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Some Properties of the Potential Field of an Almost Ricci Soliton

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Abstract: In this article, we are interested in finding necessary and sufficient conditions for a compact almost Ricci soliton to be a trivial Ricci soliton. As a first result, we show that positive Ricci curvature and, for a nonzero constant c, the integral of $\mathrm{Ric}(c\xi, c\xi)$ satisfying a generic inequality on an n-dimensional compact and connected almost Ricci soliton (M^n, g, ξ, σ) are necessary and sufficient conditions for it to be isometric to the n-sphere $S^n(c)$. As another result, we show that, if the affinity tensor of the soliton vector field ξ vanishes and the scalar curvature τ of an n-dimensional compact almost Ricci soliton (M^n, g, ξ, σ) satisfies $\tau(n\sigma - \tau) \geq 0$, then (M^n, g, ξ, σ) is a trivial Ricci soliton. Finally, on an n-dimensional compact almost Ricci soliton (M^n, g, ξ, σ), we consider the Hodge decomposition $\xi = \overline{\xi} + \nabla h$, where $\operatorname{div} \overline{\xi} = 0$, and we use the bound on the integral of $\mathrm{Ric}(\overline{\xi}, \overline{\xi})$ and an integral inequality involving the scalar curvature to find another characterization of the n-sphere.

Keywords: Ricci soliton; almost Ricci soliton; trivial Ricci soliton; Einstein manifold

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1. Introduction

One of the richest structures on a Riemannian manifold (M, g) is provided by the *Ricci soliton*, which comprises a smooth vector field ξ and a real constant λ satisfying

$$\frac{1}{2}\pounds_{\xi} g + \mathrm{Ric} = \lambda g,$$

where $\pounds_{\xi}$ is the Lie derivative in the direction of ξ, and Ric is the Ricci tensor of g. We denote a Ricci soliton by (M, g, ξ, λ). A Ricci soliton (M, g, ξ, λ) is said to be a *trivial Ricci soliton* if the soliton vector field ξ is a Killing vector field, that is, $\pounds_{\xi} g = 0$, and in this case, the Ricci soliton is an Einstein manifold. This is one of the reasons that a Ricci soliton is considered to be a generalization of an Einstein manifold. Then, the notion of *almost Ricci soliton* is introduced in (cf. [1]) in an attempt to generalize Ricci solitons, by replacing the soliton constant λ with a smooth function σ. Thus, for an almost Ricci soliton (M, g, ξ, σ), we have

$$\frac{1}{2}\pounds_{\xi} g + \mathrm{Ric} = \sigma g. \tag{1}$$

The geometry of Ricci solitons and almost Ricci solitons has been a subject of immense interest owing to their elegant geometry as well as to their nice applications (cf. [1–15]). Given an almost Ricci soliton (M, g, ξ, σ), we call the vector field ξ the *soliton vector field* and the smooth function σ the *potential function*. An almost Ricci soliton (M, g, ξ, σ) is said to be a *trivial almost Ricci soliton* if it is a Ricci soliton, that is, the potential function σ is a real constant, and it is said to be *nontrivial* if the potential function σ is nonconstant. For an almost Ricci soliton (M, g, ξ, σ), we denote by η the smooth 1-form dual to the soliton vector field ξ. This leads to a skew-symmetric tensor F defined by

$$g(FX,Y)=\frac{1}{2}d\eta(X,Y),\quad X,Y\in\mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the algebra of smooth vector fields on M. The skew-symmetric tensor F is called the *associated tensor* to the soliton vector field.

A well-known example of an n-dimensional nontrivial almost Ricci soliton is provided by the n-sphere $S^n(c)$ of constant curvature c, on taking its eigenfunction f satisfying $\Delta f=-ncf$, where f is a nonconstant function, and these are in abundance (for instance, the height functions on $S^n(c)$). This function f satisfies

$$\mathrm{Hess}(f)=-cfg, \tag{2}$$

where $\mathrm{Hess}(f)$ is the Hessian of f, and g is the canonical metric of constant curvature on $S^n(c)$. On taking $\xi=\nabla f$, the gradient of f, we see that $\frac{1}{2}\pounds_\xi g=\mathrm{Hess}(f)$, and, consequently, we have

$$\frac{1}{2}\pounds_\xi g+\mathrm{Ric}=\mathrm{Hess}(f)+\mathrm{Ric}=-cfg+(n-1)cg=\sigma g, \tag{3}$$

where $\sigma=c(-f+n-1)$ and we have used Equation (2) and the expression for the Ricci tensor $\mathrm{Ric}=(n-1)cg$ of the sphere $S^n(c)$. Since σ is a nonconstant function (as f is), we see that $(S^n(c),g,\xi,\sigma)$ is a nontrivial almost Ricci soliton. For further examples of compact and noncompact nontrivial almost Ricci solitons, we refer to [1,3].

Though it appears as if there are hundreds of results in differential geometry mentioning solitons, the results on the Ricci soliton and almost Ricci soliton are not that many. In order to understand the spirit behind the introduction of the Ricci soliton and its natural generalization, the almost Ricci soliton, we have the following heat equation called Hamilton's Ricci flow (cf. [7]):

$$\frac{\partial}{\partial t}g_{ij}(t)=-2R_{ij}(t) \tag{4}$$

and a Ricci soliton (M,g,ξ,λ) is a stable solution of the above heat equation (Hamilton's Ricci flow) of the form $g_{ij}(t)=f(t)\varphi_t^*g_{ij}$, satisfying $g_{ij}(0)=g_{ij}$, where $\varphi_t:M\to M$ is a diffeomorphism for $t\in\mathbb{R}$ such that the one parameter group $\{\varphi_t\}$ induces the vector field ξ, and $f(t)$ is the scaling function. Similarly, an almost Ricci soliton (M,g,ξ,σ) is a stable solution of the above Hamilton's Ricci flow (4) of the form

$$g_{ij}(t)=f(t,x^\alpha)\varphi_t^*g_{ij}, \tag{5}$$

where the diffeomorphisms $\varphi_t:M\to M$ are generated by the family of vector fields $\xi(t)$, and $f(t,x^\alpha)$ is the scaling, and it is a function of t as well as the local coordinates x^α. Naturally, the initial conditions $g_{ij}(0)=g_{ij}$, $\varphi_0=I$ imply $f(0,x^\alpha)=1$. On differentiating Equation (5) with respect to t and using Hamilton's Ricci flow Equation (4) at $t=0$, we find

$$-2R_{ij}=\left(\frac{\partial}{\partial t}f(t,x^\alpha)\right)_0 g_{ij}+\pounds_{\xi(0)}g_{ij},$$

which, on taking $\xi(0)=\xi$ and the function $\left(\frac{\partial}{\partial t}f(t,x^\alpha)\right)_0=-2\sigma$, the above equation takes the form of Equation (1), which defines an almost Ricci soliton. It is in this spirit that both the Ricci soliton and almost Ricci soliton being stable solutions of the heat Equation (4) is important. It is like heat diffusion, where the Hamilton's Ricci flow changes the Riemannian metric to yield the desired curvature that results in the uniform curvature distribution across the manifold. Therefore, it is worth mentioning that those solitons, which come through the solution of the heat equation, have a concrete physical significance. It is this power, inherited through the heat equation, that Ricci solitons used to settle the famous Poincare conjecture. They are not limited to this role in geometry, as Ricci solitons are also highly useful in brain mapping (cf. [16]), as well as having a role to play in the study of gravity in physics (cf. [17]). Since almost Ricci solitons are a natural generalization of Ricci solitons, they automatically carry an importance similar to that of Ricci solitons.

Two natural questions associated to the geometry of an almost Ricci soliton are the following:

(i) What are the conditions under which an n-dimensional compact almost Ricci soliton (M^n, g, ξ, σ) is isometric to $(S^n(c), g, \xi, \sigma)$?

(ii) What are the conditions under which an almost Ricci soliton is a trivial Ricci soliton?

In this article, we answer these questions by proving two results with respect to question (i) and one result with respect to question (ii). In the first result, we show that, if the Ricci curvature Ric of an n-dimensional compact and connected almost Ricci soliton (M^n, g, ξ, σ) is positive, and for a nonzero constant c, $\mathrm{Ric}(c\xi, c\xi)$ satisfies a generic inequality, then these conditions guarantee that (M^n, g, ξ, σ) is isometric to $(S^n(c), g, \xi, \sigma)$ (cf. Theorem 2). Similarly, we use the Hodge decomposition of the soliton vector field ξ to find another characterization of $(S^n(c), g, \xi, \sigma)$ (cf. Theorem 4). Finally, with respect to question (ii), we show that, if the affinity tensor of the soliton vector field ξ (introduced in Section 3) vanishes and a certain inequality involving the scalar curvature holds, then these conditions imply that a compact and connected almost Ricci soliton is a trivial Ricci soliton (cf. Theorem 3).

As the proofs of our results of using integration on a compact Riemannian manifold, here we would like to add a small note on the use of integration in differential geometry. The use of integral formulas goes back to the year 1848, and it appeared through the following very well-known formula.

Theorem 1. *(Gauss–Bonnet) [18]: Suppose that M is a compact two-dimensional Riemannian manifold with boundary ∂M. Let K be the Gaussian curvature of M, and let κ_g be the geodesic curvature of ∂M. Then,*

$$\int_M K dA + \int_{\partial M} \kappa_g ds = 2\pi \mathcal{X}(M),$$

where dA is the area element of M, ds is the line element of the boundary ∂M, and $\mathcal{X}(M)$ is the Euler characteristic of M.

The Gauss–Bonnet Theorem is an outstanding result, as it connects the geometry of M (the Gaussian curvature K and the geodesic curvature κ_g) to the topology of M (the Euler characteristic $\mathcal{X}(M)$). In particular, if M is a compact surface without boundary, then $\int_{\partial M} \kappa_g ds$ from the above formula disappears, and the formula states that the integral of the Gaussian curvature is equal to 2π times the Euler characteristic $\mathcal{X}(M) = 2(1 - g_0)$, where g_0 is the genus of the surface. Let us note that this innocent looking statement for a compact surface without boundary is very significant in the following sense. The Euler characteristic $\mathcal{X}(M)$ remains the same under diffeomorphic changes, where the Gaussian curvature K is not preserved under these diffeomorphisms (unless the metric is preserved, that is, unless the diffeomorphisms are isometries). The Gauss–Bonnet Theorem for compact surfaces without boundaries states that though K is not preserved under a diffeomorphism, the total Gaussian curvature $\int_M K dA$ is preserved. This formula has been generalized to higher dimensions by Chern [19] and also by Buzano and Nguyen in [20], and was later developed into an independent branch of differential topology.

Another important integral formula that is widely used is Stokes's theorem, which, for a compact Riemannian manifold (M, g) without boundary, states [18]

$$\int_M (\mathrm{div}\ X) dV_g = 0,$$

for a smooth vector field X on M, and dV_g is the volume form of M with respect to metric g. In particular, for a smooth function $f : M \to \mathbb{R}$, on taking $X = \nabla f$, the above integral formula takes the form

$$\int_M (\Delta f) dV_g = 0.$$

There are different variants of Stokes's formula that are used in differential geometry, namely, Minkowski's formula for closed hypersurfaces in Euclidean spaces as well as in spaces of constant curvature (cf. [21]). As a consequence of Minkowski's formula, we derive a global result, namely, that there does not exist a compact minimal hypersurface in a Euclidean space.

2. Preliminaries

Let (M^n, g, ξ, σ) be an n-dimensional almost Ricci soliton with associated tensor F. Then, on using the dual 1-form η of the soliton vector field ξ with Equation (1) and equation

$$2g(\nabla_X \xi, Y) = \left(\pounds_\xi g\right)(X, Y) + d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M^n),$$

where ∇ is the Riemannian connection, we obtain

$$\nabla_X \xi = \sigma X - QX + FX, \quad X \in \mathfrak{X}(M^n), \tag{6}$$

where Q is the symmetric $(1,1)$-tensor field called the *Ricci operator* defined by

$$g(QX, Y) = \mathrm{Ric}(X, Y), \quad X, Y \in \mathfrak{X}(M^n).$$

On denoting the scalar curvature by τ, and using Equation (6), we have

$$\mathrm{div}\, \xi = n\sigma - \tau. \tag{7}$$

Also, using

$$\mathrm{div}((n\sigma - \tau)\xi) = \xi(n\sigma - \tau) + (n\sigma - \tau)\,\mathrm{div}\, \xi$$

in the above equation, we deduce

$$\xi(n\sigma - \tau) = \mathrm{div}((n\sigma - \tau)\xi) - (n\sigma - \tau)^2. \tag{8}$$

On using the following expression of the Riemannian curvature tensor

$$R(X, Y)\xi = [\nabla_X, \nabla_Y]\xi - \nabla_{[X,Y]}\xi, \quad X, Y \in \mathfrak{X}(M^n)$$

together with Equation (6), we arrive at

$$\begin{aligned} R(X, Y)\xi = {} & X(\sigma)Y - Y(\sigma)X - (\nabla_X Q)(Y) + (\nabla_Y Q)(X) \\ & + (\nabla_X F)(Y) - (\nabla_Y F)(X), \quad X, Y \in \mathfrak{X}(M^n), \end{aligned}$$

where, for a $(1,1)$-tensor Ψ, the covariant derivative is given by

$$(\nabla_X \Psi)(Y) = \nabla_X \Psi Y - \Psi(\nabla_X Y), \quad X, Y \in \mathfrak{X}(M^n).$$

Now, using a local orthonormal frame $\{E_k\}_{k=1}^n$ and the expression of the Ricci curvature tensor

$$\mathrm{Ric}(X, Y) = \sum_{k=1}^{n} g(R(E_k, X)Y, E_k), \quad X, Y \in \mathfrak{X}(M^n)$$

together with Equation (8), we immediately confirm

$$\mathrm{Ric}(Y, \xi) = -(n-1)Y(\sigma) + \frac{1}{2}Y(\tau) - g\left(Y, \sum_{k=1}^{n} (\nabla_{E_k} F)(E_k)\right), \quad Y \in \mathfrak{X}(M^n), \tag{9}$$

where we use the symmetry of Q and the skew-symmetry of F together with the following relation (cf. [7,22])

$$\frac{1}{2}\nabla\tau = \sum_{k=1}^{n} (\nabla_{E_k} Q)(E_k). \tag{10}$$

Then, Equation (9) has the following implication:

$$Q\xi = -(n-1)\nabla\sigma + \frac{1}{2}\nabla\tau - \sum_{k=1}^{n} (\nabla_{E_k} F)(E_k). \tag{11}$$

We recall that, for a smooth function f on a Riemannian manifold (M^n, g), the Hessian operator H_f is defined by

$$H_f X = \nabla_X f, \quad X \in \mathfrak{X}(M^n),$$

which is a symmetric operator that is related to $\mathrm{Hess}(f)$ by

$$g\Big(H_f X, Y\Big) = \mathrm{Hess}(f)(X,Y), \quad X, Y \in \mathfrak{X}(M^n),$$

and we also have trace $H_f = \Delta f$.

3. Almost Ricci Solitons Isometric to a Sphere

Let (M^n, g, ξ, σ) be an n-dimensional compact almost Ricci soliton. In this section, we find conditions under which (M^n, g, ξ, σ) is isometric to $S^n(c)$. Indeed, we prove the following result.

Theorem 2. *An n-dimensional compact and connected almost Ricci soliton (M^n, g, ξ, σ), where $n > 2$, with scalar curvature τ and positive Ricci curvature, satisfies*

$$\int_{M^n} \left(\mathrm{Ric}(c\xi, c\xi) + \frac{n-1}{n}(\Delta\sigma)^2 \right) \le c \int_{M^n} \Big(2(n-1)\|\nabla\sigma\|^2 + \tau\Delta\sigma \Big),$$

for a nonzero constant c, if and only if $c > 0$ and (M^n, g, ξ, σ) is isometric to $S^n(c)$.

Proof. Let (M^n, g, ξ, σ) be an n-dimensional compact and connected almost Ricci soliton $n > 2$ with positive Ricci curvature that satisfies

$$\int_{M^n} \left(\mathrm{Ric}(c\xi, c\xi) + \frac{n-1}{n}(\Delta\sigma)^2 \right) \le c \int_{M^n} \Big(2(n-1)\|\nabla\sigma\|^2 + \tau\Delta\sigma \Big). \tag{12}$$

We intend to compute $\mathrm{div}\, F(\nabla\sigma)$; therefore, we choose a local orthonormal frame $\{E_k\}_{k=1}^{n}$ and proceed as follows:

$$\mathrm{div}\, F(\nabla\sigma) = \sum_{k=1}^{n} g(\nabla_{E_k} F(\nabla\sigma), E_k) = \sum_{k=1}^{n} g\big((\nabla_{E_k} F)(\nabla\sigma) + F(H_\sigma E_k), E_k\big). \tag{13}$$

The Hessian operator H_σ is symmetric and F is skew-symmetric; therefore,

$$\sum_{k=1}^{n} g(H_\sigma E_k, F E_k) = 0.$$

Using the above equation as well as Equation (11) in Equation (13), we are led to

$$\begin{aligned}\operatorname{div} F(\nabla\sigma) &= -\sum_{k=1}^{n} g\big(\nabla\sigma, (\nabla_{E_k}F)(E_k)\big) \\ &= -g\left(\nabla\sigma, -(n-1)\nabla\sigma + \frac{1}{2}\nabla\tau - Q\xi\right) \\ &= \operatorname{Ric}(\nabla\sigma,\xi) + (n-1)\|\nabla\sigma\|^2 - \frac{1}{2}g(\nabla\sigma,\nabla\tau).\end{aligned}$$

Integrating the above equation leads to

$$\int_{M^n}\left(\operatorname{Ric}(\nabla\sigma,\xi) + (n-1)\|\nabla\sigma\|^2 - \frac{1}{2}g(\nabla\sigma,\nabla\tau)\right) = 0.$$

Next, we use

$$\operatorname{div}(\tau\nabla\sigma) = g(\nabla\tau,\nabla\sigma) + \tau\operatorname{div}(\nabla\sigma) = g(\nabla\tau,\nabla\sigma) + \tau\Delta\sigma$$

in the above equation, and we obtain

$$\int_{M^n}\left(\operatorname{Ric}(\nabla\sigma,\xi) + (n-1)\|\nabla\sigma\|^2 + \frac{\tau}{2}\Delta\sigma\right) = 0,$$

that is,

$$\int_{M^n}\operatorname{Ric}(\nabla\sigma,\xi) = -\int_{M^n}\left((n-1)\|\nabla\sigma\|^2 + \frac{\tau}{2}\Delta\sigma\right). \tag{14}$$

We also have the following Bochner's formula:

$$\int_{M^n}\operatorname{Ric}(\nabla\sigma,\nabla\sigma) = \int_{M^n}\left((\Delta\sigma)^2 - \|H_\sigma\|^2\right). \tag{15}$$

Now, for a nonzero constant c, we have

$$\operatorname{Ric}(\nabla\sigma + c\xi, \nabla\sigma + c\xi) = \operatorname{Ric}(\nabla\sigma,\nabla\sigma) + 2c\operatorname{Ric}(\nabla\sigma,\xi) + \operatorname{Ric}(c\xi,c\xi),$$

which, on integration and using Equations (14) and (15), leads to

$$\int_{M^n}\operatorname{Ric}(\nabla\sigma + c\xi,\nabla\sigma + c\xi) = \int_{M^n}\left((\Delta\sigma)^2 - \|H_\sigma\|^2 - 2(n-1)c\|\nabla\sigma\|^2 - c\tau\Delta\sigma + \operatorname{Ric}(c\xi,c\xi)\right).$$

Rearranging the above equation, we have

$$\begin{aligned}\int_{M^n}\operatorname{Ric}(\nabla\sigma + c\xi,\nabla\sigma + c\xi) + \int_{M^n}\left(\|H_\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2\right) \\ = \int_{M^n}\left(\frac{n-1}{n}(\Delta\sigma)^2 - 2(n-1)c\|\nabla\sigma\|^2 - c\tau\Delta\sigma + \operatorname{Ric}(c\xi,c\xi)\right),\end{aligned}$$

that is,

$$\begin{aligned}\int_{M^n}\operatorname{Ric}(\nabla\sigma + c\xi,\nabla\sigma + c\xi) + \int_{M^n}\left(\|H_\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2\right) \\ = \int_{M^n}\left(\operatorname{Ric}(c\xi,c\xi) + \frac{n-1}{n}(\Delta\sigma)^2\right) \\ - c\int_{M^n}\left(2(n-1)\|\nabla\sigma\|^2 + \tau\Delta\sigma\right).\end{aligned}$$

Due to inequality (12), the right-hand side of the above equation is nonpositive, whereas the almost Ricci soliton has positive Ricci curvature and the second integrand on the left-hand side of the above equation is non-negative owing to the Schwartz's inequality

$$\|H_\sigma\|^2 \geq \frac{1}{n}(\Delta\sigma)^2, \tag{16}$$

and consequently, we conclude

$$\nabla\sigma + c\xi = 0 \tag{17}$$

and the inequality in (16) becomes the equality $\|H_\sigma\|^2 = \frac{1}{n}(\Delta\sigma)^2$. The holding of this inequality requires

$$H_\sigma = \frac{\Delta\sigma}{n} I. \tag{18}$$

Now, differentiating Equation (17) and using Equation (6), we reach

$$H_\sigma X + c(\sigma X - QX + FX) = 0, \quad X \in \mathfrak{X}(M^n),$$

that is,

$$H_\sigma X + c(\sigma X - QX) = -cFX, \quad X \in \mathfrak{X}(M^n).$$

Since the constant $c \neq 0$ and the left-hand side is symmetric, while the right-hand side is skew-symmetric, we immediately conclude $F = 0$ and

$$H_\sigma X = c(QX - \sigma X), \quad X \in \mathfrak{X}(M^n). \tag{19}$$

Also, taking the divergence in Equation (17), while using Equation (7), we conclude

$$\Delta\sigma = -c(n\sigma - \tau),$$

and inserting it in Equation (18) leads to

$$H_\sigma = \left(-c\sigma + \frac{c\tau}{n}\right) I. \tag{20}$$

Combining Equations (19) and (20), it infers

$$Q = \frac{\tau}{n} I,$$

that is,

$$(\nabla_X Q)(Y) = \frac{1}{n} X(\tau) Y, \quad X, Y \in \mathfrak{X}(M^n),$$

and summing the above equation over a local orthonormal frame $\{E_k\}_{k=1}^n$ gives

$$\sum_{k=1}^{n} (\nabla_{E_k} Q)(E_k) = \frac{1}{n} \nabla\tau.$$

Now, using Equation (10), we have

$$\frac{1}{2}\nabla\tau = \frac{1}{n}\nabla\tau,$$

and, as $n > 2$, we conclude that τ is a constant. We note that the almost Ricci soliton (M^n, g, ξ, σ) is nontrivial; therefore, the potential function σ is nonconstant. From Equation (20), we have

$$H_\sigma = -c\left(\sigma - \frac{\tau}{n}\right) I. \tag{21}$$

We define a function $\rho = \sigma - \frac{\tau}{n}$, which is a nonconstant function (as σ is), and we have $\nabla\rho = \nabla\sigma$; therefore, $H_\rho = H_\sigma$. Consequently, Equation (21) takes the form

$$H_\rho = -c\rho I. \tag{22}$$

Taking the trace in the above equation yields $\Delta\rho = -nc\rho$, that is, $\rho\Delta\rho = -nc\rho^2$, which, on integration by parts, leads to

$$\int_{M^n} \|\nabla\rho\|^2 = nc\int_{M^n} \rho^2,$$

and, as ρ is nonconstant, through the above equation, we conclude $c > 0$. This confirms that Equation (22) is Obata's differential equation (cf. [22–24]) and, therefore, (M^n, g, ξ, σ) is isometric to $S^n(c)$.

Conversely, suppose that (M^n, g, ξ, σ) is isometric to $S^n(c)$. Then, through Equation (3), it follows that $(S^n(c), g, \xi, \sigma)$ is a nontrivial almost Ricci soliton, where the potential function is $\sigma = c(-f + n - 1)$, f is an eigenfunction, $\Delta f = -ncf$, and the soliton vector field is $\xi = \nabla f$. We have

$$\int_{S^n(c)} \|\nabla f\|^2 = nc\int_{S^n(c)} f^2 \tag{23}$$

and $\mathrm{Ric}(c\xi, c\xi) = (n-1)c^3\|\xi\|^2 = (n-1)c^3\|\nabla f\|^2$. Moreover, $\nabla\sigma = -c\nabla f$ and we have $\Delta\sigma = -c\Delta f = nc^2 f$. Thus, we obtain

$$\int_{S^n(c)} \left(\mathrm{Ric}(c\xi, c\xi) + \frac{n-1}{n}(\Delta\sigma)^2\right) = \int_{S^n(c)} \left((n-1)c^3\|\nabla f\|^2 + n(n-1)c^4 f^2\right),$$

and using Equation (23), we find

$$\int_{S^n(c)} \left(\mathrm{Ric}(c\xi, c\xi) + \frac{n-1}{n}(\Delta\sigma)^2\right) = 2n(n-1)c^4\int_{S^n(c)} f^2. \tag{24}$$

Also, as $\tau = n(n-1)c$ is a constant, we have

$$\int_{S^n(c)} \tau\Delta\sigma = 0,$$

and using $\nabla\sigma = -c\nabla f$, we obtain

$$c\int_{S^n(c)} \left(2(n-1)\|\nabla\sigma\|^2 + \tau\Delta\sigma\right) = 2(n-1)c\int_{S^n(c)} \|\nabla\sigma\|^2 = 2(n-1)c^3\int_{S^n(c)} \|\nabla f\|^2,$$

which, in view of Equation (23), yields

$$c\int_{S^n(c)} \left(2(n-1)\|\nabla\sigma\|^2 + \tau\Delta\sigma\right) = 2n(n-1)c^4\int_{S^n(c)} f^2. \tag{25}$$

Thus, Equations (24) and (25) confirm that one of the given conditions is satisfied. Also, the almost Ricci soliton $(S^n(c), g, \xi, \sigma)$ has positive Ricci curvature; therefore, the second of the given conditions is also satisfied. Hence, the converse implication holds. □

Next, we are interested in finding conditions under which an almost Ricci soliton (M^n, g, ξ, σ) is trivial, that is, it is a Ricci soliton, and to achieve this, we will ask if the affinity tensor $\pounds_\xi\nabla$ of the soliton vector field ξ vanishes.We note that the affinity tensor of a vector field is a useful tool in studying the geometry of a Riemannian manifold (cf. [25,26]). We recall that the affinity tensor of a vector field ξ is given by (cf. [25], p. 109)

$$(\pounds_\xi\nabla)(X, Y) = \pounds_\xi\nabla_X Y - \pounds_{\nabla_\xi X} Y - \nabla_X \pounds_\xi Y, \quad X, Y \in \mathfrak{X}(M^n),$$

which is equivalent to

$$(\pounds_\xi \nabla)(X,Y) = R(\xi,X)Y + \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi, \quad X,Y \in \mathfrak{X}(M^n). \tag{26}$$

We shall impose the condition that the affinity tensor $\pounds_\xi \nabla$ of the soliton vector field ξ vanishes and some additional inequality for the scalar curvature holds for a compact almost Ricci soliton (M^n, g, ξ, σ) to force it to not only be a trivial almost Ricci soliton but a trivial Ricci soliton. Indeed, we prove the following result.

Theorem 3. *Let (M^n, g, ξ, σ) be an n-dimensional compact and connected almost Ricci soliton, $n > 2$, with scalar curvature τ, vanishing the affinity tensor of the soliton vector field ξ, satisfying the inequality*

$$\tau(n\sigma - \tau) \geq 0.$$

Then, (M^n, g, ξ, σ) is a trivial Ricci soliton.

Proof. Suppose that the affinity tensor of the soliton vector field ξ vanishes. Then, using Equations (6) and (26), we conclude

$$R(\xi,X)Y + X(\sigma)Y - (\nabla_X Q)(Y) + (\nabla_X F)(Y) = 0, \quad X,Y \in \mathfrak{X}(M^n). \tag{27}$$

For a local orthonormal frame $\{E_k\}_{k=1}^n$, we have

$$Q\xi = \sum_{k=1}^n R(\xi, E_k)E_k,$$

and thus, summing Equation (27) and using Equation (10), it gives

$$Q\xi + \nabla\sigma - \frac{1}{2}\nabla\tau + \sum_{k=1}^n (\nabla_{E_k} F)(E_k) = 0. \tag{28}$$

Now, combining Equations (11) and (28), we arrive at

$$(n-2)\nabla\sigma = 0,$$

and that, by virtue of the fact $n > 2$, makes σ a constant; therefore, at this stage, (M^n, g, ξ, σ) is a Ricci soliton. Next, we proceed to show that, with the remaining condition $\tau(n\sigma - \tau) \geq 0$, the soliton vector field ξ is a Killing vector field. It is trivial to check that

$$\|Q\|^2 - \frac{1}{n}\tau^2 = \left|\mathrm{Ric} - \frac{\tau}{n} g\right|^2, \tag{29}$$

and we have the following equality (cf. [2,12])

$$\int_{M^n} \left|\mathrm{Ric} - \frac{\tau}{n} g\right|^2 = \frac{n-2}{2n}\int_{M^n} \xi(\tau). \tag{30}$$

We use a local orthonormal frame $\{E_k\}_{k=1}^n$ and compute $\operatorname{div} Q\xi$ with the aid of Equation (6) as follows:

$$\begin{aligned}
\operatorname{div} Q\xi &= \sum_{k=1}^n g(\nabla_{E_k} Q\xi, E_k) \\
&= \sum_{k=1}^n g((\nabla_{E_k} Q)(\xi) + Q(\nabla_{E_k}\xi), E_k) \\
&= g\left(\xi, \sum_{k=1}^n (\nabla_{E_k} Q)(E_k)\right) + \sum_{k=1}^n g(\sigma E_k - QE_k + FE_k, QE_k),
\end{aligned}$$

which, in view of Equation (10), the symmetry of Q and the skew-symmetry of F, yields

$$\operatorname{div} Q\xi = \frac{1}{2}\xi(\tau) + \sigma\tau - \|Q\|^2.$$

On integrating the above equation, we arrive at

$$\int_{M^n} \left(\|Q\|^2 - \frac{1}{n}\tau^2 \right) = \frac{1}{2}\int_{M^n} \xi(\tau) + \int_{M^n} \left(\sigma\tau - \frac{1}{n}\tau^2 \right).$$

Inserting from Equations (29) and (30) into the above equation, while using $n > 2$, it leads to

$$\int_{M^n} \left| \mathrm{Ric} - \frac{\tau}{n} g \right|^2 = \frac{n}{n-2} \int_{M^n} \left| \mathrm{Ric} - \frac{\tau}{n} g \right|^2 + \frac{1}{n} \int_{M^n} \tau(n\sigma - \tau),$$

that is,

$$\frac{-2}{n-2} \int_{M^n} \left| \mathrm{Ric} - \frac{\tau}{n} g \right|^2 = \frac{1}{n} \int_{M^n} \tau(n\sigma - \tau).$$

The above equation and the condition in the hypothesis $\tau(n\sigma - \tau) \geq 0$ forces the conclusion

$$\mathrm{Ric} = \frac{\tau}{n} g. \tag{31}$$

Since $n > 2$, the above equation implies that τ is a constant. We have already proved that σ is a constant, and feeding this information in Equation (8), we have

$$\operatorname{div}((n\sigma - \tau)\xi) = (n\sigma - \tau)^2,$$

which, on integration, gives $n\sigma = \tau$, and Equation (31) in turn gives

$$\mathrm{Ric} = \sigma g,$$

that is, $\pounds_\xi g = 0$. Hence, (M^n, g, ξ, σ) is a trivial Ricci soliton. □

4. Hodge Decomposition of Soliton Vector Field and Applications

It is known that on an n-dimensional compact almost Ricci soliton (M^n, g, ξ, σ), the soliton vector field ξ admits a Hodge decomposition (cf. [27])

$$\xi = \overline{\xi} + \nabla h, \tag{32}$$

where $\operatorname{div} \overline{\xi} = 0$ and ∇h is the gradient of a smooth function h. We call the vector field $\overline{\xi}$ the *Hodge potential field* and the function h the *Hodge function* of the soliton vector field ξ. Then, using Equations (6) and (7), it follows that

$$\nabla_X \overline{\xi} = \sigma X - QX + FX - H_h X, \quad X \in \mathfrak{X}(M^n), \tag{33}$$

and we obtain the Poisson equation

$$\Delta h = n\sigma - \tau. \tag{34}$$

Using Equation (33), we have

$$\left(\pounds_{\overline{\xi}} g \right)(X, Y) = 2\sigma g(X, Y) - 2\,\mathrm{Ric}(X, Y) - 2\,\mathrm{Hess}(h)(X, Y), \quad X, Y \in \mathfrak{X}(M^n),$$

which, on using a local orthonormal frame $\{E_k\}_{k=1}^n$, gives

$$\left| \pounds_{\overline{\xi}} g \right|^2 = 4\left(n\sigma^2 + \|Q\|^2 + \|H_h\|^2 - 2\sigma\tau - 2\sigma\Delta h + 2\sum_{k=1}^{n} g(QE_k, H_h E_k) \right). \tag{35}$$

Also, using Equation (33), the symmetry of Q and H_h, and the skew-symmetry of F, we obtain

$$\left\|\nabla\overline{\xi}\right\|^2 = n\sigma^2 + \|Q\|^2 + \|F\|^2 + \|H_h\|^2 - 2\sigma\tau - 2\sigma\Delta h + 2\sum_{k=1}^{n} g(QE_k, H_hE_k). \tag{36}$$

Now, we shall use the bound on the integral of $\mathrm{Ric}\left(\overline{\xi},\overline{\xi}\right)$ and an inequality satisfied by the scalar curvature τ to find the following characterization of the sphere $S^n(c)$.

Theorem 4. *An n-dimensional compact and connected nontrivial almost Ricci soliton (M^n, g, ξ, σ), where $n > 2$, with associated tensor F, scalar curvature τ, and Hodge decomposition $\xi = \overline{\xi} + \nabla h$, satisfies*

$$\int_{M^n} \mathrm{Ric}\left(\overline{\xi},\overline{\xi}\right) \geq \int_{M^n} \|F\|^2 \quad \textit{and} \quad \int_{M^n} \tau(n\sigma - \tau) \geq 0,$$

if and only if (M^n, g, ξ, σ) is isometric to $S^n(c)$, where $c > 0$.

Proof. Suppose that (M^n, g, ξ, σ) is an n-dimensional compact and connected nontrivial almost Ricci soliton, where $n > 2$, with the Hodge decomposition $\xi = \overline{\xi} + \nabla h$, with $\mathrm{Ric}\left(\overline{\xi},\overline{\xi}\right)$ and τ, satisfying

$$\int_{M^n} \mathrm{Ric}\left(\overline{\xi},\overline{\xi}\right) \geq \int_{M^n} \|F\|^2 \tag{37}$$

and

$$\int_{M^n} \tau(n\sigma - \tau) \geq 0. \tag{38}$$

Since $\mathrm{div}\,\overline{\xi} = 0$, using the integral formula (cf. [28])

$$\int_{M^n} \left(\mathrm{Ric}\left(\overline{\xi},\overline{\xi}\right) + \frac{1}{2}\left|\pounds_{\overline{\xi}}g\right|^2 - \left\|\nabla\overline{\xi}\right\|^2\right) = 0$$

and Equations (35) and (36), we obtain

$$\int_{M^n} \left(2\sigma\tau + 2\sigma\Delta h - 2\sum_{k=1}^{n} g(QE_k, H_hE_k)\right) = \int_{M^n} \left(\mathrm{Ric}\left(\overline{\xi},\overline{\xi}\right) - \|F\|^2\right) + \int_{M^n} \left(n\sigma^2 + \|Q\|^2 + \|H_h\|^2\right). \tag{39}$$

On using Equation (10), we compute

$$\begin{aligned}
\mathrm{div}\, Q(\nabla h) &= \sum_{k=1}^{n} g\left(\nabla_{E_k} Q(\nabla h), E_k\right) \\
&= \sum_{k=1}^{n} g\left(\left(\nabla_{E_k} Q\right)(\nabla h) + Q(H_hE_k), E_k\right) \\
&= \frac{1}{2} g(\nabla h, \nabla\tau) + \sum_{k=1}^{n} g(QE_k, H_hE_k) \\
&= \frac{1}{2}(\mathrm{div}(\tau\nabla h) - \tau\Delta h) + \sum_{k=1}^{n} g(QE_k, H_hE_k),
\end{aligned}$$

that is,

$$\sum_{k=1}^{n} g(QE_k, H_hE_k) = \mathrm{div}\, Q(\nabla h) - \frac{1}{2}\,\mathrm{div}(\tau\nabla h) + \frac{1}{2}\tau\Delta h.$$

Inserting this equation in Equation (39) yields

$$\int_{M^n}(2\sigma\tau + 2\sigma\Delta h - \tau\Delta h) = \int_{M^n}\left(\mathrm{Ric}(\overline{\xi},\overline{\xi}) - \|F\|^2\right) + \int_{M^n}\left(n\sigma^2 + \|Q\|^2 + \|H_h\|^2\right). \quad (40)$$

We note that

$$n\sigma^2 + \|Q\|^2 + \|H_h\|^2 = \left(\|Q\|^2 - \frac{1}{n}\tau^2\right) + \left(\|H_h\|^2 - \frac{1}{n}(\Delta h)^2\right) + n\sigma^2 + \frac{1}{n}\left(\tau^2 + (\Delta h)^2\right).$$

Using Equation (34) in the above equation, we have

$$\int_{M^n}\left(n\sigma^2 + \|Q\|^2 + \|H_h\|^2\right) = \int_{M^n}\left(\|Q\|^2 - \frac{1}{n}\tau^2\right) + \int_{M^n}\left(\|H_h\|^2 - \frac{1}{n}(\Delta h)^2\right) + \int_{M^n}\left(2n\sigma^2 + \frac{2}{n}\tau^2 - 2\sigma\tau\right). \quad (41)$$

Also, on using Equation (34), we compute

$$2\sigma\tau + 2\sigma\Delta h - \tau\Delta h = 2n\sigma^2 - n\sigma\tau + \tau^2,$$

that is,

$$\int_{M^n}(2\sigma\tau + 2\sigma\Delta h - \tau\Delta h) = \int_{M^n}\left(2n\sigma^2 - n\sigma\tau + \tau^2\right). \quad (42)$$

Using inequality (37) in Equation (40), we conclude

$$\int_{M^n}(2\sigma\tau + 2\sigma\Delta h - \tau\Delta h) \geq \int_{M^n}\left(n\sigma^2 + \|Q\|^2 + \|H_h\|^2\right).$$

Inserting Equations (41) and (42) in the above inequality, we have

$$-\frac{n-2}{n}\int_{M^n}\tau(n\sigma - \tau) \geq \int_{M^n}\left(\|Q\|^2 - \frac{1}{n}\tau^2\right) + \int_{M^n}\left(\|H_h\|^2 - \frac{1}{n}(\Delta h)^2\right). \quad (43)$$

Since the integrands in the right-hand side of the above inequality are non-negative by virtue of the Schwartz's inequalities

$$\|Q\|^2 \geq \frac{1}{n}\tau^2 \text{ and } \|H_h\|^2 \geq \frac{1}{n}(\Delta h)^2$$

and the left-hand side is nonpositive owing to inequality (38), by inequality (43), we conclude

$$\|Q\|^2 = \frac{1}{n}\tau^2 \text{ and } \|H_h\|^2 = \frac{1}{n}(\Delta h)^2$$

and these equalities in Schwartz's inequalities hold if and only if

$$Q = \frac{\tau}{n}I \text{ and } H_h = \frac{\Delta h}{n}I. \quad (44)$$

As $n > 2$, the first equation in (44), by virtue of Equation (10), yields that τ is a constant. Moreover, the second equation in (44) and Equation (34) allows

$$H_h = \left(\sigma - \frac{\tau}{n}\right)I, \quad (45)$$

which, in view of the fact that τ is a constant, yields

$$(\nabla_X H_h)(Y) = X(\sigma)Y, \quad X, Y \in \mathfrak{X}(M^n) \quad (46)$$

and using the identity

$$R(X,Y)\nabla h = (\nabla_X H_h)(Y) - (\nabla_Y H_h)(X), \quad X,Y \in \mathfrak{X}(M^n)$$

with Equation (46), we arrive at

$$R(X,Y)\nabla h = X(\sigma)Y - Y(\sigma)X, \quad X,Y \in \mathfrak{X}(M^n).$$

On taking the trace in the above equation, we conclude

$$\text{Ric}(Y,\nabla h) = -(n-1)Y(\sigma), \quad Y \in \mathfrak{X}(M^n),$$

that is,

$$Q\nabla h = -(n-1)\nabla\sigma,$$

and combining it with the first equation in (44) yields

$$\nabla\sigma = -\frac{\tau}{n(n-1)}\nabla h.$$

Choosing the constant c such that $\tau = n(n-1)c$, we have $\nabla\sigma = -c\nabla h$. Differentiating the last equation gives

$$H_\sigma X = -cH_h X, \quad X \in \mathfrak{X}(M^n),$$

and combining it with Equation (45), we conclude

$$H_\sigma X = -c\left(\sigma - \frac{\tau}{n}\right)X, \quad X \in \mathfrak{X}(M^n). \tag{47}$$

Since (M^n, g, ξ, σ) is a nontrivial almost Ricci soliton, the potential function σ is nonconstant. Now, we define $\rho = \sigma - \frac{\tau}{n}$, which is a nonconstant function, too, and we have $\nabla\rho = \nabla\sigma$, and $H_\rho = H_\sigma$, and Equation (47) becomes

$$H_\rho X = -c\rho X, \quad X \in \mathfrak{X}(M^n). \tag{48}$$

Taking the trace in the above equation, we conclude $\Delta\rho = -nc\rho$, that is, $\rho\Delta\rho = -nc\rho^2$. Integrating by parts the last equation, yields

$$\int_{M^n} \|\nabla\rho\|^2 = nc\int_{M^n}\rho^2,$$

which proves that c is a positive constant. Thus, by Equation (48), we obtain that (M^n, g, ξ, σ) is isometric to $S^n(c)$ (cf. [22–24]).

Conversely, suppose that (M^n, g, ξ, σ) is isometric to $S^n(c)$. Then, we see that for the nontrivial almost Ricci soliton $(S^n(c), g, \xi, \sigma)$, the soliton vector field $\xi = \nabla f$ has the Hodge decomposition $\xi = \overline{\xi} + \nabla f$, where the Hodge potential field is $\overline{\xi} = 0$. Moreover, the soliton vector field ξ being closed, we have $F = 0$. Hence, the condition

$$\int_{S^n(c)} \text{Ric}(\overline{\xi}, \overline{\xi}) = \int_{S^n(c)} \|F\|^2$$

holds. Also, $\tau = n(n-1)c$ and $\sigma = c(-f + n - 1)$, and we have $n\sigma - \tau = -ncf = \Delta f$. Then

$$\int_{S^n(c)} \tau(n\sigma - \tau) = n(n-1)c\int_{S^n(c)} \Delta f = 0.$$

Hence, all the conditions are met for the converse implication to hold. □

5. Conclusions

In Section 4, we have considered the Hodge decomposition of the soliton vector field ξ of an n-dimensional compact almost Ricci soliton (M^n, g, ξ, σ) given by Equation (32), and, in Theorem 4, we have utilized the Hodge potential field $\bar{\xi}$ to find a characterization of the sphere $S^n(c)$. A natural component of the Hodge decomposition is the Hodge function h, which satisfies the Poisson Equation (34), namely

$$\Delta h = n\sigma - \tau. \tag{49}$$

As we have seen that the Hodge potential field is used to find a characterization of the sphere $S^n(c)$, one would be interested in seeing the role of the Hodge function h. It is known that the Poisson equation plays a crucial role in constraining the geometry of (M^n, g, ξ, σ) (cf. [10]); in particular, it is known that, on a compact almost Ricci soliton (M^n, g, ξ, σ), the solution of the above Poisson equation is unique up to a constant, that is, two solutions differ by a constant. Natural functions defined on (M^n, g, ξ, σ) other than h are the potential function σ and the scalar curvature τ. It will be worth seeking the following questions:

(i) Assuming, for a positive constant c, that $\frac{1}{c}\sigma$ is a solution of the Poisson Equation (49), is a compact nontrivial almost Ricci soliton (M^n, g, ξ, σ) isometric to the sphere $S^n(c)$?

(ii) Assuming that the scalar curvature τ is a solution of the Poisson Equation (49), then, in this case, the compact nontrivial almost Ricci soliton (M^n, g, ξ, σ) will not be isometric to the sphere $S^n(c)$, as this will require that τ is a constant whereas σ is not a constant. So, as a point of interest, what will be the impact of the scalar curvature τ being a solution of the Poisson Equation (49) on the geometry of almost Ricci soliton (M^n, g, ξ, σ)?

These two questions are worth exploring, and the answers to them will add new insights to the geometry of the almost Ricci soliton.

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