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Fuzzy Assessment Mechanisms under Multi-Objective Considerations

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Abstract: In many operational environments, it is essential to conduct comprehensive minimal assessments of the effects arising from various operational causes, with the goal of achieving effective outcomes. For instance, the aim might be to meet basic production targets in the shortest time possible, using the least cost and minimal labor. Given that actual operational behaviors are often vague and unpredictable, this study proposes a mechanism to assess the minimal effects generated by various operational causes under multi-objective and fuzzy behavior considerations. By considering the relative significance of operational causes or its behaviors under different environments, several weighted extensions are further developed. The mathematical correctness and practical applicability of these assessment mechanisms are analyzed by using an axiomatic characterization.

Keywords: multi-objective; fuzzy behavior; assessment mechanism; weighted extension; axiomatic characterization

MSC: 91A12; 91A40; 91B06; 91B16

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1. Introduction

Recently, effect-related issues have received heightened attention due to drastic environmental change, depletion of available utility, and other factors, leading to the emergence of related studies on subjects such as cost control, toxin mitigation, and pollution suppression. The effect inflicted on operational environments by inefficient assessments has become an undeniable fact, with some effects even irreversible. Related operational causes have also become crucial aspects of effect-related research. Assessing effects typically requires simultaneously focusing on multiple objectives, which might sometimes conflict. For example, a factory producing eco-friendly tableware must adhere to carbon emission standards aligned with sustainability goals. It needs to consider how to achieve quality production targets within the shortest time frame while minimizing costs, equipment wear, and pollution emissions. However, these considerations can sometimes conflict, necessitating an effective assessment to achieve equilibrium. In the field of mathematics, multi-objective optimization or game-theoretical analysis aims to achieve such outcomes within any operational environment. Related studies can be found in Peldschus [1], Atanassov et al. [2], Bednarczuk et al. [3], Goli et al. [4], Tirkolaee et al. [5], Cheng et al. [6], Wu et al. [7], and so on.

In real-world situations, most operational environments are composed of numerous operational causes, and the utilities derived from these causes are closely interrelated, often influencing one another, thereby forming *transferable-utility* (TU) conditions. For instance, in a company that manufactures eco-friendly tableware, hiring decisions made by the human resources department impact the production efficiency in the manufacturing department, while production efficiency and product quality, in turn, affect the operational performance of the sales department. Under standard TU conditions, each operational cause is typically categorized as either fully active or entirely uninvolved in other operational causes. However, in most cases, the behavior associated with these operational causes is ambiguous and difficult to assess. For example, an employee's work performance may fluctuate due to emotions, health issues, or other factors. Such performance cannot be accurately evaluated in terms of mere participation or non-participation; rather, it follows a continuously variable, fuzzy pattern. Therefore, many operational environments can be modeled as *fuzzy* TU *situations*. Based on fuzzy TU situations (Aubin [8,9]), operational causes are allowed to engage with infinite variations in energy behavior. Various assessment mechanisms for fuzzy TU situations have been widely applied across numerous fields, as explored in results from Branzei et al. [10], Nishizaki and Sakawa [11], Muto et al. [12], Hwang [13], Li and Zhang [14], Meng and Zhang [15], Khorram et al. [16], Borkotokey and Mesiar [17], Hwang and Liao [18], Masuya and Inuiguchi [19], Liao and Chung [20], Debnath and Mohiuddine [21], among others.

Consistency is an important characteristic within axiomatic concepts for traditional assessment mechanisms. It claims that an outcome remains independent if certain operational causes are fixed with its assigned effects. In other words, this property indicates that outcomes generated for any issue should align with those generated for sub-issues where the effects of particular operational causes are evaluated. Consistency has been introduced via different concepts relying upon how the effects of operational causes that "exit the bargaining" are considered. This property has been extensively applied across diverse subjects through the relative notion of *reduced situations*, including bargaining and cost evaluation processes. Applying the marginal concept, the equal allocation of non-separable costs (EANSC, Ransmeier [22]) and the normalized index have been proposed, respectively, for assessing utility under traditional TU situations. Moulin [23] utilized the concept of complement reduction to illustrate that the EANSC could generate an impartial mechanism for assessing utility.

The findings stated above prompt a key inquiry:

- Could the marginal concept and its associated results be expanded to assess effects by simultaneously considering multiple objectives and fuzzy TU situations?
- Could the reduction and related consistency be extended to characterize assessment mechanisms under multi-objective fuzzy TU considerations?

Based on the above motivation, this study aims to establish the necessary game-theoretical foundations to analyze assessment issues with multiple objectives and a fuzzy behavior simultaneously.

1. Departing from the contexts of standard and fuzzy TU situations, this study considers the framework of *multi-objective fuzzy TU situations*. Two new assessments are also considered in Section 2: the *minimized equal assessment of inalienable effects* (MAIE) and the *normalized MAIE*. In short, the MAIE involves operational causes assessing minimized individual effects and then equally assessing the remaining effects. Conversely, the normalized MAIE assesses the entire effect proportionally by applying the relative minimized individual effects of all operational causes. Based on fuzzy multi-objective considerations, these two assessments generalize the concept of the minimized individual effects.
2. To analyze these two assessments by axiomatic processes, an extended reduction and related characterizations are introduced in Sections 3 and 4:
    - The MAIE is the unique assessment satisfying the properties of *multi-objective standardness for fuzzy situations* and *multi-objective bilateral consistency*.
    - The MAIE is the unique assessment satisfying the properties of *multi-objective completeness*, *multi-objective covariance*, *multi-objective symmetric equality*, and *multi-objective bilateral consistency*.
    - Since the normalized MAIE violates multi-objective bilateral consistency, it adheres to the properties of *normalized standardness for fuzzy situations* and *modified bilateral consistency*.
3. Under practical situations, assessments may emerge as unrealistic due to varying operational sizes or bargaining abilities. Moreover, asymmetry may emerge when

modeling distinct bargaining abilities among operational causes and their energy behavior. To address these conditions, this study introduce a *weight function for operational causes* and a *weight function for energy behavior* and propose different assessments in Section 5 whereby an arbitrary extra fixed effect is assessed among operational causes and their energy behavior proportionally to relative weights.

Throughout the study, some more discussions and interpretations regarding these axiomatic properties and related characterizations are provided to further elucidate its implications.

2. Preliminaries

2.1. Definitions and Notations

Let $\mathbb{P}$ denote the universe of operational causes. Any $i \in \mathbb{P}$ is identified as an operational cause of $\mathbb{P}$. For each operational cause $i \in \mathbb{P}$ and $e_i \in (0,1]$, we define $E_i = [0, e_i]$ as the energy behavior space of operational cause i, with $E_i^+ = (0, e_i]$ indicating active participation, and 0 indicating non-participation.

Consider $P \subseteq \mathbb{P}$ as the largest set encompassing all operational causes of an operational environment within $\mathbb{P}$. Let $E^P = \prod_{i \in P} E_i$ denote the Cartesian product set of energy behavior spaces for operational causes in P. Let 0_P represent the zero vector in $\mathbb{R}^P$. For $m \in \mathbb{N}$, 0_m denotes the zero vector in $\mathbb{R}^m$, and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **fuzzy transferable-utility (TU) situation** is characterized as a triple (P, e, d), where P denotes a non-empty and finite set of operational causes, $e = (e_i)_{i \in P} \in (0,1]^P$ represents the vector indicating the highest energy levels for each operational cause $i \in P$, and $d : E^P \to \mathbb{R}$ is a function satisfying $d(0_P) = 0$ that presents the utility generated by corresponding energy behavior vector $\eta = (\eta_i)_{i \in P} \in E^P$ if each $i \in P$ engages with energy behavior η_i. A **multi-objective fuzzy TU situation** is defined as a triple (P, e, D^m), where $m \in \mathbb{N}$, $D^m = (d^t)_{t \in \mathbb{N}_m}$, and (P, e, d^t) represents a fuzzy TU situation for all $t \in \mathbb{N}_m$. The class encompassing all multi-objective fuzzy TU situations is denoted as $\mathbb{MF}$.

An **assessment** is defined as a mapping τ that assigns to each $(P, e, D^m) \in \mathbb{MF}$ an element

$$\tau(P, e, D^m) = \left(\tau^t(P, e, D^m)\right)_{t \in \mathbb{N}_m},$$

where $\tau^t(P, e, D^m) = \left(\tau_i^t(P, e, D^m)\right)_{i \in P} \in \mathbb{R}^P$, and $\tau_i^t(P, e, D^m)$ represents the output of operational cause i when i engages with (P, e, d^t).

In order to define new assessments under multi-objective fuzzy considerations, some more notations are needed. For $(P, e, D^m) \in \mathbb{MF}$, $H \subseteq P$, and $\eta \in \mathbb{R}^P$, $NE(\eta) = \{i \in P | \eta_i \neq 0\}$ is defined to denote the set of operational causes with non-zero energy behavior, and $\eta_H \in \mathbb{R}^H$ represents the restriction of η to H. For a given $i \in P$, the notation η_{-i} is introduced to denote $\eta_{P \setminus \{i\}}$, and $\mu = (\eta_{-i}, t) \in \mathbb{R}^P$ is defined by $\mu_{-i} = \eta_{-i}$ and $\mu_i = t$. Next, we provide two generalized assessments under multi-objective fuzzy considerations.

Definition 1.

1. *The* **minimizing assessment of inalienable effects (MAIE)**, $\overline{\Delta}$, *is the assessment on* $\mathbb{MF}$ *which is defined by*

$$\overline{\Delta_i^t}(P, e, D^m) = \Delta_i^t(P, e, D^m) + \frac{1}{|P|} \cdot \left[d^t(e) - \sum_{k \in P} \Delta_k^t(P, e, D^m)\right] \quad (1)$$

for all $(P, e, D^m) \in \mathbb{MF}$, *for all* $t \in \mathbb{N}_m$, *and for all* $i \in P$. *The quantity* $\Delta_i^t(P, e, D^m) = \inf_{j \in E_i^+} \{d^t(0_{-i}, j)\}$ *represents the* **minimized individual effect** *incurred by operational cause* i *in* (P, e, d^t). *Within the framework of* $\overline{\Delta}$, *each operational cause initially assesses their minimized individual effects, following which the remaining effects are assessed equally among them.*

2. *The* **normalized MAIE**, $\overline{\Gamma}$, *is the assessment on* $\mathbb{MF}^*$ *which is defined by*

$$\overline{\Gamma^t_i}(P,e,D^m) = \frac{d^t(e)}{\sum\limits_{k\in P} \Delta^t_k(P,e,D^m)} \cdot \Delta^t_i(P,e,D^m) \tag{2}$$

for all $(P,e,D^m) \in \mathbb{MF}^*$, *for all* $t \in \mathbb{N}_m$, *and for all* $i \in P$, *where* $\mathbb{MF}^* = \{(P,e,D^m) \in \mathbb{MF} \mid \sum\limits_{i\in P} \Delta^t_i(P,e,D^m) \neq 0$ *for all* $t \in \mathbb{N}_m\}$. *Under the concept of* $\overline{\Gamma}$, *all operational causes assess the effect of the overall fuzzy coalition proportionally by utilizing the minimized individual effects of all operational causes.*

Remark 1.

- *A fuzzy TU situation, defined initially by Aubin [8,9], is represented as a pair* (P,d^a), *where* P *is a coalition, and* v^a *is a mapping such that* $d^a : [0,1]^P \longrightarrow \mathbb{R}$ *and* $d^a(0_P) = 0$.
- *From this point onward, our focus is limited to bounded fuzzy TU situations, defined as those situations* (P,e,d^t) *for which there exists* $M_v \in \mathbb{R}$ *such that* $d^t(\eta) \leq M_v$ *for all* $\eta \in E^P$. *This restriction ensures that* $\Delta_i(P,e,d^t)$ *is well defined.*

2.2. Practical Examples

Here, a concise application of multi-objective fuzzy TU situations in the context of an "operational system" is formulated as follows. Consider a set $P = \{1,2,\cdots,n\}$ representing all operational causes in a comprehensive operational system (P,e,D^m). The function d^t serves as an effect-assessing function, assessing the relative effect for each behavior vector $\eta = (\eta_i)_{i\in P} \in E^P$, indicating the relative effects operational causes can achieve if each operational cause i engages in an operational behavior $\eta_i \in E_i$ within each sub-operational system (P,e,d^t). Conceptualized in this manner, the comprehensive operational system (P,e,D^m) can be regarded as a multi-objective fuzzy TU situation, where d^t represents each characteristic function, and E_i denotes the set of all operational behavior available to operational cause i.

In an operational system, assessments are often conducted with the objective of minimizing multiple criteria simultaneously. As mentioned in the Introduction, an operational system must adhere to carbon emission standards that align with sustainability goals. It must consider how to meet production quality targets within the shortest possible time frame, while minimizing costs, equipment wear, and pollution emissions.

- Given that the effects derived from various operational causes within the operational system are closely interrelated and mutually influential, the assessment begins by assessing the minimized marginal effect for each operational cause, as defined by the assessment Δ in Definition 1. Subsequently, all operational causes are collectively and evenly assessed for the remaining effects, which correspond to the MAIE $\overline{\Delta}$ in Definition 1.
- From an alternative perspective, the overall effect of the operational system is evaluated based on the proportion formed by the minimized marginal effects of all contributors, as outlined in the normalized MAIE $\overline{\Gamma}$ of Definition 1.

In subsequent sections, we aim to demonstrate that the MAIE and the normalized MAIE can be treated as efficient assessment mechanisms for all operational causes, ensuring that all the operational system can derive balance and stable effects via each combination of operational behaviors under multi-objective fuzzy considerations.

Subsequently, a numerical example is provided as follows. Let $(P,e,D^m) \in \mathbb{MF}$ with $P = \{p,q,t\}$, $E_p = [0,0.93]$, $E_q = [0,0.79]$, $E_t = [0,1]$, and $m = 2$. Thus, $(P,e,D^m) = \Big(\{p,q,t\},(0.93,0.79,1),D^2\Big)$. Further, let $d^1((k_p,k_q,k_t)) = \{-2k_p + k_q - \frac{1}{4}k_t - 0.21 | (k_p,k_q,k_t) \in E^P,\ (k_p,k_q,k_t) \neq (0,0,0)\}$, $d^2((k_p,k_q,k_t)) = \{\frac{-1}{3}k_p - k_q + 2k_t + 1.9 | (k_p,k_q,k_t) \in E^P,\ (k_p,k_q,k_t) \neq (0,0,0)\}$, and $d^1(0,0,0) = d^2(0,0,0) = 0$. By Definition 1,

$$\begin{array}{rclrclrcl}
\Delta^1_p(P,e,D^m) &=& -2.07, & \Delta^1_q(P,e,D^m) &=& -0.21, & \Delta^1_t(P,e,D^m) &=& -0.46,\\
\overline{\Delta^1_p}(P,e,D^m) &=& \frac{-5}{3}, & \overline{\Delta^1_q}(P,e,D^m) &=& \frac{0.58}{3}, & \overline{\Delta^1_t}(P,e,D^m) &=& \frac{-0.17}{3},\\
\overline{\Gamma^1_p}(P,e,D^m) &=& -1.155, & \overline{\Gamma^1_q}(P,e,D^m) &=& -0.117, & \overline{\Gamma^1_t}(P,e,D^m) &=& 0.256,\\
\Delta^2_p(P,e,D^m) &=& 1.59, & \Delta^2_q(P,e,D^m) &=& 1.11, & \Delta^2_t(P,e,D^m) &=& 3.9,\\
\overline{\Delta^2_p}(P,e,D^m) &=& 0.323, & \overline{\Delta^2_q}(P,e,D^m) &=& -0.156, & \overline{\Delta^2_t}(P,e,D^m) &=& 2.633,\\
\overline{\Gamma^2_p}(P,e,D^m) &=& 0.674, & \overline{\Gamma^2_q}(P,e,D^m) &=& 0.47, & \overline{\Gamma^2_t}(P,e,D^m) &=& 1.654.
\end{array}$$

3. Axiomatic Results for the MAIE

To analyze the rationality of the MAIE, an extended reduction and some axioms are utilized to generate some axiomatic results.

- An assessment τ satisfies **multi-objective completeness (MOCOM)** if for all $(P,e,D^m) \in \mathbb{MF}$ and for all $t \in \mathbb{N}_m$, $\sum_{i\in P} \tau^t_i(P,e,D^m) = d^t(e)$. The MOCOM roperty stipulates that all operational causes assess the entire effect comprehensively.
- An assessment τ satisfies **multi-objective standardness for fuzzy situations (MOSFS)** if $\tau(P,e,D^m) = \overline{\Delta}(P,e,D^m)$ for all $(P,e,D^m) \in \mathbb{MF}$ with $|P| \leq 2$. The MOSFS property is an analogue of the assessing standard for two-person situations due to Hart and Mas-Colell [24] for characterizing the Shapley value [25].
- An assessment τ satisfies **multi-objective symmetric equality (MOSEQ)** if $\tau_i(P,e,D^m) = \tau_k(P,e,D^m)$ for all $(P,e,D^m) \in \mathbb{MF}$ with $\Delta^t_i(P,e,D^m) = \Delta^t_k(P,e,D^m)$ for some $i,k \in P$ and for all $t \in \mathbb{N}_m$. The MOSEQ property states that the outcomes should be equal if the minimized individual effects are equal.
- An assessment τ satisfies **multi-objective covariance (MOCOV)** if $\tau(P,e,D^m) = \tau(P,e,Q^m) + (h^t)_{t\in\mathbb{N}_m}$ for all $(P,e,D^m),(P,e,Q^m) \in \mathbb{MF}$ with $d^t(\eta) = q^t(\eta) + \sum_{i\in NE(\eta)} h^t_i$ for some $h^t \in \mathbb{R}^P$, for all $t \in \mathbb{N}_m$ and for all $\eta \in E^P$. The MOCOV property can be regarded as an extremely weak analogue of additivity.

By the related statement of MOSFS, the MAIE satisfies MOSFS absolutely. Next, we show that the MAIE satisfies MOCOM, MOSEQ, and MOCOV.

Lemma 1. *On $\mathbb{MF}$, the MAIE $\overline{\Delta}$ satisfies multi-objective completeness.*

Proof. By Definition 1, for all $(P,e,D^m) \in \mathbb{MF}$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned}
\sum_{i\in P} \overline{\Delta^t_i}(P,e,D^m) &= \sum_{i\in P} \Delta^t_i(P,e,D^m) + \sum_{i\in P} \frac{1}{|P|} \cdot \Big[d^t(e) - \sum_{k\in P} \Delta^t_k(P,e,D^m)\Big]\\
&= \sum_{i\in P} \Delta^t_i(P,e,D^m) + \frac{|P|}{|P|} \cdot \Big[d^t(e) - \sum_{k\in P} \Delta^t_k(P,e,D^m)\Big]\\
&= \sum_{i\in P} \Delta^t_i(P,e,D^m) + d^t(e) - \sum_{k\in P} \Delta^t_k(P,e,D^m)\\
&= d^t(e).
\end{aligned}$$

Hence, the MAIE $\overline{\Delta}$ satisfies MOCOM. □

Lemma 2. *On $\mathbb{MF}$, the MAIE $\overline{\Delta}$ satisfies multi-objective symmetric equality.*

Proof. Let $(P,e,D^m) \in \mathbb{MF}$ with $\Delta^t_i(P,e,D^m) = \Delta^t_k(P,e,D^m)$ for some $i,k \in P$ and for all $t \in \mathbb{N}_m$. By Definition 1,

$$\begin{aligned}
\overline{\Delta^t_i}(P,e,D^m) &= \Delta^t_i(P,e,D^m) + \frac{1}{|P|} \cdot \Big[d^t(e) - \sum_{p\in P} \Delta^t_p(P,e,D^m)\Big]\\
&= \Delta^t_k(P,e,D^m) + \frac{1}{|P|} \cdot \Big[d^t(e) - \sum_{p\in P} \Delta^t_p(P,e,D^m)\Big]\\
&= \overline{\Delta^t_k}(P,e,D^m).
\end{aligned}$$

Hence, the MAIE $\overline{\Delta}$ satisfies MOSEQ. □

Lemma 3. *On $\mathbb{MF}$, the MAIE $\overline{\Delta}$ satisfies multi-objective covariance.*

Proof. Let $(P,e,D^m),(P,e,Q^m) \in \mathbb{MF}$ with $d^t(\eta) = q^t(\eta) + \sum_{k\in NE(\eta)} h^t_k$ for all $t \in \mathbb{N}_m$ and for all $\eta \in E^P$, where $h^t \in \mathbb{R}^P$. By Definition 1, for all $p \in P$ and for all $t \in \mathbb{N}_m$,

$$\Delta^t_p(P,e,D^m) = \inf_{j\in E^+_p}\{d^t(0_{-p},j)\} = \inf_{j\in E^+_p}\{q^t(0_{-p},j) + h^t_p\} = \inf_{j\in E^+_p}\{q^t(0_{-p},j)\} + h^t_p = \Delta^t_p(P,e,D^m) + h^t_p. \tag{3}$$

By Definition 1 and Equation (3), for all $i \in P$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned}
\overline{\Delta^t_i}(P,e,D^m) &= \Delta^t_i(P,e,D^m) + \tfrac{1}{|P|}\cdot\Big[d^t(e) - \sum_{p\in P}\Delta^t_p(P,e,D^m)\Big] \\
&= \Delta^t_k(P,e,Q^m) + h^t_i + \tfrac{1}{|P|}\cdot\Big[q^t(e) + \textstyle\sum_{k\in NE(e)} h^t_k - \displaystyle\sum_{p\in P}[\Delta^t_p(P,e,Q^m) + h^t_p]\Big] \\
&\big(\textbf{by Equation (3)}\big) \\
&= \Delta^t_k(P,e,Q^m) + h^t_i + \tfrac{1}{|P|}\cdot\Big[q^t(e) + \textstyle\sum_{k\in P} h^t_k - \displaystyle\sum_{p\in P}\Delta^t_p(P,e,Q^m) - \sum_{p\in P} h^t_p\Big] \\
&\big(\textbf{since } NE(e) = P\big) \\
&= \Delta^t_k(P,e,Q^m) + \tfrac{1}{|P|}\cdot\Big[q^t(e) - \sum_{p\in P}\Delta^t_p(P,e,Q^m)\Big] + h^t_i \\
&= \overline{\Delta^t_i}(P,e,Q^m) + h^t_i.
\end{aligned}$$

Hence, the MAIE $\overline{\Delta}$ satisfies MOCOV. □

To characterize the EANSC, Moulin [23] defined a specific reduced situation, where each coalition in an arbitrary subgroup could achieve effects for its operational causes only if they aligned with the initial effects of all operational causes outside the subgroup. A natural analogue of Moulin's reduction can be considered on multi-objective fuzzy TU situations as follows.

The following reduction is based on the concept of "re-assessing upon disagreement". If there is a group or individual H within P that disagrees with assessment τ in a fuzzy TU situation (P,e,D^m), a re-assessment is conducted. If none of the dissenting causes wish to engage in the re-assessment, then the resulting utility from the re-assessment is naturally zero. If there are multiple dissenting causes but only one seeks a re-assessment, the process is simplified to avoid unnecessary complexity, and the cause is granted the utility it would derive from acting alone. In other cases, when dissenting causes opt for a re-assessment, all agreeing causes cooperate fully with the process. Upon completion, the agreeing causes receive the utility allocated by assessment τ and then exit. The remaining utility is allocated to the dissenting causes based on the outcome of the re-assessment.

Definition 2.

- *Let $(P,e,D^m) \in \mathbb{MF}$, $H \subseteq P$ and τ be an assessment. The* **reduced situation** $(S,e_H,D^m_{H,\tau})$ *is defined by $D^m_{H,\tau} = (d^t_{H,\tau})_{t\in\mathbb{N}_m}$, and for all $\eta \in E^H$,*

$$d^t_{H,\tau}(\eta) = \begin{cases} 0 & ,\ \eta = 0_H, \\ d^t(\eta) & ,\ |H| \geq 2,\ |NE(\eta)| = 1, \\ d^t(\eta, e_{P\setminus H}) - \sum\limits_{i\in P\setminus H} \tau^t_i(P,e,D^m) & ,\ \textit{otherwise}. \end{cases} \tag{4}$$

- *An assessment τ is said to be multi-objective bilateral consistent if any group in a game that disagrees with assessment τ undergoes the aforementioned re-assessment and receives a coincident utility value as it did before the re-assessment. Formally, an assessment τ adheres to the principle of* **multi-objective bilateral consistency (MOBCSY)** *if $\tau^t_i(H,e_H,D^m_{H,\tau}) = \tau^t_i(P,e,D^m)$ for all $(P,e,D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, for all $H \subseteq P$ with $|H| = 2$, and for all $i \in H$.*

Lemma 4. *On $\mathbb{MF}$, the MAIE $\overline{\Delta}$ satisfies the property of multi-objective bilateral consistency.*

Proof. Let $(P, e, D^m) \in \mathbb{MF}$, $H \subseteq P$, and $t \in \mathbb{N}_m$. Assume that $|P| \geq 2$ and $|H| = 2$. By Definition 1,

$$\overline{\Delta_i^t}(H, e_H, D^m_{H,\overline{\Delta}}) = \Delta_i^t(H, e_H, D^m_{H,\overline{\Delta}}) + \frac{1}{|H|} \cdot \left[d^t_{H,\overline{\Delta}}(e_H) - \sum_{k \in H} \Delta_k^t(H, e_H, D^m_{H,\overline{\Delta}})\right] \quad (5)$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. By Definitions 1 and 2,

$$\begin{aligned} \Delta_i^t(H, e_H, D^m_{H,\overline{\Delta}}) &= \inf_{j \in E_i^+} \{d^t_{H,\overline{\Delta}}(e_{H\setminus\{i\}}, j) - d^t_{H,\overline{\Delta}}(e_{H\setminus\{i\}}, 0)\} \\ &\quad \big(\textbf{by Definition 1}\big) \\ &= \inf_{j \in E_i^+} \{d^t(e_{-i}, j) - d^t(e_{-i}, 0)\} \\ &\quad \big(\textbf{by Definition 2}\big) \\ &= \Delta_i^t(P, e, D^m). \end{aligned} \quad (6)$$

By Equations (5) and (6) and Definitions 1 and 2,

$$\begin{aligned} \overline{\Delta_i^t}(H, e_H, D^m_{H,\overline{\Delta}}) &= \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|} \cdot \left[d^t_{H,\overline{\Delta}}(e_H) - \textstyle\sum_{k \in H} \Delta_k^t(P, e, D^m)\right] \\ &\quad \big(\textbf{by Equations (5) and (6)}\big) \\ &= \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|} \cdot \left[d^t(e) - \textstyle\sum_{k \in P\setminus H} \overline{\Delta_k^t}(P, e, D^m) - \textstyle\sum_{k \in H} \Delta_k^t(P, e, D^m)\right] \\ &\quad \big(\textbf{by Definition 2}\big) \\ &= \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|} \cdot \left[\textstyle\sum_{k \in H} \overline{\Delta_k^t}(P, e, D^m) - \textstyle\sum_{k \in H} \Delta_k^t(P, e, D^m)\right] \\ &\quad \big(\textbf{by MOCOM of } \overline{\Delta}\big) \\ &= \Delta_i^t(P, e, D^m) + \tfrac{1}{|H|} \cdot \left[\tfrac{|H|}{|P|} \cdot \left[d^t(e) - \textstyle\sum_{k \in P} \Delta_k^t(P, e, D^m)\right]\right] \\ &\quad \big(\textbf{by Definition 1}\big) \\ &= \Delta_i^t(P, e, D^m) + \tfrac{1}{|P|} \cdot \left[d^t(e) - \textstyle\sum_{k \in P} \Delta_k^t(P, e, D^m)\right] \\ &= \overline{\Delta_i^t}(P, e, D^m) \\ &\quad \big(\textbf{by Definition 1}\big) \end{aligned}$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. Thus, the MAIE satisfies MOBCSY. □

Next, we characterize the MAIE by means of multi-objective bilateral consistency.

Theorem 1. *On $\mathbb{MF}$, the MAIE is the only assessment satisfying MOSFS and MOBCSY.*

Proof. By Lemma 4, $\overline{\Delta}$ satisfies MOBCSY. Clearly, $\overline{\Delta}$ satisfies MOSFS.

To prove uniqueness, suppose τ satisfies MOSFS and MOBCSY. By MOSFS and MOBCSY of τ, it is easy to derive that τ also satisfies MOCOM, hence we omit it. Let $(P, e, D^m) \in \mathbb{MF}$. By the MOSFS of τ, $\tau(P, e, D^m) = \overline{\Delta}(P, e, D^m)$ if $|P| \leq 2$. For the case $|P| > 2$: Let $i \in P$, $t \in \mathbb{N}_m$, and $H = \{i, k\}$ for some $k \in P \setminus \{i\}$.

$$
\begin{aligned}
& \tau_i^t(P,e,D^m) - \tau_k^t(P,e,D^m) \\
= \; & \tau_i^t(H,e_H,D^m_{H,\tau}) - \tau_k^t(H,e_H,D^m_{H,\tau}) \\
& (\textbf{by the MOBCSY of } \tau) \\
= \; & \overline{\Delta_i^t}(H,e_H,D^m_{H,\tau}) - \overline{\Delta_k^t}(H,e_H,D^m_{H,\tau}) \\
& (\textbf{by the MOSFS of } \tau) \\
= \; & \Delta_i^t(H,e_H,D^m_{H,\tau}) - \Delta_k^t(H,e_H,D^m_{H,\tau}) \\
& (\textbf{by Definition 1}) \\
= \; & \inf_{j\in E_i^+}\{d^t_{H,\tau}(e_{H\setminus\{i\}},j) - d^t_{H,\tau}(e_{H\setminus\{i\}},0)\} - \inf_{j\in E_k^+}\{d^t_{H,\tau}(e_{H\setminus\{k\}},j) - d^t_{H,\tau}(e_{H\setminus\{k\}},0)\} \\
= \; & \inf_{j\in E_i^+}\{d^t(e_{-i},j) - d^t(e_{-i},0)\} - \inf_{j\in E_k^+}\{d^t(e_{-k},j) - d^t(e_{-k},0)\} \\
= \; & \Delta_i^t(P,e,D^m) - \Delta_k^t(P,e,D^m) \\
= \; & \overline{\Delta_i^t}(P,e,D^m) - \overline{\Delta_k^t}(P,e,D^m).
\end{aligned}
$$

Thus,

$$\tau_i^t(P,e,D^m) - \tau_k^t(P,e,D^m) = \overline{\Delta_i^t}(P,e,D^m) - \overline{\Delta_k^t}(P,e,D^m). \tag{7}$$

By Equation (7) and the MOCOM of τ and $\overline{\Delta}$,

$$
\begin{aligned}
|P|\cdot\tau_i^t(P,e,D^m) - d^t(e) & = \sum_{k\in P}\tau_i^t(P,e,D^m) - \sum_{k\in P}\tau_k^t(P,e,D^m) \\
& (\textbf{by the MOCOM of } \tau) \\
& = \sum_{k\in P}[\tau_i^t(P,e,D^m) - \tau_k^t(P,e,D^m)] \\
& = \sum_{k\in P}[\overline{\Delta_i^t}(P,e,D^m) - \overline{\Delta_k^t}(P,e,D^m)] \\
& (\textbf{by Equation (7)}) \\
& = |P|\cdot\overline{\Delta_i^t}(P,e,D^m) - d^t(e). \\
& (\textbf{by the MOCOM of } \overline{\Delta})
\end{aligned}
$$

Hence, $\tau_i^t(P,e,D^m) = \overline{\Delta_i^t}(P,e,D^m)$ for all $i \in P$ and for all $t \in \mathbb{N}_m$. □

Inspired by Moulin [23] and Maschler and Owen [26], we characterize the MAIE by means of the related properties of MOCOM, MOSEQ, MOCOV, and MOBCSY.

Lemma 5. *On $\mathbb{MF}$, an assessment τ satisfies MOSFS if it satisfies MOCOM, MOSEQ, and MOCOV.*

Proof. Assume that an assessment τ satisfies MOCOM, MOSEQ, and MOCOV on $\mathbb{MF}$. Let $(P,e,D^m) \in \mathbb{MF}$. The proof is completed by the MOCOM of τ if $|P| = 1$. Let $(P,e,D^m) \in \mathbb{MF}$ with $P = \{i,k\}$ for some $i \neq k$. We define a situation (P,e,Q^m) where $q^t(\eta) = d^t(\eta) - \sum_{i\in NE(\eta)}\Delta_i^t(P,e,D^m)$ for all $\eta \in E^P$ and for all $t \in \mathbb{N}_m$. By definition of Q^m,

$$
\begin{aligned}
\Delta_i^t(P,e,Q^m) & = \inf_{j\in E_i^+}\{q^t(j,e_k) - q^t(0,e_k)\} \\
& = \inf_{j\in E_i^+}\{d^t(j,e_k) - d^t(0,e_k) - \Delta_i^t(P,e,D^m)\} \\
& (\textbf{by definition of } q) \\
& = \inf_{j\in E_i^+}\{d^t(j,e_k) - d^t(0,e_k)\} - \Delta_i^t(P,e,D^m) \\
& = \Delta_i^t(P,e,D^m) - \Delta_i^t(P,e,D^m) \\
& = 0.
\end{aligned}
$$

Similarly, $\Delta_k^t(P,e,Q^m) = 0$. Therefore, $\Delta_i^t(P,e,Q^m) = \Delta_k^t(P,e,Q^m)$. By the MOSEQ of τ,

$$\tau_i^t(P,e,Q^m) = \tau_k^t(P,e,Q^m).$$

By the MOCOM of τ,

$$q^t(e) = \tau_i^t(P, e, Q^m) + \tau_k^t(P, e, Q^m) = 2 \cdot \tau_i^t(P, e, Q^m).$$

Therefore,

$$\begin{aligned} \tau_i^t(P, e, Q^m) &= \tfrac{q^t(e)}{2} \\ &= \tfrac{1}{2} \cdot \left[d^t(e) - \Delta_i(P, e, D^m) - \Delta_k(P, e, D^m) \right]. \end{aligned} \tag{8}$$

By Equation (8) and the MOCOV of τ,

$$\begin{aligned} \tau_i^t(P, e, D^m) &= \underline{\Delta}_i^t(P, e, D^m) + \tfrac{1}{2} \cdot \left[d^t(e) - \Delta_i^t(P, e, D^m) - \Delta_k^t(P, e, D^m) \right] \\ &= \overline{\Delta}_i^t(P, e, D^m). \end{aligned}$$

Similarly, $\tau_k^t(P, e, D^m) = \overline{\Delta}_k^t(P, e, D^m)$. Hence, τ satisfies MOSFS. □

Theorem 2. *On $\mathbb{MF}$, the MAIE $\overline{\Delta}$ is the only assessment satisfying MOCOM, MOSEQ, MOCOV, and MOBCSY.*

Proof. By Lemmas 1–4, the MAIE $\overline{\Delta}$ satisfies MOCOM, MOSEQ, MOCOV, and MOBCSY. The remaining proofs follow from Theorem 1 and Lemma 5. □

The subsequent examples illustrate the logical independence of each axiom utilized in Theorems 1 and 2 from the remaining axioms.

Example 1. *Define an assessment τ where for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, and for all $i \in P$,*

$$\tau_i^t(P, e, D^m) = \begin{cases} \overline{\Delta}_i^t(P, e, D^m) & \text{if } |P| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ satisfies MOSFS, but it violates MOBCSY.

Example 2. *Define an assessment τ where*

$$\tau_i^t(P, e, D^m) = \Delta_i^t(P, e, D^m)$$

for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, and for all $i \in P$. Clearly, τ satisfies MOSEQ, MOCOV, and MOBCSY, but it violates MOCOM and MOSFS.

Example 3. *Define an assessment τ where*

$$\tau_i^t(P, e, D^m) = \frac{d^t(e)}{|P|}$$

for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, and for all $i \in P$. Clearly, τ satisfies MOCOM, MOSEQ, and MOBCSY, but it violates MOCOV.

Example 4. *Define an assessment τ where for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$ and for all $i \in P$,*

$$\tau_i^t(P, e, D^m) = \left[d^t(e) - d^t(e_{-i}, 0) \right] + \frac{1}{|P|} \cdot \left[d^t(e) - \sum_{k \in P} \left[d^t(e) - d^t(e_{-k}, 0) \right] \right].$$

Clearly, τ satisfies MOCOM, MOCOV, and MOBCSY, but it violates MOSEQ.

Example 5. *Define an assessment τ where for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, and for all $i \in P$,*

$$\tau_i^t(P, e, D^m) = \Delta_i^t(P, e, D^m) + \frac{f^t(i)}{\sum_{k \in P} f^t(k)} \cdot \left[d^t(e) - \sum_{k \in P} \Delta_k^t(P, e, D^m)\right],$$

where for all $(P, e, D^m) \in \mathbb{MF}$, $f^t : P \to \mathbb{R}^+$ is defined by $f^t(i) = f^t(k)$ if $\Delta_i^t(P, e, D^m) = \Delta_k^t(P, e, D^m)$. Define an assessment ψ where for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, and for all $i \in P$,

$$\psi_i^t(P, e, D^m) = \begin{cases} \overline{\Delta_i^t}(P, e, D^m) & \text{if } |P| \leq 2, \\ \tau_i^t(P, e, D^m) & \text{otherwise.} \end{cases}$$

Clearly, ψ satisfies MOCOM, MOSEQ, and MOCOV, but it violates MOBCSY.

4. The Axiomatic Results for the Normalized MAIE

It is easy to show that the normalized MAIE satisfies MOCOM and MOSEQ, but it violates MOCOV. Therefore, similar to Theorem 1, we aim to characterize the normalized MAIE through the lens of multi-objective bilateral consistency. However, it becomes apparent that $(H, e_H, D^m_{H,\tau})$ does not exist if $\sum_{i \in H} \Delta_i^t(P, e, D^m) = 0$. Consequently, we introduce the concept of modified bilateral consistency (MBCSY) as follows. An assessment τ adheres to modified bilateral consistency (MBCSY) if $(H, e_H, D^m_{H,\tau}) \in \mathbb{MF}^*$ for some $(P, e, D^m) \in \mathbb{MF}$ and for some $H \subseteq P$ with $|H| = 2$, such that $\tau_i^t(H, e_H, D^m_{H,\tau}) = \tau_i^t(P, e, D^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in H$.

Lemma 6. *The normalized MAIE satisfies MBCSY on $\mathbb{MF}^*$.*

Proof. Let $(P, e, D^m) \in \mathbb{MF}^*$. If $|P| \leq 2$, then the proof is completed. Assume that $|P| \geq 3$ and $H \subseteq P$ with $|H| = 2$. Similar to Equation (6),

$$\Delta_i^t(H, e_H, D^m_{H,\overline{\Gamma}}) = \Delta_i^t(P, e, D^m). \tag{9}$$

for all $i \in H$ and for all $t \in \mathbb{N}_m$. Define $\sigma^t = \frac{d^t(e)}{\sum_{p \in P} \Delta_p^t(P,e,D^m)}$. For all $i \in H$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned} \overline{\Gamma_i^t}(H, e_H, D^m_{H,\overline{\Gamma}}) &= \frac{d^t_{H,\overline{\Gamma}}(e_H)}{\sum_{k \in H} \Delta_k^t(H,e_H,D^m_{H,\overline{\Gamma}})} \cdot \Delta_i^t(H, e_H, D^m_{H,\overline{\Gamma}}) \\ &= \frac{d^t(e) - \sum_{h \in P \setminus H} \overline{\Gamma_h^t}(P,e,D^m)}{\sum_{k \in H} \Delta_k^t(P,e,D^m)} \cdot \Delta_i^t(P, e, D^m) \\ &\quad \big(\textbf{by Equations (4) and (9)}\big) \\ &= \frac{\sum_{h \in H} \overline{\Gamma_h^t}(P,e,D^m)}{\sum_{k \in H} \Delta_k^t(P,e,D^m)} \cdot \Delta_i^t(P, e, D^m) \\ &\quad \big(\textbf{by MOCOM of } \overline{\Gamma}\big) \\ &= \sigma^t \cdot \Delta_i^t(P, e, D^m) \\ &= \overline{\Gamma_i^t}(P, e, D^m). \\ &\quad \big(\textbf{by Definition 1}\big) \end{aligned}$$

Hence, the assessment $\overline{\Gamma}$ satisfies MBCSY. □

An assessment τ satisfies **normalized standardness for fuzzy situations (NSFS)** if $\tau(P, e, d) = \overline{\Gamma}(P, e, d)$ for all $(P, e, d) \in \mathbb{MF}$, $|P| \leq 2$.

Theorem 3. *On $\mathbb{MF}^*$, the assessment $\overline{\Gamma}$ is the only assessment satisfying NSFS and MBCSY.*

Proof. By Lemma 6, $\overline{\Gamma}$ satisfies MBCSY. Clearly, $\overline{\Gamma}$ satisfies NSFS.

To prove uniqueness, suppose τ satisfies MBCSY and NSFS on $\mathbb{MF}^*$. By the NSFS and MBCSY of τ, it is easy to derive that τ also satisfies MOCOM, hence we omit it. Let $(P, e, D^m) \in \mathbb{MF}^*$. We complete the proof by induction on $|P|$. If $|P| \le 2$, it is trivial that $\tau(P, e, D^m) = \overline{\Gamma}(P, e, D^m)$ by NSFS. Assume that it holds if $|P| \le p - 1$, $p \le 3$. For the case $|P| = p$: Let $i, j \in P$ with $i \ne j$ and $t \in \mathbb{N}_m$. By Definition 1, $\overline{\Gamma}^t_k(P, e, D^m) = \frac{d^t(e)}{\sum_{h\in P} \Delta^t_h(P,e,D^m)} \cdot \Delta^t_k(P, e, D^m)$ for all $k \in P$. Assume that $\eta^t_k = \frac{\Delta^t_k(P,e,D^m)}{\sum_{h\in P} \Delta^t_h(P,e,D^m)}$ for all $k \in P$. Therefore,

$$
\begin{aligned}
\tau^t_i(P, e, D^m) &= \tau^t_i\big(P \setminus \{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big) \\
&\big(\textbf{by the MBCSY of } \tau\big) \\
&= \overline{\Gamma}^t_i\big(P \setminus \{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big) \\
&\big(\textbf{by the NSFS of } \tau\big) \\
&= \frac{d^t_{P\setminus\{j\},\tau}(e_{P\setminus\{j\}})}{\sum_{k\in P\setminus\{j\}} \Delta^t_k\big(P\setminus\{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big)} \cdot \Delta^t_i\big(P \setminus \{j\}, e_{P\setminus\{j\}}, D^m_{P\setminus\{j\},\tau}\big) \\
&= \frac{d^t(e) - \tau^t_i(P,e,D^m)}{\sum_{k\in P\setminus\{j\}} \Delta^t_k(P,e,D^m)} \cdot \Delta^t_i(P, e, D^m) \\
&\big(\textbf{by Equation (4)}\big) \\
&= \frac{d^t(e) - \tau^t_i(P,e,D^m)}{-\Delta^t_j(P,e,D^m) + \sum_{k\in P} \Delta^t_k(P,e,D^m)} \cdot \Delta^t_i(P, e, D^m).
\end{aligned}
\tag{10}
$$

By Equation (10),

$$
\begin{aligned}
& \tau^t_i(P, e, D^m) \cdot [1 - \eta^t_j] = [d^t(e) - \tau^t_j(P, e, D^m)] \cdot \eta^t_j \\
\Longrightarrow\ & \sum_{i\in P} \tau^t_i(P, e, D^m) \cdot [1 - \eta^t_j] = [d^t(e) - \tau^t_j(P, e, D^m)] \cdot \sum_{i\in P} \eta^t_j \\
\Longrightarrow\ & d^t(e) \cdot [1 - \eta^t_j] = [d^t(e) - \tau^t_j(P, e, D^m)] \cdot 1 \\
& \big(\textbf{by the MOCOM of } \tau\big) \\
\Longrightarrow\ & d^t(e) - d^t(e) \cdot \eta^t_j = d^t(e) - \tau^t_j(P, e, D^m) \\
\Longrightarrow\ & \overline{\Gamma}^t_j(P, e, D^m) = \tau^t_j(P, e, D^m).
\end{aligned}
$$

The proof is completed. □

The subsequent examples illustrate the logical independence of each axiom utilized in Theorem 3 from the remaining axioms.

Example 6. *Define an assessment τ where for all $(P, e, D^m) \in \mathbb{MF}^*$, for all $t \in \mathbb{N}_m$, and for all $i \in P$,*

$$\tau^t_i(P, e, D^m) = 0.$$

Clearly, τ satisfies MBCSY, but it violates NSFS.

Example 7. *Define an assessment τ where for all $(P, e, D^m) \in \mathbb{MF}^*$, for all $t \in \mathbb{N}_m$, and for all $i \in P$,*

$$\tau^t_i(P, e, D^m) = \begin{cases} \overline{\Gamma}^t_i(P, e, D^m) & , \textit{if } |P| \le 2, \\ 0 & , \textit{otherwise.} \end{cases}$$

Clearly, τ satisfies NSFS, but it violates MBCSY.

5. Two Weighted Extensions

In various contexts, operational causes and their associated energy behaviors may be assigned different weights, which serve as a priori measures of importance. These weights capture considerations that extend beyond those represented by characteristic functions. For example, when evaluating costs among investment plans, weights might correspond to each plan's profitability. Similarly, when assessing travel costs among visited institutions, as discussed by Shapley [27], weights could reflect the duration of stay at each institution.

If $f : U \to \mathbb{R}^+$ is a positive function, then f is called a **weight function for operational causes**. If $w : E^U \to \mathbb{R}^+$ is a positive function, then w is called a **weight function for energy behavior**. By these two types of weight functions, two weighted revisions of the MAIE are defined as follows.

Definition 3.

- *The* **one-weighted minimized assessment of inalienable effects (1-WMAIE)**, Γ^f, *is the assessment on* $\mathbb{MF}$ *which is defined for all* $(P, e, D^m) \in \mathbb{MF}$, *for all weight functions for operational causes* f, *for all* $t \in \mathbb{N}_m$, *and for all operational cause* $i \in P$ *by*

$$\Gamma_i^{f,t}(P, e, D^m) = \Delta_i^t(P, e, D^m) + \frac{f(i)}{\sum\limits_{k \in P} f(k)} \cdot \Big[d^t(e) - \sum_{k \in P} \Delta_k^t(P, e, D^m)\Big]. \tag{11}$$

- *The* **two-weighted minimized assessment of inalienable effects (2-WMAIE)**, Γ^w, *is the assessment on* $\mathbb{MF}$ *which is defined for all* $(P, e, D^m) \in \mathbb{MF}$, *for all weight functions for operational causes* w, *for all* $t \in \mathbb{N}_m$, *and for all operational cause* $i \in P$ *by*

$$\Gamma_i^{w,t}(P, e, D^m) = \Delta_i^{w,t}(P, e, D^m) + \frac{1}{|P|} \cdot \Big[d^t(e) - \sum_{k \in P} \Delta_k^{w,t}(P, e, D^m)\Big], \tag{12}$$

where $\Delta_i^{w,t}(P, e, D^m) = \inf\limits_{j \in E_i^+} \{w(j) \cdot d^t(0_{-i}, j)\}$.

Remark 2. *It is easy to show that the 1-WMAIE violates MOSEQ. Similarly, the 2-WMAIE violates MOSEQ and MOCOV. Therefore, these two assessments could not be characterized by means of MOSEQ and MOCOV.*

An assessment τ satisfies **one-weighted standardness for fuzzy situations (1WSFS)** if $\tau(P, e, D^m) = \Gamma^f(P, e, D^m)$ for all $(P, e, D^m) \in \mathbb{MF}$ with $|P| \leq 2$ and for all weight functions for operational causes f. An assessment τ satisfies **two-weighted standardness for fuzzy situations (2WSFS)** if $\tau(P, e, D^m) = \Gamma^w(P, e, D^m)$ for all $(P, e, D^m) \in \mathbb{MF}$ with $|P| \leq 2$ and for all weight functions for energy behavior w. Similar to the proofs of Lemmas 1, 3, 4, and Theorem 1, we propose the analogs of Lemmas 1, 3, 4, and Theorem 1 as follows.

Lemma 7. *On* $\mathbb{MF}$, *the 1-WMAIE* Γ^f *and the 2-WMAIE* Γ^w *satisfy the properties of multi-objective completeness and multi-objective bilateral consistency.*

Lemma 8. *On* $\mathbb{MF}$, *the 1-WMAIE satisfies the property of multi-objective covariance.*

Theorem 4.

- *On* $\mathbb{MF}$, *the 1-WMAIE* Γ^f *is the only assessment satisfying 1WSFS and MOBCSY.*
- *On* $\mathbb{MF}$, *the 2-WMAIE* Γ^w *is the only assessment satisfying 2WSFS and MOBCSY.*

The following examples are to show that each of the axioms used in above axiomatizations is logically independent of the remaining axioms.

Example 8. *Define an assessment* τ *where for all* $(P, e, D^m) \in \mathbb{MF}$, *for all* $t \in \mathbb{N}_m$, *for all weight functions* w, *and for all* $i \in P$,

$$\tau_i^t(P, e, D^m) = 0.$$

Clearly, τ *satisfies MOBCSY, but it violates 1WSFG and 2WSFG.*

Example 9. *Define an assessment τ where for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, for all weight functions for operational causes d, and for all $i \in P$,*

$$\tau_i^t(P, e, D^m) = \begin{cases} \Gamma_i^{d,t}(P, e, D^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ satisfies 1WSFG, but it violates MOBCSY.

Example 10. *Define an assessment τ where for all $(P, e, D^m) \in \mathbb{MF}$, for all $t \in \mathbb{N}_m$, for all weight functions for energy behavior w and for all $i \in P$,*

$$\tau_i^t(P, e, D^m) = \begin{cases} \Gamma_i^{w,t}(P, e, D^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ satisfies 2WSFG, but it violates MOBCSY.

6. Conclusions

1. Under numerous conditions, each operational cause is granted the flexibility to engage with an infinite range of energy behavior. Operational causes are increasingly required to efficiently address multiple objectives throughout operating processes. Consequently, we concurrently addressed fuzzy behavior and multi-objective considerations. Weighting naturally plays a crucial role under the framework of effect assessment. For instance, in assessing the impact among investment plans, weights could be tied to the profitability of each plan. Therefore, this study also investigated the concept of weighted assessments. Unlike previous studies on standard TU situations and fuzzy TU situations, this study presented several significant contributions.
    - Simultaneously focusing on fuzzy behavior and multi-objective considerations, we applied the framework of multi-objective fuzzy TU situations.
    - By assessing minimized individual effects under fuzzy multi-objective considerations, we proposed the MAIE, the normalized MAIE, and related axiomatic processes.
        - On $\mathbb{MF}$, the MAIE $\overline{\Delta}$ was the only assessment satisfying MOSFS and MOBCSY.
        - On $\mathbb{MF}$, the MAIE $\overline{\Delta}$ was the only assessment satisfying MOCOM, MOSEQ, MOCOV, and MOBCSY.
        - On $\mathbb{MF}^*$, the normalized MAIE $\overline{\Gamma}$ was the only assessment satisfying NSFS and MBCSY. Furthermore, the normalized MAIE satisfied MOCOM and MOSEQ but violated MOBCSY, MOSFS, and MOCOV.
    - To mitigate relative influences among operational causes and their energy behavior, we introduced the 1-WMAIE, the 2-WMAIE, and related axiomatic processes.
        - On $\mathbb{MF}$, the 1-WMAIE Γ^f was the only assessment satisfying 1WSFS and MOBCSY. Furthermore, the 1-WMAIE satisfied MOCOM and MOCOV but violated MOSFS and MOSEQ.
        - On $\mathbb{MF}$, the 2-WMAIE Γ^w was the only assessment satisfying 2WSFS and MOBCSY. Furthermore, the 2-WMAIE satisfied MOCOM but violated MOSFS, MOSEQ, and MOCOV.
    - All assessments and related results were initially presented within the frameworks of standard TU situations and fuzzy TU situations.
2. The following presents a comparison between several different game-theoretical conditions and the model proposed in this paper.
    - Under standard TU conditions, each operational cause is typically classified as either fully active or entirely uninvolved with other operational causes. The effects and assessed benefits derived from these operational causes and the groups they form are represented as single-valued outcomes.

- Under multi-choice TU conditions, operational causes are allowed to engage with a finite range of energy behaviors. The effects and assessed benefits arising from the operational causes and their corresponding participation scenarios are also represented as single-valued outcomes.
- Under interval TU conditions, each operational cause is similarly classified as either fully active or entirely uninvolved with other operational causes. However, the effects and assessed benefits derived from these causes and the groups they form are expressed as interval values.
- The model considered in this paper integrates fuzzy TU conditions and multi-objective considerations. Operational causes are allowed to engage with an infinite range of energy behaviors. The effects and assessed benefits derived from these causes and their participation scenarios are represented as single-valued outcomes.

3. Based on the findings of this paper, as well as the previous comparisons and conclusions, several avenues for future research can be identified:
   - Can assessments under other game-theoretical conditions, along with their associated axiomatic results, be extended to contexts involving fuzzy behavior and multi-objective considerations?
   - Under the framework of fuzzy behavior and multi-objective considerations proposed in this paper, is it possible to assess the effects and allocated benefits of operational causes and the groups by applying interval uncertainty?

   These issues are left for further investigation by interested readers.

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