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On Nilpotent Elements and Armendariz Modules

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Abstract: For a left module ${}_RM$ over a non-commutative ring R, the notion for the class of nilpotent elements ($nil_R(M)$) was first introduced and studied by Sevviiri and Groenewald in 2014 (*Commun. Algebra*, *42*, 571–577). Moreover, Armendariz and semicommutative modules are generalizations of reduced modules and $nil_R(M) = 0$ in the case of reduced modules. Thus, the nilpotent class plays a vital role in these modules. Motivated by this, we present the concept of nil-Armendariz modules as a generalization of reduced modules and a refinement of Armendariz modules, focusing on the class of nilpotent elements. Further, we demonstrate that the quotient module M/N is nil-Armendariz if and only if N is within the nilpotent class of ${}_RM$. Additionally, we establish that the matrix module $M_n(M)$ is nil-Armendariz over $M_n(R)$ and explore conditions under which nilpotent classes form submodules. Finally, we prove that nil-Armendariz modules remain closed under localization.

Keywords: nilpotent element; Armendariz module; Armendariz ring; nil-Armendariz module

MSC: 16D10; 16S36; 16S50

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1. Introduction

In this article, R represents a ring with identity, and ${}_RM$ represents a unital left R-module. Recall that for some $n \in \mathbb{N}$ and $a \in R$, if $a^n = 0$, then a is said to be a nilpotent element in R. The notation $Nil(R)$ denotes the set of all nilpotent elements in R. If $Nil(R) = \{0\}$, R is called a reduced ring. For a polynomial ring $R[x]$ over R, Armendariz [1] proved a very interesting result: if R is reduced, then the coefficients $u_l m_k = 0$ for each l, k whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} u_k x^k$ with coefficients in R satisfy $p(x)m(x) = 0$. Inspired by this result, Rege and Chhawchharia [2] introduced a new class of rings named Armendariz rings as a generalization of reduced rings and provided a sufficient class of rings that are Armendariz but not reduced. A ring R is called Armendariz if $u_l v_k = 0$, whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} v_k x^k$ in $R[x]$ satisfy $p(x).m(x) = 0$. R. Antoine [3] introduced nil-Armendariz rings and extensively studied a nilpotent class's structure in non-commutative rings. A ring R is called nil-Armendariz if $u_l v_k \in Nil(R)$, whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} v_k x^k$ in $R[x]$ satisfy $p(x).m(x) \in Nil(R)[x]$. The classes of Armendariz and nil-Armendariz rings and their relation with other classes of rings are briefly studied in [1,3–5]. In [4], Liu and Zhao introduced weak Armendariz rings to generalize nil-Armendariz rings. A ring R is weak Armendariz if $u_l v_k \in Nil(R)$, whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $m(x) = \sum_{k=0}^{q} v_k x^k$ in $R[x]$ satisfy $p(x).m(x) = 0$. Thus, we have the following chain: reduced ring $\Longrightarrow$

Armendariz ring $\Longrightarrow$ nil-Armendariz ring $\Longrightarrow$ weak Armendariz, but the converse is not necessarily true. Moreover, Lee and Zhou expanded the concept of the reduced property to modules in their work [6]. A module ${}_RM$ is reduced if it satisfies one of the following equivalent conditions:

(1) If $u^2v = 0$ for some $u \in R$ and $v \in M$, then $uRv = 0$.
(2) If $uv = 0$ for some $u \in R$ and $v \in M$, then the $uM \cap Rv = 0$.

Similarly ${}_RM$ is called rigid if $uv = 0$ holds true whenever $u^2v = 0$ for $u \in R$ and $v \in M$. A module ${}_RM$ is called Armendariz if $u_lv_k = 0$ whenever $p(x) = \sum_{l=0}^{n} u_lx^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_kx^k \in M[x]$ satisfy $p(x).m(x) = 0$. Lee and Zhou recorded many examples of Armendariz modules [6], as well as Rege and Buhphang [7]. They also conducted a comparative study on Armendariz, reduced, and semicommutative modules. A module ${}_RM$ is semicommutative if, for any $u \in R$ and $v \in M$ that satisfy $uv = 0$, it follows that $uRv = 0$. Over the past few decades, many algebraists have generalized concepts defined for non-commutative rings to modules. In this context, as early as 2014, Ssevvirri and Groenewald [8] proposed the idea of nilpotent elements for modules. An element $v \in {}_RM$ is called nilpotent if either $v = 0$ or $u^tv = 0$ but $uv \neq 0$ for some $u \in R$ and $t \in \mathbb{N}$. The set of all nilpotent elements in ${}_RM$ is denoted by $nil_R(M)$. In 2019, Ansari and Singh carried out a comparative study of nilpotent elements and established crucial relationships between nilpotent elements and other classes of modules. They showed that if ${}_RM$ is reduced, then ${}_RM$ contains no non-zero nilpotent elements. Since the concepts of nil-Armendariz and weak Armendariz depend on nilpotency conditions on elements, extending these concepts to modules becomes straightforward. In this direction, Ansari and Singh [9] defined a weak Armendariz module. A module ${}_RM$ is called weak Armendariz if whenever $p(x) = \sum_{l=0}^{n} u_lx^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} u_kx^k \in M[x]$ satisfy $p(x).m(x) = 0$, then $u_lv_k \in nil_R(M)$ for each l, k. This new concept further helped study the structure of nilpotent elements and their connection with other subclasses of modules. Recall that an element $v \in {}_RM$ is a torsion element if $uv = 0$ for some non-zero $u \in R$. We denote by $Tor(M)$ the set containing all torsion elements of ${}_RM$. In [8], Ssevvirri and Groenewald raise an important question regarding the conditions under which the set of nilpotent elements forms a submodule of ${}_RM$. In this article, we note some conditions on the ring that help make the set of nilpotent elements a submodule.

Many researchers have conducted extensive research on the generalization of reduced rings, including Armendariz and semi-commutative rings. However, the absence of various subclass definitions has prevented advancements in these areas from extending to modules. Thus, in this article we present a new concept known as nil-Armendariz modules as a different category within the Armendariz module class. This concept aims to generalize reduced modules in the context of the nilpotent class. We delve into various properties of this extension and perform a comparative analysis between the concepts developed in rings and their module counterparts.

Among the significant results, we demonstrate the existence of a large class of nil-Armendariz modules but not Armendariz, and vice versa. Additionally, we establish that for a submodule N of ${}_RM$, the quotient module M/N is nil-Armendariz if and only if N is a subset of the nilpotent class of ${}_RM$. We also prove that for a module ${}_RM$, the matrix module $M_n(M)$ is nil-Armendariz over $M_n(R)$. Furthermore, we explore the structure of the nilpotent class and identify certain conditions under which these classes form a submodule. Additionally, we demonstrate that nil-Armendariz modules maintain closure under localizations.

2. Results on Nil-Armendariz Modules

We begin with the following definition.

Definition 1. *A left R-module M is called nil-Armendariz if whenever $f(x)m(x) \in nil_R(M)[x]$ for $f(x) = \sum_{i=0}^{n} r_ix^i \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_jx^j \in M[x]$, then $r_im_j \in nil_R(M)$.*

Based on Definition 1, we conclude that the class of nil-Armendariz modules is closed under submodules and that every reduced module is nil-Armendariz. Moreover, we find that all nil-Armendariz modules are weak Armendariz. However, Propositions 1 and 3, presented later in this article, demonstrate that the converse does not hold in either case. In the case of ring theory, we easily verify that all Armendariz rings are nil-Armendariz. When extending these concepts to module theory, one might assume that all Armendariz modules are nil-Armendariz. However, this assumption is incorrect. To illustrate, consider a module ${}_RM$. We recognize $M_n(M)$ as a module over $M_n(R)$. We can express any matrix $K = [m_{ij}]_{n\times n} \in M_n(M)$ as $K = \sum_{i,j=1}^{n} E_{ij}m_{ij}$, where E_{ij} denotes the elementary matrices.

Lemma 1. *Let M be a left R-module. Then, $nil_R(M_n(M)) = M_n(M)$.*

Proof. Consider any non-zero matrix $[m_{ij}]_{n\times n}$. This implies at least one $m_{ij} \neq 0$ for some $1 \leq i, j \leq n$. Thus, we have two cases as follows:

(a) Suppose $m_{ij} \neq 0$ for $i \neq j$. Then, we can take $r = E_{ji}$. Thus, we can easily see that $r^2K = (E_{ji})^2K = 0$, but $rK = e_{ji}K \neq 0$.

(b) Suppose $m_{ij} \neq 0$ for $i = j$. Then, we can take $r = E_{li}$ such that $l \neq i$ and $1 \leq l, i \leq n$. Thus, we can easily see that $r^2K = (E_{li})^2K = 0$, but $rK = E_{li}K \neq 0$.

□

Proposition 1. *For a module ${}_RM$, the matrix module $M_n(M)$ is nil-Armendariz over $M_n(R)$ for $n \geq 2$, but it is not Armendariz.*

Proof. From Lemma 1, it is clear that ${}_{M_n(R)}M_n(M)$ is nil-Armendariz for $n \geq 2$. Now, consider $p(x) = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1_R \\ 0 & 0 \end{pmatrix} x \in R[x]$ and for any $0 \neq m \in M$, $m(x) = \begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix} + \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} x \in M[x]$. We observe that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \neq 0$, but $p(x)m(x) = 0$. Therefore, $M_2(M)$ is not Armendariz over $M_2(R)$. Since ${}_{M_2(R)}M_2(M)$ is embedded as a submodule in ${}_{M_n(R)}M_n(M)$ for $n \geq 2$, we can conclude that ${}_{M_n(R)}M_n(M)$ is not Armendariz. □

Next, we note a significant result concerning the nilpotency of $\mathbb{Z}_{p^2}$ as a $\mathbb{Z}$-module.

Proposition 2 ([10], Proposition 2.3)**.** *Let m be an element of left R-module M. Then, the following conditions are equivalent:*

(i) There exist $r \in R$ and $t \geq 2$ such that $r^tm = 0$ but $r^{t-1}m \neq 0$.

(ii) There exists $k \in R$ such that $k^2m = 0$ but $km \neq 0$.

Proof. We note that the implication $(ii) \Rightarrow (i)$ is trivial. For $(i) \Rightarrow (ii)$ choose $k = r^{t-1}$. Since $t \geq 2$, we have $2t - 2 \geq t$. Hence, $k^2m = r^{2t-2}m = 0$ while $km \neq 0$. □

We note that if $m \in {}_RM$ satisfies any of the equivalent conditions of Proposition 2, then m is a nilpotent element of ${}_RM$.

Lemma 2. *For any prime p, $\overline{p} \notin nil_{\mathbb{Z}}(\mathbb{Z}_{p^2})$.*

Proof. Let us suppose that $\overline{p} \in nil_{\mathbb{Z}}(\mathbb{Z}_{p^2})$. Then, there exists some non-zero $r \in \mathbb{Z}$ such that $r^2\overline{p} = \overline{0}$ but $r.\overline{p} \neq \overline{0}$. This implies that $p^2 | r^2.p \Rightarrow r = pl$ for some $l \in \mathbb{Z}$. Thus, $p^2 | r.p$, which implies that $r.\overline{p} = \overline{0}$. Hence, a contradiction. □

Proposition 3. *For any prime p, the $\mathbb{Z}$-module $\mathbb{Z}_{p^2}$ is Armendariz but not nil-Armendariz.*

Proof. Consider $q(x) = 1 + px \in \mathbb{Z}[x]$ and $m(x) = \overline{1} - \overline{p}x \in \mathbb{Z}_{p^2}[x]$. Then, we have $q(x) \cdot m(x) = \overline{1}$. Clearly, $p^2 \cdot \overline{1} = 0$ and $p \cdot \overline{1} \neq 0$. Thus, $\overline{1} \in nil_{\mathbb{Z}}(\mathbb{Z}_{p^2})$ and, hence, $q(x) \cdot m(x) \in nil_{\mathbb{Z}}(\mathbb{Z}_{p^2})[x]$. However, by Lemma 2, $\overline{p} \notin nil_{\mathbb{Z}}(\mathbb{Z}_{p^2})$. Thus, $\mathbb{Z}_{p^2}$ is not a nil-Armendariz module, but it is an Armendariz module (see Lemma 2.6 in [1]). □

Next, we record some conditions under which the above newly defined concept is equivalent to an Armendariz module.

Proposition 4. *For a reduced module $_RM$, the statements given below are equivalent:*

(1) $_RM$ is Armendariz.
(2) $_RM$ is nil-Armendariz.
(3) $_RM$ is weak Armendariz.

Proof. Since the module $_RM$ is reduced, by Corollary 2.11 in [10], we have $nil_R(M) = 0$. Hence, the proof follows straightforwardly. □

Proposition 5. *Let R be a reduced ring. If $_RM$ is torsion-free, then the statements given below are equivalent:*

(1) $_RM$ is Armendariz.
(2) $_RM$ is nil-Armendariz.
(3) $_RM$ is weak Armendariz.

Proof. The proof follows easily from Proposition 2.7 in [10]. □

Next, for a module $_RM$, we provide a large class of submodules of the matrix module $_{M_n(R)}M_n(M)$, which are both Armendariz and nil-Armendariz. For this purpose, we denote $T_n(R)$ as the ring of $n \times n$ upper triangular matrices over R. For a left R-module $_RM$ and $K = (a_{ij}) \in M_n(R)$, let $KM = \{(a_{ij}m) : m \in M\}$. For elementary matrices E_{ij}, let $U = \sum_{i=0}^{n} E_{i,i+1}$ for $n \geq 2$. We consider $U_n(R) = RI_n + RU + RU^2 + \cdots + RU^{n-1}$ and $U_n(M) = I_nM + UM + U^2M + \cdots + U^{n-1}M$. Then, $U_n(R)$ forms a ring, and $U_n(M)$ forms a left module over $U_n(R)$.

There exists a ring isomorphism $\phi : U_n(R) \to \frac{R[x]}{(x^n)}$ defined as $\phi(r_0I_n + r_1U + r_2U^2 + \cdots + r_{n-1}U^{n-1}) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1} + (x^n)$, and an abelian group isomorphism $\theta : U_n(M) \to \dfrac{M[x]}{(M[x](x^n))}$ defined as $\theta(m_0I_n + m_1U + m_2U^2 + \cdots + m_{n-1}U^{n-1}) = m_0 + m_1x + \cdots + m_{n-1}x^{n-1} + M[x](x^n)$, such that $\theta(AW) = \phi(A)\theta(W)$ for all $A \in U_n(R)$ and $W \in U_n(M)$.

In [11], Corollary 3.7, Zhang and Chen prove that $_RM$ is a reduced module if and only if $U_n(M)$ is Armendariz over $U_n(R)$. Thus, for a reduced module $_RM$, we find a larger class of Armendariz submodules of $T_n(M)$ over $T_n(R)$. We recall the following notations from [12].

Let $k \in \mathbb{N}$, and for $n = 2k \geq 2$, consider

$$A_n^e(M) = \sum_{i=1}^{k} \sum_{j=k+i}^{n} E_{i,j}M$$

and for $n = 2k + 1 \geq 3$

$$A_n^o(M) = \sum_{i=1}^{k+1} \sum_{j=k+i}^{n} E_{i,j}M.$$

Let

$$A_n(M) = I_nM + UM + \cdots + U^{k-1} + A_n^e(M) \text{ for } n = 2k \geq 2$$

and

$$A_n(M) = I_nM + UM + \cdots + U^{k-1} + A_n^o(M) \text{ for } n = 2k+1 \geq 3.$$

For example,

$$A_4(M) = \left\{ \begin{pmatrix} v_1 & v_2 & v & w \\ 0 & v_1 & v_2 & z \\ 0 & 0 & v_1 & v_2 \\ 0 & 0 & 0 & v_1 \end{pmatrix} : v_1, v_2, v, w, z \in M \right\}$$

$$A_5(M) = \left\{ \begin{pmatrix} a_1 & a_2 & a & b & c \\ 0 & a_1 & a_2 & d & e \\ 0 & 0 & a_1 & a_2 & f \\ 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_1 \end{pmatrix} : a_1, a_2, a, b, c, d, e, f \in M \right\}.$$

For $A = (a_{ij})$, $B = (b_{ij})$, we write $[A.B]_{ij} = 0$ to mean that $a_{il}b_{lj} = 0$ for $l = 0, \ldots, n$.

Lemma 3 ([12], Lemma 1.2)**.** *For $r(x) = A_0 + A_1x + \cdots + A_px^p \in M_n(R)[x]$ and $m(x) = B_0 + B_1x + \cdots + B_qx^q \in M_n(M)[x]$, let $f_{ij} = a^0_{ij} + a^1_{ij}x + \cdots + a^p_{ij}x^p$ and $g_{ij} = b^0_{ij} + b^1_{ij}x + \cdots + b^q_{ij}x^q$, where a^l_{ij} are the (i,j)-entries of A_l for $l = 0, 1, \ldots, p$ and b^s_{ij} are the (i,j)-entries of B_s for $s = 0, 1, \ldots, q$. Then, $r(x) = (f_{ij}(x)) \in M_n(R[x])$ and $m(x) = (g_{ij}(x)) \in M_n(M[x])$. If ${}_RM$ is Armendariz and $[r(x).m(x)]_{ij} = 0$ for all i, j, then $A_iB_j = 0$ for all i, j.*

The first main result of this paper is the following:

Theorem 1. *Let ${}_RM$ be a reduced module. For $n = 2k + 1 \geq 3$, the following statements are true:*

(1) ${}_{A_n(R)}A_n(M)$ is an Armendariz module.
(2) ${}_{A_n(R)}A_n(M)$ is a nil-Armendariz module.

Proof. (1) Let $r(x) = A_0 + A_1x + \cdots + A_px^p \in A_n(R)[x]$ and $m(x) = B_0 + B_1x + \cdots + B_qx^q \in A_n(M)[x]$ satisfy $r(x).m(x) = 0$. Here, we identify $A_n(R)[x]$ with $A_n(R[x])$ and $A_n(M)[x]$ with $A_n(M[x])$ canonically. Then, $r(x) = (f_{ij}(x)) \in A_n(R[x])$ and $m(x) = (g_{ij}(x)) \in A_n(M[x])$, where $f_{ij} = a^{(0)}_{ij} + a^{(1)}_{ij}x + \cdots + a^{(p)}_{ij}x^p$ and $g_{ij} = b^{(0)}_{ij} + b^{(1)}_{ij}x + \cdots + b^{(q)}_{ij}x^q$. We show that $[r(x).m(x)]_{ij} = 0$ for all i, j. Firstly, notice that $r(x)$ and $m(x)$ have following properties:

$$\begin{aligned} f_1 &:= f_{11} = f_{22} = \cdots = f_{nn} & g_1 &:= g_{11} = g_{22} = \cdots = g_{nn} \\ f_2 &:= f_{12} = f_{23} = \cdots = f_{n-1,\, n} & g_2 &:= g_{12} = g_{23} = \cdots = g_{n-1,\, n} \\ &\vdots & &\vdots \\ f_k &:= f_{1k} = f_{2,\, k+1} = \cdots = f_{n-k+1,\, n} & g_k &:= g_{1k} = g_{2,\, k+1} = \cdots = g_{n-k+1,\, n} \\ f_{i,\, j} &:= 0, \ i > j & g_{i,\, j} &:= 0, \ i > j. \end{aligned}$$

Now, $r(x).m(x) = 0$ implies

$$\sum_{i+j=t} f_ig_j = 0 \text{ for } t = 2, 3, \ldots, k+1. \quad (1)$$

We know that ${}_RM$ is a reduced module if and only if ${}_{R[x]}M[x]$ is reduced ([6], Theorem 1.6). Thus, from $f_1g_1 = 0$, we obtain $f_1^2g_1 = 0$ and, hence, $f_1R[x]g_1 = 0$. Multiplying $f_1g_2 + f_2g_1 = 0$ by f_1 from the left side, we obtain $f_1^2g_2 = 0$, which implies $f_1g_2 = 0$, thus $f_2g_1 = 0$. Similarly, multiplying $f_1g_3 + f_2g_2 + f_3g_1 = 0$ by f_1 from the left, we obtain $f_1^2g_3 + f_1f_2g_2 + f_1f_3g_1 = 0$, hence $f_1^2g_3$, which implies $f_1g_3 = 0$. Again, multiplying f_2 with the same equation, we obtain $f_2^2g_2 + f_2f_3g_1 = 0$, which implies $f_2^2g_2 = 0$ and, hence, $f_3g_1 = 0$. Similarly continuing this process, we obtain

$$f_ig_j = 0 \, \forall \ i + j \leq k + 1. \quad (2)$$

This implies that $[r(x).m(x)]_{ij} = 0$ for all i, j with $(i,\ j) \notin \Gamma$, where $\Gamma = \{(u,\ k+u) : u = 1, \ldots, k+1\} \cup \{(u,\ k+u+1) : u = 1, \ldots, k\} \cup \cdots \cup \{(u,\ u+n-2) : u = 1,\ 2\} \cup \{(u,\ n-1+u) : u = 1\}$.
Again from $r(x).m(x) = 0$, we have

$$\begin{aligned} f_1 g_{1,\ k+1} + f_2 g_k + f_3 g_{k-1} + \cdots + f_k g_2 + f_{1,\ k+1} g_1 &= 0 \\ f_1 g_{2,\ k+2} + f_2 g_k + f_3 g_{k-1} + \cdots + f_k g_2 + f_{2,\ k+2} g_1 &= 0 \\ &\vdots \\ f_1 g_{k+1,\ 2k+1} + f_2 g_k + \cdots + f_{k-1} g_3 + f_k g_2 + f_{k+1,\ 2k+1} g_1 &= 0. \end{aligned}$$

By applying the same process of left multiplication and using the earlier results obtained in Equation (2), we conclude that for $u = 1, 2, \ldots, k+1$,

$$f_1 g_{u,\ k+u} = f_{u,\ k+u} g_1 = 0 \tag{3}$$

and with $i + j = k + 2$ for i, j,

$$f_i g_j = 0. \tag{4}$$

Thus, from Equations (3) and (4), we obtain $[r(x).m(x)]_{u,\ u+k} = 0$ for $1 \le u \le k+1$.
Now, for some $1 \le l \le k$, assume the condition $[r(x).m(x)]_{u,\ k+u+t} = 0$ holds true for $0 \le t \le l-1$ and $1 \le u \le k-t+1$. Thus, it is sufficient to show that for each $u = 1, \ldots, k-t+1$, the equation $[r(x).m(x)]_{u,\ k+u+l} = 0$ holds true. Again, $r(x).m(x) = 0$ gives

$$\sum_{j=1}^{n} f_{u,\ j} g_{j,\ k+u+l} = 0 \text{ for } u = 1, \ldots, k-l+1.$$

Thus,

$$\begin{aligned} f_1 g_{u,\ k+u+l} + \cdots + f_{l+1} g_{u+l,\ k+u+l} + f_{l+2} g_k + \cdots + f_k g_{l+2} + f_{u,\ k+u} g_{l+1} \\ + \cdots + f_{u,\ k+u+l-1} g_2 + f_{u,\ k+u+l} g_1 = 0. \end{aligned} \tag{5}$$

Again, by induction hypothesis and using results obtained in (2)–(4), we obtain the following:

(i) (a) $f_1 g_{u,\ k+u+t} = f_{u,\ k+u+t} g_1 = 0$, for $1 \le u \le k-t+1; 0 \le t \le l-1$.

(b) $f_2 g_{u+1,\ k+u+t} = f_{u,\ k+u+t-1} g_2 = 0$, for $1 \le u \le k-t+1; 1 \le t \le l-1$.

$\vdots$

(c) $f_{t+1} g_{u+t,\ k+u+t} = f_{u,\ k+u} g_{l+1} = 0$, for $1 \le u \le k-t+1; t = l-1$.

(ii) $f_i g_j = 0$, for $i + j = u + k$, $i, j \ge u$ and $1 \le u \le l+1$.

Thus, (i), (ii), and the left multiplication process imply each left-side component of Equation (5) is equal to zero. Hence, $[r(x).m(x)]_{u,\ k+u+t} = 0$ for $1 \le u \le k-l+1$. Hence, mathematical induction gives $[r(x).m(x)] = 0 \ \forall\ (i, j) \in \Gamma$. Thus, ${}_{A_n(R)}A_n(M)$ is an Armendariz module.

(2) By using the calculations in Lemma 1, it is easy to verify that $A_n(M)$ is a nil module over $A_n(R)$. Thus, it is nil-Armendariz.
□

Theorem 2. *Let ${}_RM$ be a reduced module. For $n = 2k \ge 2$, the following statements are true:*

(1) $A_n(M) + E_{1,k}M$ is an Armendariz module over $A_n(R) + E_{1,k}R$.
(2) $A_n(M) + E_{1,k}M$ is a nil-Armendariz module over $A_n(R) + E_{1,k}R$.

Proof. The proof of this theorem is almost similar to that of Theorem 1(1) above. However, for further illustration, we demonstrate it as follows:

(1) Consider $r(x).m(x) = 0$ for some $r(x) = (f_{ij}) \in A_n(R)$ and $m(x) = (g_{ij}) \in A_n(M)$. Firstly, we notice that $r(x)$ and $m(x)$ have the following properties:

$$\begin{array}{ll} f_1 := f_{11} = f_{22} = \cdots = f_{nn} & g_1 := g_{11} = g_{22} = \cdots = g_{nn} \\ f_2 := f_{12} = f_{23} = \cdots = f_{n-1,\, n} & g_2 := g_{12} = g_{23} = \cdots = g_{n-1,\, n} \\ \vdots & \vdots \\ f_k := f_{1k} = f_{2,\, k+1} = \cdots = f_{k+1,\, n} & g_k := g_{1k} = g_{2,\, k+1} = \cdots = g_{k+1,\, n} \\ f_0 = f_{1,\, k} & g_0 = g_{1,\, k} \\ f_{i,\, j} := 0,\ i > j & g_{i,\, j} := 0,\ i > j. \end{array}$$

Now, we have

$$\sum_{i+j=t} f_i g_j = 0 \ for \ t = 2, 3, \ldots, k+1 \tag{6}$$

$$f_1 g_0 + f_2 g_{k-1} + \cdots + f_{k-1} g_2 + f_0 g_1 = 0. \tag{7}$$

By applying a similar left multiplication to Equations (6) and (7), we obtain

$$f_i g_j = 0 \, \forall \ i + j \leq k + 1. \tag{8}$$

and

$$f_1 g_0 = f_0 g_1 = 0. \tag{9}$$

This implies that $[r(x).m(x)]_{ij} = 0$ for all i, j with $(i, j) \notin \Gamma$, where $\Gamma = \{(u,\ k+u) : u = 1, \ldots, k+1\} \cup \{(u,\ k+u+1) : u = 1, \ldots, k\} \cup \cdots \cup \{(u,\ u+n-2) : u = 1, 2\} \cup \{(u,\ n-1+u) : u = 1\}$.

Again, from $r(x).m(x) = 0$, we have

$$f_1 g_{1,\, k+1} + f_2 g_k + f_3 g_{k-1} + \cdots + f_{k-1} g_3 + f_0 g_2 + f_{1,\, k+1} g_1 = 0 \tag{10}$$

and

$$\begin{aligned} f_1 g_{2,\, k+2} + f_2 g_k + \cdots + f_k g_2 + f_{2,\, k+2} g_1 &= 0 \\ &\vdots \\ f_1 g_{k,\, 2k} + f_2 g_k + \cdots + f_{k-1} g_3 + f_k g_2 + f_{k,\, 2k} g_1 &= 0. \end{aligned}$$

By applying the same process of left multiplications and using the earlier results obtained in Equations (6)–(9), we conclude that for $u = 1, 2, \cdots, k+1$

$$f_1 g_{u,\, k+u} = f_{u,\, k+u} g_1 = 0 \tag{11}$$

and with $i + j = k + 2$ for i, j

$$f_i g_j = 0 \tag{12}$$

and

$$f_0 g_2 = f_2 g_0 = 0.$$

Thus, from Equations (11) and (12), we obtain $[r(x).m(x)]_{u,\, u+k} = 0$ for $u = 1, 2, \cdots, k$. Now, for some $1 \leq l \leq k$, assume the condition $[r(x).m(x)]_{u,\, k+u+t} = 0$ holds true for $0 \leq t \leq l$ and $1 \leq u \leq k - t$. Thus, it is sufficient to show that for each $u = 1, \ldots, k-l$, the equation $[r(x).m(x)]_{u,\, k+u+l} = 0$ holds true. For these, consider $r(x).m(x) = 0$. This implies that

$$\begin{cases} f_1 g_{u,\, k+u+l} + \cdots + f_{l+1} g_{u+l,\, k+u+l} + f_{l+2} g_k + \cdots + f_k g_{l+2} + f_{u,\, k+u} g_{l+1} + \\ \cdots + f_{u,\, k+u+l-1} g_2 + f_{u,\, k+u} g_{l+1} + \cdots + f_{u,\, k+u+l-1} g_2 + f_{u,\, k+u+l} g_1 = 0 \end{cases} \tag{13}$$

and

$$\begin{cases} f_1 g_{1,\,k+l+1} + \cdots + f_{l+1} g_{l+1,\,k+1} + f_{l+2} g_k + \cdots + f_{k-1} g_{l+3} + f_0 g_{l+2} + f_{1,\,k+1} g_{l+1} + \\ \cdots + f_{1,\,k+1} g_2 + f_{1,\,k+l+1} g_1 = 0. \end{cases} \tag{14}$$

Again, by an induction hypothesis and using the results obtained in (8)–(12), we obtain the following:

(i) (a) $f_1 g_{u,\,k+u+t} = f_{u,\,k+u+t} g_1 = 0$, for $1 \le u \le k-t$; $0 \le t \le l-1$.

(b) $f_2 g_{u+1,\,k+u+t} = f_{u,\,k+u+t-1} g_2 = 0$, for $1 \le u \le k-t$; $1 \le t \le l-1$.

$\vdots$

(c) $f_{t+1} g_{u+t,\,k+u+t} = f_{u,\,k+u} g_{t+1} = 0$, for $1 \le u \le k-t$; $t = l-1$.

(ii) $f_i g_j = 0$ for $i+j = u+k,\ i,j \ge u$ for $1 \le u \le l+1$.

(iii) $f_0 g_u = 0$ and $1 \le u \le l+1$.

Thus, from (i), (ii), (iii), and the left multiplication process, we find that each component of Equations (13) and (14) is equal to zero. Hence, $[r(x).m(x)]_{u,\,k+u+t} = 0$ for $u = 1, \ldots, k-l$. Hence, mathematical induction gives $[r(x).m(x)] = 0\,\forall\,(i,j) \in \Gamma$.

(2) By using the calculations in Lemma 1, it is easy to verify that $A_n(M) + E_{1,k}M$ is a nil module over $A_n(R) + E_{1,k}R$. Thus, it is nil-Armendariz.

□

Proposition 6. *Let R be a commutative ring. If $nil_R(M) \subseteq {}_RM$, then the quotient module $M/nil_R(M)$ is rigid.*

Proof. Let $u^2\overline{m} = \overline{0}$ in $M/nil_R(M)$. This implies that $u^2 m \in nil_R(M)$. Thus, there exists some $r \in R$ such that $r^2u^2m = 0$ and $ru^2m \neq 0$. Since R is commutative, $r^2u^2m = 0$ implies that $(ru)^2m = 0$, but $ru^2m \neq 0$. This implies that $um \in nil_R(M)$. Therefore, $u\overline{m} = \overline{0}$ in $M/nil_R(M)$. Hence, $M/nil_R(M)$ is a rigid module. □

Proposition 7. *Let R be a commutative ring and ${}_RM$ be a torsion-free module. If $nil_R(M) \subseteq {}_RM$, then $M/nil_R(M)$ is torsion-free.*

Proof. Suppose that $\overline{0} \neq \overline{m} \in \mathrm{Tor}(M/nil_R(M))$. Thus, there exists a non-zero $t \in R$ such that $t\overline{m} = \overline{0}$. This means that $tm \in nil_R(M)$. Therefore, there exists some $l \in R$ such that $l^2tm = 0$ and $ltm \neq 0$. This implies that $(lt)^2m = 0$ but $ltm \neq 0$. Hence, $m \in nil_R(M)$. Therefore, $\overline{m} = \overline{0}$ in $M/nil_R(M)$. □

Proposition 8. *Let R be a commutative ring. If $nil_R(M) \subseteq {}_RM$, then ${}_RM$ is nil-Armendariz.*

Proof. Recall from [5] that if a module ${}_RM$ is both rigid and semi-commutative, then it is Armendariz. We observe, as per Proposition 6, that $M/\mathrm{nil}_R(M)$ constitutes a rigid module. Since R is commutative, this implies that $M/\mathrm{nil}_R(M)$ is semi-commutative. Thus, $M/\mathrm{nil}_R(M)$ is an Armendariz module. Let us consider $p(x)m(x) \in \mathrm{nil}_RM[x]$. Clearly, $p(x)\overline{m(x)} = \overline{0}$, where $\overline{m(x)}$ signifies the corresponding polynomial in $M/\mathrm{nil}_R(M)[x]$. Consequently, $r\overline{m} = \overline{0}$ for all $r \in \mathrm{coef}(p(x))$ and $\overline{m} \in \mathrm{coef}(\overline{m(x)})$. This suggests that rm is a nilpotent element for all $r \in \mathrm{coef}(p(x))$ and $m \in \mathrm{coef}(m(x))$. □

Proposition 9. *Let N be a submodule of ${}_RM$. If N is a subset of $nil_R(M)$, then ${}_RM$ is nil-Armendariz if and only if M/N is nil-Armendariz over R.*

Proof. Let $f(x) = \sum_{i=0}^{n} r_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$. We denote $\overline{M} = M/N$. Since N is a nil submodule, $nil(\overline{M}) = \overline{nil(M)}$. Hence, $f(x)m(x) \in nil_R(M)[x]$ if and only if $\overline{f(x)} \cdot \overline{m(x)} \in nil_R(\overline{M})[x]$. Therefore, we conclude that $am \in nil_R(M)$ if and only if $a\overline{m} \in nil(\overline{M})$. Thus, M is nil-Armendariz if and only if $\overline{M}$ is nil-Armendariz. □

For a module $_RM$, recall that if R is a commutative domain, then $Tor(M)$ is a submodule and $M/Tor(M)$ is torsion-free. However, the same is not true if R contains a non-zero zero divisor, as illustrated by $M = R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Here, $Tor(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0,0), (1,0), (0,1)\}$, which is not a submodule. Next, we identify some conditions for the nil-Armendariz property in the context of the torsion class.

Proposition 10. *Let R be a commutative domain. Then, $_RM$ is nil-Armendariz if and only if its torsion submodule $Tor(M)$ is nil-Armendariz.*

Proof. Let $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_k x^k \in M[x]$ satisfy $p(x)m(x) \in nil_R(M)[x]$. Then, we have

$$\begin{aligned} u_0v_0 &\in nil_R(M) \\ u_1v_0 + u_0v_1 &\in nil_R(M) \\ u_2v_0 + u_1v_1 + u_0v_2 &\in nil_R(M) \\ &\vdots \\ u_nv_q &\in nil_R(M). \end{aligned}$$

R being a commutative domain implies that $nil_R(M) \subseteq Tor(M)$. We can assume that $u_0 \neq 0$. Hence, from the first equation, we obtain $u_0v_0 \in nil_R(M) \Rightarrow l^2u_0v_0 = 0$ for some $l \in R$. Thus, $v_0 \in Tor(M)$. $Tor(M)$ is a submodule of $_RM$, implying that $u_1v_0 \in Tor(M)$. Thus, from the second equation, it is clear that $u_0v_1 \in Tor(M)$, which, again, implies that $v_1 \in Tor(M)$. Thus, by repeating the same process finitely many times, we conclude that $m(x) \in Tor(M)[x]$. Therefore, M is a nil-Armendariz module. □

Proposition 11. *Let R be a commutative domain. If $_RM$ is a nil-Armendariz module, then $M/Tor(M)$ is a nil-Armendariz module.*

Proof. We denote the quotient $M/Tor(M)$ by $\overline{M}$. Since $\overline{M}$ is torsion-free, it is sufficient to show that $\overline{M}$ is Armendariz. Let $\overline{m(x)} = \sum_{k=0}^{q} \overline{v}_i x^i \in \overline{M}[x]$ and $p(x) = \sum_{l=0}^{n} u_j x^j \in R[x]$ satisfy $p(x)\overline{m}(x) = \overline{0}$ in $\overline{M}[x]$. Then, we have

$$\begin{aligned} u_0\overline{v}_0 &= \overline{0} \\ u_0\overline{v}_1 + u_1\overline{v}_0 &= \overline{0} \\ &\vdots \\ u_n\overline{v}_q &= \overline{0}. \end{aligned}$$

Now, from first equation, we have $u_0v_0 \in Tor(M)$, which further implies $v_0 \in Tor(M)$. Since $Tor(M)$ is a submodule of M, $u_1v_0 \in Tor(M)$. Thus, from the second equation, we obtain $u_0v_1 \in Tor(M)$. Thus, repeating the same process finitely many times, we conclude that $u_lv_k \in Tor(M)$ for $0 \leq l \leq n$ and $0 \leq k \leq q$. Thus, $M/Tor(M)$ is an Armendariz module. □

Here, we record a "change of rings" result.

Proposition 12. *Let $_AM$ be a module over a ring A and $\phi : R \to A$ be a ring homomorphism. By defining $uv = \phi(u)v$, M can be made an R-module. If ϕ is onto, then the following are equivalent.*

(1) $_RM$ is nil-Armendariz.
(2) $_AM$ is nil-Armendariz.

Proof. Firstly, we show that if $rm \in nil_R(M)$, then $\theta(r)m \in nil_A(M)$. So, let $rm \in nil_R(M)$. Thus, there exists some $k \in R$ such that $k^2rm = 0$ and $krm \neq 0$. Now, $0 = k^2rm =$

$\theta(k^2r)m = (\theta(k))^2\theta(r)m$ and $0 \neq krm = \theta(k)\theta(r)m$. Thus, $\theta(r)m \in nil_A(M)$ and vice verse. Thus, the remaining part of the proof easily follows. □

Recall that for a multiplicative closed subset S of the center C of the ring R, the set $S^{-1}M$ has a left module structure over $S^{-1}R$. In the next proposition, we study the localization.

Lemma 4. *For a module $_RM$, an element $v \in nil_R(M)$ if and only if $d^{-1}v \in nil_{S^{-1}R}(S^{-1}M)$ for some $d \in S$.*

Proof. Suppose $d^{-1}m \in nil_{S^{-1}R}(S^{-1}M)$, where $d \in S$ and $v \in M$. Thus, $\exists\ s^{-1}r \in S^{-1}R$ such that $(s^{-1}r)^2d^{-1}m = 0$ but $s^{-1}rd^{-1}v \neq 0$. This implies that $r^2v = 0$ but $rv \neq 0$. Hence $v \in nil_R(M)$. For the converse part, suppose $v \in nil_R(M)$. Thus, $t^2v = 0$ but $tv \neq 0$ for some $t \in R$. Hence, $(r^2/1).v/d = 0$, but $(r/1).v/d \neq 0$. □

Theorem 3. *For a module $_RM$, the following conditions are equivalent.*

(1) *$_RM$ is nil-Armendariz.*

(2) *$S^{-1}M$ is a nil-Armendariz $S^{-1}R$-module for each multiplicatively closed subset S of C.*

Proof. (1)⇒(2) Let $f(x) = \sum_{i=0}^{m} \xi_i x^i \in S^{-1}R[x]$ and $m(x) = \sum_{j=0}^{n} \eta_j x^j \in S^{-1}M[x]$ such that $f(x) \cdot m(x) \in nil_{S^{-1}R}S^{-1}M[x]$. Here, $\xi_i = s_i^{-1}x^i \in S^{-1}R$ and $\eta_j = t_j^{-1}m_j \in S^{-1}M$. Thus, we have

$$\begin{cases} \xi_0\eta_0 & \in nil_{S^{-1}R}S^{-1}M \\ \xi_0\eta_1 + \xi_1\eta_0 & \in nil_{S^{-1}R}S^{-1}M \\ \vdots & \\ \xi_m\eta_n & \in nil_{S^{-1}R}S^{-1}M. \end{cases} \tag{15}$$

Let us take $s = (s_0s_1\cdots s_m)$ and $t = (t_0t_1\cdots t_n)$ and consider $\overbrace{f(x)} = s.f(x) = \sum_{i=0}^{m} s\xi_i x^i$, $\overbrace{m(x)} = tm(x) = \sum_{j=0}^{n} t\eta_j x^j$. Clearly, $\overbrace{f(x)} \in R[x]$ and $\overbrace{m(x)} \in M[x]$, and $\overbrace{f(x)}.\overbrace{m(x)} = s\xi_0t\eta_0 + (s\xi_0t\eta_1 + s\xi_1t\eta_0)x + (s\xi_2t\eta_0 + s\xi_1t\eta_1 + s\xi_0t\eta_2)x^2 + \cdots + (s\xi_mt\eta_n)x^{m+n}$. From the first equation, we have $\xi_0\eta_0 \in nil_{S^{-1}R}S^{-1}M \Rightarrow (s_0^{-1}a_0)(t_0^{-1}m_0) \in nil_{S^{-1}R}(S^{-1}M) \Rightarrow$ $\exists q^{-1}r \in S^{-1}R$ such that $(q^{-1}r)^2(s_0^{-1}a_0)(t_0^{-1}m_0) = 0$ but $(q^{-1}r)(s_0^{-1}a_0)(t_0^{-1}m_0) \neq 0$. Thus, $r^2a_0m_0 = 0$ but $ra_0m_0 \neq 0$, which implies that $r^2(s_1\cdots s_m)a_0(t_1\cdots t_n)m_0 = 0$ and $r(s_1\cdots s_m)a_0(t_1\cdots t_n)m_0 \neq 0$. Otherwise, suppose $r(s_1\cdots s_m)a_0(t_1\cdots t_n)m_0 = 0$, then

$$(s_1\cdots s_m)^{-1}(t_1\cdots t_n)^{-1}r(s_1\cdots s_m)a_0(t_1\cdots t_n)m_0 = 0 \Rightarrow ra_0m_0 = 0,$$

which is not possible. Thus, $s\xi_0t\eta_0 \in nil_R(M)$. Similarly, we can show that $s\xi_mt\eta_n \in nil_R(M)$. Proceeding in a similar way, again from the first equation, $\xi_0\eta_1 + \xi_1\eta_0 \in nil_{S^{-1}R}S^{-1}M$; we have $s_0^{-1}a_0t_1^{-1}m_1 + s_1^{-1}a_1t_0^{-1}m_0 \in nil_{S^{-1}R}(S^{-1}M)$, which implies that $s_1t_0a_0m_1 + s_0t_1a_1m_0 \in nil_R(M)$. Also, we can see that $s\xi_0t\eta_1 + s\xi_1t\eta_0 = (s_1\cdots s_m)(t_1\cdots t_n)$ $(s_1a_0t_0m_1 + s_0a_1t_1m_0) \in nil_R(M)$. Thus, similarly, we can show that all the coefficients of x^i in $\overbrace{f(x)}.\overbrace{m(x)}$ are in $nil_R(M)$. Since $_RM$ is nil-Armendariz, this implies that $st\xi_i\eta_j \in nil_R(M) \forall\ i, j$. Thus, by Lemma 4, $\xi_i\eta_j \in nil_{S^1R}(S^{-1}M)$. (2)⇒(1) Let $f(x).m(x) \in nil_R(M)[x]$, where $f(x) = \sum_{i=0}^{m} a_ix^i \in R[x]$ and $m(x) = \sum_{j=0}^{n} m_jx^j \in M[x]$. Since $f(x) \in S^{-1}R[x]$ and $m(x) \in S^{-1}M[x]$, $a_im_j \in nil_{S^{-1}M}(S^{-1}M)$, by Lemma 4, $a_im_j \in nil_R(M)$. □

Theorem 4. *Let R be a commutative domain. Then, for a module $_RM$, the following are equivalent:*

(1) *$_RM$ is nil-Armendariz.*

(2) *$_QM$ is nil-Armendariz, where Q is the field of fraction of R.*

Proof. The proof of this theorem follows similarly to that of Theorem 3. □

3. Results on Nilpotent Class of Modules

In ring theory, the class of nilpotent elements forms an ideal, provided the ring is commutative, semi-commutative, or even nil-Armendariz. However, the same is not true for the class of nilpotent elements in modules. A finite sum of nilpotent elements of a module ${}_RM$ is not necessarily nilpotent in ${}_RM$, even when ${}_RM$ is defined over a commutative ring R. For example, $\overline{1}$ and $\overline{3}$ are nilpotent elements in ${}_{\mathbb{Z}}\mathbb{Z}_8$ since $\overline{2}^3.\overline{1} = 0$ but $\overline{2}.\overline{1} = \overline{2} \neq \overline{0}$ and $\overline{2}^3.\overline{3} = 0$ but $\overline{2}.\overline{3} = \overline{6} \neq \overline{0}$. However, their sum $\overline{4}$ is not nilpotent. Additionally, the class of nilpotent elements is not closed under left multiplication by R, even if R is commutative. For instance, $\overline{2} \in nil_{\mathbb{Z}}\mathbb{Z}_8$, but $2.\overline{2} = 4 \notin nil_{\mathbb{Z}}(\mathbb{Z}_8)$. In [8], SSevviiri and Groenewald posed the question of the conditions under which $nil_R(M)$ forms a submodule. Here, we find some conditions under which $nil_R(M)$ may form a submodule.

Lemma 5. *Let ${}_RM$ be nil-Armendariz. Then, the following are true.*
(1) If $u \in Nil(R)$ and $v \in nil_R(M)$, then $uv \in nil_R(M)$.
(2) If $v, w \in nil_R(M)$, then $v + w \in nil_R(M)$.
(3) If $u, y \in Nil(R)$ and $v \in nil_R(M)$, then $(u + y)v \in nil_R(M)$.

Proof. (1) Suppose $u \in nil(R)$ and $u^t = 0$. Then,

$$(1 + ux + ux^2 + \cdots + u^{t-1}x^{t-1}).(v - uvx) = v \in nil_R(M)[x].$$

which further implies that $uv \in nil_R(M)$.

(2) Suppose $v, w \in nil_R(M)$.

$$(1 - x).(w + (u + w)x + vx^2) = w + ux - wx^2 - ux^3 \in nil_R(M)[x].$$

Now, since ${}_RM$ is nil-Armendariz, from each polynomial, we can select the suitable coefficients to obtain $1.(v + w) \in nil_R(M)$.

(3) Suppose $u, y \in Nil(R)$, then $u^k = y^l = 0$. Then,

$$(1 + ux + \cdots + u^{k-1}x^{k-1})(1 - ux)(1 - yx)(1 + yx + \cdots + y^{l-1}x^{l-1})v = v.$$

Multiplying the intermediate polynomials yields

$$(1 + ux + \cdots + u^{k-1}x^{k-1})(1 - (u + y)x + uyx^2)(1 + yx + \cdots + y^{l-1}x^{l-1})v = v.$$

Now, since ${}_RM$ is nil-Armendariz, and $m \in nil_R(M)[x]$, from each polynomial, we can select the suitable coefficients to obtain $(u + y)v \in nil_R(M)$. □

Proposition 13. *Let ${}_RM$ be a nil-Armendariz module. If R is a nil ring, then $nil_R(M)$ is a submodule of ${}_RM$.*

Proof. Since R is a nil ring, it follows directly from (5) that $nil_R(M)$ is a submodule of ${}_RM$. □

Proposition 14. *Let ${}_RM$ be a nil-Armendariz module over a finitely generated commutative ring R. Then, $nil_R(M)$ is a submodule of ${}_RM$ if every proper ideal of R is nil ideal.*

Proof. Since every proper ideal is nil ideal, by Theorem 2.1 in [13], it follows that R is a nil ring. Hence, by Lemma 5, $nil_R(M)$ is a submodule of ${}_RM$. □

For a left R-module M, we generally have $Tor(M) \nsubseteq nil_R(M)$ as $\overline{2} \in Tor(\mathbb{Z}_4)$ while $\overline{2} \notin nil_{\mathbb{Z}_4}(\mathbb{Z}_4)$. Considering the definitions of $Tor(M)$ and $nil_R(M)$, one could suspect $nil_R(M)$ to be a subset of $Tor(M)$. However, the example given below refutes this possibility.

Example 1. *Consider the module ${}_{\mathbb{Z}}\mathbb{Z}$. Then, by Lemma 1, the matrix module $M_3(\mathbb{Z})$ is a nil module over $M_3(\mathbb{Z})$. On the other hand, consider $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. If possible, let us suppose that A is a torsion element in $M_3(\mathbb{Z})$. Then, by definition, there exits a non-zero $L = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ in $M_3(\mathbb{Z})$, satisfying $LA = 0$. However, solving $LA = 0$ implies that $a_{ij} = 0 \; \forall \; i, j \Rightarrow L = 0$. Thus, $A \notin Tor(M_3(\mathbb{Z}))$.*

Theorem 5. *If a module ${}_RM$ is torsion-free, then $nil_R(M)$ is a submodule of ${}_RM$.*

Proof. If the ring R is reduced, then it is obvious that $nil_R(M) \subseteq Tor(M) = \{0\}$, so $nil_R(M)$ is a submodule. On the other hand, consider that R is non-reduced; then, there exits some non-zero $a \in R$ such that $a^2 = 0$. Thus, by hypothesis that ${}_RM$ is torsion-free, we have $a^2m = 0$ for every $m \in {}_RM$ and $am \neq 0$ in ${}_RM$. Hence, $nil_R(M)$ is a submodule. □

4. Conclusions and Future Scope

In this paper, we delve into the structure and properties of nil-Armendariz modules, providing a new perspective on how nilpotent elements influence module theory. Examining quotient and matrix modules within this framework reveals intricate relationships between nilpotency and module operations, enriching our understanding of module behavior over non-commutative rings. Our findings on the formation of submodules from nilpotent classes uncover additional layers of structure that have been unexplored, underscoring the significance of these classes in the broader algebraic context. The preservation of nil-Armendariz properties under localization further reinforces the utility of these modules in various algebraic settings.

Looking ahead, there are several promising avenues for future research. One potential direction is to explore the interaction of nil-Armendariz modules with other module-theoretic properties, such as injectivity or projectivity, and to investigate how these interactions can inform the classification of modules over different types of rings. Another area worth exploring is the extension of nil-Armendariz concepts to modules over more complex algebraic structures, such as graded rings or rings with additional topological properties. Additionally, the potential applications of nil-Armendariz modules in computational algebra and their role in solving problems related to ring extensions or module homomorphisms present exciting challenges. Finally, a further study into the connections between nil-Armendariz modules and homological dimensions could lead to new insights in both algebra and homological algebra.

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