mathematics

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Instability of Standing Waves for INLS with Inverse Square Potential

Saleh Almuthaybiri and Tarek Saanouni *

Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; s.almuthybiri@qu.edu.sa
* Correspondence: t.saanouni@qu.edu.sa

Abstract: This work studies an inhomogeneous generalized Hartree equation with inverse square potential. The purpose is to prove the existence and strong instability of inter-critical standing waves. This means that there are infinitely many data near to the ground state, such that the associated solution blows-up in finite time. The proof combines a variational analysis with the standard variance identity. The challenge is to deal with three difficulties: the singular potential $|x|^{-2}$, an inhomogeneous term $|x|^{-\lambda}$, and a non-local source term.

Keywords: inhomogeneous Hartree equation; inverse square potential; nonlinear equations; instability; approximation; existence; uniqueness; ground states

MSC: 35Q55

check for updates

Citation: Almuthaybiri, S.; Saanouni, T. Instability of Standing Waves for INLS with Inverse Square Potential. *Mathematics* **2024**, *12*, 2999. https://doi.org/10.3390/math12192999

Academic Editors: Calogero Vetro and Omar Bazighifan

Received: 30 July 2024
Revised: 24 September 2024
Accepted: 24 September 2024
Published: 26 September 2024

1. Introduction

This paper addresses the Cauchy problem related to an inhomogeneous generalized Hartree equation,

$$\begin{cases} i\partial_t u - \mathcal{J}_\nu u = -|x|^{-\lambda}|u|^{p-2}(R_\rho * |\cdot|^{-\lambda}|u|^p)u; \\ u(0,\cdot) = u_0, \end{cases} \tag{1}$$

where the wave function is $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ for some $N \geq 3$. The inhomogeneous singular term is $|\cdot|^{-\lambda}$ for a certain $\lambda > 0$. The Riesz-potential is defined for $x \in \mathbb{R}^N$ by

$$R_\rho(x) := C_{N,\rho}|x|^{\rho-N}, \quad C_{N,\rho} := \pi^{-\frac{N}{2}}2^{-\rho}\frac{\Gamma(\frac{N-\rho}{2})}{\Gamma(\frac{\rho}{2})}.$$

Here and hereafter, we assume that

$$0 < \rho, \lambda < N, \quad \rho > 2(\lambda - 1). \tag{2}$$

Moreover, in order to ensure that the extension of $-\Delta + \frac{\nu}{|x|^2}$, labeled as $\mathcal{J}_\nu$, is a non-negative operator, we consider the case $\nu > -\frac{(N-2)^2}{4}$. If $-\frac{(N-2)^2}{4} < \nu < 1 - \frac{(N-2)^2}{4}$, we can obtain more than one possible extension, so we decide on the extension due to Friedrichs [1–3].

The Hartree Equation (1) is of physical root [4]. If the source term vanishes, the equation being examined is significant in quantum mechanics [1]. Otherwise, particularly for $\nu = 0$, the Choquard Equation (1) captures the non-relativistic bosonic molecules and atoms [5–9].

The inhomogeneous generalized Hartree Equation (1) was first studied by the second author [10]. The existence of L^2 solutions in the mass-sub-critical non-linearity and H^1 solutions for an energy-sub-critical source term were obtained thanks to an adapted

Gagliardo–Nirenberg-type inequality. Then, the second author established the locally well-posedness of (1) in $\dot{H}^1 \cap \dot{H}^{s_c}$; $0 < s_c < 1$. The long time asymptotics under the ground state energy was obtained by the second author [11], and then [12,13] removed the spherically symmetric assumption. The energy-critical well-posedness was investigated recently [14,15].

To the authors' knowledge, this article is the first one dealing with the instability of ground states to inhomogeneous Hatree equation with an inverse square potential, precisely (1) with $\nu \neq 0$. The key innovation here is the inclusion of the strongly singular potential $|x|^{-2}$, which maintains the same scaling as the Laplacian operator. The proof combines a variational analysis and the standard variance identity; this follows ideas in [16].

The purpose of this paper is to establish the existence and strong instability of inter-critical ground states to the inhomogeneous generalized Hartree Equation (1). This complements a recent work by the second author [17], where a local well-posedness was developed in the energy space.

The rest of this paper is organized as follows. The next section contains the main result and some technical estimates. The last section proves the main result.

2. Notation and Preliminary

In the following, we will simplify the notation for certain standard spaces and norms.

$$L^r(\mathbb{R}^N) := L^r, \quad W^{s,r}(\mathbb{R}^N) := W^{s,r}, \quad \dot{W}^{s,r}(\mathbb{R}^N) := \dot{W}^{s,r}, \quad H^1 := W^{1,2}, \quad \dot{H}^1 := \dot{W}^{1,2};$$
$$\|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2.$$

Also, we define Sobolev spaces by taking account of the operator $\mathcal{J}_\nu$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norms

$$\|\cdot\|_{\dot{W}_\nu^{1,r}} := \|\sqrt{\mathcal{J}_\nu}\cdot\|_r, \quad \|\cdot\|_{W_\nu^{1,r}} := \|\left\langle\sqrt{\mathcal{J}_\nu}\right\rangle\cdot\|_r,$$

where $\langle\cdot\rangle := \left(1+|\cdot|^2\right)^{\frac{1}{2}}$. Also take this for short $\dot{H}_\nu^1 := \dot{W}_\nu^{1,2}$ and $H_\nu^1 := W_\nu^{1,2}$. Note that, by the definition of the operator $\mathcal{J}_\nu$ and Hardy estimate, we have

$$\|\cdot\|_{\dot{H}_\nu^1} := \|\sqrt{\mathcal{J}_\nu}\cdot\| = \left(\|\nabla\cdot\|^2 + \nu\|\frac{\cdot}{|x|}\|^2\right)^{\frac{1}{2}} \simeq \|\cdot\|_{\dot{H}^1}.$$

Take also, for $\omega > 0$, the norm

$$\|\cdot\|_{H_{\nu,\omega}^1} := \left(\|\nabla\cdot\|^2 + \nu\|\frac{\cdot}{|x|}\|^2 + \omega\|\cdot\|^2\right)^{\frac{1}{2}} \simeq \|\cdot\|_{H^1}.$$

Let us denote also the real numbers

$$\mathcal{Z} := -\mathcal{Y} + 2p \quad \text{and} \quad \mathcal{Y} := Np - N - \rho + 2\lambda.$$

If $u \in H^1$, we define some quantities to be used later, and are called, respectively, potential energy, mass, and energy.

$$\mathcal{T}[u] := \int_{\mathbb{R}^N} |x|^{-\lambda}(R_\rho * |\cdot|^{-\lambda}|u|^p)|u|^p\,dx;$$
$$M[u] := \int_{\mathbb{R}^N} |u(x)|^2\,dx;$$
$$E[u] := \|\sqrt{\mathcal{J}_\nu}u\|^2 - \frac{1}{p}\mathcal{T}[u].$$

Equation (1) enjoys the scaling invariance,

$$u_\alpha := \alpha^{\frac{2-2\lambda+\rho}{2(p-1)}} u(\alpha^2\cdot, \alpha\cdot), \quad \alpha > 0. \tag{3}$$

The critical exponent s_c keeps the following homogeneous Sobolev norm invariant:

$$\|u_\alpha(t)\|_{\dot H^\mu} = \alpha^{\mu-(\frac{N}{2}-\frac{2-2\lambda+\rho}{2(p-1)})}\|u(\alpha^2 t)\|_{\dot H^\mu} := \alpha^{\mu-s_c}\|u(\alpha^2 t)\|_{\dot H^\mu}.$$

Now, we define two regimes. The first one is called mass-critical, and is given by $p = p_c := 1 + \frac{\rho+2-2\lambda}{N}$ or $s_c = 0$. The second one is called energy-critical, and is given by $p = p^c := 1 + \frac{2-2\lambda+\rho}{N-2}$ or $s_c = 1$. In the focusing case, $e^{i\omega t}\varphi$, where $\omega \in \mathbb{R}$ is a frequency, gives a global periodic solution of (1), called the standing wave.

$$-\Delta\varphi + \frac{\nu}{|x|^2}\varphi + \omega\varphi = |x|^{-\lambda}|\varphi|^{p-2}(R_\rho * |\cdot|^{-\lambda}|\varphi|^p)\varphi, \quad 0 \neq \varphi \in H^1_\nu. \tag{4}$$

Solutions to (4) are critical points of the action

$$\mathcal{S}_\omega := E + \omega M. \tag{5}$$

Let us denote the set of non-trivial solutions to (4),

$$\mathcal{A}_\omega := \{0 \neq u \in H^1_\nu, \quad \mathcal{S}'_\omega[u] = 0\}. \tag{6}$$

We define the set of ground states of (4) by

$$\mathcal{G}_\omega := \{u \in \mathcal{A}_\omega, \quad \mathcal{S}_\omega[u] \leq \mathcal{S}_\omega(v), \quad \forall v \in \mathcal{A}_\omega\}. \tag{7}$$

We denote the scaling

$$u^{\gamma,\mu}_\kappa := \kappa^\gamma u(\kappa^\mu\cdot), \quad \gamma,\mu \in \mathbb{R}, \,, \kappa > 0. \tag{8}$$

Moreover, if $F : H^1_\nu \to \mathbb{R}$, we denote the useful operator

$$\mathcal{L}_{\gamma,\mu}(F[u]) := \partial_\kappa\Big(F[u^{\gamma,\mu}_\kappa]\Big)_{|\kappa=1}.$$

Now, let the so-called constraint when equal to zero

$$\begin{aligned} \mathcal{K}^\omega_{\gamma,\mu}[u] &:= \mathcal{L}_{\gamma,\mu}\Big(\mathcal{S}_\omega[u]\Big) \\ &= \partial_\kappa\Big(\kappa^{2\gamma+\mu(2-N)}\|\nabla u\|^2 + \omega\kappa^{2\gamma+\mu(-N)}\|u\|^2 + \nu\kappa^{2\gamma+\mu(2-N)}\|\frac{u}{|x|}\|^2 - \frac{1}{p}\kappa^{2p\gamma+\mu(2\lambda-N-\rho)}\mathcal{T}[u]\Big)_{|\kappa=1} \\ &= (2\gamma+\mu(2-N))\|\sqrt{\mathcal{J}_\nu}u\|^2 + \omega(2\gamma-N\mu)\|u\|^2 - \frac{1}{p}(2p\gamma+\mu(2\lambda-N-\rho))\mathcal{T}[u]. \end{aligned} \tag{9}$$

Here and hereafter, we define the quantities

$$\begin{aligned} \underline{\rho} &:= \min\Big\{2\gamma+\mu(2-N), \omega(2\gamma-N\mu)\Big\}; \\ \bar{\rho} &:= \max\Big\{2\gamma+\mu(2-N), \omega(2\gamma-N\mu)\Big\}; \\ \mathcal{H}^\omega_{\gamma,\mu} &:= \mathcal{S}_\omega - \frac{1}{\bar{\rho}}\mathcal{K}^\omega_{\gamma,\mu}; \\ \mathcal{A} &:= \Big\{(\gamma,\mu) \in \mathbb{R}^*_+ \times \mathbb{R} \quad \text{s.t} \quad \underline{\rho} \geq 0 \quad \text{and} \quad \bar{\rho} > 0\Big\}. \end{aligned}$$

Let us also state the minimizing problem,

$$m^{\omega}_{\gamma,\mu} := \inf_{0\neq u\in H^1}\Big\{\mathcal{S}_\omega[u], \quad \mathcal{K}^{\omega}_{\gamma,\mu}[u]=0\Big\}. \tag{10}$$

Finally, we define some stable sets under the flow of (1),

$$\mathcal{A}^{\omega,-}_{\gamma,\mu} := \Big\{u\in H^1_\nu \quad \text{s.t} \quad \mathcal{S}_\omega[u]<m^{\omega}_{\gamma,\mu} \quad \text{and} \quad \mathcal{K}^{\omega}_{\gamma,\mu}[u]<0\Big\}.$$

The next section contains the main results and some useful estimates.

3. Background and Main Results

In the following sub-section, we list the contribution of this article.

3.1. Main Result

The main result of this article is about the existence of ground states and instability of standing waves.

Theorem 1. *Let $N\geq 3$, $\nu > -\frac{(N-2)^2}{4}$, $0<\omega<1$, λ,ρ satisfying (2) and $(\gamma,\mu)\in\mathcal{A}$. Then,*

1. *$m := m^{\omega}_{\gamma,\mu}$ is nonzero and independent of (γ,μ);*
2. *There is a ground state solution to (4)–(10), provided that one of the following cases holds:*

1. *$p_c < p < p^c$ and (20) is satisfied.*
2. *$p = p_c$ and (31) is satisfied.*

Moreover, if $p_c\leq p<p^c$, any ground state is strongly unstable.

Given the results outlined in the above theorem, some observations are warranted.

- Following lines in Lemma 3.3 of [18], a ground state solution to (4)–(10) satisfies

$$|\varphi_\omega| + |\nabla\varphi_\omega| \lesssim e^{-\delta|x|}, \quad \delta>0. \tag{11}$$

In particular,

$$\varphi_\omega\in L^2(|x|^2\,dx). \tag{12}$$

This property of finite variance allows to use the standard variance identity (Proposition 2) in order to prove the finite time blow-up of solutions to (1) with data near to the ground state.

- $(1,0)\in\mathcal{A}$ satisfies (31).
- The local existence of energy solutions to (1) follows by [17].
- The strong instability of standing waves means that there is infinitely many data near to the ground state such that the associated local solution to (1) blows-up in finite time.

3.2. Useful Estimates

In what follows, we present some essential tools that will be used later. We begin with the Hardy–Littlewood–Sobolev inequality [19].

Lemma 1. *Let $N\geq 1$ and $0<\rho<N$.*

1. *Let $a,b>1$, such that $\frac{1}{a}=\frac{1}{b}+\frac{\rho}{N}$. Then,*

$$\|R_\rho * g\|_b \leq C_{N,b,\rho}\|g\|_a, \quad \forall g\in L^a.$$

2. *Let $1<a,b,c<\infty$ be such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{\rho}{N}$. Then,*

$$\|f(R_\rho * g)\|_c \leq C_{N,b,\rho}\|f\|_a\|g\|_b, \quad \forall (f,g)\in L^a\times L^b.$$

Since $\|\cdot\|_{H^1} \simeq \|\cdot\|_{H^1_\nu}$ yields a compact embedding; see Lemma 3.1 in [20].

Lemma 2. *Let $0 < \lambda < 2$, $N \geq 3$ and $2 < r < \frac{2(N-\lambda)}{N-2}$. Then,*

$$H^1_\nu \hookrightarrow\hookrightarrow L^r(|x|^{-\lambda}\,dx) \quad \textit{is compact}.$$

The following Gagliardo–Nirenberg-type inequality [17] is tailored for the problem (1).

Proposition 1. *Let $N \geq 3$, $\nu > -\frac{(N-2)^2}{4}$, λ, ρ, satisfying (2) and $1 + \frac{\rho}{N} < p < p^c$. Then,*

1. *The minimization problem*

$$\frac{1}{C_{N,p,\lambda,\rho,\nu}} = \inf\left\{\frac{\|u\|^{\mathcal{Z}}\|\sqrt{\mathcal{J}_\nu}u\|^{\mathcal{Y}}}{\mathcal{T}[u]}, \quad 0 \neq u \in H^1_\nu\right\}$$

is attained in some $\zeta \in H^1_\nu$ satisfying $C_{N,p,\lambda,\rho,\nu} = \mathcal{T}[\zeta]$ and

$$\mathcal{Y}\mathcal{J}_\nu\zeta + \mathcal{Z}\zeta - \frac{2p}{C_{N,p,\lambda,\rho,\nu}}|x|^{-\lambda}|\zeta|^{p-2}(R_\rho * |\cdot|^{-\lambda}|\zeta|^p)\zeta = 0; \tag{13}$$

2. *There exists $\varphi \in H^1_\nu$ a ground state solution to (4), ensuring that*

$$C_{N,p,\lambda,\rho,\nu} = \frac{2p}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\mathcal{Y}}\right)^{\frac{\mathcal{Y}}{2}}\|\varphi\|^{-2(p-1)}, \tag{14}$$

3. *We have the Pohozaev identities*

$$\mathcal{T}[\varphi^\omega] = \frac{2p}{\mathcal{Z}}M[\varphi^\omega] = \frac{2p}{\mathcal{Y}}\|\sqrt{\mathcal{J}_\nu}\varphi\|^2. \tag{15}$$

To examine the non-global existence of solutions to (1), we require the variance type identity [17].

Proposition 2. *Let $N \geq 3$, $\nu > -\frac{(N-2)^2}{4}$, λ, ρ, satisfying (2), $2 \leq p < p^c$ and $u \in C_T(H^1_\nu)$ be a local solution to (1). We define the real function*

$$V : t \mapsto \|xu(t)\|^2, \quad t \in [0,T].$$

Then, $V \in C^2[0,T)$ and

$$V'' \quad : \quad t \mapsto 8\Big(\|\sqrt{\mathcal{J}_\nu}u(t)\|^2 - \frac{\mathcal{Y}}{2p}\mathcal{T}[u(t)]\Big) \quad t \in [0,T].$$

In the following sections, we will demonstrate the main results of this article.

4. Proof of Theorem 1

First, we establish the existence of ground states with variational methods.

4.1. Existence of Ground States

Let us start with the case $p_c < p < p^c$.

4.1.1. Inter-Critical Regime

The proof is based on several auxiliary results.

Lemma 3. *Let $(\gamma,\mu) \in \mathcal{A}$. Then,*

1. $\min\left\{\mathcal{H}^\omega_{\gamma,\mu}(u), \mathcal{L}_{\gamma,\mu}\left(\mathcal{H}^\omega_{\gamma,\mu}(u)\right)\right\} > 0$ *for all $0 \neq u \in H^1_\nu$;*

2. $\kappa \mapsto \mathcal{H}^{\omega}_{\gamma,\mu}[u^{\gamma,\mu}_{\kappa}]$ *is increasing.*

Proof. Let us write

$$\begin{aligned}
\mathcal{H}^{\omega}_{\gamma,\mu}[u] &= \frac{1}{\bar{\rho}}\Big(\bar{\rho}\mathcal{S}_{\omega}[u] - \mathcal{K}^{\omega}_{\gamma,\mu}[u]\Big) \\
&= \frac{1}{\bar{\rho}}\Big[\Big(\bar{\rho} - (2\gamma + \mu(2-N))\Big)\|\sqrt{\mathcal{J}_{\nu}}u\|^2 + \Big(\bar{\rho} - \omega(2\gamma - N\mu)\Big)\|u\|^2 \\
&+ \frac{1}{p}\Big(2\gamma p + \mu(2\lambda - N - \rho) - \bar{\rho}\Big)\mathcal{T}[u]\Big] \\
&\gtrsim \Big(2\gamma p + \mu(2\lambda - N - \rho) - \bar{\rho}\Big)\mathcal{T}[u].
\end{aligned}$$

• First case: $\mu > 0$. Since $\omega \in (0,1)$, we have

$$\omega(2\gamma - N\mu) \leq 2\gamma - N\mu < 2\gamma + \mu(2-N). \tag{16}$$

This implies that $\bar{\rho} = 2\gamma + \mu(2-N)$. Moreover, $\underline{\rho} \geq 0$ gives $2\gamma \geq \mu N$. Thus,

$$\begin{aligned}
2\gamma p + \mu(2\lambda - N - \rho) - \bar{\rho} &= 2\gamma p + \mu(2\lambda - N - \rho) - (2\gamma + \mu(2-N)) \\
&= 2\gamma(p-1) - \mu(2 - 2\lambda + \rho) \\
&\geq 2\gamma(p-1) - \frac{2\gamma}{N}(2 - 2\lambda + \rho) \\
&\geq 2\gamma(p - p_c) \\
&> 0.
\end{aligned} \tag{17}$$

• Second case: $\mu < 0$. Here, there are two sub-cases. First, if $\bar{\rho} = 2\gamma + \mu(2-N)$, we have

$$\begin{aligned}
2\gamma p + \mu(2\lambda - N - \rho) - \bar{\rho} &= 2\gamma(p-1) - \mu(2 - 2\lambda + \rho) \\
&> 0.
\end{aligned} \tag{18}$$

Second, if $\bar{\rho} = \omega(2\gamma - N\mu)$, we have

$$\begin{aligned}
2\gamma p + \mu(2\lambda - N - \rho) - \bar{\rho} &= 2\gamma p + \mu(2\lambda - N - \rho) - \omega(2\gamma - N\mu) \\
&> 2\gamma p + \mu(2\lambda - N - \rho) - (2\gamma - N\mu) \\
&> 2\gamma(p-1) + \mu(2\lambda - \rho) \\
&> 0,
\end{aligned} \tag{19}$$

provided that

$$2\gamma(2 - 2\lambda + \rho) - N\mu(\rho - 2\lambda) > 0. \tag{20}$$

So, $\mathcal{H}^{\omega}_{\gamma,\mu}[u] > 0$. Now, we write

$$\begin{aligned}
\mathcal{L}_{\gamma,\mu}\mathcal{H}^{\omega}_{\gamma,\mu}[u] &= \underline{\rho}\big(1 - \frac{\mathcal{L}_{\gamma,\mu}}{\bar{\rho}}\big)\mathcal{S}_{\omega}[u] - \frac{1}{\bar{\rho}}\big(\mathcal{L}_{\gamma,\mu} - \bar{\rho}\big)\big(\mathcal{L}_{\gamma,\mu} - \underline{\rho}\big)\mathcal{S}_{\omega}[u] \\
&= \underline{\rho}\mathcal{H}^{\omega}_{\gamma,\mu}[u] - \frac{1}{\bar{\rho}}\big(\mathcal{L}_{\gamma,\mu} - \bar{\rho}\big)\big(\mathcal{L}_{\gamma,\mu} - \underline{\rho}\big)\mathcal{S}_{\omega}[u] \\
&\geq \frac{1}{\bar{\rho}}\big(\mathcal{L}_{\gamma,\mu} - \bar{\rho}\big)\big(\mathcal{L}_{\gamma,\mu} - \underline{\rho}\big)\Big(\frac{1}{p}\mathcal{T}[u]\Big) \\
&\geq \frac{1}{p\bar{\rho}}\Big(2\gamma p + \mu(2\lambda - N - \rho) - \bar{\rho}\Big)\Big(2\gamma p + \mu(2\lambda - N - \rho) - \underline{\rho}\Big)\mathcal{T}[u] \\
&> 0.
\end{aligned} \tag{21}$$

Indeed, we used (17), (18) and (19) via the identities

$$\big(\mathcal{L}_{\gamma,\mu} - \bar{\rho}\big)\big(\mathcal{L}_{\gamma,\mu} - \underline{\rho}\big)\|u\|^2_{H^1_\nu} = 0; \tag{22}$$

$$\mathcal{L}_{\gamma,\mu} P[u] = (2\gamma p + \mu(2\lambda - N - \rho))\mathcal{T}[u]. \tag{23}$$

The last point is a consequence of the equality $\partial_\kappa\Big(\mathcal{H}^\omega_{\gamma,\mu}[u^{\gamma,\mu}_\kappa]\Big) = \mathcal{L}_{\gamma,\mu}\Big(\mathcal{H}^\omega_{\gamma,\mu}[u^\kappa_{\gamma,\mu}]\Big)$. □

The constraint is positive if it is quadratic part vanishes.

Lemma 4. *Let* $(\gamma,\mu) \in \mathcal{A}$ *and* $0 \neq \varphi_n \in H^1_\nu$*, such that*

$$\lim_n \Big(\mathcal{K}^{Q,\omega}_{\gamma,\mu}[\varphi_n]\Big) = 0; \tag{24}$$

$$\sup_n \|\varphi_n\|_{H^1} < \infty. \tag{25}$$

Then, there exists $n_0 \in \mathbb{N}$*, such that* $\mathcal{K}^\omega_{\gamma,\mu}[\varphi_n] > 0$ *for all* $n \geq n_0$.

Proof. The assumption

$$\mathcal{K}^{Q,\omega}_{\gamma,\mu}[\varphi_n] = \Big((2\gamma - \mu(N-2))\|\sqrt{\mathcal{J}_\nu}\varphi_n\|^2 + \omega(2\gamma - N\mu)\|\varphi_n\|^2\Big) \to 0, \quad \text{as} \quad n \to \infty,$$

implies that

$$\|\sqrt{\mathcal{J}_\nu}\varphi_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Indeed, otherwise, $0 < 2\gamma = \mu(N-2)$ contradicts $0 < \bar{\rho} = 2\gamma - \mu N = -2\mu$. Using Proposition 1 via $\mathcal{Y} > 2$, we write when $n \to \infty$,

$$\begin{aligned} \mathcal{T}[\varphi_n] &\leq C\|\varphi_n\|^{\mathcal{Z}}\|\sqrt{\mathcal{J}_\nu}\varphi_n\|^{\mathcal{Y}} \\ &= o\Big(\mathcal{K}^{Q,\omega}_{\gamma,\mu}[\varphi_n]\Big). \end{aligned}$$

Thus, when $n \to \infty$,

$$\mathcal{K}^\omega_{\gamma,\mu}[\varphi_n] \simeq \mathcal{K}^{Q,\omega}_{\gamma,\mu}[\varphi_n] > 0.$$

□

The minimizing problem (10) can be written as follows.

Lemma 5. *Let* $(\gamma,\mu) \in \mathcal{A}$*. Then,*

$$m^\omega_{\gamma,\mu} = \inf_{0 \neq u \in H^1_\nu}\big\{\mathcal{H}^\omega_{\gamma,\mu}[u] \quad s.\ t \quad \mathcal{K}^\omega_{\gamma,\mu}[u] \leq 0\big\}.$$

Proof. It is sufficient to prove the inequality in the above requested equality. Take $u \in H^1_\nu$ such that $\mathcal{K}^\omega_{\gamma,\mu}[u] < 0$. Because $\lim_{\kappa\to 0}\mathcal{K}^{Q,\omega}_{\gamma,\mu}[u^{\gamma,\mu}_\kappa] = 0$. By Lemma 4, there exists $\kappa \in (0,1)$, such that $\mathcal{K}^\omega_{\gamma,\mu}[u^{\gamma,\mu}_\kappa] > 0$. So, by a continuity argument, there exists $\kappa_0 \in (0,1)$ such that $\mathcal{K}^\omega_{\gamma,\mu}[u^{\gamma,\mu}_{\kappa_0}] = 0$. Now, since $\kappa \mapsto \mathcal{H}^\omega_{\gamma,\mu}[u^{\gamma,\mu}_\nu]$ is increasing, the proof is closed by the following line:

$$m^\omega_{\gamma,\mu} \leq \mathcal{H}^\omega_{\gamma,\mu}[u^{\gamma,\mu}_{\kappa_0}] \leq \mathcal{H}^\omega_{\gamma,\mu}[u].$$

□

Proof of the existence of inter-critical ground states. Let the sequence of minimizer

$$0 \neq \varphi_n \in H^1_\nu, \quad \mathcal{K}^\omega_{\gamma,\mu}[\varphi_n] = 0 \quad \text{and} \quad \lim_n \mathcal{H}^\omega_{\gamma,\mu}[\varphi_n] = \lim_n \mathcal{S}_\omega[\varphi_n] = m^\omega_{\gamma,\mu}. \tag{26}$$

- Step 1: $\sup_n \|\varphi_n\|_{H^1_\nu} < \infty$. Let us discuss some cases.

1. First case $\mu < 0$. Denoting $\theta := -\frac{\mu}{2\gamma}$ yields

$$\|\varphi_n\|^2_{H^1_{\nu,\omega}} - \mathcal{T}[\varphi_n] = \theta\Big(2\|\sqrt{\mathcal{J}_\nu}\varphi_n\|^2 - N\|\varphi_n\|^2_{H^1_{\nu,\omega}} + \frac{\rho+N-2\lambda}{p}\mathcal{T}[\varphi_n]\Big); \quad (27)$$

$$\|\varphi_n\|^2_{H^1_{\nu,\omega}} - \frac{1}{p}\mathcal{T}[\varphi_n] \to m^\omega_{\gamma,\mu}. \quad (28)$$

So, the following sequence is bounded

$$-2\theta\|\sqrt{\mathcal{J}_\nu}\varphi_n\|^2 + \|\varphi_n\|^2_{H^1_{\nu,\omega}} - (1+\theta\frac{\rho-2\lambda}{p})\mathcal{T}[\varphi_n].$$

Thus, the next sequence is bounded, for all $\beta \in \mathbb{R}$

$$2\theta\|\sqrt{\mathcal{J}_\nu}\varphi_n\|^2 + (\beta-1)\|\varphi_n\|^2_{H^1_{\nu,\omega}} + (1+\frac{\theta(\rho-2\lambda)-\beta}{p})\mathcal{T}[\varphi_n].$$

Taking account of (2) and (20), we pick $1 < \beta < p_c + \theta(\rho - 2\lambda)$. Thus, (φ_n) is bounded in H^1_ν.

2. Second case $0 \le \mu \le \frac{2\gamma}{N}$. Taking account of (16), ones writes

$$\begin{aligned}(\bar{\rho} - \mathcal{L}_{\gamma,\mu})\mathcal{S}_\omega[\varphi_n] &= 2\mu\|\varphi_n\|^2 + \Big(2\gamma(p-1) - \mu(\rho+2-2\lambda)\Big)\frac{1}{p}\mathcal{T}[\varphi_n] \\ &\geq \Big(2\gamma(p-1) - \mu(\rho+2-2\lambda)\Big)\frac{1}{p}\mathcal{T}[\varphi_n].\end{aligned}$$

Moreover, as previously, since $\underline{\rho} \ge 0$ and $p > p_c$ yields $2\gamma(p-1) - \mu(\rho+2-2\lambda) > 0$, and $\mathcal{K}^\omega_{\gamma,\mu}[\varphi_n] = 0$ gives

$$\begin{aligned}&\Big(\bar{\rho} + 2\gamma(p-1) - \mu(\rho+2-2\lambda)\Big)\mathcal{S}_\omega[\varphi_n] \\ =\ & (\bar{\rho} - \mathcal{L}_{\gamma,\mu})S_\omega[\varphi_n] + \Big(2\gamma(p-1) - \mu(\rho+2-2\lambda)\Big)S[\varphi_n] + \mathcal{L}_{\gamma,\mu}\mathcal{S}_\omega[\varphi_n] \\ \geq\ & \Big(2\gamma(p-1) - \mu(\rho+2-2\lambda)\Big)\|\varphi_n\|^2_{H^1_\nu}.\end{aligned}$$

it follows that (φ_n) is bounded in H^1_ν.

• Step 2: $\varphi_n \to \varphi \neq 0$ and $m^\omega_{\gamma,\mu} > 0$.
Thanks to Lemma 2, let

$$\varphi_n \rightharpoonup \varphi \quad \text{in} \quad H^1_\nu; \quad (29)$$

$$\varphi_n \to \varphi \quad \text{in} \quad L^r(|x|^{-\lambda}\,dx), \quad \text{for all} \quad 2 < r < \frac{2(N-\lambda)}{N-2}. \quad (30)$$

Assume that $\varphi = 0$. Using Lemma 1 via (30) and $p < p^c$, we write

$$\mathcal{T}[\varphi_n] \le \||\cdot|^{-\frac{\lambda}{p}}\varphi_n\|^{2p}_{\frac{2Np}{\rho+N}} \to \||\cdot|^{-\frac{\lambda}{p}}\varphi\|^{2p}_{\frac{2Np}{\rho+N}} = 0.$$

The equality $\mathcal{K}^\omega_{\gamma,\mu}[\varphi_n] = 0$ via Lemma 4 gives $\mathcal{K}^\omega_{\gamma,\mu}[\varphi_n] > 0$ for large n. This contradiction implies that

$$\varphi \neq 0.$$

The lower semi continuity of the H^1_ν norm via Lemma 1, gives

$$\begin{aligned}0 &= \liminf_n \mathcal{K}^\omega_{\gamma,\mu}[\varphi_n] \\ &\geq (2\gamma - (N-2)\mu)\liminf_n \|\sqrt{\mathcal{J}_\nu}\varphi_n\|^2 + \omega(2\gamma - N\mu)\liminf_n \|\varphi_n\|^2\end{aligned}$$

$$
\begin{aligned}
&- \frac{2\gamma p + \mu(2\lambda - N - \rho)}{p}\mathcal{T}[\varphi^\omega] \\
&\geq \mathcal{K}^\omega_{\gamma,\mu}[\varphi^\omega].
\end{aligned}
$$

Similarly, we have $\mathcal{H}^\omega_{\gamma,\mu}[\varphi^\omega] \leq m^\omega_{\gamma,\mu}$. Moreover, thanks to Lemma 5, we assume that $\mathcal{K}^\omega_{\gamma,\mu}[\varphi^\omega] = 0$ and $\mathcal{S}_\omega[\varphi^\omega] = \mathcal{H}^\omega_{\gamma,\mu}[\varphi^\omega] \leq m^\omega_{\gamma,\mu}$. So, φ is a minimizer satisfying (26) and using Lemma 5,

$$
m^\omega_{\gamma,\mu} = \mathcal{H}^\omega_{\gamma,\mu}[\varphi^\omega] > 0.
$$

• Step 3: φ satisfies (4).
Let a Lagrange multiplier $\lambda \in \mathbb{R}$, such that $\mathcal{S}'_\omega[\varphi^\omega] = \lambda {\mathcal{K}^\omega_{\gamma,\mu}}'[\varphi^\omega]$. Thus,

$$
0 = \mathcal{L}_{\gamma,\mu}\mathcal{S}_\omega[\varphi^\omega] = \langle \mathcal{S}'_\omega[\varphi^\omega], \mathcal{L}_{\gamma,\mu}[\varphi^\omega]\rangle = \lambda\langle {\mathcal{K}^\omega_{\gamma,\mu}}'[\varphi^\omega], \mathcal{L}_{\gamma,\mu}[\varphi^\omega]\rangle = \lambda\mathcal{L}_{\gamma,\mu}\mathcal{K}^\omega_{\gamma,\mu}[\varphi^\omega] = \lambda\mathcal{L}^2_{\gamma,\mu}\mathcal{S}_\omega[\varphi^\omega].
$$

Now, (21) gives

$$
-\mathcal{L}^2_{\gamma,\mu}\mathcal{S}_\omega[\varphi^\omega] - \bar{\rho}\underline{\rho}\mathcal{S}_\omega[\varphi^\omega] = -(\mathcal{L}_{\gamma,\mu} - \bar{\rho})(\mathcal{L}_{\gamma,\mu} - \underline{\rho})\mathcal{S}_\omega[\varphi^\omega] > 0.
$$

Therefore, $\mathcal{L}^2_{\gamma,\mu}\mathcal{S}_\omega[\varphi^\omega] < 0$. Thus, $\lambda = 0$ and $\mathcal{S}'_\omega[\varphi^\omega] = 0$. So, φ is a ground state and $m^\omega_{\gamma,\mu}$ is autonomous from (γ, μ). □

4.1.2. Mass Critical Regime

In such a case, we assume that

$$
\underline{\rho} > 0 \quad \text{and} \quad 2\gamma\rho + N\mu(2\lambda - \rho)\chi_{\mu<0} - N\mu(2 - 2\lambda + \rho)\chi_{\mu>0} > 0. \tag{31}
$$

It is sufficient to follow the proof in the case of the inter-critical regime; indeed, (17), (18) and (19) hold. Moreover, Lemma 4 holds. Indeed, $\underline{\rho} > 0$ implies that $\|\varphi_n\| \to 0$, and so

$$
\begin{aligned}
\mathcal{T}[\varphi_n] &\leq C\|\varphi_n\|^{2(p-1)}\|\sqrt{\mathcal{J}_\nu}\varphi_n\|^2 \\
&= o\Big(\mathcal{K}^{Q,\omega}_{\gamma,\mu}[\varphi_n]\Big).
\end{aligned}
$$

4.2. Strong Instability of Inter-Critical Standing Waves

Let us prepare the proof.

Lemma 6. *Let $u_0 \in \mathcal{A}^{\omega,-}_{1,\frac{2}{N}}$ and a local solution to (1) denoted by $u \in C_T(H^1_\nu)$. Thus, for certain $\delta > 0$, yields*

$$
\mathcal{K}^\omega_{1,\frac{2}{N}}[u] < -\delta \quad on \quad [0, T].
$$

Proof. If $\mathcal{K}^\omega_{1,\frac{2}{N}}[u(\tau_n)] \to 0$, where $0 < \tau_n < T$, by Lemma 5, it follows that

$$
m^\omega \leq \big(\mathcal{S}_\omega - \frac{N}{4}\mathcal{K}^\omega_{1,\frac{2}{N}}\big)[u(\tau_n)] \leq \mathcal{S}_\omega[u_0] - \frac{N}{4}\mathcal{K}^\omega_{1,\frac{2}{N}}[u(\tau_n)] \to \mathcal{S}_\omega[u_0] < m^\omega.
$$

This absurdity finishes the proof. □

For $u \in H^1_\nu$ and $\kappa > 0$, define the scaling

$$
u_\kappa := u_\kappa^{\frac{N}{2},1} = \kappa^{\frac{N}{2}}u(\kappa\cdot).
$$

Let us outline the variations of various functions under the scaling discussed earlier.

Lemma 7. *Let $u \in H^1_\nu$ be such that $\mathcal{K}_{1,\frac{2}{N}}[u] \leq 0$. Then, there exists a unique $\kappa_0 \in (0, 1]$, such that*

1. $\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u_{\kappa_0}] = 0$ *and* $\kappa_0 = 1$ *if and only if* $\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u] = 0$;
2. $\frac{\partial}{\partial\kappa}\mathcal{S}_{\omega}[u_\kappa] > 0$ *for* $\kappa \in (0, \kappa_0)$ *and* $\frac{\partial}{\partial\kappa}\mathcal{S}_{\omega}[u_\kappa] < 0$ *for* $\kappa \in (\kappa_0, \infty)$;
3. $\frac{\partial}{\partial\kappa}\mathcal{S}_{\omega}[u_\kappa] = \frac{N}{2\kappa}\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u_\kappa]$.

Proof. In order to prove the last point, it is sufficient to write

$$\begin{aligned}\frac{\partial}{\partial\kappa}\mathcal{S}_{\omega}[u_\kappa] &= \frac{\partial}{\partial\kappa}\Big(\kappa^2\|\sqrt{\mathcal{J}_\nu}u\|^2 + \omega\|u\|^2 - \frac{\kappa^{\mathcal{Y}}}{p}\mathcal{T}[u]\Big)\\ &= 2\kappa\|\sqrt{\mathcal{J}_\nu}u\|^2 - \frac{\mathcal{Y}}{p}\kappa^{\mathcal{Y}-1}\mathcal{T}[u]\\ &= \frac{N}{2\kappa}\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u_\kappa].\end{aligned}$$

Moreover, $\mathcal{Y} > 2$ and $\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u] \le 0$, via the equality

$$\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u_\kappa] = \frac{4}{N}\kappa^2\Big(\|\sqrt{\mathcal{J}_\nu}u\|^2 - \frac{\mathcal{Y}}{2p}\kappa^{N(\mathcal{Y}-2)}\mathcal{T}[u]\Big).$$

imply that the real function $\kappa \mapsto \kappa^{-2}\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u_\kappa]$ defined on $[0,1]$ is decreasing from a positive real number to a negative one. This closes the proof. □

Now, we turn attention to the evolution problem (1).

Lemma 8. *Let* $\varphi^{\omega} \in H^1_\nu$ *be a ground state solution to* (4), $\kappa > 1$ *a real number and* u_κ *be the solution to* (1) *with data* $\varphi^{\omega}_{\kappa}$. *Thus,*

$$\mathcal{S}_{\omega}[u_\kappa] < \mathcal{S}_{\omega}[\varphi^{\omega}] \quad \textit{and} \quad \mathcal{K}^{\omega}_{1,\frac{2}{N}}[u_\kappa] < 0 \quad \textit{on} \quad [0, T^*).$$

Proof. By Lemma 7, we write

$$\mathcal{S}_{\omega}[\varphi^{\omega}_{\kappa}] < \mathcal{S}_{\omega}[\varphi^{\omega}] \quad \text{and} \quad \mathcal{K}^{\omega}_{1,\frac{2}{N}}[\varphi^{\omega}_{\kappa}] < 0.$$

The proof is completed by demonstrating the stability of the following set under the flow of (1).

$$\mathcal{A}^{\omega,-}_{1,\frac{2}{N}} := \{u \in H^1_\nu, \quad \mathcal{S}_{\omega}[u] < m^{\omega}_{1,\frac{2}{N}}, \quad \mathcal{K}^{\omega}_{1,\frac{2}{N}}[u] < 0\}. \tag{32}$$

Indeed, if $u_0 \in \mathcal{A}^{\omega,-}_{1,\frac{2}{N}}$ and u is the energy solution to (1). Then, by the conservation laws $\mathcal{S}_{\omega}[u] < m^{\omega}_{1,\frac{2}{N}}$. Moreover, if there is $t_0 > 0$ such that $\mathcal{K}^{\omega}_{1,\frac{2}{N}}[u] = 0$, this contradicts the definition of $m^{\omega}_{1,\frac{2}{N}}$.
□

Now, we are ready to prove the last point of Theorem 1.

Let φ^{ω}, a ground state solution to (4) and $u_\kappa \in C_{T^*}(H^1_\nu)$, be the maximal solution to (1) with data $\varphi^{\omega}_{\kappa}$. Lemma 8 gives

$$u_\kappa(t) \in \mathcal{A}^{\omega,-}_{1,\frac{2}{N}}, \quad \text{for any} \quad t \in [0, T^*).$$

Then, using Proposition 2 via (12) and Lemma 6, it follows that

$$\limsup_{t\to T^*}\|u_\kappa(t)\|_{H^1_\nu} = \infty.$$

The instability of inter-critical standing waves is proved.

4.3. Instability of Mass-Critical Standing Waves

Take the scaling $u_\kappa^{1+\frac{N}{2},1} := u_\kappa$. Thus, by direct calculus, we obtain

$$\|u_\kappa\|^2 = \kappa^2\|u\|^2; \tag{33}$$
$$\|\nabla u_\kappa\|^2 = \kappa^4\|\nabla u\|^2; \tag{34}$$
$$\||x|^{-1}u_\kappa\|^2 = \kappa^4\||x|^{-1}u\|^2; \tag{35}$$
$$\mathcal{T}[u_\kappa] = \kappa^{2p_c+2}\mathcal{T}[u]. \tag{36}$$

Then, $\varphi_\kappa^\omega \to \varphi^\omega$ in H^1 if $\kappa \to 1$. Let the datum $u_0 := \varphi_\kappa^\omega$, such that $\kappa = 1^+$. So, taking account of (15), we obtain

$$\begin{aligned} E[u] &= \|\sqrt{\mathcal{J}_\nu}\varphi_\kappa^\omega\|^2 - \frac{1}{p}\mathcal{T}[\varphi_\kappa^\omega] \\ &= \kappa^4\Big(\|\sqrt{\mathcal{J}_\nu}\varphi^\omega\|^2 - \frac{1}{p}\kappa^{2(p_c-1)}\mathcal{T}[\varphi^\omega]\Big) \\ &= \kappa^4\Big(1-\kappa^{2(p_c-1)}\Big)\|\sqrt{\mathcal{J}_\nu}\varphi^\omega\|^2 \\ &< 0. \end{aligned} \tag{37}$$

By (37) and the variance identity in Proposition 2, it follows that

$$V'' \lesssim E[u] < -c < 0.$$

Integrating in time, we finish the proof.

Author Contributions: S.A. and T.S. confirm the equal responsibility for the following: study conception and design, data collection, analysis and interpretation of results, and manuscript preparation. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data-sets were generated or analyzed during the current study.

Acknowledgments: The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2024-9/1).

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Kalf, H.; Schmincke, U.-W.; Walter, J.; Wüst, R. On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials. In *Spectral Theory and Differential Equations (Proceedings Symposium Dundee, 1974; Dedicated to Konrad Jörgens)*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1975; Volume 448, pp. 182–226.
2. Planchon, F.; Stalker, J.; Tahvildar-Zadeh, A.S. L^p estimates for the wave equation with the inverse- square potential. *Discrete Contin. Dyn. Syst.* **2003**, *9*, 427–442. [CrossRef]
3. Titchmarsh, E.C. *Eigenfunction Expansions Associated with Second-Order Differential Equations*; University Press: Oxford, UK, 1946.
4. Hartree, D.R. The Wave Mechanics of an Atom with a Non-Coulomb Central Field. Part I. Theory and Methods. *Math. Proc. Camb. Philos. Soc.* **1928**, *24*, 89–110. [CrossRef]
5. Alkhidhr, H.A. Closed-form solutions to the perturbed NLSE with Kerr law nonlinearity in optical fibers. *Result. Phys.* **2021**, *22*, 103875. [CrossRef]
6. Fröhlich, J.; Lenzmann, E. Mean-field limit of quantum Bose gases and nonlinear Hartree equation. In *Séminaire: Equations aux Dérivées Partielles 2003–2004*; Ecole Polytechnique: Palaiseau, France, 2004; Exp. No. XIX; p. 26.
7. Gross, E.P.; Meeron, E. *Physics of Many-Particle Systems*; Gordon Breach: New York, NY, USA, 1966; Volume 1, pp. 231–406.
8. Moroz, I.M.; Penrose, R.; Tod, P. Spherically-symmetric solutions of the Schrödinger-Newton equations. *Class. Quantum Gravity* **1998**, *15*, 2733–2742. [CrossRef]

9. Spohn, H. On the Vlasov hierarchy. *Math. Method Appl. Sci.* **1981**, *3*, 445–455. [CrossRef]
10. Alharbi, M.G.; Saanouni, T. Sharp threshold of global well-posedness vs finite time blow-up for a class of inhomogeneous Choquard equations. *J. Math. Phys.* **2019**, *60*, 081514. [CrossRef]
11. Saanouni, T.; Xu, C. Scattering Theory for a Class of Radial Focusing Inhomogeneous Hartree Equations. *Potential Anal* **2021**, *58*, 617–643. [CrossRef]
12. Saanouni, T.; Peng, C. Scattering for a class of inhomogeneous generalized Hartree equations. *Appl. Anal* **2023**, *103*, 790–806. [CrossRef]
13. Xu, C. Scattering for the non-radial focusing inhomogeneous nonlinear Schrödinger-choquard equation. *arXiv* **2021**, arXiv:2104.09756.
14. Kim, S. On well-posedness for inhomogeneous Hartree equations in the critical case. *Comm. Pur. Appl. Anal.* **2023**, *22*, 2132–2145. [CrossRef]
15. Kim, S.; Lee, Y.; Seo, I. Sharp weighted Strichartz estimates and critical inhomogeneous Hartree equations. *Nonlinear Anal.* **2024**, *240*, 113463. [CrossRef]
16. Berestycki, H.; Cazenave, T. Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires. *C. R. Acad. Sci. Paris Sér. I Math.* **1981**, *293*, 489–492.
17. Saanouni, T.; Boubaker, A. Inhomogeneous generalized Hartree equation with inverse square potential. *SeMA* **2023**, *6*, 1–28. [CrossRef]
18. Dinh, V.D. On instability of standing waves for the nonlinear Schrödinger equation with inverse- square potential. *Complex Var. Elliptic Equ.* **2021**, *66*, 1699–1716. [CrossRef]
19. Lieb, E.; Loss, M. *Analysis*, 2nd ed.; Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA, 2001; Volume 14.
20. Campos, L.; Guzman, C.M. On the inhomogeneous NLS with inverse-square potential. *Z. Angew. Math. Phys.* **2021**, *72*, 143. [CrossRef]