$\inf_n\Big(\inf_{t\in(t_n,t_n+T)}\|e^{\mathbf{B}(t)}u(t,x)\|_{L^2(\mathcal{Q}_{x_n}(2))}^2\Big) \gtrsim \epsilon_1^2, \tag{39}$$

for some $\epsilon_1 > 0$. Notice that the previous estimate (39) provides, in combination with the Strauss radial inequality (7), that the sequence of centers $(x_n)_{n\in\mathbb{N}}$ is uniformly bounded. Observe also that since $t_n \to \infty$, we can assume that the intervals (eventually passing to a subsequence) $(t_n, t_n + T)$ are disjoint. In particular, we acquire the following for $d \geq 4$:

$$\sum_n T\epsilon_1^2 \lesssim \sum_n \int_{t_n}^{t_n+T} \int_{Q^d_{x_n}(2)} |e^{\mathbf{B}(t)}u(t,x)|^2\, dxdt \tag{40}$$

$$\lesssim \int_0^\infty \sup_{\tilde{x}\in\mathbb{R}^d} \int_{Q^d_{\tilde{x}}(2)} \frac{1}{|x|^3} |e^{\mathbf{B}(t)}u(t,x)|^2\, dxdt, \tag{41}$$

So, we obtain a contradiction because the right-hand side of the above (40) is bounded by (30). □

Remark 1. *Note that* (10) *and* (12) *introduce*

$$\|e^{\mathbf{B}(t)}u(t)\|_{H^1} \lesssim \|f\|_{H^1} + \int_0^t \|e^{\mathbf{B}(\tau}u(\tau)\|^2_{\dot{H}^{\frac{\gamma}{2}}} \|e^{\mathbf{B}(\tau)}u(\tau)\|_{H^1} d\tau$$
$$\lesssim \|e^{\mathbf{B}(t)}u(t)\|_{H^1} + C\int_0^t \|e^{\mathbf{B}(\tau}u(\tau)\|_{H^1}\, \mathrm{d}\tau.$$

Then, by Gronwall's inequality, we have

$$\left\|e^{\mathbf{B}(t)}u(t)\right\|_{H^1_x} \lesssim \|f\|_{H^1_x} e^{Kt},$$

with $K > 0$ *depending on* $\|f\|_{L^2_x}$ *and* $H(u(0))$. *Then, the Sobolev embedding and interpolation with the conservation of mass in* (11) *lead to*

$$\|e^{\mathbf{B}(t)}u(t)\|_{L^r_x} \lesssim e^{\varepsilon Kt},$$

with $0 < \varepsilon < 1$ *and* $2 < r \leq \frac{2d}{d-2}$, *which is not sufficient to guarantee a behavior like the one disclosed by* (6) *in Theorem 1 upon letting* $t \to +\infty$.

Remark 2. *We also highlight that one can achieve an exponential decay just by interpolating the conservation of mass in* (11) *and the estimate arising from the Sobolev embedding and* (17). *Namely, one has*

$$\|e^{\mathbf{B}(t)}u(t)\|_{L^r_x} \lesssim 1.$$

However, it is not enough to ensure a behavior such as in (6) *in Theorem 1, which is a stronger property of the solutions to* (1). *Moreover, the case* $\Re\mathbf{B}(t) = 0$, *that is, when* $ib(t)$ *is a real function, cannot be included in the previous analysis.*

Remark 3. *It is important to notice that our results can be used to deal with a class of damped nonlinearities fulfilling* (5), *particularly when*

$$b(t) \sim \frac{\tilde{a}}{1+t}, \quad \textit{for} \quad t > 1,\ \tilde{a} > 0,$$

as considered in [25]. Also, we can address more general damping terms, leading to equations of the form

$$i\partial_t u + \Delta_x^A u + \frac{i\tilde{a}}{(1+t)^\alpha} u + \tilde{b}(t) - k[|\cdot|^{-(d-\gamma)} * |u|^2]u = 0.$$

with $\alpha \geq 0$ *and where* $\tilde{b}(t) \in \mathcal{C}([0,\infty))$ *is a real-valued function (see [22]).*

6. Conclusions

We broaden the outcomes achieved in [17–20] to the damped scenario. The assumptions formulated for the time-depending function $ib(t)$ are more general than the ones found for the example in [21–23]. This is because we include an oscillatory part in the perturbed propagator $e^{\mathbf{B}(t)+it\Delta_x^A}$, which can not be treated if one uses the techniques developed in the aforementioned works. We underline that assumptions (2) and (3) related to the operator ∇_x^A and the function $A(x)$ are less restrictive than those imposed in [17,19,20]. This is because our well-posedness analysis relies solely on the energy estimate for (1). We are not using any Strichartz estimates here, forcing the constraint $d-2 \leq \gamma < d$. The obstacle here is that the multipliers utilized in the papers previously mentioned are not well suited to handle a non-local nonlinearity because their application cannot guarantee the non-negativity of the last term in (24). This aspect also determines the radial assumption we made on the initial data. A second problem is the equivalence of the norm result of Proposition 1, valid only in the L^2 framework. We are confident that we will overcome all these issues and shed light on the lower regularity frame $d-4 \leq \gamma < d-2$ in a future paper. We are also confident that our decay result can greatly simplify the scattering theory associated with (1), as well as shed light on the case $\Re\mathbf{B}(t) < 0$.

7. Open Problems and Further Developments

The theory established in this paper is general and allows us to obtain, in a straightforward manner, the decay in the energy spaces of the solutions to the damped magnetic Schrödinger equation with non-local nonlinearity. We believe that it can be used for the following open problems:

- The analysis of the scattering in the energy space for the solution to (1);
- The investigation of the decay properties (and eventually scattering) for the solutions to the generalized Schrödinger–Hartree equation, that is

$$i\partial_t u + \Delta_x^A u + ib(t)u + u - [|\cdot|^{-(d-\gamma)} * |u|^p]|u|^{p-2}u = 0,$$

  where the positive nonlinear parameter is either $p = 2$ or satisfies $\frac{d+\gamma+2}{d} < p < \frac{d+\gamma}{d-2}$;
- The exploration of the decaying and scattering properties of the solutions on other nonlinear dispersive equations such as the nonlinear Beam Equation

$$\partial_{tt} u + (\Delta_x^A)^2 u + ib(t)u + u - [|\cdot|^{-(d-\gamma)} * |u|^p]|u|^{p-2}u = 0,$$

  where $(\Delta_x^A)^2 = \Delta_x^A(\Delta_x^A)$ is the magnetic bi-Laplacian operator, or the nonlinear Klein–Gordon equation

$$\partial_{tt} u - \Delta_x^A u + ib(t)u + u + [|\cdot|^{-(d-\gamma)} * |u|^p]|u|^{p-2}u = 0,$$

  with the nonlinear parameter p defined as above, including the special case when $A(x) = 0$.

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