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Idempotent-Aided Factorizations of Regular Elements of a Semigroup

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Abstract: In the present paper, we introduce the concept of idempotent-aided factorization (I.-A. factorization) of a regular element of a semigroup, which can be understood as a semigroup-theoretical extension of full-rank factorization of matrices over a field. I.-A. factorization of a regular element d is defined by means of an idempotent e from its Green's $\mathscr{D}$-class as decomposition into the product $d = uv$, so that the element u belongs to the Green's $\mathscr{R}$-class of the element d and the Green's $\mathscr{L}$-class of the idempotent e, while the element v belongs to the Green's $\mathscr{L}$-class of the element d and the Green's $\mathscr{R}$-class of the idempotent e. The main result of the paper is a theorem which states that each regular element of a semigroup possesses an I.-A. factorization with respect to each idempotent from its Green's $\mathscr{D}$-class. In addition, we prove that when one of the factors is given, then the other factor is uniquely determined. I.-A. factorizations are then used to provide new existence conditions and characterizations of group inverses and (b, c)-inverses in a semigroup. In our further research, these factorizations will be applied to matrices with entries in a field, and efficient algorithms for realization of such factorizations will be provided.

Keywords: Green's equivalences; trace product; factorization; group inverse; (b, c)-inverse

MSC: 20M10; 20M99; 15A09

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1. Introduction

Factorization is the general mathematical idea of representing a mathematical object as a product of other objects, which are usually simpler objects of the same kind. Over hundreds of years, this idea has proven very successful in its application to numbers, polynomials, matrices, and other types of mathematical objects. Factorization problems in semigroups, discussed in numerous papers, have been mainly motivated by corresponding problems in number theory and the fundamental theorem of arithmetic that asserts that every natural number greater than one can be decomposed into a product of prime numbers, in an essentially unique way. In one direction, this led to an extensive study of decompositions of elements of semigroups and monoids into products with irreducible or atomic factors (for a review of classical results in that area and their recent generalizations, we refer to [1–3]), and in another, to consideration of the uniqueness problem for such decompositions (see, for instance, [4,5]). The ideas, results, and methods of factorization theories for elements of monoids have also been successfully used in the decomposition of various algebraic structures, such as monoids, groups, rings, or modules, into the product or sum of their proper substructures (cf. [6–11]). In addition to factorizations inspired by the factorization of integers, different factorizations are also considered, such as, for example, idempotent factorizations, where an element of a semigroup is decomposed into a product of idempotents. Such factorizations are inspired by the classical result which states that every square non-invertible matrix can be decomposed into a product of idempotent matrices (see [12], and for information on recent advances in the field, see [13]).

In this paper, we approach the factorization problem in semigroups with a different motivation, which comes from linear algebra and matrix theory. Namely, we introduce the concept of *idempotent-aided factorization* (I.-A. factorization), where an arbitrary regular element d of a semigroup S is decomposed into the product $d = uv$ in such a way that the position of the factors u and v in the structure of the Green's equivalence classes of that semigroup is determined by the element d, which is factorized, and an idempotent e, which additionally specifies those locations. More precisely, u is required to belong to the $\mathscr{H}$-class $R_d \cap L_e$ and v to belong to the $\mathscr{H}$-class $L_d \cap R_e$. For instance, in the semigroup containing matrices over fields, the I.-A. factorization of the matrix D with regard to a given idempotent matrix E of the same rank as D, is its representation as the product $D = UV$, where the matrix U has an identical range (image, column space) to D and identical null-space (kernel) to E, while the matrix V has the same range as E and the same null-space as D. Arbitrary I.-A. factorization of a matrix D with respect to the identity matrix of the same rank as D is just a full-rank factorization (F.-R. factorization) of D, so the concept of I.-A. factorization can be understood as an extension of the classical matrix F.-R. factorization.

The key result of the current research is achieved in Theorem 3, which asserts that any regular element d of a semigroup possesses a factorization with respect to any idempotent e from the $\mathscr{D}$-class of d. In addition, the same theorem asserts that for every $u \in R_d \cap L_e$ there is a unique $v \in R_e \cap L_d$ such that (u, v) is a factorization pair for d with respect to e, and conversely, for every $v \in R_e \cap L_d$ there is a unique $u \in R_d \cap L_e$ such that (u, v) is a factorization pair for d with respect to e. Theorem 3 also says that for every factorization $d = uv$ such that $u \in R_d$, $v \in L_d$, there exists an idempotent e so that this factorization is an I.-A. factorization with respect to e. In Theorems 4 and 5, the existence conditions and characterizations of group inverses and (b, c)-inverses, based on I.-A. factorizations, have been given. The basic idea behind these results is that the group invertibility issue of an element d is solved in the $\mathscr{H}$-class of an arbitrary idempotent from the $\mathscr{D}$-class of d, instead of in the $\mathscr{H}$-class of d, while the problem of the (b, c)-invertibility is solved in the $\mathscr{H}$-class of an arbitrary idempotent from the $\mathscr{D}$-class of b and c, instead of in the $\mathscr{H}$-classes $R_b \cap L_c$ and $L_b \cap R_c$. In particular, in the semigroup of matrices over a field, these problems can be transferred to the $\mathscr{H}$-class of the identity matrix of the same rank as the matrix D or the matrices B and C under consideration.

It is worth noting that I.-A. factorizations originate from the concept of a trace product of elements of a semigroup, which was first studied in [14]. In [15–17], trace products were used in the construction of regular semigroups, and in [18] they were used in the construction of pseudo-inverse semigroups. As shown there, the structure of a pseudo-inverse semigroup is completely determined if we know trace products and the natural partial order on it, and to determine the structure of a regular semigroup we need to know trace products and the biordered set of idempotents. In [19], trace products were used in the study of generalized inverses in a semigroup.

The paper's organization is subject to the following global structure. After this prelusive part, in Section 2 we introduce some fundamental terms and present some basic results that will be utilized below. These are mainly concepts and results concerning Green's relations in a semigroup. For the sake of completeness, we also list some well-known results that can be found in most basic books on general semigroup theory, including the books [20,21]. In the same section, we give examples that show what Green's relations look like on principal semigroups considered in later examples—the semigroup of matrices over a field and the full transformation semigroup on a set. Section 3 involves the basic result on I.-A. factorizations as well as examples to illustrate proposed factorizations. Finally, in Section 4 theorems are proved that provide existence constraints and procedures for computing group inverses and (b, c)-inverses. Examples are also given to illustrate the presented theory. In the examples dealing with matrices over a field, algorithms for performing a particular F.-R. factorization, known as column–row factorization, were used.

2. Preliminaries

According to standard notation, $\mathbb{N}$ symbolizes the set of all natural numbers (without zero), and $\mathbb{R}$ stands for the set of all real numbers.

Let S be a semigroup. If S does not have an identity, then S^1 denotes a new semigroup created from S by joining an element 1 (which is not in S) and extending the multiplication so that 1 is an identity in S^1. Otherwise, if S has an identity, we write $S^1 = S$.

For an element $a \in S$, $L(a) = S^1 a = Sa \cup \{a\}$ is the least left ideal of S containing a, termed as the *principal left ideal* of S generated by a, while $R(a) = aS^1 = aS \cup \{a\}$ is the least right ideal of S containing a, termed as the *principal right ideal* of S generated by a. *Green's equivalences* $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, and $\mathscr{D}$ on S are defined as

$$p \,\mathscr{L}\, q \;\Leftrightarrow\; L(p) = L(q) \;\Leftrightarrow\; (\exists s, t \in S^1)\; p = sq,\; q = tp$$
$$p \,\mathscr{R}\, q \;\Leftrightarrow\; R(p) = R(q) \;\Leftrightarrow\; (\exists u, v \in S^1)\; p = bq,\; q = pv$$

for all $a, b \in S$, and $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$. Since $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ is an equivalence relation (where $\circ$ is the composition of binary relations), $\mathscr{D}$ is defined by $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. The equivalence classes of $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, and $\mathscr{D}$ containing $a \in S$ are denoted by L_a, R_a, H_a, and D_a, respectively.

Because $\mathscr{L} \subseteq \mathscr{D}$ and $\mathscr{R} \subseteq \mathscr{D}$, each $\mathscr{D}$-class is the union of $\mathscr{L}$-classes contained in it, and also, is the union of $\mathscr{R}$-classes contained in it. The subsequent equivalencies are valid for any $p, q \in S$:

$$p \,\mathscr{D}\, q \;\Leftrightarrow\; R_p \cap L_q \neq \emptyset \;\Leftrightarrow\; L_p \cap R_q \neq \emptyset. \tag{1}$$

The result (1) enables us to visualize the $\mathscr{D}$-class in a diagram that Clifford and Preston [20] termed an 'eggbox', in which rows represent $\mathscr{R}$-classes, columns represent $\mathscr{L}$-classes, and cells represent $\mathscr{H}$-classes. Figure 1a gives an illustration.

For a semigroup S, the *inner left translation* λ_a and the *inner right translation* ϱ_a determined by $a \in S$ are mappings of S into itself determined by $\lambda_a(x) = ax$ and $\varrho_a(x) = xa$, for each $x \in S$. The restriction of a mapping χ with a domain D to arbitrary subset $X \subseteq D$ is designated as $\chi|_X$.

Some basic results relevant to Green's equivalences are highlighted in the subsequent results. Proofs of these results can be found in [20,21].

Lemma 1 (Green's lemma [22])**.** *Let p, q be $\mathscr{R}$-related elements of a semigroup S, and let $s, t \in S^1$ satisfy $ps = q$ and $qt = p$. The inner right translations $\varrho_s|_{L_p}$ and $\varrho_t|_{L_q}$ are mutually inverse bijections from L_p onto L_q and L_q onto L_p, in that order.*

Moreover, these bijections preserve $\mathscr{R}$-classes, that is, $x \,\mathscr{R}\, \varrho_s(x)$, for every $x \in L_p$, and $y \,\mathscr{R}\, \varrho_t(y)$, for every $y \in L_q$.

Lemma 2 (Green's lemma [22])**.** *Let p, q be $\mathscr{L}$-related elements of a semigroup S, and let $u, v \in S^1$ satisfy $up = q$ and $qb = p$. Then, the inner left translations $\lambda_u|_{R_p}$ and $\lambda_v|_{R_q}$ are mutually inverse bijections from R_p onto R_q and R_q onto R_p, respectively.*

Moreover, these bijections preserve $\mathscr{L}$-classes, that is, $x \,\mathscr{L}\, \lambda_u(x)$, for every $x \in R_p$, and $y \,\mathscr{L}\, \lambda_v(y)$, for every $y \in R_q$.

Visualizations of Lemmas 1 and 2 are given in Figure 1b and Figure 1c, respectively.

Lemma 3 ([21])**.** *Every idempotent e of a semigroup S is a left identity for elements from R_e and a right identity for elements from L_e.*

Theorem 1 (Green's Theorem [22])**.** *If H is an $\mathscr{H}$-class of a semigroup S, then either $H^2 \cap H = \emptyset$ or H represents a maximal subgroup of S.*

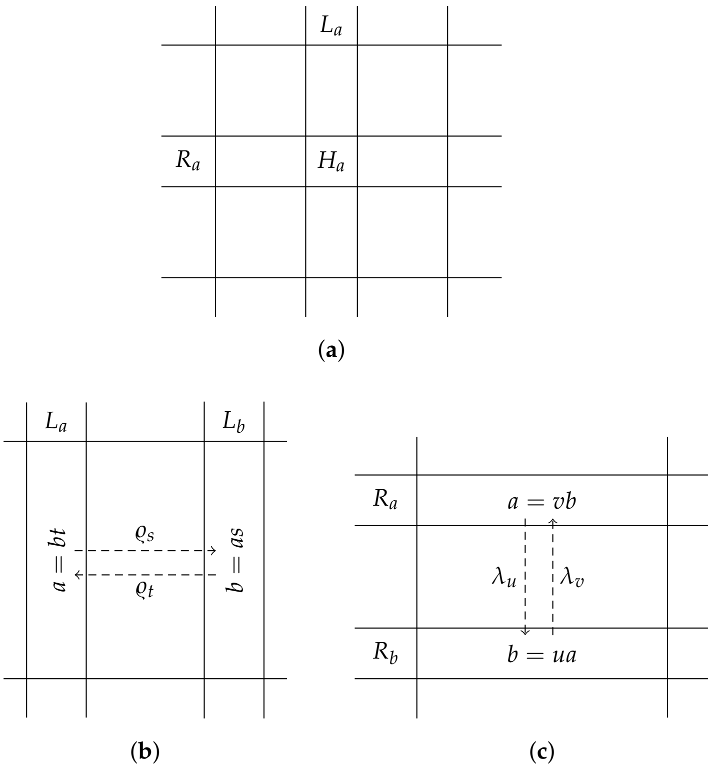

Figure 1. (**a**) Visualizations of the 'eggbox'; (**b**,**c**) Green's lemmas.

Let us note that for an arbitrary idempotent e of a semigroup S, the maximal subgroup of S with e as its identity, i.e., the $\mathscr{H}$-class H_e of e, can be characterized as follows:

$$H_e = \{a \in S \mid a \in R(e) \cap L(e),\ e \in R(a) \cap L(a)\} = \{a \in S \mid a \in eS \cap Se,\ e \in aS \cap Sa\}. \quad (2)$$

Theorem 2 (Miller–Clifford's theorem [14])**.** *Let $p, q \in S$ be arbitrary in a semigroup S. Then, $pq \in R_p \cap L_q$ if and only if $R_q \cap L_p$ contains an idempotent.*

If $pq \in R_p \cap L_q$ holds, then pq is called a *trace product* ([18]; see also [14,19]). This is an old and standard notion of semigroup theory, and when dealing with matrices it should not be confused with the trace of a matrix, which is a completely independent notion of matrix theory.

Example 1. *Let $M(\mathbb{S})$ be the set of all matrices of arbitrary dimensions and with entries from a semiring $\mathbb{S}$, i.e.,*

$$M(\mathbb{S}) = \bigcup_{m,n\in\mathbb{N}} \mathbb{S}^{m\times n},$$

where $\mathbb{S}^{m\times n}$ denotes the set of all $m \times n$-matrices with entries over $\mathbb{S}$. In addition, let $\varnothing \in \mathbb{S}$ not belong to $M(\mathbb{S})$, and consider $M_\varnothing(\mathbb{S}) = M(\mathbb{S}) \cup \{\varnothing\}$.

The multiplication in $M_\varnothing(\mathbb{S})$ is defined as their ordinary matrix product in the case of proper dimensions, and $\varnothing$ otherwise. Supplied by this multiplication, $M_\varnothing(\mathbb{S})$ becomes a semigroup with the zero $\varnothing$. Such a structure will be called the semigroup of matrices over $\mathbb{S}$.

In the following, let $\mathbb{F}$ denote a field. The range $\mathcal{R}(M)$ and null space $\mathcal{N}(M)$ of $M \in \mathbb{F}^{m\times n}$ are subspaces of the vector spaces $\mathbb{F}^m$ and $\mathbb{F}^n$, defined as follows:

$$\mathcal{R}(M) = \{\boldsymbol{q} \in \mathbb{F}^m \mid (\exists \boldsymbol{p} \in \mathbb{F}^n)\, M\boldsymbol{p} = \boldsymbol{q}\}, \quad \mathcal{N}(M) = \{\boldsymbol{p} \in \mathbb{F}^n \mid M\boldsymbol{p} = \boldsymbol{0}\}$$

(cf. [23,24]). It is well known that $\mathcal{R}(M)$ *is a subspace of* $\mathbb{F}^m$ *generated by the column vectors of* M*, and it is also called the column space of* M*. The dimension of* $\mathcal{R}(A)$ *is denoted by* $rank(M)$ *and called the rank of* M*. The set of all* $m \times n$*-matrices over* $\mathbb{F}$ *of rank* r *is denoted by* $\mathbb{F}_r^{m\times n}$*.*

For any $A, B \in M(\mathbb{F})$ *it follows that*

$$\begin{aligned} M\,\mathscr{R}\,N &\Leftrightarrow \mathcal{R}(M) = \mathcal{R}(N); \\ M\,\mathscr{L}\,N &\Leftrightarrow \mathcal{N}(M) = \mathcal{N}(N); \\ M\,\mathscr{H}\,N &\Leftrightarrow \mathcal{R}(M) = \mathcal{R}(N) \ \wedge \ \mathcal{N}(M) = \mathcal{N}(N); \\ M\,\mathscr{D}\,N &\Leftrightarrow rank(M) = rank(N). \end{aligned}$$

Example 2. *Let* $\mathcal{T}_X$ *denote the full transformation semigroup on a set* X*, whose carrier set consists of all mappings from* X *into* X *and the multiplication operation is the composition of mappings. If* $X = \{1, \dots, n\}$ *is a finite set of* n *elements, then* $\mathcal{T}_n$ *is used instead of* $\mathcal{T}_X$*, while a mapping* $\zeta \in \mathcal{T}_n$ *is represented as* $\zeta = (\zeta_1 \dots \zeta_n)$*, such that* $\zeta_i = \zeta(i)$*, for each* $i \in X$*.*

The composition of mappings $\zeta, \theta \in \mathcal{T}_X$ *is defined by* $(\zeta\theta)(x) = \zeta(\theta(x))$*, for every* $x \in X$*.*

For $\zeta \in \mathcal{T}_X$*,* $\mathrm{Im}(\zeta)$ *stands for the image of* ζ *and* $\ker(\zeta)$ *means the kernel of* ζ*, defined by*

$$\mathrm{Im}(\zeta) = \{\delta \in X \,|\, (\exists\gamma \in X)\, \zeta(\gamma) = \delta\}, \quad \ker(\zeta) = \{(\gamma, \delta) \in X \times X \,|\, \zeta(\gamma) = \zeta(\delta)\}.$$

Further, $rank\,(\zeta) = |\,\mathrm{Im}(\zeta)|$ *denotes the rank of* ζ*, which is defined as the cardinality of* $\mathrm{Im}(\zeta)$*. It is evident that* $rank(\zeta) = |X/ker(\zeta)|$*, where* $X/ker(\zeta)$ *denotes the factor set of* X *with respect to* $ker(\zeta)$*. The following inclusions and inequalities are valid for arbitrary* $\zeta, \theta \in \mathcal{T}_X$*:*

$$im(\zeta\theta) \subseteq im(\zeta), \ \ ker(\theta) \subseteq ker(\zeta\theta), \ \ rank(\zeta\theta) \leqslant \min\{rank(\zeta), rank(\theta)\}.$$

According to [25] (Theorem 4.5.1) (or [20] (§2.2) or [21] (Exercise 2.6.16)) it follows that

$$\begin{aligned} \gamma\,\mathscr{R}\,\delta &\Leftrightarrow \mathrm{Im}(\gamma) = \mathrm{Im}(\delta); \\ \gamma\,\mathscr{L}\,\delta &\Leftrightarrow \ker(\gamma) = \ker(\delta); \\ \gamma\,\mathscr{H}\,\delta &\Leftrightarrow \mathrm{Im}(\gamma) = \mathrm{Im}(\delta) \ \wedge \ \ker(\gamma) = \ker(\delta); \\ \gamma\,\mathscr{D}\,\delta &\Leftrightarrow \gamma\,\mathscr{J}\,\delta \Leftrightarrow rank(\gamma) = rank(\delta). \end{aligned} \tag{3}$$

Consider a semigroup S and $a \in S$. If the $\mathscr{H}$-class H_a of a contains an idempotent e, then it is a group with identity e and we say that a is *group invertible* and the inverse of a in this group is denoted by $a^\#$ and known as the *group inverse* of a. If $x \in S$ fulfills $axa = a$, then x is an *inner inverse* of a, and if $x = xax$, it is said that x is an *outer inverse* of a. An element of a semigroup which has an inner inverse is called a *regular element*. It is well known that for any regular element a there is an element which is both an inner and outer inverse of a. If a $\mathscr{D}$-class D of a semigroup S contains a regular element, then all elements from D are regular, and D is termed as a *regular* $\mathscr{D}$*-class*. For a subset $T \subseteq S$, $E(T)$ denotes the set of all idempotents from T.

3. Idempotent-Aided Factorizations

Theorem 3 can be interpreted as a generalization of the F.-R. factorization theorem for matrices.

Theorem 3. *Let* D *be a regular* $\mathscr{D}$*-class of a semigroup* S *and* $d \in D$*. In this case, the subsequent statements hold:*

(a) *For each idempotent* $e \in E(D)$*, there exist* $u \in R_d \cap L_e$ *and* $v \in L_d \cap R_e$ *such that* $d = uv$*.*
(b) *For each* $e \in E(D)$ *and every* $u \in R_d \cap L_e$*, there exists a unique* $v \in L_d \cap R_e$ *such that* $d = uv$*.*
(c) *For each* $e \in E(D)$ *and every* $v \in L_d \cap R_e$*, there exists a unique* $u \in R_d \cap L_e$ *such that* $d = uv$*.*

(d) *For each $u, v \in S$ satisfying $d = uv$, $R_d = R_u$, $L_d = L_v$, there exists an idempotent $e \in E(D)$ that fulfills $u \in L_e$ and $v \in R_e$.*

Proof. (a) Take an idempotent $e \in E(D)$. Then, there exists $u \in S$ such that $e \mathscr{L} u$ and $u \mathscr{R} d$, and hence, $u \in L_e$. By $u \mathscr{R} d$, it follows that $d = us$ for some $s \in S^1$, so $d = us = (ue)s = u(es) = uv$, where $v = es$. According to Green's lemma, $\varrho_s|_{L_u} : L_u \to L_d$ is a bijection which preserves $\mathscr{R}$-classes, so $e \mathscr{R} \varrho_s(e) = es = v$, and thus, $v \in R_e$. Finally, the $\mathscr{H}$-class $L_u \cap R_v = L_e \cap R_e = H_e$ contains an idempotent, and by Theorem 2 $d = uv \in R_u \cap L_v$, i.e., $R_d = R_u$ and $L_d = L_v$.

(b) Let $u \in R_d \cap L_e$. By $u \in R_d$ and the regularity of d it follows that equations $ux = d$ and $dy = u$, with unknowns x and y, have solutions in S, and by (a) it follows that the equation $ux = d$ has a solution in $L_d \cap R_e$. It will be verified that this equation is uniquely solvable in the larger set $R(e) = eS$, which will imply that it has a unique solution in $L_d \cap R_e$ as well.

Consider $p \in S$ such that $dp = u$. Suppose that $s, t \in R(e)$ such that $us = ut = d$. By the first Green's lemma (Lemma 1), the inner right translations $\varrho_s|_{L_u}$ and $\varrho_p|_{L_d}$ are mutually inverse bijections from $L_u = L_e$ onto L_d and L_d onto $L_u = L_e$, respectively. However, this also holds for $\varrho_t|_{L_u}$ and $\varrho_p|_{L_d}$, so we conclude that

$$\varrho_s|_{L_u} = (\varrho_p|_{L_d})^{-1} = \varrho_t|_{L_u}.$$

By $s, t \in R(e)$, it follows that $es = s$ and $et = t$, and therefore,

$$s = es = \varrho_s|_{L_u}(e) = \varrho_t|_{L_u}(e) = et = t.$$

Hence, we have proved the uniqueness of the solution of the equation $ux = d$ in $R(e)$, and thus, also in $L_d \cap R_e$.

(c) This can be proved similarly as in (b).

(d) If $u, v \in S$ satisfy $d = uv$, $R_d = R_u$, and $L_d = L_v$, then by Theorem 2 there exists an idempotent $e \in R_v \cap L_u$, which means $u \in L_e$ and $v \in R_e$. □

The requirements in Theorem 3 are illustrated in Figure 2.

	L_d		L_e	
R_d	$d = uv$		u	
R_e	v		e	

Figure 2. I.-A. factorization of d with respect to e.

The representation of $d \in S$ in the form $d = uv$, with $e \in E(D_d)$, $u \in R_d \cap L_e$, and $v \in L_d \cap R_e$, will be called a *factorization of d with respect to e*, or shortly, an *e-factorization* of d. Factorizations of this type will also be called *idempotent-aided factorizations* (I.-A. factorization). The pair (u, v) will be called an *e-factorization pair* for d.

This theorem implies Corollary 1.

Corollary 1. *Every regular element of a semigroup possesses a factorization with respect to any idempotent from its $\mathscr{D}$-class.*

It should be pointed out that I.-A. factorizations of an element d of a semigroup make sense only if d is a regular element, because otherwise, if d is not regular, the $\mathscr{D}$-class D_d does not contain any idempotents.

The assertions (b) and (c) of Theorem 3 can also be stated as in Corollary 2.

Corollary 2. *Let d be a regular element of a semigroup S and $e \in E(D)$. In this case,*

$$\phi(u) = v \quad \Leftrightarrow \quad (u, v) \text{ is an } e\text{-factorization pair for } d$$

defines a bijective mapping from the $\mathscr{H}$-class $R_d \cap L_e$ onto the $\mathscr{H}$-class $L_d \cap R_e$.

Let us also note that $u \in L_e$ and $v \in R_e$ implies that e is a right identity for u and a left identity for v, i.e., $ue = u$ and $ev = v$, and also, u is left invertible and v is right invertible with respect to e, i.e., $e = su$ and $e = vt$, for some $s, t \in S$.

Theorem 3 is illustrated in the following example considering a full transformation semigroup.

Example 3. *Observe the full transformation semigroup $\mathcal{T}_4$ on $X = \{1, 2, 3, 4\}$, and its elements $\delta = (1124)$ and $\varepsilon = (4234)$. These elements satisfy*

$$\mathrm{rank}(\delta) = |\,\mathrm{Im}(\delta)| = |\{1, 2, 4\}| = 3 = |\{2, 3, 4\}| = |\,\mathrm{Im}(\varepsilon)| = \mathrm{rank}(\varepsilon),$$

which means that $\delta\, \mathscr{D}\, \varepsilon$. It is easy to verify the idempotency of ε. The $\mathscr{H}$-classes $R_\delta \cap L_\delta$, $R_\delta \cap L_\varepsilon$, $R_\varepsilon \cap L_\delta$, and $R_\varepsilon \cap L_\varepsilon$ are shown in the 'eggbox' diagram in Figure 3.

	L_δ			L_ε	
R_δ	**(1124)**	(1142)		(1241)	(1421)
	(2214)	(2241)		(2142)	(2412)
	(4412)	(4421)		(4124)	(4214)
R_ε	(2234)	(2243)		(2342)	(2432)
	(3324)	(3342)		(3243)	(3423)
	(4423)	(4432)		**(4234)**	(4324)

Figure 3. $\mathscr{H}$-classes $R_\delta \cap L_\delta$, $R_\delta \cap L_\varepsilon$, $R_\varepsilon \cap L_\delta$, and $R_\varepsilon \cap L_\varepsilon$ of $\mathcal{T}_4$ for $\delta = (1124)$ and $\varepsilon = (4234)$.

Let us note that the class $R_\varepsilon \cap L_\delta$ contains the idempotent (2234), and the class $R_\varepsilon \cap L_\varepsilon$ contains the idempotent $\varepsilon = (4234)$, so these two classes are groups. The other two classes $R_\delta \cap L_\delta$ and $R_\delta \cap L_\varepsilon$ do not contain idempotents, so they are not groups.

By a straightforward verification, we obtain that the ε-factorization pairs for δ are

$$\begin{array}{lll} ((4214), (3324)), & ((2412), (3342)), & ((4124), (2234)), \\ ((1421), (4432)), & ((2142), (2243)), & ((1241), (4423)). \end{array} \tag{4}$$

Note also that an overview of all Green's equivalence classes of the semigroup $\mathcal{T}_4$ is given in [20] (Section 2.2). However, it should be kept in mind that in that book the symbol denoting a mapping is written on the right side of the argument, which is why the full transformation semigroups from that book are dual (anti-isomorphic) to those which we deal with in this paper. This means that $\mathscr{R}$-classes from the book [20] are $\mathscr{L}$-classes here, and $\mathscr{L}$-classes from [20] are $\mathscr{R}$-classes here.

As said earlier, the notion of the I.-A. factorization can be treated as a semigroup extension of the classical F.-R. matrix factorization (cf. [23,24]). This will be explained below. Consider an arbitrary matrix $D \in \mathbb{F}^{m \times n}$ of rank r, where $\mathbb{F}$ is a field. As we noted in Example 1, in the semigroup of matrices $M_\varnothing(\mathbb{F})$, Green's $\mathscr{D}$-class of D consists of all matrices having the rank r. Take an arbitrary idempotent matrix $E \in \mathbb{F}^{k\times k}$ with the same rank r, and let $D = UV$, where $U \in \mathbb{F}^{m\times k}$ and $V \in \mathbb{F}^{k\times n}$, be an E-factorization of D. According to Theorem 3, the matrices U and V are also of rank r, and in the case $k = r$, i.e., when E possesses full rank, U is of full column rank and V is of full row rank. Therefore, I.-A. factorizations with respect to a full-rank idempotent matrix are nothing but F.-R. factorizations. Recall that the only full-rank idempotent matrix of rank r is the identity $r \times r$-matrix I_r.

On the other hand, if E is rank-deficient (not of full rank), it is clear that the E-factorization $D = UV$ is not an F.-R. factorization. However, it is such a factorization where U is of the same range as D and has the same null space (kernel) as E, while V is of the same kernel as D and the same range (column space) as E.

The next example illustrates I.-A. factorizations over a field.

Example 4. *Consider a matrix $D \in \mathbb{R}^{4\times 5}$ given by*

$$D = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

and of rank $\operatorname{rank}(D) = 3$. *The pair of matrices*

$$U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

form a factorization pair for D with respect to the 3×3-identity matrix I_3; that is, the factorization $D = UV$ is an F.-R. factorization of D. More precisely, it is a factorization known as the column–row factorization (cf. [26] (Section 7.6)).

On the other hand, the pair of matrices

$$U' = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -2 & -5 & -3 & -1 \\ 0 & -3 & -3 & 4 \\ 3 & 6 & 3 & -7 \end{bmatrix}, \qquad V' = \begin{bmatrix} 0 & -1 & 1 & 0 & -1 \\ 1 & 2 & 0 & 0 & 3 \\ -1 & -1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

form a factorization pair for D with respect to an idempotent matrix

$$E = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since E is a 4×4-matrix of rank 3, i.e., it is rank-deficient, the factorization $D = U'V'$ is not an F.-R. factorization.

4. Existence and Characterizations of Group Inverses and (b,c)-Inverses

In the following, we give several applications of I.-A. factorizations. First, we consider the group invertibility in terms of I.-A. factorizations. The following theorem extends the well-known result on the existence of the group inverse in a field based on the F.-R. factorization (cf. [23], Ch. 4, Theorem 3).

Theorem 4. *Let d belong to a semigroup S, and let D be the $\mathscr{D}$-class of d. The following statements are equivalent under these assumptions:*

(i) *d is group invertible;*
(ii) *There exist $e \in E(D)$ and an e-factorization pair (u, v) for d such that $vu \in H_e$;*
(iii) *$E(D) \neq \emptyset$ and for any $e \in E(D)$ and any e-factorization pair (u, v) for d we have $vu \in H_e$.*

If (ii) *holds, then*

$$d^{\#} = u((vu)^{\#})^2 v = usv = utv = upv = uqv, \tag{5}$$

for arbitrary $s, t, p, q \in S$ *which satisfy* $e = s(vu)^2 = (vu)^2 t$ *and* $vu = p(vu)^3 = (vu)^3 q$.

Proof. (i)⇒(iii). Let d be group invertible, and consider an arbitrary idempotent $e \in E(D)$ and an arbitrary e-factorization pair (u, v) for d. Then, $R_u \cap L_v = R_d \cap L_d = H_d$ contains an idempotent, and by Theorem 2 we obtain that $vu \in R_v \cap L_u = R_e \cap L_e = H_e$.

(iii)⇒(ii). This implication is evident.

(ii)⇒(i). Let there exist an idempotent $e \in E(D)$ and an e-factorization pair (u, v) for d such that $vu \in H_e = R_v \cap L_u$. Then, from Theorem 2 it follows that $R_u \cap L_v = H_d$ contains an idempotent, so d is group invertible.

Further, the validity of (5) is proved. For this purpose, consider $f = u(vu)^{\#}v$. Let us note that from $u \in L_e$ and $v \in R_e$ it follows that $ue = u$ and $ev = v$. This leads to the conclusion

$$f^2 = u(vu)^{\#}vu(vu)^{\#}v = ue(vu)^{\#}v = u(vu)^{\#}v = f,$$

i.e., f is an idempotent. It is clear that $f \in R(u) \cap L(v)$. Moreover,

$$u = ue = u(vu)^{\#}vu = fu \in R(f), \quad v = ev = vu(vu)^{\#}v = vf \in L(f).$$

Therefore, $f \in R_u \cap L_v = H_d$, i.e., f represents the identity in the group H_d.

The subsequent conclusion is

$$d \cdot u((vu)^{\#})^2 v = uvu(vu)^{\#}(vu)^{\#}v = ue(vu)^{\#}v = u(vu)^{\#}v = f,$$

and also,

$$u((vu)^{\#})^2 v \cdot d = u(vu)^{\#}(vu)^{\#}vuv = u(vu)^{\#}ev = u(vu)^{\#}v = f.$$

Consequently, $u((vu)^{\#})^2 v = d^{\#}$.

Next, let $e = s(vu)^2 = (vu)^2 t$ and $vu = p(vu)^3 = (vu)^3 q$, for some $s, t, p, q \in S$. Such elements exist since e, vu, $(vu)^2$ and $(vu)^3$ are $\mathscr{H}$-related. Then, we obtain

$$d^{\#} = u((vu)^{\#})^2 v = ue((vu)^{\#})^2 v = us(vu)^2((vu)^{\#})^2 v = usev = usv,$$

and in an analogous way $e = (vu)^2 t$ gives $d^{\#} = utv$. On the other hand,

$$d^{\#} = u((vu)^{\#})^2 v = ue((vu)^{\#})^2 v = uvu(vu)^{\#}((vu)^{\#})^2 v = up(vu)^3((vu)^{\#})^3 v = upev = upv,$$

and similarly, $vu = (vu)^3 q$ gives $d^{\#} = uqv$. □

Let us emphasize once again that if for some idempotent $e \in E(D)$ and some e-factorization pair (u, v) for d it holds that $vu \in H_e$, then this also holds for every factorization pair for d, with respect to any idempotent from $E(D)$. This means that we can verify the fulfillment of the conditions of the previous theorem by examining an arbitrary idempotent from $E(D)$ and an arbitrary factorization pair for d with respect to that idempotent, which we can also use to compute the group inverse of d, if the mentioned conditions are fulfilled.

Example 5. *Examine once more the full transformation semigroup* $\mathcal{T}_4$ *over* $X = \{1, 2, 3, 4\}$, *and the elements* $\delta = (1124)$ *and* $\varepsilon = (4234)$ *already examined in Example 3.*

For any ε*-factorization pair* (u, v) *for* δ *listed in* (4), *we can easily verify that* $vu \notin H_\varepsilon$. *Moreover,* vu *is of rank 2 and does not even belong to the* $\mathscr{D}$*-class of* δ. *This means that* δ *is not group invertible.*

Example 6. *Consider again the semigroup $\mathcal{T}_4$, and assume that $\delta = (2421)$. This element has rank 3, and we choose the idempotent $\varepsilon = (4234)$, also of rank 3, and the pair (α, β), where $\alpha = (1241)$ and $\beta = (2324)$, which forms an ε-factorization pair for δ. Then, we obtain*

$$\beta\alpha = (2324)(1241) = (2342) \in H_\varepsilon$$

(since (2342) has the same image and kernel as ε), so we conclude that δ is group invertible.

Using the procedure proposed in Theorem 4, we first solve the equation $\varepsilon = \sigma\beta\alpha$ for σ, and obtain that $\sigma = (i423)$, for an arbitrary $i \in \{1,2,3,4\}$, and then, we obtain

$$\begin{aligned} \delta^{\#} &= \alpha(\sigma\varepsilon)^2\beta = (1241)\big((i423)(4234)\big)^2(2324) = (1241)(3423)^2(2324) \\ &= (1241)(2342)(2324) = (4142). \end{aligned}$$

Theorem 4 actually says that testing the group invertibility of an element d and computing its group inverse, if it exists, can be moved from the $\mathscr{H}$-class of that element to the $\mathscr{H}$-class of some idempotent e from the same $\mathscr{D}$-class. This fact could be very useful because sometimes it can happen that it is easier to work within the class H_e than within the class H_d. That is best illustrated by the next example with matrices over a field, where testing the group invertibility and computing the group inverse of the matrix D is reduced to testing the ordinary invertibility and computing the ordinary inverse of the matrix VU.

Example 7. *Let us examine the matrix $D \in \mathbb{R}^{4\times 4}$ given by*

$$D = \begin{bmatrix} 1 & 2 & 2 & 0 \\ -2 & -5 & -1 & 1 \\ 0 & -3 & 4 & 3 \\ 3 & 6 & -7 & 0 \end{bmatrix},$$

whose rank is 3. The matrices $U \in \mathbb{R}^{4\times 3}$ and $V \in \mathbb{R}^{3\times 4}$, given by

$$U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

form a factorization pair for D with respect to the identity matrix I_3, and

$$VU = \begin{bmatrix} 7 & 14 & -12 \\ -5 & -11 & 6 \\ 0 & -3 & 4 \end{bmatrix}.$$

It is not hard to verify that VU is an invertible matrix, and therefore, it is group invertible. Its (group) inverse is given by

$$(VU)^{\#} = (VU)^{-1} = \begin{bmatrix} \frac{13}{41} & \frac{10}{41} & \frac{24}{41} \\ -\frac{10}{41} & -\frac{14}{41} & -\frac{9}{41} \\ -\frac{15}{82} & -\frac{21}{82} & \frac{7}{82} \end{bmatrix}.$$

Based on Theorem 4, the matrix D is group invertible, with the group inverse given by

$$D^{\#} = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix} \begin{bmatrix} \frac{13}{41} & \frac{10}{41} & \frac{24}{41} \\ -\frac{10}{41} & -\frac{14}{41} & -\frac{9}{41} \\ -\frac{15}{82} & -\frac{21}{82} & \frac{7}{82} \end{bmatrix}^2 \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{13}{3362} & \frac{379}{3362} & -\frac{263}{3362} & -\frac{353}{3362} \\ -\frac{587}{6724} & -\frac{1855}{6724} & \frac{755}{6724} & \frac{681}{6724} \\ -\frac{15}{82} & -\frac{21}{82} & \frac{7}{82} & -\frac{9}{82} \\ \frac{1053}{6724} & \frac{441}{6724} & \frac{2231}{6724} & \frac{1665}{6724} \end{bmatrix}.$$

Further, I.-A. factorizations can be used in testing (b,c)-invertibility and computing (b,c)-inverses in semigroups.

Let $a, b, c \in S$ be entries in a semigroup S. Drazin, in [27], defined $x \in S$ to be a (b,c)-*inverse* of $a \in S$ if the following conditions hold:

(D1) $x \in bSx \cap xSc$;
(D2) $xab = b \wedge cax = c$.

As noted in [28], if (D2) is satisfied, then (D1) can be substituted by a simpler constraint $x \in bS \cap Sc$, or by

(D1′) $x \in R(b) \cap L(c)$.

Drazin, in [27], showed that a (b,c)-inverse of a exists if and only if $cab \in R_c \cap L_b$, and that it is unique whenever it exists. On the other hand, it was shown in [27] (see also [28]) that x is the (b,c)-inverse of a if and only if it is the unique outer inverse of a from the $\mathscr{H}$-class $R_b \cap L_c$. The (b,c)-inverse of a, in the case of its existence, will be marked with $a^{(b,c)}$.

Mary, in [19], introduced essentially the same concept. For a semigroup S and elements $a, d \in S$, he defined an *inverse of a along d* as an element $x \in S$ satisfying the following constraints:

(M1) $x \in R(d) \cap L(d)$;
(M2) $xad = d \wedge dax = d$.

He also showed that x is an inverse of a along d if and only if it represents an outer inverse of a in the $\mathscr{H}$-class H_d, and that it is unique whenever it exists. The inverse of a along d is denoted by a^{-d}.

Theorem 5 gives representations of (b,c)-inverses by means of I.-A. factorizations. The equivalence (i)⇔(ii) generalizes [23] (Ch. 1, Theorem 5), as well as the well-known representation theorem for the outer inverse of complex matrices with a predefined image and kernel (cf. [29]), while (i)⇔(iv) can be considered as a generalization of [23] (Ch. 2, Theorem 5).

Theorem 5. *Let S be a semigroup, let D be a $\mathscr{D}$-class of S, and let $a \in S$ and $b, c \in D$ be arbitrary elements. Then, the following statements are equivalent:*

(i) *a is (b,c)-invertible;*
(ii) *There exist $e \in E(D)$, $u \in L_e \cap R_b$, and $v \in R_e \cap L_c$ satisfying $vau \in H_e$;*
(iii) *$E(D) \neq \emptyset$ and for arbitrary $e \in E(D)$, $u \in L_e \cap R_b$, $v \in R_e \cap L_c$ it follows that $vau \in H_e$;*
(iv) *$E(D) \neq \emptyset$ and for arbitrary $e \in E(D)$ there exist $u \in L_e \cap R_b$, $v \in R_e \cap L_c$ satisfying $vau = e$.*

If (ii) *holds, then*

$$a^{(b,c)} = u(vau)^{\#}v = usv = utv = upv = uqv, \tag{6}$$

for arbitrary $s, t, p, q \in S$ such that $e = svau = vaut$ and $vau = p(vau)^2 = (vau)^2 q$, and if (iv) *holds, then $a^{(b,c)} = uv$.*

Proof. (i)⇒(iii). Let a be (b,c)-invertible and let x be the (b,c)-inverse of a. Then, $x = xax$ implies $ax, xa \in E(D)$, which implies $E(D) \neq \emptyset$.

Consider $e \in E(D)$, $u \in R_b \cap L_e$, and $v \in L_c \cap R_e$ (see Figure 4). Since x is the unique outer inverse of a from $\mathscr{H}$-class $R_b \cap L_c = R_u \cap L_v$, it is a (u,v)-inverse of a too, and from (vi) of Theorem 4.5 [28] it follows that

$$vau \in R_v \cap L_u = R_e \cap L_e = H_e.$$

(iii)⇒(ii). This implication is obvious.

(ii)⇒(i). Suppose the existence of $e \in E(D)$, $u \in L_e \cap R_b$, and $v \in R_e \cap L_c$ satisfying $vau \in H_e$. Then, vau is group invertible and

$$vau(vau)^{\#} = (vau)^{\#}vau = e,$$

while from $u \in L_e$ and $v \in R_e$, it follows that $ue = u$ and $ev = v$.

Set $x = u(vau)^{\#}v$. It is understandable that $x = u(vau)^{\#}v \in R(u) \cap L(v)$. On the other hand,

$$u = ue = u(vau)^{\#}vau = xau \in R(x), \qquad v = ev = vau(vau)^{\#}v = vax \in L(x),$$

whence $x \mathscr{R} u$ and $x \mathscr{L} v$, that is, $x \in R_u \cap L_v = R_b \cap L_c$.

In addition,

$$xax = u(vau)^{\#}vau(vau)^{\#}v = ue(vau)^{\#}v = u(vau)^{\#}v = x.$$

Therefore, x is an outer inverse of a belonging to $\mathscr{H}$-class $R_b \cap L_c$, which guarantees that x is a (b,c)-inverse of a.

(i)⇒(iv). Let x be a (b,c)-inverse of a. As in the verification of (i)⇒(iii), we conclude that $E(D) \neq \emptyset$. Consider an arbitrary $e \in E(D)$, and let (u,v) be an arbitrary e-factorization pair for x. This means that $u \in L_e \cap R_x = L_e \cap R_b$, $v \in R_e \cap L_x = R_e \cap L_c$, and $x = uv$. Moreover, as in the proof of (i)⇒(iii) the relation $vau \in H_e$ is obtained. Since we have already proved that (i)⇔(ii), then, based on (ii)⇒(i), it follows that $x = u(vau)^{\#}v$. Now,

$$vau = vau(vau)^{\#}vau = vaxau = vauvau = (vau)^2,$$

and since $vau \in H_e$ and e is the unique idempotent in H_e, we conclude that $vau = e$, which was to be proved.

(iv)⇒(ii). This is evident.

Further, suppose that (ii) holds, i.e., assume the existence of $e \in E(D)$, $u \in L_e \cap R_b$, and $v \in R_e \cap L_c$ satisfying $vau \in H_e$. In the part (ii)⇒(i), we have proved that $a^{(b,c)} = u(vau)^{\#}v$. Consider arbitrary $s,t,p,q \in S$ for which $e = svau = vaut$ and $vau = p(vau)^2 = (vau)^2q$. Such elements exist because e, vau and $(vau)^2$ are $\mathscr{H}$-related. In this case,

$$a^{(b,c)} = u(vau)^{\#}v = ue(vau)^{\#}v = usvau(vau)^{\#}v = usev = usv,$$

and in the same way we show that $e = vaut$ gives $a^{(b,c)} = utv$.

On the other hand, from $vau = p(vau)^2$ we obtain that

$$a^{(b,c)} = u(vau)^{\#}v = ue(vau)^{\#}v = uvau((vau)^{\#})^2v = up(vau)^2((vau)^{\#})^2v = upev = upv,$$

and similarly, from $vau = (vau)^2q$ we obtain $a^{(b,c)} = uqv$.

Finally, if (iv) is true, then $a^{(b,c)} = u(vau)^{\#}v = ue^{\#}v = uev = uv$. This finalizes the proof. □

It is worth noting that, in the notation from (i)⇒(iii) of Theorem 5, from $u \in R_b \cap L_e$ and $v \in L_c \cap R_e$ it follows that $R_v \cap L_u = R_e \cap L_e = H_e$, and according to Theorem 2 we obtain that $uv \in R_u \cap L_v = R_b \cap L_c$. This means that $a^{(b,c)}$ and uv are in the same $\mathscr{H}$-class $R_b \cap L_c$, for all $e \in D^{\bullet}$, $u \in R_b \cap L_e$, and $v \in L_c \cap R_e$ (cf. Figure 4). If we assume that $d = uv$, then it is clear that (u,v) is an e-factorization pair for d.

Directly from Theorem 5, we obtain the following consequence.

Corollary 3. *Let S be a semigroup, D be a $\mathscr{D}$-class of S, and let $a \in S$ and $d \in D$ be arbitrary elements. The subsequent claims are mutually equivalent:*

(i) *a is invertible along d;*
(ii) *There exist $e \in E(D)$, $u \in L_e \cap R_d$ and $v \in R_e \cap L_d$ satisfying $vau \in H_e$;*
(iii) *$E(D) \neq \emptyset$ and for arbitrary $e \in E(D)$, $u \in L_e \cap R_d$, and $v \in R_e \cap L_d$ we have $vau \in H_e$;*
(iv) *$E(D) \neq \emptyset$ and for every $e \in E(D)$ there exist $u \in L_e \cap R_d$ and $v \in R_e \cap L_d$ satisfying $vau = e$.*

If (ii) *holds, then*

$$a^{-d} = u(vau)^{\#}v = uetv = usev. \tag{7}$$

for arbitrary $t, s \in S$ *such that* $e = vaut = svau$, *and if* (iv) *holds, then* $a^{-d} = uv$.

	L_b		L_e		L_c	
R_b	b		u		$a^{(b,c)}$ uv	
R_e			e vau		v	
R_c	cab				c	

Figure 4. Visualization of the situation considered in Theorem 5.

The situation considered in Corollary 3 is shown in Figure 5.

	L_d		L_e	
R_d	d, uv a^{-d}		u	
R_e	v		e vau	

Figure 5. Visualisation of the environment required in Corollary 3.

Example 8. *Choose* $A \in \mathbb{R}^{5\times4}$, $B \in \mathbb{R}^{4\times5}$, *and* $C \in \mathbb{R}^{4\times5}$ *given by*

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ -2 & -5 & -1 & -7 & 1 \\ 0 & -3 & 4 & -10 & 4 \\ 3 & 6 & -7 & 16 & -10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

The matrices B *and* C *have rank* 3, *which means that they are in the same* $\mathscr{D}$*-class of the semigroup* $M_{\varnothing}(\mathbb{R})$. *Consider the matrices*

$$U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is easy to check that $\mathcal{R}(U) = \mathcal{R}(B)$, $\mathcal{N}(U) = \mathcal{N}(I_3)$, $\mathcal{N}(V) = \mathcal{N}(C)$, *and* $\mathcal{R}(V) = \mathcal{R}(I_3)$. *Moreover, the matrix*

$$VAU = \begin{bmatrix} -4 & 3 & -29 \\ -10 & -19 & -26 \\ -1 & 1 & -27 \end{bmatrix}$$

is invertible, i.e., it belongs to the $\mathscr{H}$*-class of* I_3 *in* $M_{\varnothing}(\mathbb{R})$. *Now, according to Theorem* 5, *the matrix* A *is* (B, C)*-invertible and its* (B, C)*-inverse is given by*

$$A^{(B,C)} = U(VAU)^{\#}V = U(VAU)^{-1}V$$
$$= \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix} \begin{bmatrix} -\frac{539}{2047} & -\frac{52}{2047} & \frac{629}{2047} \\ \frac{244}{2047} & -\frac{79}{2047} & -\frac{186}{2047} \\ \frac{29}{2047} & -\frac{1}{2047} & -\frac{106}{2047} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7}{2047} & -\frac{212}{2047} & \frac{226}{2047} & \frac{45}{2047} & -\frac{160}{2047} \\ -\frac{171}{2047} & \frac{500}{2047} & -\frac{842}{2047} & -\frac{222}{2047} & \frac{107}{2047} \\ -\frac{616}{2047} & \frac{233}{2047} & -\frac{1465}{2047} & \frac{134}{2047} & -\frac{249}{2047} \\ -\frac{4}{23} & -\frac{7}{23} & -\frac{1}{23} & \frac{17}{23} & \frac{6}{23} \end{bmatrix}$$

If instead of the matrix A we take the matrix

$$A' = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 2 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 4 & -3 & 1 \\ -1 & 3 & -2 & 1 \end{bmatrix},$$

then we obtain that

$$VA'U = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 2 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 4 & -3 & 1 \\ -1 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix} = \begin{bmatrix} -6 & -5 & -5 \\ -6 & -9 & -45 \\ -10 & -12 & -45 \end{bmatrix},$$

and since the last matrix is not invertible, we conclude that A' is not (B,C)-invertible.

Example 9. *Take into account $\mathcal{T}_4$ on a set $X = \{1,2,3,4\}$, and the transformations $\alpha = (3144)$, $\beta = (1442)$, and $\gamma = (2213)$. It is clear that all these transformations have rank 3.*

Choose the idempotent transformation $\varepsilon = (4234)$. It also has rank 3, and therefore, it belongs to the same $\mathscr{D}$-class of the semigroup $M_{\emptyset}(\mathbb{R})$ that contains β and γ. The $\mathscr{H}$-classes obtained as intersections of $\mathscr{L}$-classes and $\mathscr{R}$-classes determined by β, γ, and ε are shown in Figure 6.

Next, we take the transformations $(4214) \in R_\beta \cap L_\varepsilon$ and $(3324) \in R_\varepsilon \cap L_\gamma$. We have that

$$(3324)\alpha(4214) = (3324)(3144)(4214) = (4324) \in H_\varepsilon,$$

and according to Theorem 5, one concludes that α is (β,γ)-invertible. Its (β,γ)-inverse is given by

$$\alpha^{(\beta,\gamma)} = (4214)(4324)^{\#}(3324) = (4214)(4324)(3324) = (2214) \in R_\beta \cap L_\gamma,$$

since $(4324)^2 = \varepsilon$, so $(4324)^{\#} = (4324)$.

On the other hand, if instead of α we take $\alpha' = (3242)$, then

$$(3324)\alpha'(4214) = (3324)(3242)(4214) = (3323) \notin H_\varepsilon,$$

whence we conclude that α' is not (β,γ)-invertible.

	L_β			L_ε			L_γ		
R_β	(1224) (2114) (4112)	**(1442)** (2441) (4221)		(1241) (2142) (4124)	(1421) (2412) **(4214)**		(1124) (2214) (4412)	(1142) (2241) (4421)	
R_ε	(2334) (3224) (4223)	(2443) (3442) (4332)		(2342) (3243) **(4234)**	(2432) (3423) (4324)		(2234) **(3324)** (4423)	(2243) (3342) (4432)	
R_γ	(1223) (2113) (3112)	(1332) (2331) (3221)		(1231) (2132) (3123)	(1321) (2312) (3213)		(1123) **(2213)** (3312)	(1132) (2231) (3321)	

Figure 6. Visualization of the situation considered in Example 9.

5. Concluding Remarks

In this paper, we introduced a new type of factorization of elements of a semigroup, which is called I.-A. factorization. It is a factorization in which the location of factors in the structure of Green's equivalence classes is precisely determined by the element being factorized and the idempotent by which the factorization is performed. We proved that every regular element of a semigroup possesses such a factorization with respect to an arbitrary idempotent from the same Green's $\mathscr{D}$-class. I.-A. factorizations have been used to provide the existence criteria and characterizations of group inverses and (b,c)-inverses of elements of a semigroup.

In further research, it is possible to develop algorithms for efficient I.-A. factorization for matrices over a field, as well as the corresponding algorithms for verifying the existence and calculating group inverses and the (B,C)-inverse of matrices.

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Abbreviations

The following abbreviations are used in this manuscript:

I.-A.	Idempotent-aided
F.-R.	Full-rank

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