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Characterization of Bach and Cotton Tensors on a Class of Lorentzian Manifolds

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Abstract: In this work, we aim to investigate the characteristics of the Bach and Cotton tensors on Lorentzian manifolds, particularly those admitting a semi-symmetric metric ω-connection. First, we prove that a Lorentzian manifold admitting a semi-symmetric metric ω-connection with a parallel Cotton tensor is quasi-Einstein and Bach flat. Next, we show that any quasi-Einstein Lorentzian manifold admitting a semi-symmetric metric ω-connection is Bach flat.

Keywords: Bach tensor; Cotton tensor; Lorentzian manifolds; semi-symmetric metric connection

MSC: 53C15; 53C40; 53D10

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1. Introduction

Consider an n-dimensional pseudo-Riemannian manifold denoted as M. When the torsion tensor $\widetilde{\mathcal{T}}$ associated with a linear connection $\widetilde{\nabla}$ vanishes, i.e., $\widetilde{\mathcal{T}}(\zeta_1,\zeta_2) = \widetilde{\nabla}_{\zeta_1}\zeta_2 - \widetilde{\nabla}_{\zeta_2}\zeta_1 - [\zeta_1,\zeta_2] = 0$, then $\widetilde{\nabla}$ is referred to as a symmetric connection. Conversely, if the torsion tensor does not vanish, the connection is termed non-symmetric. Many geometers classify linear connection $\widetilde{\nabla}$ into different categories based on distinct forms. For example, a connection is referred to as semi-symmetric (abbreviated as $\mathcal{SS}$) if it satisfies the following relation:

$$\widetilde{\mathcal{T}}(\zeta_1,\zeta_2) = \omega^\sharp(\zeta_2)\zeta_1 - \omega^\sharp(\zeta_1)\zeta_2, \ \forall \zeta_1,\zeta_2 \in \Gamma(TM) \tag{1}$$

where the one-form $\omega^\sharp$ and the associated vector field ω are linked through a pseudo-Riemannian metric g by $g(\cdot,\omega) = \omega^\sharp(\cdot)$. If $\widetilde{\nabla} g = 0$, the connection $\widetilde{\nabla}$ is referred to as a metric connection. Otherwise, it is categorized as a non-metric connection [1]. A connection is symmetric and metric if and only if it is a Levi–Civita connection. Hayden [2] investigated a metric connection $\widetilde{\nabla}$ with a non-vanishing torsion on a Riemannian manifold, which later became known as the Hayden connection. Following Pak's work [3], which showed that it is a semi-symmetric metric connection (abbreviated as $\mathcal{SSM}$-connection), numerous questions emerged about this type of connection. Yano [4] subsequently initiated an investigation into Riemannian manifolds equipped with an $\mathcal{SSM}$-connection, finding that such manifolds become conformally flat when the curvature tensor vanishes. Recently, Chaubey et al. [5] commenced the investigation of the concept of semi-symmetric metric ω-connection (briefly, $\mathcal{SSM}\omega$-connection) on a Riemannian manifold and explored its geometric properties. This concept was later extended to Lorentzian manifolds by Chaubey et al. [6], where they studied its geometric and physical properties within specific classifications. Duggal studied a class of almost Ricci solitons (ARSs), he showed three mathematical models of conformally flat almost-Ricci-soliton manifolds, leading to several significant findings [7]. In recent years, Kumar and Colney et al. considered the tangent bundles with NSNMC and QSNMC [8–10]. The properties, theorems, and results of the curvature tensor and Ricci tensor relevant to the connection on the tangent bundles were obtained in [11–13]. Moreover, Li and Mihai et al. conducted research relevant to inequalities [14,15], solitons [16,17], submanifolds [18–20], classical differential geometry [21–23], non-Euclidean geomgery [24–27], etc., under the viewpoint of soliton theory, singularity theory, submanifold theory, and other theories [28–30]. The results and methods of those papers motivated us to write this paper and helped us in future research.

In Bach's work [31], a novel type of conformally invariant tensor was introduced as a tool for investigating conformal relativity. The analysis involves considering a specific function based on the squared norm of the Weyl conformal curvature denoted by $\mathcal{W}(g)$, which is evaluated over a Riemannian compact manifold of dimension 4. In the context of critical points of functionals, the Bach tensor (briefly, $\mathcal{B}_g$-tensor) plays a crucial role in the study of Einstein metrics, which are Riemannian metrics that satisfy the Einstein field equations. The $\mathcal{B}_g$-tensor is used to characterize the critical points of functionals that arise from the Einstein–Hilbert action, a functional that defines the action of a physical system. For any pseudo-Riemannian manifold (M, g) of dimension n, the $\mathcal{B}_g$-tensor is defined as

$$\begin{aligned}\mathcal{B}_g(\zeta_1,\zeta_2) &= \tfrac{1}{n-3}\textstyle\sum_{i,j=1}^{n}(\nabla_{e_i}\nabla_{e_j}\mathcal{W})(\zeta_1,e_i,e_j,\zeta_2)\\ &\quad+\tfrac{1}{n-2}\textstyle\sum_{i,j=1}^{n}\mathrm{Ric}_g(e_i,e_j)\mathcal{W}(\zeta_1,e_i,e_j,\zeta_2),\end{aligned} \tag{2}$$

where $e_i,\ i = 1, 2, \cdots, n$ represents a local orthonormal frame on (M, g), Ric_g is the Ricci tensor of type $(0, 2)$, and $\mathcal{W}$ is the Weyl tensor of type $(0, 4)$. The Weyl conformal tensor of type $(1, 3)$ is defined by

$$\begin{aligned}\mathcal{W}(\zeta_1,\zeta_2)\zeta_3 &= R(\zeta_1,\zeta_2)\zeta_3-\frac{1}{n-2}\{\mathrm{Ric}_g(\zeta_2,\zeta_3)\zeta_1-\mathrm{Ric}_g(\zeta_1,\zeta_3)\zeta_2+g(\zeta_2,\zeta_3)Q\zeta_1\\ &\quad-g(\zeta_1,\zeta_3)Q\zeta_2\}+\frac{r}{(n-1)(n-2)}\{g(\zeta_2,\zeta_3)\zeta_1-g(\zeta_1,\zeta_3)\zeta_2\},\end{aligned} \tag{3}$$

where R is the Riemannian curvature tensor, Q is the Ricci operator defined by $\mathrm{Ric}_g = g(Q\cdot, \cdot)$, and r is the scalar curvature of g. Utilizing the properties of the Weyl conformal tensor $\mathcal{W}$ and the contraction of the Bianchi second identity, it can be inferred that $div\mathcal{W} = (\frac{n-3}{n-2})\mathcal{C}$, where $\mathcal{C}$ is the Cotton tensor (briefly, $\mathcal{C}_g$-tensor) of type $(0, 3)$ defined by (see [32])

$$\begin{aligned}\mathcal{C}(\zeta_1,\zeta_2)\zeta_3 &= (\nabla_{\zeta_1}\mathrm{Ric}_g)(\zeta_2,\zeta_3)-(\nabla_{\zeta_2}\mathrm{Ric}_g)(\zeta_1,\zeta_3)\\ &\quad-\frac{1}{2(n-1)}[(\zeta_1 r)g(\zeta_2,\zeta_3)-(\zeta_2 r)g(\zeta_1,\zeta_3)].\end{aligned} \tag{4}$$

A pseudo-Riemannian manifold (M, g) is called conformally flat if its conformally transformed pseudo-Riemannian metric is locally Euclidean. For $n > 3$, this implies $\mathcal{W} = 0$, and for $n = 3$, that $\mathcal{C} = 0$. Furthermore, the vanishing of the Weyl tensor also implies that the divergence of $\mathcal{W}$ is zero, which is equivalent to $\mathcal{C} = 0$. Thus, the condition $\mathcal{C} = 0$ generalizes both Einstein metrics and conformally flat pseudo-Riemannian metrics, encompassing a broader class of geometric structures. For instance, a Riemannian manifold that satisfies the condition $\mathcal{C} = 0$ is characterized by a harmonic curvature tensor, denoted as $divR = 0$. Interestingly, this condition implies that the scalar curvature r is constant. As a result, such a manifold generalizes Einstein manifolds and locally symmetric manifolds. Furthermore, the converse holds: the system of equations $divR = 0$ implies that the Levi–Civita connection ∇ of the metric g constitutes a Yang–Mills connection while the metric is fixed [33].

Utilizing Equations (3) and (4), the expression for the $\mathcal{B}_g$-tensor (2) can be expressed as shown in Chen and He [34]:

$$\mathcal{B}_g(\zeta_1, \zeta_2) = \frac{1}{(n-2)}\left[\sum_{i=1}^{n}(\nabla_{e_i}\mathcal{C})(e_i, \zeta_1)\zeta_2 + \sum_{i,j=1}^{n}\mathrm{Ric}_g(e_i, e_j)\mathcal{W}(\zeta_1, e_i, e_j, \zeta_2)\right]. \tag{5}$$

A closer examination of the $\mathcal{B}_g$-tensor in (5) has led us to realize that its final term can be expressed as

$$g(Qe_i, e_j)g(\mathcal{W}(\zeta_1, e_i)e_j, \zeta_2) = -g(\mathcal{W}(\zeta_1, e_i)\zeta_2, Qe_i) = -g(Q\mathcal{W}(\zeta_1, e_i)\zeta_2, e_i).$$

As a consequence, expression (5) takes the following form:

$$\mathcal{B}_g(\zeta_1, \zeta_2) = \frac{1}{(n-2)}\left[\sum_{i}(\nabla_{e_i}\mathcal{C})(e_i, \zeta_1)\zeta_2 - \sum_{i,j} g(QW(\zeta_1, e_i)\zeta_2, e_i)\right]. \tag{6}$$

As the Weyl tensor W becomes zero in three dimensions, the Bach tensor can be formulated as

$$\mathcal{B}_g(\zeta_1, \zeta_2) = \sum_{i=1}^{3}(\nabla_{e_i}\mathcal{C})(e_i, \zeta_1)\zeta_2. \tag{7}$$

A Riemannian metric g is called Bach flat when its associated $\mathcal{B}_g$-tensor becomes zero. Notably, a Bach-flat metric encompasses Einstein metrics and conformally flat metrics. Consequently, a Riemannian manifold endowed with a vanishing $\mathcal{B}_g$-tensor is intriguing, as it accommodates both Einstein metrics and conformally flat metrics. Given that 4-dimensional Bach-flat metrics are conformally invariant, and this property is obtained by conformally related Einstein metrics, we know that these metrics are also Bach flat. Additionally, in 4-dimensional space, the $\mathcal{B}_g$-tensor plays a special role as a symmetric, divergence-free tensor of type $(0,2)$ that is both quadratic in Riemann curvature (representing a linear combination of the products of two Riemann curvature tensors) and exhibits desirable conformal properties (for detailed insights, refer to [35]). It is well known that Einstein metrics are Bach flat, which naturally leads to the question of whether there exist Bach-flat metrics that are not Einstein metrics. Initially, this question was addressed by the authors Leistner and Nurwoski [36]. In their work, they presented 4-dimensional pp-waves that demonstrate Bach flatness. Furthermore, they provided numerous examples exhibiting Bach flatness without being Einstein metrics. Later, Ghosh [37] validated this concept within the context of an almost contact Riemannian manifold and obtained many fruitful results. Recently, Naik et al. [38] studied the $\mathcal{B}_g$-tensor and $\mathcal{C}_g$-tensor on a cosymplectic manifold and obtained many fruitful results. These works on the Bach and $\mathcal{C}_g$-tensors motivate us to investigate the validity of the converse statement in the context of Lorentzian manifolds.

The paper is structured as follows. In Section 2, we revisit the foundational definitions and formulas pertinent to Lorentzian manifolds with $\mathcal{SSM}\omega$-connections. Section 3 encompasses the derivation of auxiliary results, and subsequently, establishes the theorem that asserts the quasi-Einstein and Bach-flat nature of a Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection with a parallel $\mathcal{C}_g$-tensor. Continuing, we explore the scenario of a Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection with a purely transversal $\mathcal{B}_g$-tensor. Concretely, we prove that it is a quasi-Einstein manifold. Furthermore, we consider specific conditions on the $\mathcal{B}_g$-tensor, which are addressed in Section 4.

2. Notes on Lorentzian Manifolds

Lorentzian manifolds are one of the most important subclasses of pseudo-Riemannian manifolds, playing a vital role in mathematical physics (especially in the development of the theory of general relativity and cosmology). Let M be a connected paracompact Hausdorff smooth manifold of dimension n. Then, M is said to be a Lorentzian manifold if it admits a $(0,2)$-type non-degenerate smooth symmetric tensor $g_p : T_pM \times T_pM \to R$ that is $\forall p \in M$ of signature $(-,+,+,\cdots,+)$, where T_pM represents the tangent vector space on M at point p. A non-zero vector field $\omega \in T_pM$ is said to be time-like (resp. non-space-like, null, and space-like) if $g_p(\omega,\omega) < 0$ (resp. $\leq, =, > 0$) [39].

A one-form $\omega^\sharp$ and an associated vector field ω are connected through a pseudo-Riemannian metric g by $g(\zeta_1,\omega) = \omega^\sharp(\zeta_1)$. A linear connection $\widetilde{\nabla}$ on M is defined by

$$\widetilde{\nabla}_{\zeta_1}\zeta_2 = \nabla_{\zeta_1}\zeta_2 + \omega^\sharp(\zeta_2)\zeta_1 - g(\zeta_1,\zeta_2)\omega,\ \forall\zeta_1,\zeta_2 \in \aleph(M). \tag{8}$$

Mishra et al. [40] and Chaubey et al. [41] comprehensively studied an almost contact metric manifold admitting $\mathcal{SSM}\omega$-connection, characterizing situations where ω is a Reeb vector field ζ and $\widetilde{\nabla}\omega = 0$, and this characterization led to several significant geometric findings. Inspired by this study, Chaubey et al. [5,6] extended this concept to both Riemannian and Lorentzian manifolds. A linear connection $\widetilde{\nabla}$ defined on a Lorentzian manifold M is called a semi-symmetric metric ω-connection if it satisfies Equations (1) and (8), $\widetilde{\nabla}g = 0$, and $\widetilde{\nabla}\omega$. Consider $\widetilde{\nabla}\omega = 0$, then from Equation (8) we have

$$\nabla_{\zeta_1}\omega = \zeta_1 + \omega^\sharp(\zeta_1)\omega, \tag{9}$$

where ω is a unit-time-like vector field, that is, $g(\omega,\omega) = -1$. Now, we state the following restricted curvature with respect to the Levi–Civita connection ∇ as follows.

Lemma 1 ([6])**.** *Let M be an n-dimensional Lorentzian manifold with $\mathcal{SSM}\omega$-connection $\widetilde{\nabla}$, then*

$$\begin{aligned} R(\zeta_1,\zeta_2)\omega &= \omega^\sharp(\zeta_2)\zeta_1 - \omega^\sharp(\zeta_1)\zeta_2, && (10)\\ R(\omega,\zeta_1)\zeta_2 &= g(\zeta_1,\zeta_2)\omega - \omega^\sharp(\zeta_2)\zeta_1, && (11)\\ \omega^\sharp(R(\zeta_1,\zeta_2)\zeta_3) &= \omega^\sharp(\zeta_1)g(\zeta_2,\zeta_3) - \omega^\sharp(\zeta_2)g(\zeta_1,\zeta_3), && (12) \end{aligned}$$

for $\zeta_1,\zeta_2,\zeta_3 \in \chi(M)$.

Taking the g-trace of Equation (10) yields

$$\mathrm{Ric}_g(\zeta_1,\omega) = (n-1)\omega^\sharp(\zeta_1), \tag{13}$$

which is equivalent to

$$Q\omega = (n-1)\omega, \tag{14}$$

Taking the differentiation of $\omega^\sharp(\zeta_1) = g(\zeta_1,\omega)$ along V, together with Equation (9), we infer

$$(\nabla_V\omega^\sharp)\zeta_1 = g(\zeta_1,V) + \omega^\sharp(\zeta_1)\omega^\sharp(V). \tag{15}$$

Now, we recall the following results which are used to prove our main outcomes.

Lemma 2. *A Lorentzian manifold of dimension n equipped with $\mathcal{SSM}\omega$-connection satisfies*

$$\begin{aligned} (\nabla_{\zeta_1}Q)\omega &= (n-1)\zeta_1 - Q\zeta_1, && (16)\\ (\nabla_\omega Q)\zeta_1 &= -2Q\zeta_1 + 2(n-1)\zeta_1. && (17) \end{aligned}$$

If a Lorentzian manifold M of dimension n with non-vanishing Ricci tensor Ric_g satisfies

$$\mathrm{Ric}_g = \alpha g + \beta\omega^\sharp \otimes \omega^\sharp, \tag{18}$$

for smooth functions α and β, where $\omega^\sharp$ is a non-zero one-form, and the vector field corresponding to the one-form $\omega^\sharp$ is a unit-time-like vector field, then it is referred to as a perfect fluid spacetime. However, some geometers also call M quasi-Einstein [42]. Particularly, if $\beta = 0$ and $\alpha = constant$, then M is called Einstein.

3. Characterization of Lorentzian Manifolds Admitting $\mathcal{SSM}\omega$-Connection with Parallel $\mathcal{C}_g$-Tensor

Before proceeding to explore our main results, we first establish the following key outcome.

Lemma 3. *On a Lorentzian manifold with $\mathcal{SSM}\omega$-connection the following relations are valid:*

$$\sum_{i}^{n}(\nabla_{e_i}\mathcal{C})(e_i,\zeta_2)\omega = \frac{1}{2}\zeta_2(r) - \frac{1}{2(n-1)}[(divDr)\omega^\sharp(\zeta_2) - g(\nabla_\omega Dr,\zeta_2)], \quad (19)$$

$$\sum_{i}^{n}g(Q\mathcal{W}(\zeta_1,e_i)\omega,e_i) = \frac{(r-(n-1))^2}{(n-1)(n-2)}\omega^\sharp(\zeta_1) + \left[\frac{|Q|^2-(n-1)^2}{(n-2)}\right]\omega^\sharp(\zeta_1), \quad (20)$$

where Dr denotes the gradient of r and div denotes divergence.

Proof. Substituting ζ_3 with ω in Equation (4), we obtain

$$\mathcal{C}(\zeta_1,\zeta_2)\omega = g((\nabla_{\zeta_1}Q)\zeta_2,\omega) - g((\nabla_{\zeta_2}Q)\zeta_1,\omega) - \frac{1}{2(n-1)}\{\omega^\sharp(\zeta_2)(\zeta_1 r) - \omega^\sharp(\zeta_1)(\zeta_2 r)\}. \quad (21)$$

Utilizing Lemma 2 in Equation (21), one can easily obtain

$$\mathcal{C}(\zeta_1,\zeta_2)\omega = -\frac{1}{2(n-1)}[\omega^\sharp(\zeta_2)(\zeta_1 r) - \omega^\sharp(\zeta_1)(\zeta_2 r)]. \quad (22)$$

Differentiating Equation (22) covariantly along ζ_3 and employing Equation (9), we deduce

$$(\nabla_{\zeta_3}\mathcal{C})(\zeta_1,\zeta_2)\omega + \mathcal{C}(\zeta_1,\zeta_2)\zeta_3 + \omega^\sharp(\zeta_3)\mathcal{C}(\zeta_1,\zeta_2)\omega = -\frac{1}{2(n-1)}\{g(\zeta_2,\zeta_3)g(Dr,\zeta_1)$$
$$-g(\zeta_1,\zeta_3)g(Dr,\zeta_2) + \omega^\sharp(\zeta_3)\omega^\sharp(\zeta_2)g(Dr,\zeta_1) - \omega^\sharp(\zeta_1)\omega^\sharp(\zeta_3)g(Dr,\zeta_2)$$
$$+\omega^\sharp(\zeta_2)g(\zeta_1,\nabla_{\zeta_3}Dr) - \omega^\sharp(\zeta_1)g(\zeta_1,\nabla_{\zeta_3}Dr)\}. \quad (23)$$

Consider an orthonormal frame $e_i : i = 1,2,\cdots,n$ for the tangent space of M. Setting $\zeta_1 = \zeta_3 = e_i$ in Equation (23), and subsequently, summing over i, we obtain

$$\sum_{i}^{n}[(\nabla_{e_i}\mathcal{C})(e_i,\zeta_2)\omega + \mathcal{C}(e_i,\zeta_2)e_i] + \mathcal{C}(\omega,\zeta_2)\omega = -\frac{1}{2(n-1)}\{(2-n)(\zeta_2 r)$$
$$+\omega^\sharp(\zeta_2)(\omega r) + (divDr)\omega^\sharp(\zeta_2) - g(\nabla_\omega Dr,\zeta_2)\}. \quad (24)$$

Switching ζ_1 with ω in Equation (23), a straightforward computation gives

$$\mathcal{C}(\omega,\zeta_2)\omega = -\frac{1}{2(n-1)}\{\omega^\sharp(\zeta_2)(\omega r) + \zeta_2 r\}. \quad (25)$$

As we know, $\mathcal{C}_g$-tensor is trace-free, so employing Equation (25) in Equation (24) leads to the proof of (i).

Replacing ζ_3 with ω in Equation (3) and calling back Equations (10) and (14), we arrive at

$$Q\mathcal{W}(\zeta_1,\zeta_2)\omega = \frac{r-(n-1)}{(n-1)(n-2)}[\omega^\sharp(\zeta_2)Q\zeta_1 - \omega^\sharp(\zeta_1)Q\zeta_2]$$
$$-\frac{1}{(n-2)}[\omega^\sharp(\zeta_2)Q^2\zeta_1 - \omega^\sharp(\zeta_1)Q^2\zeta_2]. \quad (26)$$

Substituting ζ_2 with e_i in the above equation, taking the inner product with e_i, summing over i, and utilizing Equation (14), we derive (ii). □

We know that the parallelism of the $\mathcal{C}_g$-tensor encompasses both conformal flatness and Cotton flatness. So, we were intrigued to study Lorentzian manifolds admitting $\mathcal{SSM}\omega$-connection M with a parallel $\mathcal{C}_g$-tensor and proceed to prove the following outcome.

Theorem 1. *Let M be a Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection. If M has a parallel $\mathcal{C}_g$-tensor, then M is quasi-Einstein and Bach flat.*

Proof. The assumption of parallelism of the $\mathcal{C}_g$-tensor ensures its divergence-free nature. So that, from Equation (19), one can obtain

$$\frac{1}{2}(\zeta_2 r) - \frac{1}{2(n-1)}[(divDr)\omega^\sharp(\zeta_2) - g(\nabla_\omega Dr, \zeta_2)] = 0. \tag{27}$$

On the other hand,

$$Dr = -(\omega r)\omega, \qquad \text{and} \qquad (\omega r) = 2(r - n(n-1)). \tag{28}$$

Employing the foregoing equation in Equation (22) leads to $\mathcal{C}(\zeta_1, \zeta_2)\omega = 0$. Then, by taking the covariant derivative of this equation and utilizing Equation (9), we obtain

$$(\nabla_{\zeta_3}\mathcal{C})(\zeta_1, \zeta_2)\omega + \mathcal{C}(\zeta_1, \zeta_2)\zeta_3 + \omega^\sharp(\zeta_3)\mathcal{C}(\zeta_1, \zeta_2)\omega = 0.$$

Because of the parallelism of the $\mathcal{C}_g$-tensor and the fact that $\mathcal{C}(\zeta_1, \zeta_2)\omega = 0$, the preceding equation indicates that the manifold possesses a vanishing $\mathcal{C}_g$-tensor. Consequently, referring to Equation (4), we have

$$(\nabla_{\zeta_1}\mathrm{Ric}_g)(\zeta_2, \zeta_3) - (\nabla_{\zeta_2}\mathrm{Ric}_g)(\zeta_1, \zeta_3) - \frac{1}{2(n-1)}[(\zeta_1 r)g(\zeta_2, \zeta_3) - (\zeta_2 r)g(\zeta_1, \zeta_3)] = 0.$$

Replacing ζ_1 with ω in the previous equation and utilizing Lemma 2 and Equation (17), we conclude that

$$(n-1)g(\zeta_2, \zeta_3) - g(Q\zeta_2, \zeta_3) = \frac{1}{2(n-1)}[(\omega r)g(\zeta_2, \zeta_3) - (\zeta_2 r)\omega^\sharp(\zeta_3)]. \tag{29}$$

Utilizing Equation (28) in Equation (29), followed by simplification, reveals that the Ricci operator Q can be represented as

$$Q\zeta_2 = \left[\frac{r}{(n-1)} - 1\right]\zeta_2 + \left[\frac{r}{(n-1)} - n\right]\omega^\sharp(\zeta_2)\omega. \tag{30}$$

Therefore, M is quasi-Einstein. Continuing, we employ Equations(10), (14) and (30), in Equation (3), resulting in $\mathcal{W}(\zeta_1, \zeta_2)\omega = 0$. Given the parallelism of the $\mathcal{C}_g$-tensor, the first term of the $\mathcal{B}_g$-tensor (6) becomes zero. Consequently, utilizing Equation (30), the $\mathcal{B}_g$-tensor is simplified to

$$\begin{aligned}\mathcal{B}_g(\zeta_1, \zeta_2) \quad = \quad & -\frac{1}{(n-2)}\sum\{\left(\frac{r}{(n-1)} - 1\right)g(\mathcal{W}(\zeta_1, e_i)\zeta_2, e_i) \\ & + \left(\frac{r}{(n-1)} - n\right)g(\mathcal{W}(\zeta_1, \omega)\zeta_2, \omega)\}.\end{aligned} \tag{31}$$

As the Weyl tensor is trace-free and $\mathcal{W}(\zeta_1, \zeta_2)\omega = 0$, the aforementioned equation demonstrates the vanishing of $\mathcal{B}_g$ on M. This completes the proof. □

Remark 1. *It is well known that the Weyl tensor vanishes in three dimensions; however, this assertion does not hold true for higher dimensions [43]. Consequently, substituting* $\zeta_2 = \zeta_3 = \omega$*,* $n = 3$*, and* $\mathcal{W} = 0$ *into Equation (3) yields*

$$R(\zeta_1, \omega)\omega = -[2\zeta_1 + 4\omega^\sharp(\zeta_1)\omega + Q\zeta_1] + \frac{r}{2}[\zeta_1 + \omega^\sharp(\zeta_1)\omega].$$

Utilizing of (10) in the preceding equation, we obtain

$$Q\zeta_1 = \Big(\frac{r}{2} - 1\Big)\zeta_1 + \Big(\frac{r}{2} - 3\Big)\omega^\sharp(\zeta_1)\omega.$$

Under the same assumptions as stated in the preceding theorem and guided by Equation (7), the conclusion trivially holds in three dimensions.

If the unit-time-like vector field ω leaves the scalar curvature r invariant, then from the second term of Equation (28) we obtain $r = n(n-1)$. Applying this result to Equation (30), we observe that $Q\zeta_2 = (n-1)\zeta_2$. As a result of Theorem 1, the following conclusion can be drawn.

Corollary 1. *Let* M *be a Lorentzian manifold admitting* $\mathcal{SSM}\omega$*-connection with parallel Cotton tensor. If a unit-vector field* ω *leaves the scalar curvature invariant, then* M *is Einstein.*

4. $\mathcal{B}_g$-Tensor on Lorentzian Manifolds with $\mathcal{SSM}\omega$-Connection

In this section, we consider a purely transversal $\mathcal{B}_g$-tensor on a class of Lorentzian manifolds and obtain the following outcome.

Theorem 2. *A Lorentzian manifold admitting* $\mathcal{SSM}\omega$*–connection has a purely transversal* $\mathcal{B}_g$*-tensor if and only if it is quasi-Einstein.*

Proof. Firstly, we derive the expression for the transversal $\mathcal{B}_g$-tensor. To achieve this, we substitute ω into Equation (6) and with the help of Equations (19) and (20), we obtain

$$\begin{aligned}\mathcal{B}_g(\zeta_1, \omega) \quad &= \quad \frac{1}{(n-2)}\bigg\{\frac{(\zeta_1 r)}{2} - \frac{1}{2(n-1)}[div(Dr)\omega^\sharp(\zeta_1) - g(\nabla_\omega Dr, \zeta_1)] \\ &\quad + \frac{[r-(n-1)]^2}{(n-1)(n-2)}\omega^\sharp(\zeta_1) - \bigg[\frac{|Q|^2 - (n-1)^2}{(n-2)}\bigg]\omega^\sharp(\zeta_1)\bigg\}.\end{aligned} \tag{32}$$

Suppose the $\mathcal{B}_g$-tensor is purely transversal, then the foregoing equation transforms into

$$\begin{aligned}&\frac{(\zeta_1 r)}{2} - \frac{1}{2(n-1)}[div(Dr)\omega^\sharp(\zeta_1) - g(\nabla_\omega Dr, \zeta_1)] \\ &+ \frac{[r-(n-1)]^2}{(n-1)(n-2)}\omega^\sharp(\zeta_1) - \bigg[\frac{|Q|^2 - (n-1)^2}{(n-2)}\bigg]\omega^\sharp(\zeta_1) = 0.\end{aligned} \tag{33}$$

Covariant differentiate the first term of Equation (28) along ζ_1, and consider Equation (9) to obtain $\nabla_{\zeta_1} Dr = -[\zeta_1(\omega r)\omega + (\omega r)(\zeta_1 + \omega^\sharp(\zeta_1)\omega)]$. Contracting this and recalling Equation (28) gives us $div(Dr) = -(n-3)(\omega r)$. Putting this together with Equation (33), we arrive at

$$\begin{aligned}&\frac{(\zeta_1 r)}{2} + \frac{1}{2(n-1)}[(n-3)(\omega r)\omega^\sharp(\zeta_1) + 2(\omega r)\omega] \\ &+ \frac{[r-(n-1)]^2}{(n-1)(n-2)}\omega^\sharp(\zeta_1) - \bigg[\frac{|Q|^2 - (n-1)^2}{(n-2)}\bigg]\omega^\sharp(\zeta_1) = 0.\end{aligned}$$

Plugging $\zeta_1 = \omega$ into the previous equation yields

$$(n-1)[|Q|^2 - (n-1)^2] - [r-(n-1)]^2 = 0. \tag{34}$$

Utilizing Equations (14), calculating the magnitude of the tensor T becomes

$$|T|^2 = \frac{1}{(n-1)}\{(n-1)(|Q|^2 - (n-1)^2) - [r-(n-1)]^2\}. \quad (35)$$

Employing Equation (34) in Equation (35) demonstrates that $T = 0$, consequently establishing that M is quasi-Einstein. On the other hand, if we assume M is quasi-Einstein, then consequently, $\mathcal{W}(\zeta_1, \zeta_2)\omega = 0$ (as proved in Theorem 1). Differentiating this along ζ_3 and recalling Equation (10) gives

$$(\nabla_{\zeta_3}\mathcal{W})(\zeta_1, \zeta_2)\omega = -\mathcal{W}(\zeta_1, \zeta_2)\zeta_3 + \omega^\sharp(\zeta_3)\mathcal{W}(\zeta_1, \zeta_2)\omega = -\mathcal{W}(\zeta_1, \zeta_2)\zeta_3.$$

By contracting this equation over ζ_3 and taking into account that the Weyl tensor is trace-free, we deduce $(divW)(\zeta_1, \zeta_2)\omega = \mathcal{C}(\zeta_1, \zeta_2)\omega = 0$. Considering the quasi-Einstein, Equation (35) implies $(n-1)[|Q|^2 - (n-1)^2] - [r-(n-1)]^2 = 0$. By incorporating all of these results into Equation (32), we can deduce that $\mathcal{B}_g(\zeta_1, \omega) = 0$. □

As a consequence of Theorem 2, we demonstrate the following.

Corollary 2. *A quasi-Einstein Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection is Bach flat.*

Proof. As a result of Theorem 2, we know that a quasi-Einstein Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection possesses a purely transversal $\mathcal{B}_g$-tensor. Now, covariant differentiate Equation (30) along ζ_1 and invoke Equation (8) to obtain

$$\begin{aligned}(\nabla_{\zeta_1}Q)\zeta_2 &= \frac{\zeta_1 r}{(n-1)}[\zeta_2 + \omega^\sharp(\zeta_2)\omega] + \left(\frac{r}{(n-1)} - n\right) \\ &\quad \{g(\zeta_1, \zeta_2)\omega + \omega^\sharp(\zeta_2)\zeta_1 + 2\omega^\sharp(\zeta_1)\omega^\sharp(\zeta_2)\omega\}. \quad (36)\end{aligned}$$

By virtue of Equation (36), the $\mathcal{C}_g$-tensor Equation (4) can be expressed as

$$\begin{aligned}\mathcal{C}(\zeta_1, \zeta_2)\zeta_3 &= \frac{\zeta_1 r}{2(n-1)}g(\zeta_2, \zeta_3) - \frac{\zeta_2 r}{2(n-1)}g(\zeta_1, \zeta_3) + \frac{\zeta_1 r}{(n-1)}\omega^\sharp(\zeta_2)\omega^\sharp(\zeta_3) \\ &- \frac{\zeta_2 r}{(n-1)}\omega^\sharp(\zeta_1)\omega^\sharp(\zeta_3) + \left(\frac{r}{(n-1)} - n\right)\{\omega^\sharp(\zeta_2)g(\zeta_1, \zeta_3) - \omega^\sharp(\zeta_1)g(\zeta_2, \zeta_3)\}.\end{aligned}$$

Applying Equation (28) to the previous equation, we can deduce that $\mathcal{C}(\zeta_1, \zeta_2)\zeta_3 = 0$. As a result, the first term of the Bach tensor Equation (6) becomes zero. Conversely, considering M to be a quasi-Einstein manifold, we have $\mathcal{W}(\zeta_1, \zeta_2)\omega = 0$ for all vector fields ζ_1 and ζ_2 in M (a consequence of Theorem 1). The subsequent steps of the proof align with the implications arising from Equation (31). □

Combining the results of Corollary 2 with Theorem 1, we conclude the following.

Corollary 3. *In a Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection, the following conditions are equivalent:*

- *M is quasi-Einstein;*
- *M has a vanishing $\mathcal{C}_g$-tensor;*
- *M has a vanishing $\mathcal{B}_g$-tensor.*

Finally, by extending the implications of Theorem 2, we establish the following.

Theorem 3. *Let M be a Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection. If a unit-vector field ω leaves the scalar curvature invariant, then M is Einstein if one of the following conditions holds:*

- *The $\mathcal{B}_g$-tensor containing the unit-vector field ω is parallel along ω;*
- *The curvature transformation annihilates the $\mathcal{B}_g$-tensor.*

Proof. Suppose the vector field ω leaves the scalar curvature invariant, that is, $\omega(r) = 0$. Consequently, referring to Equation (28), it becomes evident that $r = n(n-1)$. As a consequence of Equation (32), it can be deduced that

$$\mathcal{B}_g(\zeta_1, \omega) = f\omega^\sharp(\zeta_1), \tag{37}$$

where $f = \frac{1}{(n-2)}[(n-1)^3 + (n-1)^2 - |Q|^2]$. Covariant differentiating Equation (37) along ω and applying the hypothesis $(\nabla_\omega B_g)(\zeta_1, \omega) = 0$, we can deduce $\nabla_\omega |Q|^2 = 0$. Utilizing Equation (17), this inference leads to

$$0 = \sum_{i=1}^{n} g((\nabla_\omega Q)e_i, Qe_i) = -2\sum_{i=1}^{n} g(Qe_i - (n-1)e_i, Qe_i).$$

Therefore, we have $|Q|^2 = (n-1)r = n(n-1)^2$. Considering this outcome, we can now proceed with the computation:

$$\begin{aligned} |Q - (n-1)I|^2 &= |Q|^2 - 2(n-1)r + n(n-1)^2 \\ &= n(n-1)^2 - 2n(n-1)^2 + n(n-1)^2 = 0. \end{aligned} \tag{38}$$

Hence, we conclude M is an Einstein manifold. This finishes the verification of (i).

Now, we assume that $R(\zeta_1, \zeta_2) \cdot \mathcal{B}_g = 0$. This equivalence can be expressed as follows: $\mathcal{B}_g(R(\zeta_1, \zeta_2)\zeta_3, U) + \mathcal{B}_g(R(\zeta_1, \zeta_2)U, \zeta_3) = 0$. By setting $\zeta_2 = \zeta_3 = \omega$, we obtain

$$\mathcal{B}_g(R(\zeta_1, \omega)\omega, U) + \mathcal{B}_g(R(\zeta_1, \omega)U, \omega) = 0. \tag{39}$$

Utilizing Equation (10), we deduce that $R(\zeta_1, \omega)\omega = -\zeta_1 - \omega^\sharp(\zeta_1)\omega$ and $R(\zeta_1, \omega)U = \omega^\sharp(U)\zeta_1 - g(\zeta_1, U)\omega$. Putting these results into Equation (39) and recalling $\mathcal{B}_g(\zeta_1, \omega) = f\omega^\sharp(\zeta_1)$, we derive $\mathcal{B}_g(\zeta_1, U) = fg(\zeta_1, U)$. Since the $\mathcal{B}_g$-tensor is trace-free, we immediately arrive at $f = 0$. This implies that

$$(n-1)^3 + (n-1)^2 - |Q|^2 = 0.$$

Consequently, by following Equation (38), we can deduce that M is an Einstein manifold. This concludes the proof of (ii). □

5. Conclusions

Rudolf Bach's introduction of the Bach tensor was a major milestone in the development of differential geometry and general relativity, and it has had far-reaching implications for the understanding of spacetime geometry and gravitational phenomena. Conformally, Einstein spaces and the Bach tensor are fundamental aspects of differential geometry and general relativity. The Bach tensor provides a powerful tool for studying the curvature properties of spacetime and has important implications for our understanding of gravitational phenomena. In the context of geometry and relativity, both the Bach and Cotton tensors play an important role in understanding spacetime geometry and its relation to matter and energy. The Bach tensor helps us to understand how matter and energy affect the curvature of spacetime, while the Cotton tensor describes how spacetime curvature responds to matter distribution. The Bach and Cotton tensors have been used in various modified gravity theories, such as Brans–Dicke theory, Jordan–Brans–Dicke theory, and $f(R)$-gravity, to describe deviations from general relativity. Our main results reveal that a Lorentzian manifold admitting $\mathcal{SSM}\omega$-connection with a parallel Cotton tensor is quasi-Einstein and Bach flat, which provides understanding in many aspects of black holes, dark energy, cosmology, and so on.

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