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Article

Shannon's Sampling Theorem for Set-Valued Functions with an Application

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Abstract: In this study, we defined a kind of Fourier expansion of set-valued square-integrable functions. In fact, we have seen that the classical Fourier basis also constitutes a basis for the Hilbert quasilinear space $L_2([-\pi,\pi],\Omega(\mathbb{C}))$ of $\Omega(\mathbb{C})$-valued square-integrable functions, where $\Omega(\mathbb{C})$ is the class of all compact subsets of complex numbers. Furthermore, we defined the quasi-Paley–Wiener space, QPW, using the Fourier transform defined for set-valued functions and thus we showed that the sequence $\{sinc(.-k)\}_{k\in\mathbb{Z}}$ form also a basis for QPW. We call this result Shannon's sampling theorem for set-valued functions. Finally, we gave an application based on this theorem.

Keywords: inner-product quasilinear spaces; non-deterministic signals; Fourier expansion of set-valued square-integrable functions; Shannon's sampling theorem for set-valued functions; Hilbert quasilinear spaces

MSC: 62D05; 42A16; 42A20; 28B20; 94A20; 94A12

1. Introduction

We know that a basis is one of the most important concepts in a vector space. A sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space, H, is a (Schauder) basis if every $f \in H$ can be represented as a (infinite) linear combinations of f_ks. In recent years, there has been increasing interest in interval algebra and interval-valued functions and their applications. Replacing a precise value by an interval value generally reflects the variability or uncertainty circumstances in an observation process. In signal processing, in general, it is very difficult to deal with a process with reliable information about the properties of the expected variations. Such uncertainties in a process lead us to set up a mathematical foundation of set-valued data and interval-based signal processing, see, for example, ref. [1] and references therein. Precisely, we want to study the mathematical analysis of set-valued and, in some special cases, interval-valued functions. In this regard, we will consider $L^2(\mathbb{R},\Omega(\mathbb{C}))$ and $L_2(\mathbb{R},\mathbb{I}_{\mathbb{C}})$, $\Omega(\mathbb{C})$-valued and $\mathbb{I}_{\mathbb{C}}$-valued square integrable functions. After saying that these are Hilbert quasilinear space structures, we will give the concept of basis in these spaces. Thus, we will talk about one-of-a-kind Fourier expansions in $L^2(\mathbb{R},\Omega(\mathbb{C}))$ and present some orthonormal basis. We will also prove the version of Shannon's sampling theorem for interval-valued functions by constructing the quasi-Paley–Wiener space.

The general motivation of this study is to show how signals with data that are inexact or corrupted to a limited extent can be re-improved by using the Hilbert quasilinear space of set-valued and, especially, interval-valued functions. Accordingly, the set-valued Shannon's sampling theorem that we have proven provides us with the advantage that such signals can be reconstructed from digital samples. Casting Representation Theorem provides a significant contribution to us in achieving this result. Thus, by using the set-valued Shannon's sampling theorem, we can reveal a method of recovering signals with inexact data (for example slightly distorted signals due to some reasons (noise, overlapping,

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etc.)). The limitation or validity of this method is that the amount of distortion must be within certain limits. The distorted signal, that is, the function, must be confined within a set-valued or interval-valued function. Otherwise, the reconstruction forecast cannot be realized.

2. Preliminaries

Any non-empty set, X, is called a quasilinear space on field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ if it is a partially ordered set (poset) with a partial order relation "$\preceq$", $(X, +)$ is an abelian ordered monoid with an algebraic sum operation $+$, and with a scalar multiplication by $\mathbb{K}$ with the following conditions: for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{K}$: $\alpha(\beta x) = (\alpha\beta)x$, $\alpha(x + y) = \alpha x + \alpha y$, $1x = x$, $0x = \theta$, $(\alpha + \beta)x \preceq \alpha x + \beta x$, $x + z \preceq y + v$ if $x \preceq y$ and $z \preceq v$, and $\alpha x \preceq \alpha y$ if $x \preceq y$. Here, θ denotes the additive unit (zero) in X.

The most popular examples are $\Omega(\mathbb{K})$ and $\Omega_C(\mathbb{K})$, which are defined as the sets of all non-empty compact and non-empty compact convex subsets of $\mathbb{K}$, respectively. The set $\mathbb{I}_{\mathbb{R}}$ of all closed intervals constitutes the basics of the interval analysis and, for $x, y \in \mathbb{I}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$, the Minkowski sum and scalar multiplication operations are defined by

$$x + y = [\underline{x}, \overline{x}] + \left[\underline{y}, \overline{y}\right] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$

and

$$\lambda x = \begin{cases} [\lambda\underline{x}, \lambda\overline{x}] & , \quad \lambda \geq 0 \\ [\lambda\overline{x}, \lambda\underline{x}] & , \quad \lambda < 0, \end{cases}$$

respectively. Further, the product of two intervals $x = [\underline{x}, \overline{x}]$ and $y = \left[\underline{y}, \overline{y}\right]$ is given by

$$x \cdot y = [\underline{x}, \overline{x}]\left[\underline{y}, \overline{y}\right] = [\min S, \max S] \tag{1}$$

where $S = \{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}$, [2].

Suppose that X is a quasilinear space and $Y \subseteq X$. Then Y is called a *subspace of* X whenever Y is a quasilinear space with the same partial order on X [3]. Y is subspace of quasilinear space X if and only if $\alpha x + \beta y \in Y$, for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{K}$ [3]. An element, x, in quasilinear space X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. An element, x, in X called *regular* whenever it has an additive inverse, that is, there exist an element, x', such that $x + x' = \theta$. A non-regular element is called a *singular*. Also, X_r denotes the set of all regular elements of X, while X_s is the set of all singular elements with the zero in X. Further, it can be easily shown that X_r, X_{sym}, and X_s are subspaces of X. They are called *regular, symmetric,* and *singular subspaces* of X, respectively [4].

A function $\|.\|$ is called a *norm* on the QLS X whenever the classical conditions of normed linear spaces are satisfied on X and following two extra conditions that are also satisfied on X:

$$\text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|, \tag{2}$$

$$\begin{gathered}\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X, \text{ such that} \\ x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon, \text{ then } x \preceq y,\end{gathered} \tag{3}$$

where $x, y, x_\varepsilon \in X$, and α is any scalar [5]. A quasilinear space, X, with a norm defined on it is called *a normed quasilinear space*. A Hausdorff metric or norm metric on X is defined by the equality

$$h_X(x, y) = \inf\left\{r \geq 0 : x \preceq y + a_1^{(r)}, y \preceq x + a_2^{(r)} \text{ and } \left\|a_i^{(r)}\right\| \leq r, i = 1, 2\right\}.$$

A norm on $\Omega(\mathbb{K})$ is defined by $\|A\|_\Omega = \sup_{a\in A}|a|$. Hence, $\Omega_C(\mathbb{K})$ and $\Omega(\mathbb{K})$ are normed quasilinear spaces [5].

Now, let us give an extended definition of the inner product given in [4]. We can say that the inner product in the following definition is a set-valued inner product on quasilinear spaces.

Definition 1 ([4])**.** *Let X be a quasilinear space on the field $\mathbb{K}$. A mapping $\langle\ ,\ \rangle : X \times X \to \Omega(\mathbb{K})$ is called an inner product on X if for any $x, y, z \in X$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied:*

$$\text{If } x, y \in X_r \text{ then } \langle x, y\rangle \in \Omega_C(\mathbb{K})_r \equiv \mathbb{K}, \tag{4}$$

$$\langle x+y, z\rangle \subseteq \langle x, z\rangle + \langle y, z\rangle, \tag{5}$$

$$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \tag{6}$$

$$\langle x, y\rangle = \langle y, x\rangle, \tag{7}$$

$$\langle x, x\rangle \geq 0 \text{ for } x \in X_r \text{ and } \langle x, x\rangle = 0 \Leftrightarrow x = 0, \tag{8}$$

$$\text{if } x \preceq y \text{ and } u \preceq v \text{ then } \langle x, u\rangle \subseteq \langle y, v\rangle, \tag{9}$$

$$\begin{gathered}\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X, \text{ such that} \\ x \preceq y + x_\varepsilon \text{ and } \langle x_\varepsilon, x_\varepsilon\rangle \subseteq S_\varepsilon(\theta), \text{ then } x \preceq y.\end{gathered} \tag{10}$$

A quasilinear space with an inner product is called as an *inner-product quasilinear space*. Every IPQLS X is a normed QLS, with the norm defined by

$$\|x\| = \sqrt{\|\langle x, x\rangle\|_{\Omega(\mathbb{C})}}$$

for every $x \in X$. This norm is called an inner-product norm. Further, $x_n \to x$ and $y_n \to y$ in an IPQLS implies $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

An IPQLS is called a *Hilbert QLS* if it is complete according to the inner-product (norm) metric. $\Omega(\mathbb{C})$ is a Hilbert QLS.

The space $L_2(\mathbb{R}, \Omega(\mathbb{C}))$ consists of all set-valued measurable functions $F : \mathbb{R} \to \Omega(\mathbb{C})$, such that the Lebesque integral

$$\int_{\mathbb{R}} \|F(x)\|^2_\Omega dx$$

exists. $L_2(\mathbb{R}, \Omega(\mathbb{C}))$ is a quasilinear space over the field $\mathbb{C}$ with the algebraic operations $(F_1 + F_2)(x) = F_1(x) + F_2(x)$, $(\lambda F)(x) = \lambda F(x)$ and the partial order relation $F_1 \preceq F_2 \Leftrightarrow F_1(x) \subseteq F_1(x)$ for almost everywhere (a.e.) $x \in \mathbb{R}$ [6]. $L_2(\mathbb{R}, \Omega(\mathbb{C}))$ is an inner-product quasilinear space with respect to the inner product

$$\langle F, G\rangle = \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), G(x)\rangle_\Omega dx.$$

Throughout the paper, $\overset{(A)}{\int_{\mathbb{R}}} F(x)$ means the Aumann integral of the set-valued function F. Any set, M, in an inner-product QLS is called *orthogonal* whenever $\langle x, y\rangle = 0$ for every

$x, y \in M$. Further, if the norm of each element in an orthogonal set is 1, the set is called *orthonormal*.

Let us give an algebraic concept from [3,4,7]. For any non-empty subset, A, of a QLS X, the span of A is given by

$$SpA = \{\sum_{k=1}^{n} \alpha_k x_k : x_1, x_2, \ldots, x_n \in A,\ \alpha_1, \alpha_2, \ldots,\ \alpha_n \in \mathbb{K},\ n \in \mathbb{N}\}.$$

However, $QspA$, the *quasispan (q-span,* for short) *of* A, is defined as

$$QspA = \{x \in X : \sum_{k=1}^{n} \alpha_k x_k \preceq x,\ x_1, x_2, \ldots, x_n \in A,\ \alpha_1, \alpha_2, \ldots,\ \alpha_n \in \mathbb{K},\ n \in \mathbb{N}\}.$$

Obviously, $SpA \subseteq QspA$. Further, $SpA = QspA$ for some linear QLS (linear space); hence, the notion of $QspA$ is redundant in linear spaces. Moreover, we say A *quasispans* X whenever $QspA = X$.

Definition 2. *A quasilinear space, X, is called* a consolidate (solid-floored) QLS *whenever* $y = \sup\{x \in X_r : x \preceq y\}$ *for each* $y \in X$. *Otherwise, X is called* a non-consolidate QLS*, briefly,* nc-QLS*.*

The supremum in this definition is taken on the order relation "$\preceq$" in the definition of a QLS. The above definition assumes $\sup\{x \in X_r : x \preceq y\}$ exists for each $y \in X$. Implicitly, we say that X is consolidate if and only if $y = \sup F_y$, for each $y \in X$.

We signify that any linear space is a consolidate QLS: indeed, $X_r = X$ for any linear space, X, and so

$$y = \sup\{x \in X_r : x \preceq y\} = \sup\{x \in X_r : x = y\} = \sup\{y\} = y$$

for any element y in X.

Definition 3. *A ql-independent subset A of a QLS X that q-spans X is called* a basis (or Hamel basis) *for X.*

Remark 1. *We know from our previous work that only consolidated quasilinear spaces can have bases; the others cannot. Also, a base of a quasilinear space is a subset of the regular subspace of the space. That is, the basis vectors of a quasilinear space must be chosen from its regular subspace.*

For any $a \in \mathbb{C}$, the singleton $\{\{a\}\}$ is a basis for $\Omega_C(\mathbb{C})$ and for $\Omega(\mathbb{C})$. Further, $B = \{1, i\}$ is a basis for $\mathbb{I}_{\mathbb{C}}$ on the field $\mathbb{R}$. In general, if $\{(a_1, b_1), (a_2, b_2)\}$ is a basis for $\mathbb{R}^2$, then $\{\{(a_1, b_1)\}, \{(a_2, b_2)\}\}$ is a basis for $\Omega_C(\mathbb{R}^2)$ and $\mathbb{I}^n_{\mathbb{R}}$. In general, any basis of $\mathbb{R}^n$ generates a basis for $\Omega_C(\mathbb{R}^n)$ and $\mathbb{I}^n_{\mathbb{R}}$. Let us give a useful basis for $\mathbb{I}^n_{\mathbb{C}}$ in signal processing and in applied mathematics.

3. Main Results

Let us start with an easy result. We are inspired by [8] for the following result.

Example 1. *(Fourier transform basis in* $\mathbb{I}^n_{\mathbb{C}}$*) for* $1, 2, \ldots, n$ *consider the set* $B = \{b_1, b_2, \ldots, b_n\}$ *of regular elements in* $\mathbb{I}^n_{\mathbb{C}}$ *such that*

$$
\begin{aligned}
b_1 &= (1/\sqrt{n})(1,1,\ldots,1) \\
b_2 &= (1/\sqrt{n})\left(1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \ldots, e^{\frac{2\pi i(n-1)}{n}}\right) \\
b_3 &= (1/\sqrt{n})\left(1, e^{\frac{4\pi i}{n}}, e^{\frac{8\pi i}{n}}, \ldots, e^{\frac{4\pi i(n-1)}{n}}\right) \\
&\vdots \\
b_n &= (1/\sqrt{n})\left(1, e^{\frac{2\pi i(n-1)}{n}}, e^{\frac{4\pi i(n-1)}{n}}, \ldots, e^{\frac{2(n-1)\pi i(n-1)}{n}}\right).
\end{aligned}
$$

Then B is a basis for $\mathbb{I}_{\mathbb{C}}^n$. In fact, we know that B is the Fourier basis for the uniter linear space $\mathbb{C}^n$, which is the regular subspace of $\mathbb{I}_{\mathbb{C}}^n$.

Remark 2. *Now, any element $x = (x_1, x_2, \ldots, x_n) \in \mathbb{I}_{\mathbb{C}}^n$ can be expressed uniquely as*

$$
\begin{aligned}
x &= \sup_{\subseteq}\{z \in (\mathbb{I}_{\mathbb{C}}^n)_r \equiv \mathbb{C}^n : z \subseteq x\} \\
&= \sup_{\subseteq}\{\sum_{k=1}^{n} \alpha_k^z b_k : z_k \subseteq x_k,\ k = 1,2,\ldots,n\}
\end{aligned}
$$

where $z = (z_1, z_2, \ldots, z_n) \in \left(\mathbb{I}_{\mathbb{C}}^n\right)_r \equiv \mathbb{C}^n$ and α_k^z, $k = 1,2,\ldots,n$, are complex scalars depending on z. This is known as the super position of each element, x, in $\mathbb{I}_{\mathbb{C}}^n$ with respect to the Fourier transform basis, B.

Remark 3. *We should note that only consolidate quasilinear spaces can have a Hamel basis. Therefore, unlike linear spaces, not every quasilinear space may have a basis. For instance*

$$(\mathbb{I}_{\mathbb{R}})_{sym} = \{[-a, a] : a \text{ is non-negative real}\},$$

the symmetric subspace of $\mathbb{I}_{\mathbb{R}}$, has no basis. This is because every subset of $(\mathbb{I}_{\mathbb{R}})_{sym}$ is quasilinear dependent. Moreover, we can easily see that the quasilinear space $(\mathbb{I}_{\mathbb{R}})_{sym}$ is not consolidate. The floors, i.e., regular subspaces, of consolidate quasilinear spaces are linear spaces, and this linear space has a basis. This regular subspace is called the linear part of the quasilinear space. We show that, if a quasilinear space has a basis, the basis elements must be chosen from the linear part. Moreover, while the elements of this Hamel basis span the linear part, they quasi-span the quasilinear space. In a consolidate space, the basis representations of regular elements are like the representations given in linear spaces. But the representations of singular elements are given in a similar way to the example above, using the partial order relation on the quasilinear space and the basis vectors.

Proposition 1. *Let X be a finite-dimensional consolidate quasilinear space and x be any element of X. Then X has a basis $B = \{b_1, b_2, \ldots, b_n\}$ and each $x \in X$ has the representation*

$$
\begin{aligned}
x &= \sup_{\preceq}\{z \in X_r : z \preceq x\} \\
&= \sup_{\preceq}\{\sum_{k=1}^{n} \alpha_k^z b_k : z \preceq x,\ k = 1,2,\ldots,n\}
\end{aligned}
$$

where (n_r, n_s) is the (binary) dimension (see [4]) of X and scalars α_k^z, $k = 1,2,\ldots,n$, are depend on z, (and are depend of course on x), and the supremum is taken on the partial order relation on X.

We know from [4] that the dimension of a finite-dimensional quasilinear space can be defined as a binary natural number (n_r, m_s), where n denotes the usual dimension of the linear part, and m is just the maximum numbers of cardinality in a quasilinear-independent subset in singular part, X_s. We know again from [4] that $m = n$ if X is a finite-dimensional consolidate quasilinear space.

Definition 4. *(Schauder basis) Let X be a normed quasilinear space. A Schauder basis is a sequence* $\{b_n\}$ *of elements of X, such that for every element* $x \in X$

$$x = \sup_{\preceq}\{z \in X_r : z \preceq x\},$$

and for every element $z \in X_r$*, with* $z \preceq x$*, there exists a unique sequence* $\{\alpha_k^z\}$ *of scalars so that*

$$z = \sum_{k \in \mathbb{Z}} \alpha_k^z b_k.$$

Remark 4. *Let X be a normed linear space, then if X is a normed space this definition turns into the classical Schauder basis definition in normed spaces. Because in this case the relation* $\leq$ *has to turn into the relation* $=$*, there is only one element that will satisfy the relation*

$$x = \sup_{\leq}\{z \in X_r : z = x\} = z.$$

Thus, there is no need for the supremum, and the first condition in the definition becomes unnecessary.

Definition 5 ([6])**.** *The set-valued Fourier transform of* $F \in L_2(\mathbb{R}, \Omega(\mathbb{C}))$ *is the set-valued function* $\hat{F}$*, defined as*

$$\hat{F}(w) = \overset{(A)}{\int_{\mathbb{R}}} F(x) e^{-2\pi i w x} dx = \{ \int_{\mathbb{R}} f(x) e^{-2\pi i w x} dx : f \in S^2(F)\} \tag{11}$$

for every $w \in \mathbb{R}$*. The set-valued Fourier transform of F is also denoted as*

$$(\mathcal{F}F)(w) = \hat{F}(w) \quad ,w \in \mathbb{R}.$$

Note that the set-valued Fourier transform of $F \in L_2(\mathbb{R}, \Omega(\mathbb{C}))$ is the set of Fourier transforms of integrable selections of F, i.e.,

$$\hat{F}(w) = \{\hat{f}(w) : \mathcal{F}(f) = \hat{f}, f \in S^2(F)\}$$

where $\mathcal{F}(F) = \hat{F}$ is the set-valued Fourier transform of the function $F : \mathbb{R} \to \Omega(\mathbb{C})$ *and* $\hat{f} = \mathcal{F}(f)$ *is the classical Fourier transform of the function* $f : \mathbb{R} \to \mathbb{C}$*. Further,* $S^2(F)$ denotes the square integrable selections of F.

Any *selection* of F is a function $\mathbb{R} \to \mathbb{C}$, such that $f(t) \in F(t)$ for all $t \in \mathbb{R}$.

Example 2 ([8])**.** *Function sequence*

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \right\}_{k \in \mathbb{Z}}$$

form an orthonormal basis for $L_2([-\pi, \pi])$*.*

Now, let us present a main result.

Theorem 1. *(Fourier series expansion of set-valued functions) The sequence*

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \right\}_{k \in \mathbb{Z}}$$

forms an orthonormal basis for $L_2([-\pi,\pi],\Omega(\mathbb{C}))$*. More precisely, each* $F \in L_2([-\pi,\pi],\Omega(\mathbb{C}))$ *can be represented uniquely as*

$$F(x) = \lim_{N\to\infty}\left(\sup_{g\in S^2(F)}\sum_{k=-N}^{N} c_k^g e^{ikx}\right) = \sup_{g\in S^2(F)}\sum_{k=-\infty}^{\infty} c_k^g e^{ikx}$$

where

$$c_k^g = \frac{1}{\sqrt{2\pi}}\int_{[-\pi,\pi]} g(x)e^{-ikx}dx$$

is the Fourier coefficient corresponding to $g \in S^2(F)$*.*

Proof. Orthonormality is just the same as in the classical case. We have to use the norm metric in calculations on $L_2([-\pi,\pi],\Omega(\mathbb{C}))$, which is defined as

$$\begin{aligned} h_L(F,G) &= \inf\left\{r \geq 0 : F \preceq G + A_1^{(r)},\, G \preceq F + A_2^{(r)} \text{ and } \left\|A_i^{(r)}\right\| \leq r, i = 1,2\right\} \\ &= \inf\left\{\begin{array}{c} r \geq 0 : F(x) \subseteq G(x) + A_1^{(r)}(x),\, G(x) \subseteq F(x) + A_2^{(r)}(x) \\ \text{and } \int_{[-\pi,\pi]} \left\|A_i^{(r)}(x)\right\|_\Omega^2 dx \leq r^2, i = 1,2. \end{array}\right\} \\ &= \left(\int_{-\pi}^{\pi}\left(h_{\Omega(\mathbb{C})}(F(x),G(x))\right)^2 dx\right)^{1/2} \end{aligned}$$

Now consider any square-integrable selections $f \in S^2(F)$ of F. Then, from the above example, f has the Fourier series expansion

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

here, the coefficient c_k is computed by

$$c_k = \frac{1}{\sqrt{2\pi}}\int_{[-\pi,\pi]} f(x)e^{-ikx}.$$

Let S_N be the Nth partial sum of the Fourier series, that is

$$S_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}.$$

By the Schauder basis definition and from the above example, any element $F \in L_2([-\pi,\pi],\Omega(\mathbb{C}))$ can be written uniquely

$$\begin{aligned} F &= \sup_{\leq}\{g \in L_2([-\pi,\pi],\Omega(\mathbb{C}))_r : g \leq F\}, \\ &= \sup\{g \in L_2[-\pi,\pi] : g \in S^2(F),\ g(x) \in F(x)\}. \end{aligned}$$

Here, for every element $g \in L_2[-\pi,\pi]$ with $g \in S^2(F)$, there exists a unique sequence $\left\{c_k^g\right\}$ of scalars, depending only on g, such that

$$g(x) = \sum_{k=-\infty}^{\infty} c_k^g e^{ikx}$$

where each

$$c_k^g = \frac{1}{\sqrt{2\pi}} \int\limits_{[-\pi,\pi]} g(x) e^{-ikx} dx$$

is the Fourier coefficient corresponding to g. Let S_N^g be the Nth partial sum of the Fourier series, that is

$$S_N^g(x) = \sum_{k=-N}^{N} c_k^g e^{ikx}.$$

Hence, for any $g \in S^2(F)$, $g(x) \in F(x)$ and

$$\left| g(x) - S_N^g(x) \right| \to 0 \text{ as } N \to \infty, \ x \in [-\pi, \pi].$$

Let $F_N(x) = \sup\{S_N^g(x) : g(x) \in F(x)\}$, where the supremum is taken on the inclusion relation. We should remark again that $g(x) \in F(x)$ means $g(x) \subseteq F(x)$. Then

$$\begin{aligned}
(h_L(F, F_N))^2 &= \int\limits_{[-\pi,\pi]} \left(h_{\Omega(\mathbb{C})}(F(x), F_N(x)) \right)^2 dx \\
&= \int\limits_{[-\pi,\pi]} \left(h_{\Omega(\mathbb{C})}\left(F(x), \sup\{S_N^g(x) : g(x) \in F(x)\} \right) \right)^2 dx \\
&= \int\limits_{[-\pi,\pi]} \left(h_{\Omega(\mathbb{C})}\left(\sup\{g(x)\}, \sup\{S_N^g(x)\} \right) \right)^2 dx \\
&\leq \sup_{g \in S^2(F)} \int\limits_{[-\pi,\pi]} \left| g(x) - S_N^g(x) \right|^2 dx \\
&\to 0 \text{ as } N \to \infty.
\end{aligned}$$

Hence

$$\begin{aligned}
F(x) &= \lim_{N\to\infty} \left(\sup\{S_N^g(x) : g(x) \in F(x)\} \right) \\
&= \lim_{N\to\infty} \left(\sup_{g \in S^2(F)} \sum_{k=-N}^{N} c_k^g e^{ikx} \right) \\
&= \lim_{N\to\infty} \left(\sup_{g \in S^2(F)} \sum_{k=-N}^{N} c_k^g e^{ikx} \right).
\end{aligned}$$

This completes the proof. □

Now let us prepare another main result.

4. Set-Valued Shannon's Sampling Theorem

Definition 6 ([8]). *Support of a function $f : \mathbb{R} \to \mathbb{C}$ is defined as*

$$Supp f = \overline{\{x \in \mathbb{R} : 0 \neq f(x)\}}.$$

The Paley–Wiener space, PW for short, is defined as

$$PW = \{f \in L_2(\mathbb{R}) : Supp \widehat{f} \subseteq [-1/2, 1/2]\}$$

where $\widehat{f}$ is the set-valued Fourier transform of $\widehat{f}$.

The *sinc*-function is given by $sinc(t) = \begin{cases} \frac{\sin \pi t}{\pi t}, & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$.

Theorem 2 ([8])**.** *(Shannon's sampling theorem) The functions* $\{sinc(.-k)\}_{k\in\mathbb{Z}}$ *form an orthonormal basis for* PW*. If* $f \in PW$ *is continuous, then*

$$f(x) = \sum_{k\in\mathbb{Z}} f(k) sinc(x-k).$$

A quick generalization of this theorem may be more useful.

Corollary 1. *(Shannon's sampling theorem) If* $Supp\widehat{f} \subseteq [-\alpha/2, \alpha/2]\}$*, then*

$$f(x) = \sum_{k\in\mathbb{Z}} f(\frac{k}{\alpha}) sinc(\alpha x - k), \;\; x \in \mathbb{R}.$$

Definition 7. *The quasi-support of a function* $F : \mathbb{R} \to \Omega(\mathbb{C})$ *is defined as*

$$QsuppF = \overline{\{x \in \mathbb{R} : 0 \notin F(x)\}}.$$

The quasi-Paley–Wiener space, QPW for short, is defined as

$$QPW = \{F \in L_2(\mathbb{R}, \Omega(\mathbb{C})) : Qsupp\hat{F} \subseteq [-1/2, 1/2]\}$$

where $\hat{F}$ *is the set-valued Fourier transform of* F*.*

Lemma 1. QPW *is a subspace of* $L_2(\mathbb{R}, \Omega(\mathbb{C}))$*.*

Proof. Suppose that $F, G \in QPW$ and $\alpha, \beta \in \mathbb{C}$. Then we can write

$$Qsupp\hat{F} \subseteq [-1/2, 1/2] \text{ and } Qsupp\hat{G} \subseteq [-1/2, 1/2],$$

i.e.,

$$\overline{\{x \in \mathbb{R} : 0 \notin \hat{F}(x)\}} \text{ and } \overline{\{x \in \mathbb{R} : 0 \notin \hat{G}(x)\}}$$

are the subsets of $[-1/2, 1/2]$. Let us take an arbitrary

$$x \in Qsupp(\alpha\hat{F} + \beta\hat{G}).$$

Then, $0 \notin \alpha\hat{F}(x) + \beta\hat{G}(x)$. Thus we observe that either $0 \notin \alpha\hat{F}(x)$ or $0 \notin \beta\hat{G}(x)$. If both $0 \in \alpha\hat{F}(x)$ and $0 \in \beta\hat{G}(x)$, then we have

$$0 \in \alpha\hat{F}(x) + \beta\hat{G}(x).$$

This contradicts the assumption that

$$x \in Qsupp(\alpha\hat{F} + \beta\hat{G}).$$

Therefore, we obtain from the hypothesis that $\alpha F + \beta G \in QPW$ since $x \in [-1/2, 1/2]$. □

Theorem 3. *(Castaing Representation theorem)* T *is a domain in* $\mathbb{R}^k$ *and* $F : T \to \Omega(\mathbb{R}^n)$ *is measurable if and only if there exists a sequence of measurable selections* $\{f_i\}_{i=1}^{\infty}$ *of* F*, such that*

$$F(x) = \overline{\bigcup_{i\in\mathbb{N}^+} f_i(x)}$$

for each $x \in T$ *[9,10].*

Proposition 2. T *is a domain in* $\mathbb{R}^k$ *and* $F : T \to \Omega(\mathbb{R}^n)$ *is measurable if it is upper or lower semi-continuous, hence if it is continuous. Further, such an* F *has a measurable selection* $f : T \to \mathbb{R}^n$ *[9,10].*

Now we will give the Shannon's sampling theorem for interval-valued functions. First, let us give the following lemma, which can be proved usually.

Lemma 2. *If $F \in QPW$, then each selection, f, of F is an element of the usual Paley–Wiener space*

$$PW = \{f \in L_2(\mathbb{R}) : supp\hat{f} \subseteq [-1/2, 1/2]\}.$$

Theorem 4. *(Shannon's sampling theorem for set-valued functions) If $F \in QPW$ is continuous, then there exists a sequence of measurable continuous selections $\{f_i\}_{i=1}^{\infty}$ of F, such that for each $i \in \mathbb{N}^+$*

$$f_i(x) = \sum_{k\in\mathbb{Z}} f_i(k) sinc(x-k)$$

where $x \in \mathbb{R}$. Therefore, F has the representation,

$$F(x) = \overline{\bigcup_{i\in\mathbb{N}^+} \{\sum_{k\in\mathbb{Z}} f_i(k) sinc(x-k)\}}$$

and this representation is unique, where $\{sinc(.-k\}_{k\in\mathbb{Z}}$ is the set of orthonormal basis functions for PW.

Proof. By the Castaing Representation theorem and by the above proposition we say that there exists a sequence of measurable selections $\{f_i\}_{i=1}^{\infty}$ of F, such that

$$F(x) = \overline{\bigcup_{i\in\mathbb{N}^+} \{f_i(x)\}}.$$

Further, the fact that $F \in QPW$ implies that each selection, f_i, of F is in the space PW by Lemma (2). These selections are continuous, since the set-valued function, F, is continuous (see, [10]). Therefore, by the classical Shannon's sampling theorem we can write that

$$f_i(x) = \sum_{k\in\mathbb{Z}} f_i(k) sinc(x-k)$$

for each $i = 1, 2, \ldots$ and $x \in \mathbb{R}$. Consequently, we obtain the representation

$$F(x) = \overline{\bigcup_{i\in\mathbb{N}^+} \{\sum_{k\in\mathbb{Z}} f_i(k) sinc(x-k)\}}$$

in the quasi-Paley–Wiener space, and this representation is unique since each selection, f_i, is unique. □

Remark 5. *In signal processing, functions that are elements of the PW are known as band-limited signals. Moreover, the classical Shannon sampling theorem expresses the following fundamental law in signal processing: "each band-limited signal can be reconstructed from its samples". Band limited means that the frequency band of the signal lays into an interval, for example in $[-1/2, 1/2]$. When we take the Fourier transform of a time signal, $f(t)$, the resulting function, $f(w)$, becomes a function of the frequency, w. Thus, the Fourier transform is a function that converts the time signal into frequency signals, or, in other words, moves it into the frequency band. The theorem we gave above can be used as follows. Interval valued functions can represent signals with inexact data. Functions whose value at a point is not exactly known, but whose value is known to be confined within an interval or a cluster, are called signals with inexact data. The theorem above states that a quasi-band-limited signal can be approximated step by step by a band-limited signal.*

Example 3. *Let us consider a piece of music that is 5 min or 300 s long. While listening to this piece of music, our ear, that is, the receiver, can make sense of frequencies of up to 20,000 Hertz. In other words, the frequency function of this 300-s $f(t)$ function, that is, the $\widehat{f}(w)$ function, which is* *the Fourier transform of f, should not exceed 20,000 Hz. We can say this to include this situation: for this function, f we can say that*

$$supp\hat{f} \subseteq [-20,000, 20,000] = [-\frac{40,000}{2}, \frac{40,000}{2}].$$

In this case, f is band-limited and $f \in PW$. Hence, from Corollary 1, f can be represented as

$$\begin{aligned} f(t) &= \sum_{k \in \mathbb{Z}} f(\frac{k}{40,000}) sinc(40,000t - k), \quad t \in [0,300], \\ &= \ldots f(\frac{-1}{40,000}) sinc(40,000t + 1) + f(0) sinc(40,000t) \\ &+ f(\frac{1}{40,000}) sinc(40,000t - 1) + \ldots. \end{aligned}$$

In other words, function f can be reconstructed from these samples created with the help of the sinc function. This rebuilding process is done by assimilating or approximating the series

$$S_n(t) = \sum_{k=-n}^{n} f(\frac{k}{40,000}) sinc(40,000t - k)$$

of partial sums to the function $f(t)$ with an acceptable error, according to the norm in the PW space. Classic CD players work according to this principle.

Now let us assume that for some reason there is distortion in this signal f (music piece) and that this distortion is in the form of a shift of at most 10% in the $f(t)$ values for each time moment, t. Such situations may occur for many reasons (for example, due to interference in the signal transmission environment, overlapping, etc.). Let us denote the distorted signal at this rate as $g(t)$ and let us create our model interval-valued function

$$F(t) = \left[f(t) - \frac{f(t)}{10}, f(t) + \frac{f(t)}{10} \right] = f(t). \left[\frac{9}{10}, \frac{11}{10} \right].$$

By the information above, we can write

$$f(t) \in F(t) \text{ and } g(t) \in \left[f(t) - \frac{f(t)}{10}, f(t) + \frac{f(t)}{10} \right].$$

Hence, $f(t) - g(t) \in \left[\frac{9}{10}, \frac{11}{10} \right]$. For any $s \in \left[\frac{9}{10}, \frac{11}{10} \right]$

$$u(s) = sf(t)$$

is a measurable selection of $F(t)$. Furthermore, all measurable selection of $F(t)$ must be of the form $u(s)$. Therefore, for any $s \in \left[\frac{9}{10}, \frac{11}{10} \right]$ we can write $g(t) = sf(t)$. Now, from Theorem 4 there exists a sequence $(s_i)_{i \in \mathbb{N}^+}$ with $s_i \in \left[\frac{9}{10}, \frac{11}{10} \right]$, such that

$$S_n^i(t) = \sum_{k=-n}^{n} s_i f(\frac{k}{40,000}) sinc(40,000t - k)$$

and

$$S_n^i(t) \to f(t) \text{ as } i, n \to \infty$$

in the norm of $L_2(\mathbb{R})$*. We thus show that there is a computable way to correct or filter the distorted signal,* $g(t)$*. These calculations are performed by calculating the samples,* $S_n^i(t)$*. Now let us guarantee that the error in these calculations will go towards zero. Again from Theorem 4, the set*

$$\left(\bigcup_{i\in\mathbb{N}^+} S_n^i(t)\right)_{n\in\mathbb{N}}$$

is dense in $F(t)$ *and in the same way we obtain*

$$\left(\bigcup_{i\in\mathbb{N}^+} S_n^i(t)\right)_{n\in\mathbb{N}} = \{S_n^1(t), S_n^2(t), \ldots, S_n^i(t)\} \to F(t) \text{ as } i, n \to \infty$$

in the norm of QPW*, hence of* $L_2(\mathbb{R}, \Omega(\mathbb{C}))$*.*

Conclusion 1. *We conclude that a certain percentage, say 10%, of faulty or noise-contaminated signals can be recovered from their digital samples via the set-valued Shannon's sampling theorem. Mathematical methods obtained with the help of quasilinear functional analysis allow signals with inexact data to be processed in signal processing. Moreover, this innovative approach can be used for many signal processing procedures, such as autocorrelation calculation or filtering, or frequency band determination. In subsequent studies, we aim to perform some other signal processing procedures for these signals in Hilbert quasilinear spaces.*

5. Applications: Energy Spectral Density Estimation

In this section, we will try to estimate the energy spectral density of a signal with inexact data, e.g., a signal that is deformed to a certain extent. For example, let us determine that the signal $f(t)$ is deformed due to noise contamination or other reasons and that the values of the deformed signal, $g(t)$, in the time interval, t, deviate by at most $1/100$. In this case, we can use interval-valued functions to estimate the energy spectral density of the deformed signal, $g(t)$, using interval-valued functions. Let us try to see a simple example of this.

Now, let us give a lemma that is necessary to solve the following problem.

Lemma 3 ([2])**.** *We define the interval integral*

$$\int_{[a,b]} F(t)dt = \bigcap_{N=1}^{\infty} S_N(F; [a,b])$$

It follows from the continuity of F *that there are two continuous real-valued functions,* $\underline{F}$ *and* $\overline{F}$*, such that, for real* t*,*

$$F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$$

Moreover, the integral defined above is equivalent to

$$\int_{[a,b]} F(t)dt = \left[\int_{[a,b]} \underline{F}(t)dt, \int_{[a,b]} \overline{F}(t)dt\right]$$

where $\underline{F}(t)$ *and* $\overline{F}(t)$ *are classical functions from* $[a,b]$ *into* $\mathbb{R}$*, which are the lower and upper bound functions of* $F(t)$*, respectively.*

From classical signal processing, we know that energy spectral density can be calculated for a signal, f, as

$$ESDf = \left|\widehat{f}(s)\right|^2$$

where $\widehat{f}(s)$ is the Fourier transform of f.

Problem: It is observed that the signal of

$$f(t)=\begin{cases} 2, & 0\le t\le 1\\ 0, & \textit{otherwise}. \end{cases}$$

is distorted due to interference or noise at the rate of at most 1/100 and assume that the resulting noisy signal is $g(t)$. Then, we can say that $g(t)\in F(t)=\left[f(t)-10^{-2}, f(t)+10^{-2}\right]$. For the solution, we have to calculate the interval-valued Fourier transform of $F(t)$. Now, firstly, in order to solve this problem we use Lemma 3.

$$\begin{aligned}
\widehat{F}(s) &= \int_0^1 \left[2-10^{-2}, 2+10^{-2}\right] e^{-2\pi i s\tau} d\tau \\
&= \int_0^1 \left[2-10^{-2}, 2+10^{-2}\right] (\cos(2\pi s\tau) - i\sin(2\pi s\tau)) d\tau \\
&= \int_0^1 \left[2-10^{-2}, 2+10^{-2}\right] \cos(2\pi s\tau) d\tau - i\int_0^1 \left[2-10^{-2}, 2+10^{-2}\right] \sin(2\pi s\tau) d\tau \\
&= \left[\int_0^1 \left(2-10^{-2}\right)\cos(2\pi s\tau) d\tau, \int_0^1 \left(2+10^{-2}\right)\cos(2\pi s\tau) d\tau\right] \\
&\quad -i\left[\int_0^1 \left(2-10^{-2}\right)\sin(2\pi s\tau) d\tau, \int_0^1 \left(2+10^{-2}\right)\sin(2\pi s\tau) d\tau\right] \\
&= \left[\left(2-10^{-2}\right)\left(\frac{\sin(2\pi s)}{2\pi s}\right), \left(2+10^{-2}\right)\left(\frac{\sin(2\pi s)}{2\pi s}\right)\right] \\
&\quad -i\left[\left(2-10^{-2}\right)\left(-\frac{\cos(2\pi s)}{2\pi s}+\frac{1}{2\pi s}\right), \left(2+10^{-2}\right)\left(-\frac{\cos(2\pi s)}{2\pi s}+\frac{1}{2\pi s}\right)\right] \\
&= \frac{\sin(2\pi s)}{2\pi s}.\left[\left(2-10^{-2}\right), \left(2+10^{-2}\right)\right] \\
&\quad -i\left(-\frac{\cos(2\pi s)}{2\pi s}+\frac{1}{2\pi s}\right)\left[\left(2-10^{-2}\right), \left(2+10^{-2}\right)\right] \\
\text{if } \cos(2\pi s\tau) &\ge 0 \text{ and } \sin(2\pi s\tau)\ge 0.
\end{aligned}$$

Now, for this case, from the interval calculus we obtain ESD

$$\begin{aligned}
ESDF(s) &= \left|\widehat{F}(s)\right|^2 \\
&= \left|\left[\left(2-10^{-2}\right), \left(2+10^{-2}\right)\right]\right|^2 \left(\frac{\sin^2(2\pi s)}{2\pi s} + \left(-\frac{\cos(2\pi s)}{2\pi s}+\frac{1}{2\pi s}\right)^2\right) \\
&= \max\left\{\left(2-10^{-2}\right)^2, \left(2+10^{-2}\right)^2\right\}\left(\frac{\sin^2(2\pi s)}{2\pi s} + \left(-\frac{\cos(2\pi s)}{2\pi s}+\frac{1}{2\pi s}\right)^2\right) \\
&= \left(2+10^{-2}\right)^2\left(\frac{\sin^2(2\pi s)}{2\pi s} + \left(-\frac{\cos(2\pi s)}{2\pi s}+\frac{1}{2\pi s}\right)^2\right)
\end{aligned}$$

For further cases of $\cos(2\pi s\tau)$ and $\sin(2\pi s\tau)$, the energy spectral density, ESD, can be calculated in a similar way. For the first case we can give the comment that the energy spectral density of the uncertain signal, $g(t)$, must be confined within the complex interval-valued function, $F(s)$. Thus, the energy spectral density of $g(t)$, i.e., each value of the ESD function, is in the interval $[u, v]$.

Conclusion 2. *We conclude that the energy spectral density function of the distorted signal, $g(t)$, cannot exceed the function $ESDF(s)$. For many signals with inexact data, it is possible to give similar estimates for the power spectral density as well as the energy spectral density. Although our results are not precise, it is crucial in many cases to determine the energy density for each value of s and subsequently an upper bound on the total signal energy. In quasilinear functional analysis, in particular using some Hilbert quasilinear spaces, we can obtain a procedure to approximate the basic concepts of classical signal processing for signals with inexact data.*

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