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# Constraint Qualifications and Optimality Conditions for Multiobjective Mathematical Programming Problems with Vanishing Constraints on Hadamard Manifolds

**Balendu Bhooshan Upadhyay [1], Arnav Ghosh [1], Savin Treanţă [2,3,4] and Jen-Chih Yao [5,6,*]**

1 Department of Mathematics, Indian Institute of Technology Patna, Patna 801103, India; bhooshan@iitp.ac.in (B.B.U.); arnav_2021ma09@iitp.ac.in (A.G.)
2 Department of Applied Mathematics, National University of Science and Technology POLITEHNICA Bucharest, 060042 Bucharest, Romania; savin.treanta@upb.ro
3 Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania
4 "Fundamental Sciences Applied in Engineering" Research Center (SFAI), National University of Science and Technology POLITEHNICA Bucharest, 060042 Bucharest, Romania
5 Center for General Education, China Medical University, Taichung 40402, Taiwan
6 Academy of Romanian Scientists, 050044 Bucharest, Romania
* Correspondence: yaojc@mail.cmu.edu.tw

**Abstract:** In this paper, we investigate constraint qualifications and optimality conditions for multiobjective mathematical programming problems with vanishing constraints (MOMPVC) on Hadamard manifolds. The MOMPVC-tailored generalized Guignard constraint qualification (MOMPVC-GGCQ) for MOMPVC is introduced in the setting of Hadamard manifolds. By employing MOMPVC-GGCQ and the intrinsic properties of Hadamard manifolds, we establish Karush–Kuhn–Tucker (KKT)-type necessary Pareto efficiency criteria for MOMPVC. Moreover, we introduce several MOMPVC-tailored constraint qualifications and develop interrelations among them. In particular, we establish that the MOMPVC-tailored constraint qualifications introduced in this paper turn out to be sufficient conditions for MOMPVC-GGCQ. Suitable illustrative examples are furnished in the framework of well-known Hadamard manifolds to validate and demonstrate the importance and significance of the derived results. To the best of our knowledge, this is the first time that constraint qualifications, their interrelations, and optimality criteria for MOMPVC have been explored in the framework of Hadamard manifolds.

**Keywords:** constraint qualifications; multiobjective programming; vanishing constraints; optimality conditions; Hadamard manifolds

**MSC:** 90C46; 90C48; 90C29; 90C30



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## 1. Introduction

In optimization theory, mathematical programming problems with vanishing constraints (MPVC) have transpired as a very interesting class of problem in the last few years. The nomenclature 'vanishing constraints' is derived from the fact that some of the constraints present in the problem may become redundant on some particular feasible elements of MPVC (see, for instance, [1]). One of the primary challenges encountered in the investigation of MPVC is the fact that the feasible set of MPVC may be non-convex and disconnected, despite the presence of convex constraint functions (see, for instance, [2]). Furthermore, in general, standard constraint qualifications, such as the Mangasarian–Fromovitz constraint qualification (MFCQ) and linear independence constraint qualification (LICQ), are violated at an arbitrary feasible point of MPVC (see, for instance, [3,4]). Recently, researchers have observed that various real-world problems emerging in several areas of engineering and technology can be modeled as MPVC. Achtziger and Kanzow [1] demonstrated that important real-world problems in the field of mechanical engineering, such as the truss optimization problem, can be formulated as an MPVC. Dai [5] modeled the problem of finding optimal pressure control in water distribution systems as an MPVC. Kirches et al. [6] investigated the problem of robot motion planning by formulating the corresponding problem as an MPVC. For more comprehensive discussions and updated surveys of MPVC, we refer the readers to [7–10] and the references provided therein.

Achtziger and Kanzow [1] established that many of the standard constraint qualifications fail to be satisfied for MPVC and developed several MPVC-tailored constraint qualifications. Hoheisel and Kanzow [9] derived necessary optimality conditions for MPVC by employing Abadie-type constraint qualification, as well as Guignard-type constraint qualification. Guu et al. [11] derived strong Karush–Kuhn–Tucker-type sufficient optimality conditions for MPVC, with infinitely many constraints. Mishra et al. [12] studied several constraint qualifications for MPVC with vector-valued objective functions. Wolfe-type, as well as the Mond-Weir type duality models for MPVC, were formulated by Mishra et al. [13], and several duality theorems were derived. Several constraint qualifications and their interrelations for MPVC in the Hadamard manifold framework were developed by Upadhyay and Ghosh [14]. One-parameter regularization methods for MPVC were explored by Hoheisel et al. [3]. Optimality conditions for nonsmooth MPVC were developed by Shirdel et al. [15] via the Dini–Hadamard derivative. Hassan et al. [16] investigated M-stationary conditions and duality for MPVC. Recently, Upadhyay et al. [17] derived necessary and sufficient Pareto efficiency criteria for semi-infinite MPVC with a vector-valued objective function on Hadamard manifolds and further investigated some duality models for MPVC. However, in this context, it is noteworthy that there is no research paper available in the literature that explores constraint qualifications and their interrelations for multiobjective MPVC in the framework of Hadamard manifolds.

In recent years, the exploration of optimization problems on manifolds has turned out to be a very significant research topic. For modeling and analyzing many practical problems of modern research, especially in data analysis, the manifold framework has been found to be more advantageous than the traditional Euclidean space setting (for reference, see [18–20]). On the other hand, generalizing various methods of optimization from Euclidean spaces to the framework of manifolds has numerous important advantages. Many challenging constrained non-convex optimization problems formulated in the Euclidean space can be reformulated as simpler, unconstrained, as well as convex problems within the Hadamard manifold framework (see [21–23]). As a direct consequence to these advantages, in the last few years, numerous scholars have generalized various key ideas related to optimization theory from the Euclidean space setting to the Hadamard manifold framework (see [24–28] and the references provided therein).

Contrary to Euclidean spaces, Hadamard manifolds, in general, do not possess a linear structure. As a result, in spite of the fact that Hadamard manifolds are globally homeomorphic to Euclidean spaces, the investigation of optimization techniques on Hadamard manifolds is associated with several challenges. Specifically, in sharp contrast to Euclidean space, the notion of a unique line segment joining any two points is not available in the Hadamard manifold setting. Moreover, the exponential function and inverse of the exponential function are nonlinear functions on Hadamard manifolds (see, for instance, [29]). Consequently, researchers have developed new techniques over the past few decades to study optimization problems on Hadamard manifolds. For instance, the concept of geodesic convexity is introduced in the Hadamard manifold setting, employing the notion of unique minimal geodesics to connect any two points in the Hadamard manifold. Moreover, the concepts of parallel transport and exponential maps on the tangent space of a Hadamard manifold (which has a vector space structure) are employed in order to deal with the nonlinearity of manifolds. The central motivation and principal objective to investigate nonsmooth MOMPVC on Hadamard manifolds, rather than Riemannian manifolds, is as follows. Firstly, the exponential function is globally diffeomorphic in the case of Hadamard manifolds (see, for instance, [30]). Nevertheless, the exponential function is locally diffeomorphic on Riemannian manifolds. Thus, the results derived in the present manuscript on Hadamard manifolds are valid within the totally normal neighborhood of each point on Riemannian manifolds.

It is noteworthy to observe that, in the last few decades, researchers have thoroughly explored optimality criteria for single-objective as well as multiobjective optimization problems in the framework of Euclidean space (see, for instance, [31–33] and the references provided therein). Furthermore, in recent times, constraint qualifications and optimality conditions for optimization problems involving scalar-valued and vector-valued objective functions in the Riemannian and Hadamard manifold framework have been investigated by several researchers (see, for instance, [25,26,34] and the references provided therein). Nevertheless, constraint qualifications and optimality criteria for a very important class of optimization problems, namely, multiobjective optimization problems with vanishing constraints (MOMPVC) have not yet been studied in the Hadamard manifold framework. The primary motivation as well as the the central objective of this paper is to address the aforementioned research gap, introduce various constraint qualifications and derive their interrelations, and further establish necessary Pareto efficiency criteria for MOMPVC in the Hadamard manifold framework.

Motivated by the results and discussions presented in the papers [9,11,12], we explore a class of MOMPVC in the Hadamard manifold framework. Firstly, the MOMPVC-tailored generalized Guignard constraint qualification (MOMPVC-GGCQ) is introduced in the framework of Hadamard manifolds. We further establish necessary optimality conditions for MOMPVC using MOMPVC-GGCQ. Thereafter, we introduce several MOMPVC-tailored constraint qualifications, for instance, Cottle-type constraint qualification, Slater-type constraint qualification, and Mangasarian–Fromovitz constraint qualification, which in turn become sufficient conditions for MOMPVC-GGCQ. Non-trivial illustrative examples have been incorporated to validate the effectiveness of the deduced results.

The primary contributions and novelty of this paper are as follows. Firsly, we extend several constraint qualifications developed by Mishra et al. [12] from the linear Euclidean space framework to the non-linear, more general space of Hadamard manifolds. Moreover, the optimality conditions deduced in this paper extend the corresponding optimality criteria developed by Hoheisel and Kanzow [9] from Euclidean space to Hadamard manifolds, and further generalize them from single objective MPVC to a more general class of mathematical programming problems, namely, MOMPVC. Secondly, the results established in this paper extend the corresponding results derived by Maeda [33] for a more general class of optimization problems, that is, MOMPVC, and further generalize them from the Euclidean space setting to the framework of a wider space, namely, Hadamard manifolds.

The rest of the article is structured as follows: Basic notations and definitions are provided in Section 2. In Section 3, we consider a class of MOMPVC in the framework of Hadamard manifolds. MOMPVC-GGCQ is introduced in the Hadamard manifold framework and KKT-type necessary criteria of optimality for MOMPVC are established. Subsequently, in Section 4, we present several MOMPVC-tailored constraint qualifications, which in turn become sufficient conditions ensuring the satisfaction of MOMPVC-GGCQ. Finally, we conclude our discussions and throw some light on our further research endeavors in Section 5.

## 2. Basic Notations, Definitions, and Mathematical Preliminaries

Throughout the paper, the symbols $\mathbb{N}$ and $\mathbb{R}$ will be employed to represent the set consisting of every natural and real number, respectively. The notation $\emptyset$ signifies the empty set. Let $n \in \mathbb{N}$. The symbol $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. The standard inner product on $\mathbb{R}^n$ is signified by $\langle \cdot, \cdot \rangle$. For $r, s \in \mathbb{R}^n$, the notation $r \geqq s$ implies that $r_i \geq s_i$, for every $i = 1, 2, \ldots, n$. Further, $r \geq s$ indicates that $r \geqq s$ and $r \neq s$. Moreover, we use the following conventions:

$$r \prec s \Longleftrightarrow r_k < s_k, \quad \forall k = 1, \ldots, n.$$

$$r \preceq s \Longleftrightarrow \begin{cases} r_k \leq s_k, & \forall k = 1, \ldots, n; \\ r_m < s_m, & \text{for at least one } m \in \{1, \ldots, n\}. \end{cases}$$

We employ the symbol $\mathcal{H}^n$ to denote an $n$-dimensional Riemannian manifold. $\mathcal{H}^n$ is said to be a Hadamard manifold, provided that it is geodesic complete, simply connected, and moreover, the sectional curvature of $\mathcal{H}^n$ is non-positive everywhere. Henceforth, in this paper, the symbol $\mathcal{H}^n$ will always indicate an $n$-dimensional Hadamard manifold, unless otherwise stated.

Let $\hat{a} \in \mathcal{H}^n$. The tangent space at $\hat{a}$ is denoted by the symbol $T_{\hat{a}}\mathcal{H}^n$. It is a well-known fact that $T_{\hat{a}}\mathcal{H}^n$ is a linear space having a dimension $n$. Let $\mathcal{B} \subset T_{\hat{a}}\mathcal{H}^n$ be non-empty. The notations $\mathrm{cl}(\mathcal{B})$ and $\mathrm{co}(\mathcal{B})$ are used to signify the closure and the convex hull of set $\mathcal{B}$, respectively.

The exponential mapping $\exp_{\hat{a}} : T_{\hat{a}}\mathcal{H}^n \to \mathcal{H}^n$ is a globally diffeomorphic map. Further, $\exp_{\hat{a}}^{-1} : \mathcal{H}^n \to T_{\hat{a}}\mathcal{H}^n$ satisfies $\exp_{\hat{a}}^{-1}(\hat{a}) = 0$. On the other hand, for every $\hat{a}_1, \hat{a}_2 \in \mathcal{H}^n$, some unique normalized minimal geodesic $\nu_{\hat{a}_1,\hat{a}_2} : [0,1] \to \mathcal{H}^n$ always exists, satisfying:

$$\nu_{\hat{a}_1,\hat{a}_2}(\tau) = \exp_{\hat{a}_1}(\tau \exp{}_{\hat{a}_1}^{-1}(\hat{a}_2)), \quad \forall \tau \in [0,1].$$

**Remark 1.** *In light of Theorem 2.1 established by Kristaly et al. [29], exponential map* $\exp : T_{\hat{a}}\mathcal{H}^n \to \mathcal{H}^n$ *on Hadamard manifolds with zero sectional curvature is a global isometry. Despite the well-known fact that Hadamard manifolds with zero sectional curvature are isometric to Euclidean spaces, there are significant problems that one encounters while investigating optimization problems in the framework of Hadamard manifolds with negative sectional curvature. For instance, due to the linear structure of Euclidean space, it is apparent that $x - y = -(y - x)$, for any $x, y \in \mathbb{R}^n$. However, in the framework of Hadamard manifold $\mathcal{H}^n$, $\exp_y^{-1} x \neq -\exp_x^{-1} y$, $x, y \in \mathcal{H}^n$, due to its nonlinear structure. As a result, the development of optimization techniques on Hadamard manifolds with non-zero sectional curvatures is significantly difficult, as compared to Euclidean spaces.*

Let $\mathcal{R}_n$ denote a complete and connected Riemannian manifold. A subset $U \subseteq \mathcal{R}_n$ is termed as a strongly convex set, provided that for any $a,\ b \in U$ some unique minimal geodesic exists in $\mathcal{R}_n$ that connects the points $a$ and $b$. We employ the symbol $\mathbb{B}^*_{\hat{a}}(w,\ \alpha)$ to represent the ball centered at $w \in T_{\hat{a}}\mathcal{R}_n$, having a radius $\alpha > 0$. Moreover, we employ the notation $\mathbb{B}_\alpha(\hat{a})$ to signify the ball centered at $\hat{a} \in \mathcal{R}_n$ with radius $\alpha > 0$. Let $\bar{0}$ denote the zero vector in $T_{\hat{a}}\mathcal{R}_n$.

We now present the following definitions from [30].

**Definition 1.** *The injectivity and convexity radius corresponding to $\hat{p} \in \mathcal{R}_n$, denoted by $s(\hat{p})$ and $r(\hat{p})$, respectively, are defined as follows:*

$$s(\hat{p}) := \sup\{\alpha > 0 |\ \exp_{\hat{p}} : \mathbb{B}^*_{\hat{p}}(\bar{0}, \alpha) \subset T_{\hat{p}}\mathcal{R}_n \to \exp_{\hat{p}}(\mathbb{B}^*_{\hat{p}}(\bar{0}, \alpha)) \textit{ is a diffeomorphism}\},$$

$$r(\hat{p}) := \sup\{\alpha > 0 |\ \textit{each ball in } \mathbb{B}_\alpha(\hat{p}) \textit{ is strongly convex and each geodesic in } \mathbb{B}_\alpha(\hat{p}) \textit{ is a minimal geodesic}\}.$$

**Remark 2.**

*(i)* *It is well-known that (see, for instance, [30]):*

$$s(\hat{p}) \geq r(\hat{p}) > 0,\ \forall \hat{p} \in \mathcal{R}_n.$$

*(ii)* *The set $\mathcal{U}_{\hat{p}} = \exp_{\hat{p}}(\mathbb{B}^*_{\hat{p}}(\bar{0}, t))$ is known as the totally normal neighborhood of $\hat{p}$. If $\mathcal{R}_n = \mathcal{H}^n$, $\mathcal{U}_{\hat{p}} = \mathcal{H}^n$, $\forall \hat{p} \in \mathcal{H}^n$. Indeed, in view of the Cartan–Hadamard theorem (see, for instance, [23]), we know that the exponential map is globally diffeomorphic in the case of Hadamard manifolds. Consequently, if $\mathcal{R}_n = \mathcal{H}^n$, then we have:*

$$r(\hat{p}) = s(\hat{p}) = +\infty, \ \forall \hat{p} \in \mathcal{H}^n.$$

*Hence, from Definition 1, it follows that, if $\mathcal{R}_n = \mathcal{H}^n$, then*

$$\mathcal{U}_{\hat{p}} = \mathcal{H}^n, \ \forall \hat{p} \in \mathcal{H}^n.$$

(iii) *Therefore, $\mathcal{H}^n$ is globally diffeomorphic to Euclidean space, while a Riemannian manifold is locally diffeomorphic to Euclidean space. Thus, the results derived in this paper in the setting of Hadamard manifolds need not be true globally for a general Riemannian manifold. In particular, the established results within the framework of Hadamard manifolds hold within the totally normal neighborhood of each point in a Riemannian manifold.*

Let $F : \mathcal{H}^n \to \mathbb{R}$ be differentiable. The gradient of $F$, denoted by grad $F$, is defined as $dF(Z) = \langle \text{grad}\, F, \ Z \rangle = Z(F)$, where $Z$ is also a vector field on $\mathcal{H}^n$.

The following definition is from Udrişte [23].

**Definition 2.** *A set $\mathcal{V}$ $(\neq \emptyset) \subset \mathcal{H}^n$ is called a geodesic convex set, if for any $\hat{a}_1, \ \hat{a}_2 \in \mathcal{V}$ $(\hat{a}_1 \neq \hat{a}_2)$, and for any geodesic $\gamma_{\hat{a}_1, \hat{a}_2} : [0,1] \to \mathcal{H}^n$ that connects $\hat{a}_1$ and $\hat{a}_2$, the following holds:*

$$\gamma_{\hat{a}_1, \hat{a}_2}(\tau) \in \mathcal{V}, \quad \forall \tau \in [0,1],$$

*where,* $\gamma_{\hat{a}_1, \hat{a}_2}(\tau) = \exp_{\hat{a}_1}\left(\tau \exp^{-1}_{\hat{a}_1}(\hat{a}_2)\right)$.

The following definition is from Rapcsák [35].

**Definition 3.** *Let $\mathcal{V}$ $(\neq \emptyset) \subset \mathcal{H}^n$ be a geodesic convex set and let $\hat{a}_1 \in \mathcal{V}$. Let $\Theta : \mathcal{V} \to \mathbb{R}$ be a differentiable function. The function $\Theta$ is referred to as a geodesic convex function at $\hat{a}_1 \in \mathcal{V}$, provided that the inequality stated below is satisfied:*

$$\Theta(\hat{a}_2) - \Theta(\hat{a}_1) \geq \left\langle \text{grad}\, \Theta(\hat{a}_1), \ \exp^{-1}_{\hat{a}_1}(\hat{a}_2) \right\rangle, \ \forall \hat{a}_2 \in \mathcal{V}.$$

*We refer to the function $\Theta$ as geodesic concave at $\hat{a}_1 \in \mathcal{V}$, if $-\Theta$ is geodesic convex at $\hat{a}_1$. A function that is both geodesic convex and geodesic concave at $\hat{a}_1 \in \mathcal{V}$ is said to be a geodesic affine function $\hat{a}_1$.*

For any $p \times q$ matrix $\mathcal{A}$, the symbol $\mathcal{A}_r$ represents the $r^{th}$ row vector of $\mathcal{A}$. In the rest of this section, we consider matrices $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \mathcal{R}^{(3)}, \mathcal{R}^{(4)}$ of order $p_1 \times n$, $p_2 \times n$, $p_3 \times n$, and $p_4 \times n$ respectively. Let $\hat{a} \in \mathcal{H}^n$. Suppose that $\mathcal{R}^{(1)}_s \in T_{\hat{a}}\mathcal{H}^n$ $(s = 1, 2, \ldots, p_1)$, $\mathcal{R}^{(2)}_s \in T_{\hat{a}}\mathcal{H}^n$ $(s = 1, 2, \ldots, p_2)$, $\mathcal{R}^{(3)}_s \in T_{\hat{a}}\mathcal{H}^n$ $(s = 1, 2, \ldots, p_3)$, and $\mathcal{R}^{(4)}_s \in T_{\hat{a}}\mathcal{H}^n$ $(s = 1, 2, \ldots, p_4)$.

The following lemmas are from [36].

**Lemma 1.** *Let us suppose that $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}$, or $\mathcal{R}^{(3)}$ is non-vacuous. The following systems,*

$$\begin{aligned}
(a) \quad & \langle \mathcal{R}^{(1)}_s, \ \overline{\omega} \rangle_{\hat{a}} \geq 0, \quad s = 1, \ldots, p_1, \\
& \langle \mathcal{R}^{(2)}_s, \ \overline{\omega} \rangle_{\hat{a}} \geq 0, \quad s = 1 \ldots, p_2, \\
& \langle \mathcal{R}^{(3)}_s, \ \overline{\omega} \rangle_{\hat{a}} \geq 0, \quad s = 1, \ldots, p_3, \\
& \langle \mathcal{R}^{(4)}_s, \ \overline{\omega} \rangle_{\hat{a}} = 0, \quad s = 1, \ldots, p_4; \\
(b) \quad & (\mathcal{R}^{(1)})^T u_1 + (\mathcal{R}^{(2)})^T u_2 + (\mathcal{R}^{(3)})^T u_3 + (\mathcal{R}^{(4)})^T u_4 = 0; \ u_1 \geqq 0, \ u_2 \geqq 0, \ u_3 \geqq 0;
\end{aligned}$$

*provide the solutions* $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$, $u_1 \in \mathbb{R}^{p_1}$, $u_2 \in \mathbb{R}^{p_2}$, $u_3 \in \mathbb{R}^{p_3}$, $u_4 \in \mathbb{R}^{p_4}$, *satisfying:*

$$\begin{aligned} &\langle \mathcal{R}_s^{(1)}, \overline{\varpi} \rangle_{\hat{a}} + u_1 > 0, \quad \forall s = 1,2,\ldots,p_1, \\ &\langle \mathcal{R}_s^{(2)}, \overline{\varpi} \rangle_{\hat{a}} + u_2 > 0, \quad \forall s = 1,2,\ldots,p_2, \\ &\langle \mathcal{R}_s^{(4)}, \overline{\varpi} \rangle_{\hat{a}} + u_3 > 0, \quad \forall s = 1,2,\ldots,p_3. \end{aligned}$$

**Lemma 2.** *Let $\mathcal{R}^{(1)}$ be a non-vacuous matrix. One of the following assertions (but not both) holds true:*

*(a) The system of inequalities:*

$$\begin{aligned} &\langle \mathcal{R}_s^{(1)}, \overline{\varpi} \rangle_{\hat{a}} > 0, \quad \forall s = 1,2,\ldots,p_1, \\ &\langle \mathcal{R}_s^{(2)}, \overline{\varpi} \rangle_{\hat{a}} \geq 0, \quad \forall s = 1,2,\ldots,p_2, \\ &\langle \mathcal{R}_s^{(3)}, \overline{\varpi} \rangle_{\hat{a}} = 0, \quad \forall s = 1,2,\ldots,p_3, \end{aligned}$$

*have a solution $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$.*

*(b) The following equation,*

$$(\mathcal{R}^{(1)})^T u_1 + (\mathcal{R}^{(2)})^T u_2 + (\mathcal{R}^{(3)})^T u_3 = 0$$

*has a solution $u_1 \in \mathbb{R}^{p_1}$, $u_2 \in \mathbb{R}^{p_2}$, $u_3 \in \mathbb{R}^{p_3}$, such that $u_1 \geq 0$, $u_2 \geqq 0$.*

The following lemma is an extension of Tucker's theorem of alternative in the Hadamard manifold framework.

**Lemma 3.** *Let $\mathcal{R}^{(1)}$ be a non-vacuous matrix. One of the following assertions (but not both) holds true:*

*(a) The system of inequalities,*

$$\begin{aligned} &\langle \mathcal{R}_s^{(1)}, \overline{\varpi} \rangle_{\hat{a}} \geq 0, \quad \forall s = 1,2,\ldots,p_1, \\ &\langle \mathcal{R}_\ell^{(1)}, \overline{\varpi} \rangle_{\hat{a}} > 0, \quad \text{for some } \ell \in \{1,2,\ldots,p_1\}, \\ &\langle \mathcal{R}_s^{(2)}, \overline{\varpi} \rangle_{\hat{a}} \geq 0, \quad \forall s = 1,2,\ldots,p_2, \\ &\langle \mathcal{R}_s^{(3)}, \overline{\varpi} \rangle_{\hat{a}} = 0, \quad \forall s = 1,2,\ldots,p_3, \end{aligned} \tag{1}$$

*has a solution $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$.*

*(b) The following equation,*

$$(\mathcal{R}^{(1)})^T u_1 + (\mathcal{R}^{(2)})^T u_2 + (\mathcal{R}^{(3)})^T u_3 = 0$$

*has a solution $u_1 \in \mathbb{R}^{p_1}$, $u_2 \in \mathbb{R}^{p_2}$, $u_3 \in \mathbb{R}^{p_3}$, satisfying $u_1 \geq 0$, $u_2 \geqq 0$.*

**Proof.** Suppose that statement (a) holds true. By reductio ad absurdum, we suppose that both of the statements (a) and (b) are valid. Consequently, some $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$ exists, satisfying (1). Further, some $u_1 \in \mathbb{R}^{p_1}$, $u_2 \in \mathbb{R}^{p_2}$, $u_3 \in \mathbb{R}^{p_3}$ exist, with $u_1 \geq 0$, $u_2 \geqq 0$, satisfying:

$$(\mathcal{R}^{(1)})^T u_1 + (\mathcal{R}^{(2)})^T u_2 + (\mathcal{R}^{(3)})^T u_3 = 0. \tag{2}$$

In light of (1), we have:

$$\langle {\mathcal{R}^{(1)}}^T u_1, \overline{\varpi} \rangle_{\hat{a}} + \langle {\mathcal{R}^{(2)}}^T u_2, \overline{\varpi} \rangle_{\hat{a}} + \langle {\mathcal{R}^{(3)}}^T u_3, \overline{\varpi} \rangle_{\hat{a}} > 0, \tag{3}$$

which contradicts (2). This establishes the fact that if statement (a) is valid, statement (b) is not satisfied.

On the other hand, we now consider that statement (a) does not hold true. Consider the following system:

$$\begin{aligned} &\langle \mathcal{R}_s^{(1)},\ \overline{\omega}\rangle_{\hat{a}} \geq 0, && \forall s = 1,2,\ldots,p_1, \\ &\langle \mathcal{R}_s^{(2)},\ \overline{\omega}\rangle_{\hat{a}} \geq 0, && \forall s = 1,2,\ldots,p_2, \\ &\langle \mathcal{R}_s^{(3)},\ \overline{\omega}\rangle_{\hat{a}} = 0, && \forall s = 1,2,\ldots,p_3. \end{aligned} \tag{4}$$

As a result, for every $\overline{\omega} \in T_{\hat{a}}\mathcal{H}^n$, we have:

$$\langle \mathcal{R}_s^{(1)},\ \overline{\omega}\rangle_{\hat{a}} \ngtr 0, \quad \forall s = 1,2,\ldots,p_1.$$

Let us now consider the equation:

$$(\mathcal{R}^{(1)})^T u_1 + (\mathcal{R}^{(2)})^T u_2 + (\mathcal{R}^{(3)})^T u_3 = 0,\ u_1,\ u_2,\ u_3 \geqq 0; \tag{5}$$

In view of Lemma 1, it follows from (4) and (5) that $u_1 > 0$. Hence, statement (b) holds true. Thus, the proof is completed. □

For more comprehensive discussions on Hadamard manifolds, we refer to [27,28,37–40] and the references cited therein.

## 3. Constraint Qualifications and Necessary Optimality Criteria for MOMPVC

In the rest of the paper, we investigate the following MOMPVC in the Hadamard framework:

$$\begin{aligned} \text{(MOMPVC)} \quad \text{Minimize} \quad & \Theta(z) = (\Theta_1(z),\ldots,\Theta_\ell(z)), \\ \text{subject to} \quad & \mathcal{B}_r(z) \leq 0, \quad \forall r \in \mathcal{R}^{\mathcal{B}} := \{1,\ldots,s\}, \\ & \mathcal{C}_r(z) = 0, \quad \forall r \in \mathcal{R}^{\mathcal{C}} := \{1,\ldots,q\}, \\ & \mathcal{U}_r(z) \geq 0, \quad \forall r \in \mathcal{R} := \{1,\ldots,p\}, \\ & \mathcal{V}_r(z)\,\mathcal{U}_r(z) \leq 0, \quad \forall r \in \mathcal{R} := \{1,\ldots,p\}. \end{aligned}$$

It is assumed that every function involved, that is, $\Theta_r$ ($r \in \mathcal{R}^{\Theta} := \{1,\ldots,\ell\}$), $\mathcal{B}_r$ ($r \in \mathcal{R}^{\mathcal{B}}$), $\mathcal{C}_r$ ($r \in \mathcal{R}^{\mathcal{C}}$), and $\mathcal{U}_r$, $\mathcal{V}_r$ ($r \in \mathcal{R}$) are smooth, scalar-valued functions defined on $\mathcal{H}^n$. The symbol $\mathcal{S}^f$ will be used throughout the rest of the article to indicate the feasible set of MOMPVC. We define a map $\mathcal{W}_r : \mathcal{H}^n \to \mathbb{R}$, as given below:

$$\mathcal{W}_r(\hat{a}) := \mathcal{V}_r(\hat{a})\,\mathcal{U}_r(\hat{a}),\ \forall \hat{a} \in \mathcal{H}^n,\ \forall r \in \mathcal{R}.$$

The next definition will be useful in the sequel (see [33]).

**Definition 4.** *A feasible element $\hat{a} \in \mathcal{S}^f$ is said to be a Pareto efficient (weak Pareto efficient, respectively) solution of MOMPVC, if no other $a \in \mathcal{S}^f$ exists, for which:*

$$\Theta(a) \preceq \Theta(\hat{a}) \quad (\Theta(a) \prec \Theta(\hat{a}),\ \textit{respectively}).$$

Let $\hat{a} \in \mathcal{S}^f$. The sets defined below will be employed frequently in this paper:

$$
\begin{aligned}
\mathcal{R}_a^{\mathcal{B}}(\hat{a}) &:= \{r \in \mathcal{R}^{\mathcal{B}} \mid \mathcal{B}_r(\hat{a}) = 0\}, \\
\mathcal{R}_+(\hat{a}) &:= \{r \in \mathcal{R} \mid \mathcal{U}_r(\hat{a}) > 0\}, \\
\mathcal{R}_0(\hat{a}) &:= \{r \in \mathcal{R} \mid \mathcal{U}_r(\hat{a}) = 0\}, \\
\mathcal{R}_{+0}(\hat{a}) &:= \{r \in \mathcal{R} \mid \mathcal{U}_r(\hat{a}) > 0, \mathcal{V}_r(\hat{a}) = 0\}, \\
\mathcal{R}_{+-}(\hat{a}) &:= \{r \in \mathcal{R} \mid \mathcal{U}_r(\hat{a}) > 0, \mathcal{V}_r(\hat{a}) < 0\}, \\
\mathcal{R}_{0+}(\hat{a}) &:= \{r \in \mathcal{R} \mid \mathcal{U}_r(\hat{a}) = 0, \mathcal{V}_r(\hat{a}) > 0\}, \\
\mathcal{R}_{00}(\hat{a}) &:= \{r \in \mathcal{R} \mid \mathcal{U}_r(\hat{a}) = 0, \mathcal{V}_r(\hat{a}) = 0\}, \\
\mathcal{R}_{0-}(\hat{a}) &:= \{r \in \mathcal{R} \mid \mathcal{U}_r(\hat{a}) = 0, \mathcal{V}_r(\hat{a}) < 0\}.
\end{aligned}
$$

**Remark 3.** *Every set defined above is dependent on the selection of $\hat{a} \in \mathcal{S}^f$. However, if such a choice is unambiguous, we shall refrain from indicating it explicitly in the sequel.*

Let us now introduce the sets $\mathscr{W}^k$ ($k \in \mathcal{R}^\Theta$) and $\mathscr{W}$, which will play a major role in the remaining portion of the present article:

$$
\begin{aligned}
\mathscr{W}^k &:= \left\{a \in \mathcal{S}^f : \Theta_r(a) \leq \Theta_r(\hat{a}),\ \forall r \in \mathcal{R}^\Theta,\ r \neq k\right\}, \\
\mathscr{W} &:= \left\{a \in \mathcal{S}^f : \Theta_r(a) \leq \Theta_r(\hat{a}),\ \forall r \in \mathcal{R}^\Theta\right\}.
\end{aligned}
$$

**Remark 4.** *From the above definitions, it is evident that* $\bigcap\limits_{k \in \mathcal{R}^\Theta} \mathscr{W}^k = \mathscr{W}$.

The following notion of contingent cone in the Hadamard manifold framework is from [39].

**Definition 5.** *Let $\mathcal{A} \subseteq \mathcal{H}^n$ and $\hat{c} \in \mathrm{cl}(\mathcal{A})$. The Bouligand tangent cone of $\mathcal{A}$ at $\hat{c}$, represented by $\mathscr{C}^T(\mathcal{A}, \hat{c})$, is defined in the following manner:*

$$
\begin{aligned}
\mathscr{C}^T(\mathcal{A}, \hat{c}) := \{u \in T_{\hat{c}}\mathcal{H}^n : \exists \{s_n\}_{n=1}^\infty \downarrow 0, \exists \{u_n\}_{n=1}^\infty,\ u_n \in T_{\hat{c}}\mathcal{H}^n,\ \{u_n\}_{n=1}^\infty \to u, \\
exp_{\hat{c}}(s_n u_n) \in \mathcal{A},\ \forall n \in \mathbb{N}\},
\end{aligned}
$$

*where $\{s_n\}_{n=1}^\infty \downarrow 0$ indicates that $s_n \geq 0$ for all $n \in \mathbb{N}$ and $\{s_n\}_{n=1}^\infty$ tends to 0 as n tends to infinity.*

The following definition is an extension of the definition of linearizing cone from Maeda [33] for MOMPVC in the Hadamard manifold framework.

**Definition 6.** *Let $\hat{a} \in \mathcal{S}^f$. The linearizing cone to the set $\mathscr{W}$ at $\hat{a}$, signified by $\mathscr{L}^C(\mathscr{W}, \hat{a})$, is defined as follows:*

$$
\begin{aligned}
\mathscr{L}^C(\mathscr{W}, \hat{a}) := \{\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n : \ &\langle \operatorname{grad} \Theta_r(\hat{a}), \overline{\varpi} \rangle \leq 0, && \forall r \in \mathcal{R}^\Theta, \\
&\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), \overline{\varpi} \rangle \leq 0, && \forall r \in \mathcal{R}_a^{\mathcal{B}}, \\
&\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), \overline{\varpi} \rangle = 0, && \forall r \in \mathcal{R}^{\mathcal{C}}, \\
&\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle \geq 0, && \forall r \in \mathcal{R}_0, \\
&\langle \operatorname{grad} \mathcal{W}_r(\hat{a}), \overline{\varpi} \rangle \leq 0, && \forall r \in \mathcal{R}_0 \cup \mathcal{R}_{+0}\}.
\end{aligned}
$$

**Remark 5.** *We observe that* $\operatorname{grad} \mathcal{W}_r(\hat{a}) = \mathcal{V}_r(\hat{a}) \operatorname{grad} \mathcal{U}_r(\hat{a}) + \mathcal{U}_r(\hat{a}) \operatorname{grad} \mathcal{V}_r(\hat{a})$. *Then, equivalently, $\mathscr{L}^C(\mathscr{W}, \hat{a})$ may be defined as:*

$$\begin{aligned}\mathscr{L}^C(\mathscr{W},\hat{a}) := \{\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n : &\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}\rangle \le 0, \quad &&\forall r \in \mathcal{R}^{\Theta},\\ &\langle \operatorname{grad}\mathcal{B}_r(\hat{a}), \overline{\varpi}\rangle \le 0, &&\forall r \in \mathcal{R}_a^{\mathcal{B}},\\ &\langle \operatorname{grad}\mathcal{C}_r(\hat{a}), \overline{\varpi}\rangle = 0, &&\forall r \in \mathcal{R}^{\mathcal{C}},\\ &\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi}\rangle = 0, &&\forall r \in \mathcal{R}_{0+},\\ &\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi}\rangle \ge 0, &&\forall r \in \mathcal{R}_{0-} \cup \mathcal{R}_{00},\\ &\langle \operatorname{grad}\mathcal{V}_r(\hat{a}), \overline{\varpi}\rangle \le 0, &&\forall r \in \mathcal{R}_{+0}\}.\end{aligned}$$

In the following definition, we introduce the definition of the linearizing cone to the set $\mathscr{W}^k$ ($k \in \mathcal{R}^{\Theta}$) at a feasible element of MOMPVC in the Hadamard manifold framework.

**Definition 7.** *Let $\hat{a} \in \mathcal{S}^f$. The linearizing cone to the set $\mathscr{W}^k$ ($k \in \mathcal{R}^{\Theta}$) at $\hat{a}$, signified by $\mathscr{L}^C\left(\mathscr{W}^k, \hat{a}\right)$, is defined as follows:*

$$\begin{aligned}\mathscr{L}^C\left(\mathscr{W}^k,\hat{a}\right) := \{\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n : &\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}\rangle \le 0, \quad &&\forall r \in \mathcal{R}^{\Theta}\setminus\{k\},\\ &\langle \operatorname{grad}\mathcal{B}_r(\hat{a}), \overline{\varpi}\rangle \le 0, &&\forall r \in \mathcal{R}_a^{\mathcal{B}},\\ &\langle \operatorname{grad}\mathcal{C}_r(\hat{a}), \overline{\varpi}\rangle = 0, &&\forall r \in \mathcal{R}^{\mathcal{C}},\\ &\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi}\rangle \ge 0, &&\forall r \in \mathcal{R}_{0},\\ &\langle \operatorname{grad}\mathcal{W}_r(\hat{a}), \overline{\varpi}\rangle \le 0, &&\forall r \in \mathcal{R}_{0} \cup \mathcal{R}_{+0}\}.\end{aligned}$$

The generalized Guignard constraint qualification (GGCQ) for MOMPVC in the Hadamard manifold framework is introduced in the next definition.

**Definition 8.** *Let $\hat{a} \in \mathcal{S}^f$. Then, the MOMPVC-tailored generalized Guignard constraint qualification (MOMPVC-GGCQ) holds at $\hat{a}$, if:*

$$\mathscr{L}^C(\mathscr{W},\hat{a}) \subseteq \bigcap_{r\in\mathcal{R}^{\Theta}} \operatorname{cl}\operatorname{co}\mathscr{C}^T(\mathscr{W}^r,\hat{a}).$$

We now deduce an important result in the next lemma, which will be directly helpful to deduce KKT-type Pareto efficiency conditions for MOMPVC.

**Lemma 4.** *Let $\hat{a} \in \mathcal{S}^f$, such that MOMPVC-GGCQ holds at $\hat{a} \in \mathcal{S}^f$. If $\hat{a}$ is a Pareto efficient solution of MOMPVC, then no $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$ exists, satisfying:*

$$\begin{aligned}\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}\rangle &\le 0, \quad &&\forall r \in \mathcal{R}^{\Theta},\\ \langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}\rangle &< 0, &&\textit{for at least one } r \in \mathcal{R}^{\Theta},\\ \langle \operatorname{grad}\mathcal{B}_r(\hat{a}), \overline{\varpi}\rangle &\le 0, &&\forall r \in \mathcal{R}_a^{\mathcal{B}},\\ \langle \operatorname{grad}\mathcal{C}_r(\hat{a}), \overline{\varpi}\rangle &= 0, &&\forall r \in \mathcal{R}^{\mathcal{C}},\\ \langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi}\rangle &\ge 0, &&\forall r \in \mathcal{R}_{0},\\ \langle \operatorname{grad}\mathcal{W}_r(\hat{a}), \overline{\varpi}\rangle &\le 0, &&\forall r \in \mathcal{R}_{0} \cup \mathcal{R}_{+0}.\end{aligned} \tag{6}$$

**Proof.** By reductio ad absurdum, we suppose that some $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$ exists, which satisfies (6). Consequently, in light of Definition 6, it is obvious that $\overline{\varpi} \in \mathscr{L}^C(\mathscr{W},\hat{a})$. Hence, without any loss of generality, we may assume the following:

$$\begin{aligned}&\langle \operatorname{grad}\Theta_1(\hat{a}), \overline{\varpi}\rangle < 0,\\ &\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}\rangle \le 0, \quad r \in \mathcal{R}^{\Theta}\setminus\{1\}.\end{aligned}$$

According to the provided hypotheses, MOMPVC-GGCQ holds at $\hat{a} \in \mathcal{S}^f$. Hence, the following holds:

$$\overline{\varpi} \in \operatorname{cl}\operatorname{co}\mathscr{C}^{T}\left(\mathscr{W}^{1}, \hat{a}\right).$$

Therefore, some sequence $\{\overline{\varpi}_m\}_{m=1}^{\infty} \subseteq \operatorname{co}\mathscr{C}^{T}\left(\mathscr{W}^{1}, \hat{a}\right)$ exists, for which $\lim_{m\to\infty} \overline{\varpi}_m = \overline{\varpi}$. Hence, for each element $\overline{\varpi}_m$ ($m \in \mathbb{N}$) of the sequence, we have some $\mathcal{M}_m \in \mathbb{N}$, satisfying:

$$\sum_{k=1}^{\mathcal{M}_m} \rho_{m_k} = 1, \quad \sum_{k=1}^{\mathcal{M}_m} \rho_{m_k}\overline{\varpi}_{m_k} = \overline{\varpi}_m,$$

where $\rho_{m_k} \in \mathbb{R}$, $\rho_{m_k} \geq 0$, and $\overline{\varpi}_{m_k} \in \mathscr{C}^{T}\left(\mathscr{W}^{1}, \hat{a}\right)$, $k = 1, 2, \ldots, \mathcal{M}_m$. Then, in view of Definition 5, there exist sequences $\left\{\overline{\varpi}^{n}_{m_k}\right\}_{n=1}^{\infty}$, $\overline{\varpi}^{n}_{m_k} \in \mathscr{W}^{1}$ for each $n \in \mathbb{N}$ and $\left\{t^{n}_{m_k}\right\}_{n=1}^{\infty}$, $t^{n}_{m_k}(>0) \in \mathbb{R}$ for each $n \in \mathbb{N}$, with $t^{n}_{m_k} \to 0$ as $n \to \infty$, such that:

$$\lim_{n\to\infty} \overline{\varpi}^{n}_{m_k} = \overline{\varpi}_{m_k}, \quad \exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k}) \in \mathscr{W}^{1}. \tag{7}$$

Let us set $x^{n}_{m_k}$ as follows:

$$x^{n}_{m_k} := \exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k}), \quad \forall n \in \mathbb{N}. \tag{8}$$

Therefore, we obtain the following inequalities for every $n \in \mathbb{N}$:

$$\begin{aligned}
\Theta_r\left(x^{n}_{m_k}\right) &= \Theta_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) \leq \Theta_r(\hat{a}),\ r \in \mathcal{R}^{\Theta} \setminus \{1\},\\
\mathcal{B}_r\left(x^{n}_{m_k}\right) &= \mathcal{B}_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) \leq 0 = \mathcal{B}(\hat{a}),\ \forall r \in \mathcal{R}^{\mathcal{B}}_{a},\\
\mathcal{C}_r\left(x^{n}_{m_k}\right) &= \mathcal{C}_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) \leq 0 = \mathcal{C}_r(\hat{a}),\ \forall r \in \mathcal{R}^{\mathcal{C}},\\
\mathcal{U}_r\left(x^{n}_{m_k}\right) &= \mathcal{U}_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) \geq 0 = \mathcal{U}_r(\hat{a}),\ \forall r \in \mathcal{R}_0,\\
\mathcal{W}_r\left(x^{n}_{m_k}\right) &= \mathcal{W}_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) \leq 0 = \mathcal{W}_r(\hat{a}),\ \forall r \in \mathcal{R}_0 \cup \mathcal{R}_{+0}.
\end{aligned} \tag{9}$$

Again, $\hat{a} \in \mathcal{S}^{f}$ is a Pareto efficient solution of MOMPVC. Therefore:

$$\Theta_1\left(x^{n}_{m_k}\right) = \Theta_1\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) \geq \Theta_1(\hat{a}), \quad \forall n \in \mathbb{N}. \tag{10}$$

By employing the Taylor series of $\Theta_r$ at $\hat{a}$, for each $r \in \mathcal{R}^{\Theta} \setminus \{1\}$, we obtain:

$$\Theta_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) = \Theta_r(\hat{a}) + t^{n}_{m_k}\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}^{n}_{m_k}\rangle + o\left(t^{n}_{m_k}\right). \tag{11}$$

Then, it follows that for every $r \in \mathcal{R}^{\Theta} \setminus \{1\}$, we have:

$$\frac{\Theta_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) - \Theta_r(\hat{a})}{t^{n}_{m_k}} = \left\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}^{n}_{m_k}\right\rangle + \frac{o\left(t^{n}_{m_k}\right)}{t^{n}_{m_k}}. \tag{12}$$

We observe that $\Theta_r\left(x^{n}_{m_k}\right) = \Theta_r\left(\exp_{\hat{a}}(t^{n}_{m_k}\overline{\varpi}^{n}_{m_k})\right) \leq \Theta_r(\hat{a})$, for every $r \in \mathcal{R}^{\Theta} \setminus \{1\}$. Hence, by letting $t^{n}_{m_k} \to 0$, it follows from equation (12) that

$$\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}_{m_k}\rangle \leq 0, \quad \forall r \in \mathcal{R}^{\Theta} \setminus \{1\}. \tag{13}$$

Continuing similarly as before, we arrive at the following:

$$\begin{aligned}
&\langle \operatorname{grad} \Theta_1(\hat{a}), \overline{\varpi}_{m_k} \rangle \geq 0, && \\
&\langle \operatorname{grad} \Theta_r(\hat{a}), \overline{\varpi}_{m_k} \rangle \leq 0, && \forall r \in \mathcal{R}^{\Theta} \setminus \{1\}, \\
&\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), \overline{\varpi}_{m_k} \rangle \leq 0, && \forall r \in \mathcal{R}^{\mathcal{B}}_a, \\
&\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), \overline{\varpi}_{m_k} \rangle = 0, && \forall r \in \mathcal{R}^{\mathcal{C}}, \\
&\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\varpi}_{m_k} \rangle \geq 0, && \forall r \in \mathcal{R}_0, \\
&\langle \operatorname{grad} \mathcal{W}_r(\hat{a}), \overline{\varpi}_{m_k} \rangle \leq 0, && \forall r \in \mathcal{R}_0 \cup \mathcal{R}_{+0}.
\end{aligned} \tag{14}$$

In light of the continuity property of inner product, we have the following:

$$\begin{aligned}
&\langle \operatorname{grad} \Theta_1(\hat{a}), \overline{\varpi} \rangle \geq 0, && \\
&\langle \operatorname{grad} \Theta_r(\hat{a}), \overline{\varpi} \rangle \leq 0, && \forall r \in \mathcal{R}^{\Theta} \setminus \{1\}, \\
&\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), \overline{\varpi} \rangle \leq 0, && \forall r \in \mathcal{R}^{\mathcal{B}}_a, \\
&\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), \overline{\varpi} \rangle = 0, && \forall r \in \mathcal{R}^{\mathcal{C}}, \\
&\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle \geq 0, && \forall r \in \mathcal{R}_0, \\
&\langle \operatorname{grad} \mathcal{W}_r(\hat{a}), \overline{\varpi} \rangle \leq 0, && \forall r \in \mathcal{R}_0 \cup \mathcal{R}_{+0},
\end{aligned} \tag{15}$$

which contradicts our initial assumption. Hence, the proof is completed. □

**Remark 6.** *Theorem 4 is a generalization of Theorem 3.1 of Maeda [33] from multiobjective optimization problems to MOMPVC, and further, extends it from the linear setting of $\mathbb{R}^n$ to the nonlinear Hadamard manifold framework.*

Now, we employ MOMPVC-GGCQ to derive KKT-type necessary Pareto efficiency criteria for MOMPVC.

**Theorem 1.** *Let $\hat{a} \in \mathcal{S}^f$ be a Pareto efficient solution of MOMPVC at which MOMPVC-GGCQ holds. Then, some real numbers, $\alpha_r$ $(r \in \mathcal{R}^{\Theta})$, $\sigma_r^{\mathcal{B}}$ $(r \in \mathcal{R}^{\mathcal{B}})$, $\sigma_r^{\mathcal{C}}$ $(r \in \mathcal{R}^{\mathcal{C}})$, $\sigma_r^{\mathcal{U}}$ $(r \in \mathcal{R})$, $\sigma_r^{\mathcal{V}}$ $(r \in \mathcal{R})$, exist, which satisfy the following:*

$$\begin{aligned}
\sum_{r \in \mathcal{R}^{\Theta}} \alpha_r \operatorname{grad} \Theta_r(\hat{a}) + \sum_{r \in \mathcal{R}^{\mathcal{B}}} \sigma_r^{\mathcal{B}} \operatorname{grad} \mathcal{B}_r(\hat{a}) + \sum_{r \in \mathcal{R}^{\mathcal{C}}} \sigma_r^{\mathcal{C}} \operatorname{grad} \mathcal{C}_r(\hat{a}) \\
- \sum_{r \in \mathcal{R}} \sigma_r^{\mathcal{U}} \operatorname{grad} \mathcal{U}_r(\hat{a}) + \sum_{r \in \mathcal{R}} \sigma_r^{\mathcal{V}} \operatorname{grad} \mathcal{V}_r(\hat{a}) = 0,
\end{aligned} \tag{16}$$

*and*

$$\begin{aligned}
&\alpha_r > 0, \quad \forall r \in \mathcal{R}^{\Theta}, \\
&\mathcal{B}_r(\hat{a}) \leq 0, \quad \sigma_r^{\mathcal{B}} \geq 0, \quad \sigma_r^{\mathcal{B}} \mathcal{B}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}^{\mathcal{B}}, \\
&\mathcal{C}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}^{\mathcal{C}}, \\
&\sigma_r^{\mathcal{U}} \text{ free}, \forall r \in \mathcal{R}_{0+}, \quad \sigma_r^{\mathcal{U}} \geq 0, \forall r \in \mathcal{R}_{00} \cup \mathcal{R}_{0-}, \quad \sigma_r^{\mathcal{U}} = 0, \forall r \in \mathcal{R}_{+}, \\
&\sigma_r^{\mathcal{V}} \geq 0, \forall r \in \mathcal{R}_{+0}, \quad \sigma_r^{\mathcal{V}} = 0, \forall r \in \mathcal{R}_0 \cup \mathcal{R}_{+-}, \\
&\sigma_r^{\mathcal{U}} \mathcal{U}_r(\hat{a}) = 0, \quad \sigma_r^{\mathcal{V}} \mathcal{V}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}.
\end{aligned} \tag{17}$$

**Proof.** According to the provided hypotheses, $\hat{a} \in \mathcal{S}^f$ is a Pareto efficient solution of MOMPVC and MOMPVC-GGCQ holds at $\hat{a}$. In light of Lemma 4, it follows that system (6) has no solution $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$. Now, by invoking Lemma 3, we infer that there exist real numbers, $\alpha_r > 0$ $(r \in \mathcal{R}^{\Theta})$, $\sigma_r^{\mathcal{B}} \geq 0$ $(r \in \mathcal{R}^{\mathcal{B}}_a)$, $\sigma_r^{\mathcal{C}+} \geq 0$, $\sigma_r^{\mathcal{C}-} \geq 0$ $(r \in \mathcal{R}^{\mathcal{C}})$, $\beta_r^{\mathcal{U}} \geq 0$ $(r \in \mathcal{R}_0)$, $\sigma_r^{\mathcal{W}} \geq 0$ $(r \in \mathcal{R}_0 \cup \mathcal{R}_{+0})$, which satisfy the following:

$$\sum_{r\in\mathcal{R}^\Theta} \alpha_r \operatorname{grad}\Theta_r(\hat{a}) + \sum_{r\in\mathcal{R}^\mathcal{B}_a} \sigma^\mathcal{B}_r \operatorname{grad}\mathcal{B}_r(\hat{a}) + \sum_{r\in\mathcal{R}^\mathcal{C}} \sigma^{\mathcal{C}+}_r \operatorname{grad}\mathcal{C}_r(\hat{a}) - \sum_{r\in\mathcal{R}^\mathcal{C}} \sigma^{\mathcal{C}-}_r \operatorname{grad}\mathcal{C}_r(\hat{a}) - \sum_{r\in\mathcal{R}_0} \beta^\mathcal{U}_r \operatorname{grad}\mathcal{U}_r(\hat{a}) + \sum_{r\in\mathcal{R}_0\cup\mathcal{R}_{+0}} \sigma^\mathcal{W}_r \operatorname{grad}\mathcal{W}_r(\hat{a}) = 0.$$

Let us set $\sigma^\mathcal{B}_r = 0$ $(\forall r \notin \mathcal{R}^\mathcal{B}_a)$, $\beta^\mathcal{U}_r = 0$ $(\forall r \notin \mathcal{R}_0)$, $\sigma^\mathcal{W}_r = 0$ $(\forall r \notin \mathcal{R}_0 \cup \mathcal{R}_{+0})$. Then, it follows that:

$$\sum_{r\in\mathcal{R}^\Theta} \alpha_r \operatorname{grad}\Theta_r(\hat{a}) + \sum_{r\in\mathcal{R}^\mathcal{B}} \sigma^\mathcal{B}_r \operatorname{grad}\mathcal{B}_r(\hat{a}) + \sum_{r\in\mathcal{R}^\mathcal{C}} \sigma^{\mathcal{C}+}_r \operatorname{grad}\mathcal{C}_r(\hat{a}) - \sum_{r\in\mathcal{R}^\mathcal{C}} \sigma^{\mathcal{C}-}_r \operatorname{grad}\mathcal{C}_r(\hat{a}) - \sum_{r\in\mathcal{R}} \beta^\mathcal{U}_r \operatorname{grad}\mathcal{U}_r(\hat{a}) + \sum_{r\in\mathcal{R}} \sigma^\mathcal{W}_r \operatorname{grad}\mathcal{W}_r(\hat{a}) = 0,$$

On the other hand, $\mathcal{B}_r(\hat{a}) = 0$ $\forall r \in \mathcal{R}^\mathcal{B}_a$, $\mathcal{C}_r(\hat{a}) = 0$, $\forall r \in \mathcal{R}^\mathcal{C}$, $\mathcal{U}_r(\hat{a}) = 0$ $\forall r \in \mathcal{R}_0$, and $\mathcal{W}_r(\hat{a}) = 0$ $\forall r \in \mathcal{R}_0 \cup \mathcal{R}_{+0}$. This entails that:

$$\begin{aligned} \sigma^\mathcal{B}_r \mathcal{B}_r(\hat{a}) &= 0, \quad r \in \mathcal{R}^\mathcal{B}, \\ \sigma^{\mathcal{C}+}_r (\mathcal{C}_r(\hat{a})) &= 0, \quad r \in \mathcal{R}^\mathcal{C}, \\ \sigma^{\mathcal{C}-}_r (-\mathcal{C}_r(\hat{a})) &= 0, \quad r \in \mathcal{R}^\mathcal{C}, \\ \beta^\mathcal{U}_r \mathcal{U}_r(\hat{a}) &= 0, \quad r \in \mathcal{R}, \\ \sigma^\mathcal{W}_r \mathcal{W}_r(\hat{a}) &= 0, \quad r \in \mathcal{R}. \end{aligned} \tag{18}$$

Let us set the following:

$$\begin{aligned} \sigma^{\mathcal{C}+}_r - \sigma^{\mathcal{C}-}_r &= \sigma^\mathcal{C}_r, \quad \forall r \in \mathcal{R}^\mathcal{C}, \\ \beta^\mathcal{U}_r - \sigma^\mathcal{W}_r \mathcal{V}_r(\hat{a}) &= \sigma^\mathcal{U}_r, \quad \forall r \in \mathcal{R}, \\ \sigma^\mathcal{W}_r \mathcal{U}_r(\hat{a}) &= \sigma^\mathcal{V}_r, \quad \forall r \in \mathcal{R}. \end{aligned} \tag{19}$$

Consequently, we arrive at the following:

$$\begin{aligned} \sum_{r\in\mathcal{R}^\Theta} \alpha_r \operatorname{grad}\Theta_r(\hat{a}) + \sum_{r\in\mathcal{R}^\mathcal{B}} \sigma^\mathcal{B}_r \operatorname{grad}\mathcal{B}_r(\hat{a}) + \sum_{r\in\mathcal{R}^\mathcal{C}} \sigma^\mathcal{C}_r \operatorname{grad}\mathcal{C}_r(\hat{a}) \\ - \sum_{r\in\mathcal{R}} \sigma^\mathcal{U}_r \operatorname{grad}\mathcal{U}_r(\hat{a}) + \sum_{r\in\mathcal{R}} \sigma^\mathcal{V}_r \operatorname{grad}\mathcal{V}_r(\hat{a}) = 0, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\alpha_r > 0, \quad \forall r \in \mathcal{R}^\Theta, \\ &\mathcal{B}_r(\hat{a}) \le 0, \quad \sigma^\mathcal{B}_r \ge 0, \quad \sigma^\mathcal{B}_r \mathcal{B}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}^\mathcal{B}, \\ &\mathcal{C}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}^\mathcal{C}, \\ &\sigma^\mathcal{U}_r \text{ free}, \forall r \in \mathcal{R}_{0+}, \quad \sigma^\mathcal{U}_r \ge 0, \forall r \in \mathcal{R}_{00} \cup \mathcal{R}_{0-}, \quad \sigma^\mathcal{U}_r = 0, \forall r \in \mathcal{R}_+, \\ &\sigma^\mathcal{V}_r \ge 0, \forall r \in \mathcal{R}_{+0}, \quad \sigma^\mathcal{V}_r = 0, \forall r \in \mathcal{R}_0 \cup \mathcal{R}_{+-}, \\ &\sigma^\mathcal{U}_r \mathcal{U}_r(\hat{a}) = 0, \quad \sigma^\mathcal{V}_r \mathcal{V}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}. \end{aligned} \tag{21}$$

Thus, the proof is completed. □

**Remark 7.**

1. *If $\mathcal{H}^n = \mathbb{R}^n$, Theorem 1 reduces to Theorem 6.2 established by Mishra et al. [12].*
2. *Theorem 1 is a generalization of Theorem 3.2 derived by Maeda [33] from multiobjective optimization problems in the setting of $\mathbb{R}^n$ to MOMPVC in the Hadamard manifold framework.*

We now provide an illustrative example to demonstrate that MOMPVC-GGCQ is not a sufficient condition for Theorem 1.

**Example 1.** *Let $\mathcal{M}^2$ signify the set of all $2 \times 2$ symmetric matrices. Let $\mathcal{M}^2_+ \subset \mathcal{M}^2$ consist of $2 \times 2$ positive definite matrices. $\mathcal{A} \in \mathcal{M}^2_+$. $\mathcal{M}^2_+$ is a well-known Hadamard manifold (see [40]), with:*

$$\langle \mathcal{W}_1, \mathcal{W}_2 \rangle_{\mathcal{A}} := \mathrm{Tr}\,(\mathcal{W}_2\, \mathcal{A}^{-1}\, \mathcal{W}_1\, \mathcal{A}^{-1}),$$

*for any arbitrary $\mathcal{W}_1, \mathcal{W}_2 \in T_{\mathcal{A}}\mathcal{M}^2_+ = \mathcal{M}^2$. Let $\mathcal{A}, \mathcal{B} \in \mathcal{M}^2_+$ and $\mathcal{W} \in T_{\mathcal{A}}\mathcal{M}^2_+$.*

*The exponential map $\exp_{\mathcal{A}}(\mathcal{W}) : T_{\mathcal{A}}\mathcal{M}^2_+ \to \mathcal{M}^2_+$ is given by*

$$\exp_{\mathcal{A}}(\mathcal{W}) = \mathcal{A}^{\frac{1}{2}}\, \mathrm{Exp}\,\left(\mathcal{A}^{-\frac{1}{2}}\, \mathcal{W}\, \mathcal{A}^{-\frac{1}{2}}\right) \mathcal{A}^{\frac{1}{2}},$$

*where Exp denotes the usual matrix exponential. The inverse exponential map $\exp^{-1}_{\mathcal{A}} : \mathcal{M}^2_+ \to T_{\mathcal{A}}\mathcal{M}^2_+$ is:*

$$\exp^{-1}_{\mathcal{A}}(\mathcal{B}) = \mathcal{A}^{\frac{1}{2}}\, \mathrm{Log}\,\left(\mathcal{A}^{-\frac{1}{2}}\, \mathcal{B}\, \mathcal{A}^{-\frac{1}{2}}\right) \mathcal{A}^{\frac{1}{2}},$$

*where Log denotes standard logarithm on $\mathcal{M}^2_+$ (see [40]). Let $F : \mathcal{M}^2_+ \to \mathbb{R}$. Then,*

$$\mathrm{grad}(F(\mathcal{A})) = \mathcal{A}\, F'(\mathcal{A})\, \mathcal{A},$$

*where $\mathcal{A} \in \mathcal{M}^+_2$ and $F'(\mathcal{U})$ signifies the Euclidean gradient of $F$ at $\mathcal{U}$ (see [40]).*

*We formulate a MOMPVC, denoted by (P) for the sake of brevity, as follows:*

$$\begin{aligned}
\text{(P) Minimize } \Theta(\mathcal{A}) = (\Theta_1(\mathcal{A}), \Theta_2(\mathcal{A})) &:= (-3\log a_1 - 3\log a_4, -5 - \log a_1 - \log a_4), \\
\text{subject to}\quad & \mathcal{B}_1(\mathcal{A}) := |a_4| - 1 \le 0, \\
& \mathcal{B}_2(\mathcal{A}) := |a_1| - 1 \le 0, \\
& \mathcal{U}(\mathcal{A}) := -a_2^2 \ge 0, \\
& \mathcal{U}(\mathcal{A})\mathcal{V}(\mathcal{A}) := (-a_2^2)(-a_3^2) \le 0,
\end{aligned}$$

*where $\Theta_i : \mathcal{M}^+_2 \to \mathbb{R}$ $(i = 1, 2)$ and $\mathcal{B}_j : \mathcal{M}^+_2 \to \mathbb{R}$ $(j = 1, 2)$, and $\mathcal{U}, \mathcal{V} : \mathcal{M}^+_2 \to \mathbb{R}$ real valued functions and $\mathcal{A} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} \in \mathcal{M}^+_2$.*

*Let $\mathcal{S}^f$ signify the feasible set of (P). Then:*

$$\mathcal{S}^f := \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \end{bmatrix} \;:\; 0 < a_1 \le 1,\ 0 < a_4 \le 1 \right\}.$$

*We choose the feasible element*

$$\mathcal{A}^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{S}^f.$$

*It can be verified that:*

$$\mathscr{L}^C(\mathscr{W}, \mathcal{A}^*) = \left\{ \begin{bmatrix} 0 & a_2 \\ a_2 & 0 \end{bmatrix} : a_2 \in \mathbb{R} \right\},$$

$$\mathscr{C}^T(\mathscr{W}^r, \mathcal{A}^*) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;:\; a_1, a_2, a_3, a_4 = 0 \right\}, \quad r = 1, 2.$$

*Therefore, MOMPVC-GGCQ is not satisfied at $\mathcal{A}^*$. Moreover, we can obtain the following:*

$$\operatorname{grad}\Theta_1(\mathcal{A}^*) = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \qquad \operatorname{grad}\Theta_2(\mathcal{A}^*) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$\operatorname{grad}\mathcal{B}_1(\mathcal{A}^*) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \operatorname{grad}\mathcal{B}_2(\mathcal{A}^*) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\operatorname{grad}\mathcal{U}(\mathcal{A}^*) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \operatorname{grad}\mathcal{V}(\mathcal{A}^*) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

*Then, there exists* $\sigma_1^{\Theta} = \frac{1}{2}$, $\sigma_2^{\Theta} = \frac{1}{2}$, $\sigma_1^{\mathcal{B}} = 1$, $\sigma_2^{\mathcal{B}} = 1$, $\sigma^{\mathcal{C}} = 1$, $\sigma^{\mathcal{D}} = 1$, *such that*

$$\sigma_1^{\Theta}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \sigma_2^{\Theta}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \sigma_1^{\mathcal{B}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \sigma_2^{\mathcal{B}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \sigma^{\mathcal{C}}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \sigma^{\mathcal{D}}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

*As a result, we infer the fact that although MOMPVC-GGCQ fails to be satisfied at* $\mathcal{A}^*$, *KKT conditions still hold.*

## 4. Sufficient Conditions for MOMPVC-GGCQ

In this section, we investigate several constraint qualifications existing in the literature in the framework of Hadamard manifolds for MOMPVC. We further derive certain sufficient conditions for MOMPVC-GGCQ.
The following definition is extended from Mishra et al. [12] for MOMPVC in the Hadamard manifold framework.

**Definition 9.** *Let* $\hat{a} \in \mathcal{S}^f$. *We say that the Cottle-type constraint qualification (CTCQ) holds at* $\hat{a}$ *if for every* $k \in \mathcal{R}^{\Theta}$, *the following system provides a solution,* $\overline{\omega} \in T_{\hat{a}}\mathcal{H}^n$:

$$\begin{aligned} \langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\omega}\rangle &< 0, \quad r \in \mathcal{R}^{\Theta} \text{ and } r \neq k, \\ \langle \operatorname{grad}\mathcal{B}_r(\hat{a}), \overline{\omega}\rangle &< 0, \quad r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}), \\ \langle \operatorname{grad}\mathcal{C}_r(\hat{a}), \overline{\omega}\rangle &= 0, \quad r \in \mathcal{R}^{\mathcal{C}}, \\ \langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\omega}\rangle &> 0, \quad r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00} \cup \mathcal{R}_{0-}, \\ \langle \operatorname{grad}\mathcal{W}_r(\hat{a}), \overline{\omega}\rangle &< 0, \quad r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00} \cup \mathcal{R}_{0-} \cup \mathcal{R}_{+0}. \end{aligned} \tag{22}$$

In the following lemma, we demonstrate that under some mild restrictions on the index sets, CTCQ is not satisfied as a feasible element of MOMPVC.

**Lemma 5.** *Let* $\hat{a} \in \mathcal{S}^f$ *and* $\mathcal{R}_{00} \cup \mathcal{R}_{0+} \neq \phi$. *Then, CTCQ does not hold at* $\hat{a}$.

**Proof.** By reductio ad absurdum, we suppose that CTCQ holds at $\hat{a}$. As a result, for every $k \in \mathcal{R}^{\Theta}$, the system (22) provides a solution, $\overline{\omega} \in T_{\hat{a}}\mathcal{H}^n$. From the definitions of the index sets, we obtain

$$\begin{aligned} \operatorname{grad}\mathcal{W}_r(\hat{a}) &= 0, \quad \forall r \in \mathcal{R}_{00}, \\ \langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\omega}\rangle = \frac{1}{\mathcal{V}_r(\hat{a})}\langle \operatorname{grad}\mathcal{W}_r(\hat{a}), \overline{\omega}\rangle &< 0, \quad \forall r \in \mathcal{R}_{0+}, \end{aligned} \tag{23}$$

which contradicts (22). Hence, the proof is completed. □

We now modify several standard constraint qualifications and introduce the following MOMPVC-tailored constraint qualifications on Hadamard manifolds. In the following definitions, we always assume that $\hat{a} \in \mathcal{S}^f$.

**Definition 10.** *The MOMPVC-tailored Abadie's constraint qualification (MOMPVC-ACQ) holds at $\hat{a}$, if:*

$$\mathscr{L}^C(\mathscr{W}, \hat{a}) \subseteq \mathscr{C}^T(\mathscr{W}, \hat{a}). \tag{24}$$

**Definition 11.** *The MOMPVC-tailored generalized Abadie's constraint qualification (MOMPVC-GACQ) holds at $\hat{a}$, if:*

$$\mathscr{L}^C(\mathscr{W}, \hat{a}) \subseteq \bigcap_{i \in \mathcal{R}^\Theta} \mathscr{C}^T\left(\mathscr{W}^i; \hat{a}\right). \tag{25}$$

**Lemma 6.** *If MOMPVC-ACQ holds at $\hat{a}$, then MOMPVC-GACQ holds at $\hat{a}$.*

**Proof.** The proof follows readily from Definitions 10 and 11. □

**Lemma 7.** *If MOMPVC-GACQ holds at $\hat{a}$, then MOMPVC-GGCQ holds at $\hat{a}$.*

**Proof.** The proof follows readily from Definitions 10 and 11. □

**Definition 12.** *The MOMPVC-tailored Cottle-type constraint qualification (MOMPVC-CTCQ) holds at $\hat{a}$, if for every $k \in \mathcal{R}^\Theta$, the following system provides a solution, $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$:*

$$\begin{aligned}
\langle \operatorname{grad} \Theta_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad r \in \mathcal{R}^\Theta \text{ and } r \neq k, \\
\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}), \\
\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), \overline{\varpi} \rangle &= 0, \quad r \in \mathcal{R}^{\mathcal{C}}, \\
\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle &= 0, \quad r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00}, \\
\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle &> 0, \quad r \in \mathcal{R}_{0-}, \\
\langle \operatorname{grad} \mathcal{V}_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad r \in \mathcal{R}_{+0}.
\end{aligned} \tag{26}$$

**Definition 13.** *The MOMPVC-tailored Slater-type constraint qualification (MOMPVC-STCQ) holds at $\hat{a}$ if $\Theta_r$ $(r \in \mathcal{R}^\Theta)$, and $\mathcal{B}_r$ $(r \in \mathcal{R}^{\mathcal{B}})$, $\mathcal{V}_r$ $(r \in \mathcal{R}_{+0})$ are geodesic convex, $\mathcal{U}_r$ $(r \in \mathcal{R}_{0-})$ are geodesic concave, $\mathcal{U}_r$ $(r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+})$, $\mathcal{C}_r$ $(r \in \mathcal{R}^{\mathcal{C}})$ are geodesic affine, and for every $k \in \mathcal{R}^\Theta$, the following system provides a solution, $z \in \mathcal{H}^n$:*

$$\begin{aligned}
\Theta_r(z) &< \Theta_r(\hat{a}), && \forall r \in \mathcal{R}^\Theta \text{ and } r \neq k, \\
\mathcal{B}_r(z) &< 0, && \forall r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}), \\
\mathcal{C}_r(z) &= 0, && \forall r \in \mathcal{R}^{\mathcal{C}}, \\
\mathcal{U}_r(z) &= 0, && \forall r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+}, \\
\mathcal{U}_r(z) &> 0, && \forall r \in \mathcal{R}_{0-}, \\
\mathcal{V}_r(z) &< 0, && \forall r \in \mathcal{R}_{+0}.
\end{aligned} \tag{27}$$

**Definition 14.** *The MOMPVC-tailored Mangasarian–Fromovitz constraint qualification (MOMPVC-MFCQ) holds at $\hat{a}$ if* $\operatorname{grad} \Theta_r$ $(r \in \mathcal{R}^\Theta)$, $\operatorname{grad} \mathcal{C}_r$ $(r \in \mathcal{R}^{\mathcal{C}})$, $\operatorname{grad} \mathcal{U}_r$ $(r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00})$ *are linearly independent and the following system provides a solution, $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$:*

$$\begin{aligned}
\langle \operatorname{grad} \Theta_r(\hat{a}), \overline{\varpi} \rangle &= 0, \quad r \in \mathcal{R}^\Theta, \\
\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}), \\
\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), \overline{\varpi} \rangle &= 0, \quad r \in \mathcal{R}^{\mathcal{C}}, \\
\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle &= 0, \quad r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00}, \\
\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle &> 0, \quad r \in \mathcal{R}_{0-}, \\
\langle \operatorname{grad} \mathcal{V}_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad r \in \mathcal{R}_{+0}.
\end{aligned} \tag{28}$$

**Definition 15.** *Let $\hat{a} \in \mathcal{S}^f$. Then the linearly independent constraint qualification (MOMPVC-LICQ) is said to hold at $\hat{a}$, if for every $k \in \mathcal{R}^{\Theta}$, the gradients* $\operatorname{grad}\Theta_r(\hat{a})$ $(r \in \mathcal{R}^{\Theta},\ r \neq k)$, $\operatorname{grad}\mathcal{B}_r\ (\hat{a})(r \in \mathcal{R}_a^{\mathcal{B}})$, $\operatorname{grad}\mathcal{C}_r\ (\hat{a})(r \in \mathcal{R}^{\mathcal{C}})$, $\operatorname{grad}\mathcal{U}_r\ (\hat{a})(r \in \mathcal{R}_0)$, $\operatorname{grad}\mathcal{V}_r\ (\hat{a})(r \in \mathcal{R}_{+0})$ *are linearly independent.*

In the following theorem, we establish that the satisfaction of MOMPVC-CTCQ is a sufficient condition for the satisfaction of MOMPVC-GGCQ under certain reasonable restriction on index sets.

**Lemma 8.** *Let $\hat{a} \in \mathcal{S}^f$. Suppose that $\mathcal{R}_{00} = \phi$. If MOMPVC-CTCQ holds at $\hat{a}$, then MOMPVC-GGCQ also holds at $\hat{a}$.*

**Proof.** According to the provided hypotheses, $\hat{a} \in \mathcal{S}^f$ and $\mathcal{R}_{00} = \phi$. Further, CTCQ holds at $\hat{a} \in \mathcal{S}^f$. As a result, some $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$ exists, such that, for every $k \in \mathcal{R}^{\Theta}$, we have:

$$\begin{aligned}
\langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad \forall r \in \mathcal{R}^{\Theta} \text{ and } r \neq k,\\
\langle \operatorname{grad}\mathcal{B}_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad \forall r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}),\\
\langle \operatorname{grad}\mathcal{C}_r(\hat{a}), \overline{\varpi} \rangle &= 0, \quad \forall r \in \mathcal{R}^{\mathcal{C}},\\
\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle &= 0, \quad \forall r \in \mathcal{R}_{0+},\\
\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi} \rangle &> 0, \quad \forall r \in \mathcal{R}_{0-},\\
\langle \operatorname{grad}\mathcal{V}_r(\hat{a}), \overline{\varpi} \rangle &< 0, \quad \forall r \in \mathcal{R}_{+0}.
\end{aligned} \tag{29}$$

Let $v \in \mathscr{L}^C(\mathscr{W},\ \hat{a})$. Then, in view of Definition 6 and the fact that $\mathcal{R}_{00} = \phi$, we get:

$$\begin{aligned}
\langle \operatorname{grad}\Theta_r(\hat{a}), v \rangle &\leq 0, \quad \forall r \in \mathcal{R}^{\Theta},\\
\langle \operatorname{grad}\mathcal{B}_r(\hat{a}), v \rangle &\leq 0, \quad \forall r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}),\\
\langle \operatorname{grad}\mathcal{C}_r(\hat{a}), v \rangle &= 0, \quad \forall r \in \mathcal{R}^{\mathcal{C}},\\
\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), v \rangle &= 0, \quad \forall r \in \mathcal{R}_{0+},\\
\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), v \rangle &\geq 0, \quad \forall r \in \mathcal{R}_{0-},\\
\langle \operatorname{grad}\mathcal{V}_r(\hat{a}), v \rangle &\leq 0, \quad \forall r \in \mathcal{R}_{+0}.
\end{aligned} \tag{30}$$

At first, we claim that $v \in \mathscr{C}^T(\mathscr{W}^1, \hat{a})$. Consider a sequence $\{\tau_n\}_{n=1}^{\infty} \downarrow 0$, where $\downarrow 0$ indicates that the sequence approaches zero from the positive direction. Correspondingly, we define $\{v_n\}_{n=1}^{\infty}$ as follows:

$$v_n := v + \tau_n \overline{\varpi}, \quad \forall n \in \mathbb{N}. \tag{31}$$

Clearly, $v_n \to v$ as $n \to \infty$. From (29), (30), and (31), it follows that

$$\begin{aligned}
\langle \operatorname{grad}\Theta_r(\hat{a}), v_n \rangle &< 0, \quad \forall r \in \mathcal{R}^{\Theta},\\
\langle \operatorname{grad}\mathcal{B}_r(\hat{a}), v_n \rangle &< 0, \quad \forall r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}),\\
\langle \operatorname{grad}\mathcal{C}_r(\hat{a}), v_n \rangle &= 0, \quad \forall r \in \mathcal{R}^{\mathcal{C}},\\
\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), v_n \rangle &= 0, \quad \forall r \in \mathcal{R}_{0+}\\
\langle \operatorname{grad}\mathcal{U}_r(\hat{a}), v_n \rangle &> 0, \quad \forall r \in \mathcal{R}_{0-},\\
\langle \operatorname{grad}\mathcal{V}_r(\hat{a}), v_n \rangle &< 0, \quad \forall r \in \mathcal{R}_{+0}.
\end{aligned} \tag{32}$$

For every element of the sequence $\{v_n\}$ $(n \in \mathbb{N})$, we consider a sequence $\{\lambda_{n_k}\}_{k=1}^{\infty} \downarrow 0$. Now, we construct a sequence $\{z_{n_k}\}_{k=1}^{\infty}$ as:

$$z_{n_k} := \exp_{\hat{a}}(\lambda_{n_k} v_n), \quad \forall k \in \mathbb{N}. \tag{33}$$

Clearly $\{z_{n_k}\} \to \hat{a}$ as $k \to \infty$. Then, for large enough $k$, we have the following for every $r \in \mathcal{R}^{\Theta} \setminus \{1\}$:

$$\Theta_r\left(z_{n_k}\right) = \Theta_r(\hat{a}) + \lambda_{n_k}\langle \operatorname{grad}\Theta_r(\hat{a}), v_n\rangle + o\left(\left|\lambda_{n_k}\right|\right) < \Theta_r(\hat{a}). \tag{34}$$

Similarly, for large enough $k$, we have the following for every $r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a})$:

$$\mathcal{B}_r\left(z_{n_k}\right) = \mathcal{B}_r(\hat{a}) + \lambda_{n_k}\langle \operatorname{grad}\mathcal{B}_r(\hat{a}), v_n\rangle + o(|\lambda_n|) < \mathcal{B}_r(\hat{a}) = 0. \tag{35}$$

Now, for every $r \notin \mathcal{R}_a^{\mathcal{B}}(\hat{a})$, it follows from the continuity of $\mathcal{B}_r$ that

$$\mathcal{B}_r\left(z_{n_k}\right) = \mathcal{B}_r\left(\exp_{\hat{a}}(\lambda_{n_k} v_n)\right) < 0, \quad \text{for sufficiently large } k. \tag{36}$$

Continuing in a similar manner, it can be shown that for large enough $k$, we have:

$$\begin{aligned} \mathcal{C}_r(z_{n_k}) &= 0,\ \forall r \in \mathcal{R}^{\mathcal{C}}, \\ \mathcal{U}_r(z_{n_k}) &= 0,\ \forall r \in \mathcal{R}, \\ \mathcal{U}_r(z_{n_k}) &\geq 0,\ \forall r \in \mathcal{R}, \\ \mathcal{V}_r(z_{n_k}) &\leq 0,\ \forall r \in \mathcal{R}. \end{aligned} \tag{37}$$

Since $\mathcal{R}_{00} = \phi$, then it follows from (34)–(37) that

$$z_{n_k} = \left(\exp_{\hat{a}}(\lambda_{n_k} v_n)\right) \in \mathscr{W}^1, \qquad \text{for sufficiently large } k. \tag{38}$$

Without any loss of generality, we may assume that $z_{n_k} \in \mathscr{W}^1$ for all $k$. This implies that

$$v \in \mathscr{C}^T\left(\mathscr{W}^1, \hat{a}\right). \tag{39}$$

By following exactly the same procedure, we can show that for every $k \in \mathcal{R}^{\Theta} \setminus \{1\}$, we have $v \in \mathscr{C}^T\left(\mathscr{W}^k, \hat{a}\right)$. Then, it follows that

$$v \in \bigcap_{k \in \mathcal{R}^{\Theta}} \mathscr{C}^T\left(\mathscr{W}^k, \hat{a}\right) \subseteq \bigcap_{k \in \mathcal{R}^{\Theta}} \operatorname{cl}\operatorname{co}\mathscr{C}^T\left(\mathscr{W}^k, \hat{a}\right). \tag{40}$$

Hence, the proof is completed. □

In the following lemma, we provide a relation between MOMPVC-CTCQ and MOMPVC-MFCQ.

**Lemma 9.** *Let $\hat{a} \in \mathcal{S}^f$. If MOMPVC-MFCQ holds at $\hat{a}$, then MOMPVC-CTCQ is also satisfied at $\hat{a}$.*

**Proof.** According to the provided hypotheses, MOMPVC-MFCQ holds at $\hat{a} \in \mathcal{S}^f$. This implies that $\operatorname{grad}\Theta_r$ $(r \in \mathcal{R}^{\Theta})$, $\operatorname{grad}\mathcal{C}_r$ $(r \in \mathcal{R}^{\mathcal{C}})$, $\operatorname{grad}\mathcal{U}_r$ $(r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00})$ are linearly independent. Moreover, some $\overline{\varpi} \in T_{\hat{a}}\mathcal{H}^n$ exists, satisfying:

$$\begin{aligned} \langle \operatorname{grad}\Theta_r(\hat{a}), \overline{\varpi}\rangle &= 0, \quad r \in \mathcal{R}^{\Theta}, \\ \langle \operatorname{grad}\mathcal{B}_r(\hat{a}), \overline{\varpi}\rangle &< 0, \quad r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}), \\ \langle \operatorname{grad}\mathcal{C}_r(\hat{a}), \overline{\varpi}\rangle &= 0, \quad r \in \mathcal{R}^{\mathcal{C}}, \\ \langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi}\rangle &= 0, \quad r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00}, \\ \langle \operatorname{grad}\mathcal{U}_r(\hat{a}), \overline{\varpi}\rangle &> 0, \quad r \in \mathcal{R}_{0-}, \\ \langle \operatorname{grad}\mathcal{V}_r(\hat{a}), \overline{\varpi}\rangle &< 0, \quad r \in \mathcal{R}_{+0}. \end{aligned} \tag{41}$$

By reductio ad absurdum, let us assume that MOMPVC-CTCQ is not satisfied at $\hat{a} \in \mathcal{S}^f$. Then, some $k \in \mathcal{R}^\Theta$ exists, such that the system given below does not provide a solution, $v \in T_{\hat{a}}\mathcal{H}^n$:

$$\begin{aligned} &\langle \operatorname{grad} \Theta_r(\hat{a}), v \rangle < 0, \quad r \in \mathcal{R}^\Theta \text{ and } r \neq k, \\ &\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), v \rangle < 0, \quad r \in \mathcal{R}_a^\mathcal{B}(\hat{a}), \\ &\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), v \rangle = 0, \quad r \in \mathcal{R}^\mathcal{C}, \\ &\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), v \rangle = 0, \quad r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00}, \\ &\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), v \rangle > 0, \quad r \in \mathcal{R}_{0-}, \\ &\langle \operatorname{grad} \mathcal{V}_r(\hat{a}), v \rangle < 0, \quad r \in \mathcal{R}_{+0}. \end{aligned} \tag{42}$$

In view of Lemma 2 and (42), some real numbers, $\alpha_r \geq 0(r \in \mathcal{R}^\Theta, r \neq k)$, $\sigma_r^\mathcal{B} \geq 0(r \in \mathcal{R}_a^\mathcal{B})$, $\sigma_r^\mathcal{U} \geq 0(r \in \mathcal{R}_{0-})$, $\sigma_r^\mathcal{V} \geq 0(r \in \mathcal{R}_{+0})$, not all zero, and $\sigma_r^\mathcal{C}(r \in \mathcal{R}^\mathcal{C})$, $\tilde{\sigma}_r^\mathcal{U}$ $(r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00})$, exist, for which:

$$\begin{aligned} &\sum_{\substack{r \in \mathcal{R}^\Theta \\ r \neq k}} \alpha_r \operatorname{grad} \Theta_r(\hat{a}) + \sum_{r \in \mathcal{R}_a^\mathcal{B}} \sigma_r^\mathcal{B} \operatorname{grad} \mathcal{B}_r(\hat{a}) + \sum_{r \in \mathcal{R}^\mathcal{C}} \sigma_r^\mathcal{C} \operatorname{grad} \mathcal{C}_r(\hat{a}) \\ &+ \sum_{r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+}} \tilde{\sigma}_r^\mathcal{U} \operatorname{grad} \mathcal{U}_r(\hat{a}) - \sum_{r \in \mathcal{R}_{0-}} \sigma_r^\mathcal{U} \operatorname{grad} \mathcal{U}_r(\hat{a}) + \sum_{r \in \mathcal{R}_{+0}} \sigma_r^\mathcal{V} \operatorname{grad} \mathcal{V}_r(\hat{a}) = 0. \end{aligned} \tag{43}$$

From (41) and (43), it follows that:

$$\begin{aligned} &\left\langle \sum_{r \in \mathcal{R}_a^\mathcal{B}} \sigma_r^\mathcal{B} \operatorname{grad} \mathcal{B}_r(\hat{a}), \overline{\omega} \right\rangle - \left\langle \sum_{r \in \mathcal{R}_{0-}} \sigma_r^\mathcal{U} \operatorname{grad} \mathcal{U}_r(\hat{a}), \overline{\omega} \right\rangle + \\ &\qquad\qquad \left\langle \sum_{r \in \mathcal{R}_{+0}} \sigma_r^\mathcal{V} \operatorname{grad} \mathcal{V}_r(\hat{a}), \overline{\omega} \right\rangle = 0. \end{aligned} \tag{44}$$

Combining (41) and (44), we yield: $\sigma_r^\mathcal{B} = 0, \forall r \in \mathcal{R}_a^\mathcal{B}$, $\sigma_r^\mathcal{U} = 0, \forall r \in \mathcal{R}_{0-}$, $\sigma_r^\mathcal{V} = 0$, $\forall r \in \mathcal{R}_{+0}$. Then, it follows from (43) that

$$\sum_{\substack{r \in \mathcal{R}^\Theta \\ r \neq k}} \alpha_r \operatorname{grad} \Theta_r(\hat{a}) + \sum_{r \in \mathcal{R}^\mathcal{C}} \sigma_r^\mathcal{C} \operatorname{grad} \mathcal{C}_r(\hat{a}) - \sum_{r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+}} \tilde{\sigma}_r^\mathcal{U} \operatorname{grad} \mathcal{C}_r(\hat{a}) = 0. \tag{45}$$

From the linearly independence of $\operatorname{grad} \Theta_r(\hat{a})$ $(r \in \mathcal{R}^\Theta)$, $\operatorname{grad} \mathcal{C}_r(\hat{a})$ $(r \in \mathcal{R}^\mathcal{C})$, $\operatorname{grad} \mathcal{U}_r(\hat{a})$ $(r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+})$, we infer that:

$$\begin{aligned} &\alpha_r = 0, \quad \forall r \in \mathcal{R}^\Theta, \ r \neq k, \\ &\sigma_r^\mathcal{C} = 0, \quad \forall r \in \mathcal{R}^\mathcal{C}, \\ &\tilde{\sigma}_r{}^\mathcal{U} = 0, \quad \forall r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+}, \end{aligned} \tag{46}$$

which is a contradiction. Hence, the proof is completed. □

In the following lemma, we provide a relation between MOMPVC-CTCQ and MOMPVC-STCQ.

**Lemma 10.** *Let $\hat{a} \in \mathcal{S}^f$. If MOMPVC-STCQ holds at $\hat{a}$, then MOMPVC-CTCQ also holds at $\hat{a}$.*

**Proof.** From the provided hypotheses, MOMPVC-STCQ holds at $\hat{a} \in \mathcal{S}^f$. Then, it follows that each of the functions $\Theta_r$ $(r \in \mathcal{R}^\Theta)$ $\mathcal{B}_r$ $(r \in \mathcal{R}^\mathcal{B})$ and $\mathcal{V}_r$ $(r \in \mathcal{R}_{+0})$ are geodesic convex, $\mathcal{U}_r$ $(r \in \mathcal{R}_{0-})$ are geodesic concave, $\mathcal{U}_r$ $(r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+})$, $\mathcal{C}_r$ $(r \in \mathcal{R}^\mathcal{C})$ are affine, and for every $k \in \mathcal{R}^\Theta$, the system of inequalities given below,

$$
\begin{aligned}
&\Theta_r(z) < \Theta_r(\hat{a}), && \forall r \in \mathcal{R}^{\Theta} \text{ and } r \neq k,\\
&\mathcal{B}_r(z) < 0, && \forall r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}),\\
&\mathcal{C}_r(z) = 0, && \forall r \in \mathcal{R}^{\mathcal{C}},\\
&\mathcal{U}_r(z) = 0, && \forall r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+},\\
&\mathcal{U}_r(z) > 0, && \forall r \in \mathcal{R}_{0-},\\
&\mathcal{V}_r(z) < 0, && \forall r \in \mathcal{R}_{+0}.
\end{aligned} \tag{47}
$$

provides a solution, $z_k \in \mathcal{H}^n$. Then, it follows that

$$
\begin{aligned}
&\left\langle \operatorname{grad} \Theta_r(\hat{a}), \exp_{\hat{a}}^{-1}(z_k) \right\rangle \leq \Theta_r(z_k) - \Theta_r(\hat{a}) < 0, \quad \forall r \in \mathcal{R}^{\Theta},\ r \neq k,\\
&\left\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), \exp_{\hat{a}}^{-1}(z_k) \right\rangle \leq \mathcal{B}_r(z_k) - \mathcal{B}_r(\hat{a}) < 0, \quad \forall r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}),\\
&\left\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), \exp_{\hat{a}}^{-1}(z_k) \right\rangle = \mathcal{C}_r(z_k) - \mathcal{C}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}^{\mathcal{C}},\\
&\left\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \exp_{\hat{a}}^{-1}(z_k) \right\rangle = \mathcal{U}_r(z_k) - \mathcal{U}_r(\hat{a}) = 0, \quad \forall r \in \mathcal{R}_{00} \cup \mathcal{R}_{0+},\\
&\left\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), \exp_{\hat{a}}^{-1}(z_k) \right\rangle \geq \mathcal{U}_r(z_k) - \mathcal{U}_r(\hat{a}) > 0, \quad \forall r \in \mathcal{R}_{0-},\\
&\left\langle \operatorname{grad} \mathcal{V}_r(\hat{a}), \exp_{\hat{a}}^{-1}(z_k) \right\rangle \leq \mathcal{V}_r(z_k) - \mathcal{V}_r(\hat{a}) < 0, \quad \forall r \in \mathcal{R}_{+0}.
\end{aligned} \tag{48}
$$

Let us define $v_k := \exp_{\hat{a}}^{-1}(z_k)$. Then, it follows that for every $k \in \mathcal{R}^{\Theta}$, we have the following

$$
\begin{aligned}
&\langle \operatorname{grad} \Theta_r(\hat{a}), v_k \rangle < 0, && r \in \mathcal{R}^{\Theta} \text{ and } r \neq k,\\
&\langle \operatorname{grad} \mathcal{B}_r(\hat{a}), v_k \rangle < 0, && r \in \mathcal{R}_a^{\mathcal{B}}(\hat{a}),\\
&\langle \operatorname{grad} \mathcal{C}_r(\hat{a}), v_k \rangle = 0, && r \in \mathcal{R}^{\mathcal{C}},\\
&\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), v_k \rangle = 0, && r \in \mathcal{R}_{0+} \cup \mathcal{R}_{00},\\
&\langle \operatorname{grad} \mathcal{U}_r(\hat{a}), v_k \rangle > 0, && r \in \mathcal{R}_{0-},\\
&\langle \operatorname{grad} \mathcal{V}_r(\hat{a}), v_k \rangle < 0, && r \in \mathcal{R}_{+0}.
\end{aligned} \tag{49}
$$

Therefore, from the above inequalities it follows that MOMPVC-CTCQ holds at $\hat{a} \in \mathcal{S}^f$. Hence, the proof is completed. □

The results derived in this section are summarized in the following theorem.

**Theorem 2.** *Let $\hat{a} \in \mathcal{S}^f$ be a Pareto efficient solution of MOMPVC and let $\mathcal{R}_{00} = \phi$. If any of the constraint qualifications as defined in Definitions 10–15 holds at $\hat{a}$, then MOMPVC-GGCQ holds at $\hat{a}$, and there exist real numbers, $\alpha_r$ $(r \in \mathcal{R}^{\Theta})$, $\sigma_r^{\mathcal{B}}$ $(r \in \mathcal{R}^{\mathcal{B}})$, $\sigma_r^{\mathcal{C}}$ $(r \in \mathcal{R}^{\mathcal{C}})$, $\sigma_r^{\mathcal{U}}$ $(r \in \mathcal{R})$, $\sigma_r^{\mathcal{V}}$ $(r \in \mathcal{R})$, which satisfy (16) and (17).*

**Remark 8.**

1. *If $\mathcal{H}^n = \mathbb{R}^n$, Theorem 2 reduces to Theorem 6.3 derived by Mishra et al. [12].*
2. *Theorem 2 generalizes Theorem 4.1 of Maeda [33] from multiobjective optimization problems to MOMPVC and extends it from the linear setting of Euclidean spaces to the nonlinear Hadamard manifold framework.*

We now provide an illustrative example to demonstrate that MOMPVC-GACQ is not a sufficient condition for Theorem 1.

**Example 2.** *Consider $\mathcal{H}^2 \subset \mathbb{R}^2$, commonly known as the positive orthant of $\mathbb{R}^2$, defined as:*

$$\mathcal{H}^2 := \{a = (a_1,\ a_2) \in \mathbb{R}^2 :\ a_1, a_2 > 0\}.$$

*Then, $\mathcal{H}^2$ is a Hadamard manifold (see [35]). Let $\hat{a} \in \mathcal{H}^2$. Then, $\langle s_1, s_2 \rangle_{\hat{a}} = \langle \mathscr{B}(\hat{a})\, s_1, s_2 \rangle$, $\forall s_1, s_2 \in T_{\hat{a}}\mathcal{H}^n = \mathbb{R}^2$, where*

$$\mathscr{B}(\hat{a}) = \begin{pmatrix} \frac{1}{\hat{a}_1^2} & 0 \\ 0 & \frac{1}{\hat{a}_2^2} \end{pmatrix}.$$

*Let $s \in T_{\hat{a}}\mathcal{H}^2$. Then, $exp_{\hat{a}} : T_{\hat{a}}\mathcal{H}^2 \to \mathcal{H}^2$ is defined as $exp_{\hat{a}}(s) = (\hat{a}_1 e^{\frac{s_1}{\hat{a}_1}}, \hat{a}_2 e^{\frac{s_2}{\hat{a}_2}})$, $s = (s_1, s_2) \in T_{\hat{a}}\mathcal{H}^2$.*

*Consider the following MOMPVC on $\mathcal{H}^2$:*

$$\begin{aligned} (P1) \quad & Minimize\ \Theta(a) = (\Theta_1(a), \Theta_2(a)) := (\sqrt{a_1}, \log a_2), \\ & subject\ to\ \ \mathcal{U}(a) := \ln a_1 - 1 \geq 0, \\ & \qquad \mathcal{V}(a)^T \mathcal{U}(a) := (e - a_2)(\ln a_1 - 1) \leq 0, \end{aligned}$$

*where $\Theta_r : \mathcal{H}^2 \to \mathbb{R}$ $(r = 1, 2)$, $\mathcal{U} : \mathcal{H}^2 \to \mathbb{R}$, $\mathcal{V} : \mathcal{H}^2 \to \mathbb{R}$ are differentiable. The feasible set of (P1), denoted by $\mathcal{S}^f$, is:*

$$\mathcal{S}^f := \{a \in \mathcal{H}^2 : \ a_1 \geq e, \ a_2 \geq e\}.$$

*Let $\hat{a} = (e, e) \in \mathcal{S}^f$. The following expressions can be easily obtained:*

$$\begin{aligned} \operatorname{grad} \Theta_1(a) &= \mathscr{B}(a)^{-1} \begin{pmatrix} \frac{1}{2\sqrt{a_1}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} a_1 \sqrt{a_1} \\ 0 \end{pmatrix}, \\ \operatorname{grad} \Theta_2(a) &= \mathscr{B}(a)^{-1} \begin{pmatrix} 0 \\ \frac{1}{a_2} \end{pmatrix} = \begin{pmatrix} 0 \\ a_2 \end{pmatrix}, \\ \operatorname{grad} \mathcal{U}(a) &= \mathscr{B}(a)^{-1} \begin{pmatrix} \frac{1}{a_1} \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, \\ \operatorname{grad} \mathcal{V}(a) &= \mathscr{B}(a)^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -a_2^2 \end{pmatrix}. \end{aligned}$$

*In view of the above expressions, one can see that:*

$$\begin{aligned} \mathscr{L}^C(\mathscr{W}, \hat{a}) &= \{s = (s_1, s_2) \in \mathbb{R}^2 : \ s_1 = 0, \ s_2 \leq 0\}, \\ \mathscr{C}^T(\mathscr{W}^1, \hat{a}) &= \{s = (s_1, s_2) \in \mathbb{R}^2 : \ s_1 \geq 0, \ s_2 = 0\}, \\ \mathscr{C}^T(\mathscr{W}^2, \hat{a}) &= \{s = (s_1, s_2) \in \mathbb{R}^2 : \ s_1 = 0, \ s_2 \geq 0\}. \end{aligned}$$

*As a result, we have:*

$$\bigcap_{i=1}^{2} \operatorname{cl} \operatorname{co} \mathscr{C}^T \left( \mathscr{W}^i, \hat{a} \right) = \{(0, 0)\}. \tag{50}$$

*Therefore, MOMPVC-GACQ fails to be satisfied at the point $\hat{a}$. However, there exist $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\sigma^{\mathcal{U}} = \frac{\sqrt{e}}{4}$, $\sigma^{\mathcal{V}} = \frac{1}{2e}$, satisfying:*

$$\sum_{r=1}^{2} \alpha_r\, grad\ \Theta_r(\hat{a}) - \sigma^{\mathcal{U}} \operatorname{grad} \mathcal{U}(\hat{a}) + \sigma^{\mathcal{V}}\, grad\ \mathcal{V}(\hat{a}) = (0, 0).$$

*Thus, it is verified that the satisfaction of MOMPVC-GACQ is not a sufficient condition for Theorem 1.*

The interrelations among the various constraint qualifications for MOMPVC is illustrated in Figure 1.

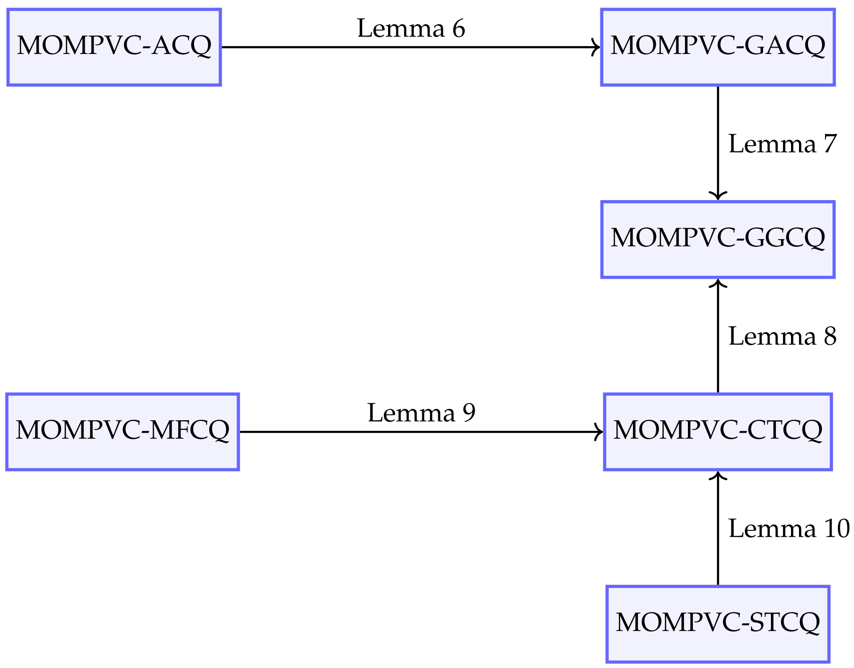


**Figure 1.** Interralations among the constraint qualifications for MOMPVC.

## 5. Conclusions and Future Research Directions

In this paper, we have investigated a class of MOMPVC in the Hadamard manifold framework. We have presented the MOMPVC-GGCQ for MOMPVC on Hadamard manifolds. Employing MOMPVC-GGCQ, we have deduced the KKT-type necessary optimality criteria for MOMPVC. Subsequently, we have introduced several MOMPVC-tailored constraint qualifications, which are sufficient criteria for MOMPVC-GGCQ. To validate the results derived in this paper, we have provided non-trivial illustrative examples in the setting of well-known Hadamard manifolds.

Several notable results currently existing in the literature are generalized and extended by the results presented in this article. For instance, the KKT-type necessary optimality conditions established in this paper extend the corresponding optimality criteria deduced by Mishra et al. [12] from the setting of Euclidean spaces to the Hadamard manifold framework. On the other hand, the results presented in this paper extend the corresponding results studied by Hoheisel and Kanzow [9] from Euclidean space to Hadamard manifolds and generalize them from MPVC to MOMPVC. Furthermore, we have extended the constraint qualifications and Pareto efficiency conditions investigated by Maeda [33] from multiobjective optimization problems to MOMPVC and have further generalized them from the Euclidean space setting to the framework of Hadamard manifolds.

The results derived in the present manuscript may be applied to solve real-life optimization problems in engineering, science, and technology. It is a well-known fact that various real-world problems emerging in various areas of modern research, such as truss optimization (see [1]), robot motion planning (see [5]), and optimal pressure control in water distribution system (see [16]), can be formulated as MPVC. In our future research work, we intend to formulate a real-world practical problem in terms of an MOMPVC on Hadamard manifolds and solve it by employing the constraint qualifications and optimality conditions for MOMPVC established in the present paper.

It is significant to observe that all the functions involved in our considered problem (MOMPVC) are assumed to be differentiable. As a result, the results of this paper cannot be applied when the functions involved in the considered problem are not necessarily differentiable. This may be considered as a limitation of this paper. Further, sufficient optimality criteria and duality results have not been investigated in the present article. We intend to address these issues in our future course of study.

It is worthwhile to note that, in the framework of Hadamard manifolds, numerical methods for solving mathematical programming problems with vanishing constraints have not yet been investigated. The framework of Hadamard manifolds is advantageous due to the fact that various constrained, non-convex optimization problems in the Euclidean space setting can be suitably transformed into unconstrained and convex optimization problems in the Hadamard manifold framework (see [23]). Moreover, it is a well-known fact that in the case of numerical methods, in order to prove convergence, the satisfaction of a constraint qualification, such as the Mangasarian–Fromovitz constraint qualification (MFCQ), must be assumed at a limit point of the sequence generated by the corresponding numerical method. In light of these facts, following the works of [2,3], the constraint qualifications for MOMPVC on Hadamard manifolds developed in this paper can be employed to investigate numerical methods, such as the relaxation approach and the one-parameter regularization methods for solving MOMPVC in the framework of Hadamard manifolds. We intend to study these numerical methods for MOMPVC in the setting of Hadamard manifolds in our future research works and, moreover, analyze the convergence of the corresponding methods.

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