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Differential Games between a Brand Manufacturer and an Internet Celebrity Regarding Fairness Concerns

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Abstract: With the rapid growth of social media and live-streaming technology, live-stream selling has become integral to the digital economy. Using differential game theory, this paper examines how fairness concerns impact the profits of internet celebrities and brand manufacturers under the "pure commission" model. We analyzed no fairness concern, gap fairness concern, and self-due fairness concern models, to investigate the optimal decisions and corresponding profits for an internet celebrity and a brand manufacturer. The results show that the internet celebrity earned the highest profits with low commission rates under the self-due fairness concern model, whereas higher commission rates yielded higher profits for the internet celebrity under the gap fairness concern model. Simultaneously, fairness concerns significantly affected the cooperation stability and long-term benefits, motivating the internet celebrity to maintain efficient collaborations with the brand manufacturer. Furthermore, the self-due fairness concern model was more practical than the gap fairness concern model.

Keywords: live streaming; differential game; fairness concern

MSC: 91A23

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1. Introduction

In recent years, with the rapid development of internet technology and the widespread use of social media platforms, live-stream selling, an emerging e-commerce model, has swiftly risen and become an integral part of the digital economy. Live-stream selling involves a brand manufacturer collaborating with an internet celebrity, who presents the brand's product information in a live-streaming format and interacts with viewers. The internet celebrity's influence stimulates and guides consumers to purchase the product, increasing the product's visibility and sales volume [1]. Live-stream selling promotes products directly to consumers via real-time interaction, live demonstrations, and immediate purchasing options, increasing product exposure and conversion rates and fostering a closer connection between the brand manufacturer and consumers.

The live-streaming industry originated in 2005. By 2016, the focus of live-streaming had shifted from PC to mobile platforms, and the content had expanded from singular show performances to various fields, such as e-commerce, sports, and education. Live-stream e-commerce has developed in four stages: inception, exploration, growth, and explosion. The rapid development in 2019 marked this as the inaugural year of live-stream e-commerce [2]. With the continuous advancement of network technology, live-stream selling has become a novel marketing model and a new growth point in the e-commerce industry. The COVID-19 pandemic in 2020 impacted traditional industries, hindering offline store operations. Leveraging the advantages of online platforms, live-stream e-commerce continued to develop, becoming a new channel for enterprises seeking sales growth and a tool for offline stores to resume operations. Diverse forms of live-streaming emerged,

such as celebrity promotions and county governors live-streaming, driving the economic recovery. The live-stream e-commerce market in China reached nearly CNY 5 trillion in 2023, significantly increasing from CNY 420 billion in 2019, and is forecast to grow to CNY 8.16 trillion by 2026 [3]. As of December 2023, the number of users is approximately 597 million, approximately 54.7% of China's internet population [4].

Internet celebrities play a crucial role in live-stream selling, leveraging their influence and appeal to recommend and sell products to their followers via live-streaming platforms. Internet celebrities can be categorized into top-tier or regular celebrities based on their level of influence. Platforms such as Douyin, Kuaishou, and Taobao classify internet celebrities into these two categories based on their fan base and sales capabilities [5]. Top-tier internet celebrities, such as Li Jiaqi, possess many followers. In addition to earning per-unit commissions from their collaborations with brand manufacturers, they also charge a fixed fee, known as a slotting fee, regardless of the sales volume. Although top-tier internet celebrities can stimulate demand due to their large follower base, their high slotting fees may erode the profits of brand manufacturers, leading to economic losses. Increasingly, brand manufacturers find collaborating with top-tier internet celebrities daunting as a result. In contrast, regular internet celebrities have a relatively lower visibility. Consequently, their bargaining power in collaborations is more modest, and they typically only charge per-unit commissions.

In traditional management fields, decision-makers are typically considered rational, meaning they are only concerned with their own gains, and not those of other participants. However, increasing research in the literature suggests that participants in supply chains may have fairness concerns. Fairness concern theory, also known as fairness preference theory, can trace its business application back to the 1960s during the Adams era. Adams introduced fairness theory, a type of motivational theory, in his 1965 paper "Inequity in Social Exchange", with the core premise that individuals experience a psychological imbalance and sense of unfairness when they perceive a discrepancy between their rewards and others' rewards. Adams [6] posited that individuals only feel satisfied and motivated when they perceive their rewards as fair. An internet celebrity may also experience disputes due to an unfair profit distribution during long-term collaborations with a brand manufacturer. In July 2022, "ShiShuyaoyao" established a cooperative relationship with Laifen to promote product sales via ShiShuyaoyao's online influence. The two parties cooperated well and the business progressed smoothly in the early stage of this cooperation; however, they experienced serious differences in the allocation of advertising expenses as the cooperation deepened [7].

Therefore, this paper explores the strategic choices of internet celebrities and brand manufacturers in different cooperation models under the pure commission model based on fairness concern theory. Considering the dynamic impact of live-streaming on product demand, we constructed dynamic differential equations to analyze the optimal decisions of an internet celebrity and a brand manufacturer under different strategies. We incorporated fairness concerns into a no fairness concern model, a gap fairness concern model, and a self-due fairness concern model, to provide insights into how fairness concerns influence optimal effort levels, pricing decisions, and profit distribution between the internet celebrity and brand manufacturer. Our study's innovation lies in three key areas:

- Innovation in Research Subjects: Our research focuses on the cooperation between an internet celebrity and a brand manufacturer—a relatively underexplored area in the existing literature. The rapid expansion of live-stream commerce on digital social platforms provides a novel perspective on how the influence of internet celebrities can affect collaborations with brand manufacturers and, consequently, impact the physical economy.
- Innovation in Research Content: We delve deeper into the cooperation models and resource allocation challenges within live-stream commerce. Unlike previous studies that employed differential game models, our research thoroughly investigated the optimal strategy choices for both the internet celebrity and the brand manufacturer

under various fairness concern models—namely, a no fairness concern model, gap fairness concern model, and self-due fairness concern model.

- Innovation in Research Methods: While fairness concerns are commonly applied in supply chain research, our study extended this concept to the context of live-stream commerce. We focused on how fairness concern parameters influenced the optimal effort levels, pricing decisions, and resulting profits for both the brand manufacturer and internet celebrity.

2. Literature Review

Our work builds upon prior research on live streaming and fairness concerns. In this section, we explore discussions about these two research topics in the literature and elaborate on our distinct contributions.

2.1. Live Streaming

There has been much literature focusing on the live-streaming industry as a research background. Cui et al. [8] studied live-stream e-commerce and found that sellers and platforms prefer live-streaming when consumer hassle costs are low and procurement costs are high. Ma et al. [9] identified the psychological mechanisms of live-stream commerce using a stimulus–organism–response framework, highlighting its impact on consumer behavior across different genders and platforms. Gong et al. [10] analyzed multichannel sales strategies, finding that live-streaming boosts profits for high-quality standardized products but may reduce profits for personalized products. Ji et al. [11] explored dynamic pricing for live-streaming platforms, showing that optimal pricing correlates with initial reference prices. Dynamic pricing boosts platform revenues but lowers broadcaster earnings. Lu et al. [12] analyzed the role of internet celebrities in live-stream e-commerce. A greater influencer impact leads to lower prices to motivate effort. Dual-channel strategies (direct sales plus live-streaming) outperform single-channel strategies. Commission models are more effective than markup models.

Internet celebrities and brand manufacturers frequently encounter complex challenges in coordinating the pursuit of their interests in the live-streaming industry. Differential game theory provides a new theoretical framework to analyze these coordination and control issues. Internet celebrities and brand manufacturers can compute equilibrium decisions, such as efforts, pricing, and profits, during the game process using differential game models. Several scholars have applied differential game theory methods in recent years to study relevant issues in the live-streaming industry. Hu et al. [13] applied differential game theory to investigate dynamic pricing between live-streaming platforms and streamers considering the reference effect. They analyzed how the reference effect and streamer influence impact pricing strategies. Wei et al. [14] explored the role of internet celebrities in live-stream e-commerce. They studied optimal pricing strategies for brand manufacturers in single- and dual-channel scenarios (direct sales and live-streaming), considering the influences of internet celebrities and market demand proportions on equilibrium decisions. Fei and Wang [15] studied dynamic strategies for food safety in live-stream e-commerce. They used a stochastic differential game model to compare brand owner, platform, and streamer behaviors under different streaming modes. Zhang et al. [16] analyzed quality management in live-stream e-commerce considering streamer types. They employed a differential game model to find that top streamers only influence quality decisions above a certain threshold. Fei and Wang [17] studied food safety strategies using a differential game model, comparing centralized and decentralized decision-making and cost-sharing modes.

Previous research has extensively explored the effectiveness of live-streaming as a sales channel. Studies have shown that live-streaming enhances consumer engagement, creates a sense of urgency, and builds trust via real-time interactions. Internet celebrities play a pivotal role in this context, leveraging their personal brands to drive sales and influence consumer behavior. Our work extends this literature by examining how varying

commission rates impact the performance and profitability of live-stream sales for the internet celebrity and the brand manufacturer.

2.2. Fairness Concern

The fairness concern model is used to study how much participants focus on fairness during decision-making processes. In this model, participants consider their own profits alongside the fairness of the distribution of profits and resources among the other participants. The fairness concern model incorporates different types of fairness standards, such as gap fairness and self-due fairness, which help understand how fairness concerns influence participants' decisions and strategic behaviors in supply chains or markets and how they impact the supply chain's overall performance and stability. Retailers concerned with gap fairness perceive unfairness due to profit gaps between themselves and manufacturers, a concept widely explored in the literature, such as by Nie and Du [18], Li and Li [19], and Huang [20]. Retailers concerned with self-due fairness perceive unfairness based on the difference between their actual profits and what they believe would be fair profits. Following Du et al.'s approach [21], Huang [20] considered Nash negotiation profits as a retailers' self-due fair profits. Retailers concerned with gap fairness use leaders' profits as their fairness reference point, whereas those concerned with self-due fairness use their own Nash negotiation profits as their reference point.

Our study is related to fairness concerns in supply chain management. Mei et al. [22] analyzed the impacts of manufacturer fairness concerns and carbon emission reduction investments on pricing decisions under a countervailing power. They found that disadvantageous fairness concerns reduce overall supply chain profits and utility, while moderate advantageous fairness concerns benefit supply chain members. Ni et al. [23] used Nash bargaining as a fairness reference in push-and-pull supply chains, finding that unilateral fairness concerns lead leaders to offer coordinated wholesale prices. Song et al. [24] explored the impact of consumer fairness concerns on supply chain decisions, finding that such concerns raise prices and profits for low-carbon products, while reducing those for traditional products. Huang et al. [25] studied sustainable remanufacturing in e-commerce supply chains alongside government policies and fairness concerns, analyzing the impacts on members, consumers, and the environment. Xue and Wang [26] investigated dual-channel supply chain coordination under risk aversion and fairness concerns, proposing a joint contract to improve the Pareto efficiency. Zhang [27] considered the spillover effect of live-streaming and agribusiness fairness concerns in a dual-channel supply chain, proposing a cost-sharing commission mechanism for coordination. Gong et al. [28] studied green supply chain strategies and fairness preferences. They found that different fairness strategies affect supply chain profits, with an RR strategy being optimal for retailers. Zhao and Wang [29] examined consumer channel preferences and retailer fairness concerns. They found that platform commission rates, fairness type, and consumer preferences affect manufacturer and platform profits, with horizontal fairness concerns improving service levels. Zhao et al. [30] investigated consumer fairness concerns in e-commerce supply chain pricing. They found that increased fairness concerns lower sales prices and profits, with platform profits first increasing and then decreasing with higher commission rates.

Fairness concerns in business transactions, especially in supply chains, have been a significant study area. Researchers have investigated how perceptions of fairness affect collaboration, motivation, and performance among business partners. Fairness models, such as the self-due fairness concern and gap fairness concern models, have been developed to understand these dynamics. Our research contributes to this field by integrating these fairness models into the live-stream sales context, analyzing how they influence the outcomes for both internet celebrities and brand manufacturers.

3. Problem Description and Assumptions

This paper assumes a Stackelberg game between an internet celebrity and a brand manufacturer under a "pure commission" model, where the brand manufacturer acts as the

leader and the internet celebrity as the follower. The brand manufacturer first determines the live-streaming price p, and then the internet celebrity determines the effort level $u(t)$ for the live-streaming service, receiving a certain percentage λ of commission from the brand manufacturer. The brand manufacturer is responsible for producing and supplying the goods, while the internet celebrity acts as the seller of the product, leveraging their influence and platform to recommend the brand manufacturer's products to fans and viewers. The internet celebrity showcases the products via live-streaming, explains their usage, interacts with the audience to stimulate purchase interest, and, ultimately, promotes sales. Additionally, we considered the fairness concern factor and constructed a differential game model for three different cooperation modes between the internet celebrity and the brand manufacturer. This paper investigated the optimal efforts and pricing equilibrium strategies for the internet celebrity and brand manufacturer under different cooperation modes, alongside their respective profits at equilibrium. The sequential order of the game is illustrated in Figure 1.

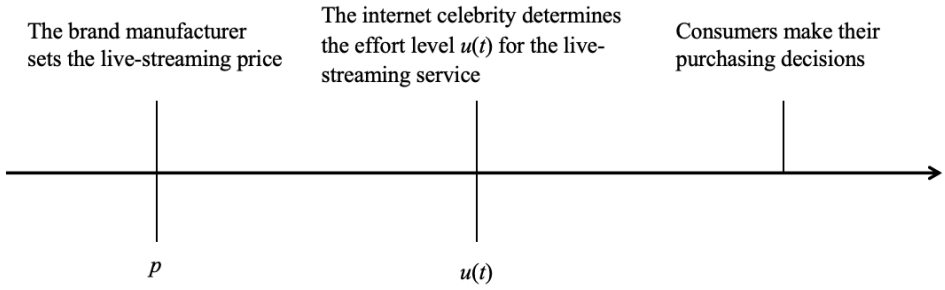

Figure 1. Sequence of events.

The influence and attractiveness of the internet celebrity during live-streaming are crucial for the brand manufacturer because they can attract potential customers and audiences, thereby increasing brand awareness and sales. This attractiveness and reputation can be considered goodwill, which positively impacts the brand's value; thus, the internet celebrity's influence and the brand manufacturer's promotional efforts for the internet celebrity are positively correlated with potential market demand. This paper draws on the goodwill model proposed by Nerlove et al. [31] as a reference, considering the correlation between the change in potential demand for internet celebrity live-streaming traffic and the change in goodwill. We constructed a differential equation to describe the live-streaming traffic of the internet celebrity based on this model, considering the decay in internet celebrity live-streaming traffic [14].

$$d[N(t)] = [\varphi u(t) - \mu N(t)]dt. \tag{1}$$

where $u(t)$ represents the effort level of the internet celebrity's live-streaming at time t, φ represents the coefficient of the influence of the internet celebrity live-streaming, $N(t)$ represents the internet celebrity live-streaming traffic, μ is the decay rate of internet celebrity live-streaming traffic, $N(0) = \varphi N_0 \geq 0$.

Based on the research of Liu et al. [32], we assumed that the demand faced by the brand manufacturer depends on a separable multiplicative form between the retail price p and live-streaming traffic N at time t. Therefore, the demand function $Q(t)$ for the brand manufacturer at time t can be represented as

$$Q(t) = (a - bp)\eta N(t). \tag{2}$$

In similar research, authors have typically used separable multiplicative forms related to price and non-price factors to model demand functions [33–35], where $(a - bp)$ represents the price factor, $\eta N(t)$ represents the non-price factor, b represents the coefficient representing the impact of the product price on the sales volume of the brand manufacturer, and

$\eta(>0)$represents the coefficient indicating the impact of internet celebrity live-streaming traffic on additional sales volume.

Drawing on Kotowitz and Mathewson's notion of convex cost [36], we determined that the coefficient of internet celebrity service cost is $\frac{1}{2}cu^2$, where c is the coefficient of the internet celebrity service cost. The larger the coefficient, the higher the cost of internet celebrity service.

In the infinite decision period $[0, \infty)$, both the brand manufacturer and the internet celebrity share the same discount rate ρ. Their common goal is to explore the best strategies during this boundless time range to maximize their respective interests.

The profit of the brand manufacturer and the internet celebrity are as follows:

$$R_M = (1-\lambda)p(a-bp)\eta N(t), \tag{3}$$

$$R_O = \lambda p(a-bp)\eta N(t) - \frac{1}{2}cu^2(t). \tag{4}$$

Table 1 introduces the notations used in the problem's definition and methodology.

Table 1. Notation definitions.

Notations	Explanation
p	Live-streaming price of the product
$N(t)$	Traffic brought by the internet celebrity live-streaming at time t.
$u(t)$	Effort level of the internet celebrity's live-streaming at time t .
λ	Commission rate paid by the brand manufacturer to the internet celebrity $(0 < \lambda < 1)$.
a	Potential sales quantity of the brand manufacturer's product from the live-streaming channel $(a > 0)$.
b	Coefficient of the influence of product price on the sales quantity of the brand manufacturer $(b > 0)$.
φ	Coefficient of the influence of the internet celebrity live-streaming.
μ	Attenuation rate of the internet celebrity live-streaming traffic.
η	Coefficient of the impact of traffic on sales quantity $(\eta > 0)$.
θ	Fairness concern coefficient of the internet celebrity towards the brand manufacturer $(0 < \theta < 1)$.
γ	Nash bargaining power parameter for the internet celebrity $(0 < \gamma < 1)$.
c	Cost coefficient of the internet celebrity's service.
ρ	Discount rate of the internet celebrity and the brand manufacturer $(\rho > 0)$.
$R_M(t), R_O(t)$	Represent the profit of the brand manufacturer and the internet celebrity, respectively.
$J_M(t), J_O(t)$	Represent the objective function of the brand manufacturer and the internet celebrity, respectively.
$V_M(t), V_O(t)$	Represent the value model of the brand manufacturer and the internet celebrity, respectively.

4. Differential Game Equilibrium Analysis

4.1. No Fairness Concern Model (NF)

Under the no fairness concern model, the internet celebrity and the brand manufacturer act as independent decision entities, selecting strategies based on their respective principles of maximizing benefits without considering fairness concerns. The profits of the internet celebrity and the brand manufacturer are discounted by the discount factor ρ over an infinite time interval. The objective functions of the internet celebrity and the brand manufacturer under the no fairness concern model are as follows:

$$\max_{p} J_M^{NF} = \int_0^{\infty} e^{-\rho t} R_M dt, \tag{5}$$

$$\max_{u(t)} J_O^{NF} = \int_0^{\infty} e^{-\rho t} R_O dt. \tag{6}$$

Substituting Equations (3) and (4) into Equations (5) and (6), we can obtain

$$\max_{p} J_M^{NF} = \int_0^{\infty} e^{-\rho t}\left[(1-\lambda)p(a-bp)\eta N(t)\right]dt, \tag{7}$$

$$\max_{u(t)} J_O^{NF} = \int_0^{\infty} e^{-\rho t}\left[\lambda p(a-bp)\eta N(t) - \frac{1}{2}cu^2(t)\right]dt. \tag{8}$$

From Equations (7) and (8), we can obtain the optimal efforts, profits, and pricing strategies for both the internet celebrity and the brand manufacturer under the no fairness concern model.

Proposition 1. *Under the no fairness concern model, the differential game equilibrium strategies for the internet celebrity and the brand manufacturer are as follows:*

(1) The optimal pricing and effort level for the internet celebrity are

$$p^{NF*} = \frac{a}{2b}, \tag{9}$$

$$u^{NF*}(t) = \frac{\lambda\eta\varphi a^2}{4bc(\rho+\mu)}. \tag{10}$$

(2) The live-streaming traffic at time t is

$$N^{NF*}(t) = \frac{\varphi u^{NF*}(t)}{\mu} + (N_0 - \frac{\varphi u^{NF*}(t)}{\mu})e^{-\rho t}. \tag{11}$$

(3) The optimal profit for the brand manufacturer and the internet celebrity at time t are

$$V_M^{NF} = A_1 N^{NF*} + B_1, \tag{12}$$

$$V_O^{NF} = A_2 N^{NF*} + B_2. \tag{13}$$

where

$$\begin{cases} A_1 = \dfrac{(1-\lambda)\eta a^2}{4b(\rho+\mu)}, \\ B_1 = \dfrac{A_1 A_2 \varphi^2}{\rho c}, \\ A_2 = \dfrac{\lambda\eta a^2}{4b(\rho+\mu)}, \\ B_2 = \dfrac{A_2^2\varphi^2}{2\rho c}. \end{cases} \tag{14}$$

Proof. According to the definition of the no fairness concern model, the objective function for both the internet celebrity and the brand manufacturer are

$$\max_{p} J_M^{NF} = \int_0^{\infty} e^{-\rho t}[(1-\lambda)p(a-bp)\eta N(t)]dt,$$

$$\max_{u(t)} J_O^{NF} = \int_0^{\infty} e^{-\rho t}[\lambda p(a-bp)\eta N(t) - \frac{1}{2}cu^2(t)]dt.$$

Let V_M^{NF} and V_O^{NF} represent the profit functions of the brand manufacturer and the internet celebrity, respectively. We can derive the following HJB equations:

$$\rho V_M^{NF} = \max_{p}\left[(1-\lambda)p(a-bp)\eta N(t) + V_M^{NF'}[\varphi u(t) - \mu N(t)]\right], \tag{15}$$

$$\rho V_O^{NF} = \max_{u(t)} \left[\lambda p(a - bp)\eta N(t) - \frac{1}{2} c u^2(t) + V_O^{NF'} [\varphi u(t) - \mu N(t)] \right]. \tag{16}$$

Taking the first-order derivatives of p and $u(t)$ in the HJB Equations (15) and (16) and setting them to zero, we obtain the optimal pricing and effort level for the internet celebrity:

$$p^{NF*} = \frac{a}{2b}, \tag{17}$$

$$u^{NF*}(t) = \frac{\varphi V_O^{NF'}}{c}. \tag{18}$$

Substituting the optimal pricing (17) and effort level for the internet celebrity (18) into HJB Equations (15) and (16), we can obtain

$$\rho V_M^{NF} = \left[\frac{(1-\lambda)\eta a^2}{4b} - \mu V_M^{NF'} \right] N(t) + \frac{\varphi^2 V_M^{NF'} V_O^{NF'}}{c}, \tag{19}$$

$$\rho V_O^{NF} = \left[\frac{(1-\lambda)\eta a^2}{4b} - \mu V_O^{NF'} \right] N(t) + \frac{\left(\varphi V_O^{NF'}\right)^2}{2c}. \tag{20}$$

By observing the structure of Equations (19) and (20), we can infer that the optimal values V_M^{NF} and V_O^{NF} in the HJB equations are linear functions of the internet celebrity live-streaming traffic N. Therefore, we set

$$V_M^{NF} = A_1 N^{NF*}(t) + B_1, V_M^{NF'} = A_1, \tag{21}$$

$$V_O^{NF} = A_2 N^{NF*}(t) + B_2, V_O^{NF'} = A_2. \tag{22}$$

Substituting Equations (21) and (22) into (19) and (20), and using the method of undetermined coefficients, we can solve for the following coefficients:

$$\begin{cases} A_1 = \dfrac{(1-\lambda)\eta a^2}{4b(\rho+\mu)}, \\ B_1 = \dfrac{A_1 A_2 \varphi^2}{\rho c}, \\ A_2 = \dfrac{\lambda \eta a^2}{4b(\rho+\mu)}, \\ B_2 = \dfrac{A_2^2 \varphi^2}{2\rho c}. \end{cases} \tag{23}$$

Substituting coefficients (23) into the expressions for $u^{NF*}(t)$ in (18), we can obtain

$$u^{NF*}(t) = \frac{\lambda \eta \varphi a^2}{4bc(\rho+\mu)}.$$

The expression for live stream traffic as a function of time is

$$\bar{N}(t) = \frac{dN(t)}{dt} = \varphi u(t) - \mu N(t).$$

Based on the general solution of the first-order differential equation, we can obtain the expression of the particular solution function:

$$N(t) = \frac{\varphi u(t)}{\mu} + \left(N_0 - \frac{\varphi u(t)}{\mu} \right) e^{-\rho t}. \tag{24}$$

Therefore, the live-streaming traffic under the no fairness concern models is

$$N^{NF*}(t) = \frac{\varphi u^{NF*}(t)}{\mu} + (N_0 - \frac{\varphi u^{NF*}(t)}{\mu})e^{-\rho t}. \tag{25}$$

The proposition is thus proved. We can derive the optimal pricing and effort level for the internet celebrity, as well as the live-streaming traffic at time t, and the optimal profits for both the brand manufacturer and the internet celebrity at time t under the no fairness concern model. □

4.2. Gap Fairness Concern Model (GF)

The sense of unfairness perceived by an internet celebrity comes from the profit gap between the internet celebrity and the brand manufacturer. When the profit earned by the brand manufacturer exceeds that earned by the internet celebrity, the internet celebrity perceives this as unfair. The greater the profit gap, the more unfair this is perceived to be by the internet celebrity.

The profits of the internet celebrity and the brand manufacturers are discounted by the discount factor ρ over an infinite time interval. The objective functions of the internet celebrity and the brand manufacturer under the gap fairness concern model are as follows:

$$\max_{p} J_M^{GF} = \int_0^{\infty} e^{-\rho t} R_M dt, \tag{26}$$

$$\max_{u(t)} J_O^{GF} = \int_0^{\infty} e^{-\rho t} \Big[R_O + \theta(R_O - R_M) \Big] dt. \tag{27}$$

where $\theta(0 < \theta < 1)$ represents the fairness concern coefficient of the internet celebrity towards brand manufacturers, π_O represents the profit obtained by the internet celebrity from the brand manufacturer, and π_M represents the profit obtained by the brand manufacturer. The term $\theta(R_O - R_M)$ represents the internet celebrity's sense of unfairness regarding the profit gap between the internet celebrity and the brand manufacturer [20].

Substituting Equations (3) and (4) into Equations (26) and (27), we can obtain

$$\max_{p} J_M^{GF} = \int_0^{\infty} e^{-\rho t} [(1-\lambda)p(a-bp)\eta N(t)] dt, \tag{28}$$

$$\max_{u(t)} J_O^{GF} = \int_0^{\infty} e^{-\rho t} \Big[[\lambda + \theta(2\lambda - 1)]p(a-bp)\eta N(t) - \frac{1}{2}(1+\theta)cu^2(t) \Big] dt. \tag{29}$$

From Equations (28) and (29), we can obtain the optimal efforts, profits, and pricing strategies for both the internet celebrity and the brand manufacturer under the gap fairness concern model.

Proposition 2. *Under the gap fairness concern model, the differential game equilibrium strategies for the internet celebrity and the brand manufacturer are as follows:*

(1) The optimal pricing and effort level for the internet celebrity are

$$p^{GF*} = \frac{a}{2b}, \tag{30}$$

$$u^{GF*}(t) = \frac{[\lambda + \theta(2\lambda - 1)]\eta\varphi a^2}{4(1+\theta)bc(\rho + \mu)}. \tag{31}$$

(2) The live-streaming traffic at time t is

$$N^{GF*}(t) = \frac{\varphi u^{GF*}(t)}{\mu} + (N_0 - \frac{\varphi u^{GF*}(t)}{\mu})e^{-\rho t}. \tag{32}$$

(3) The optimal profit for the brand manufacturer and the internet celebrity at time t are

$$V_M^{GF} = C_1 N^{GF*}(t) + D_1, \tag{33}$$

$$V_O^{GF} = C_2 N^{GF*}(t) + D_2. \tag{34}$$

where

$$\begin{cases} C_1 = \dfrac{(1-\lambda)\eta a^2}{4b(\rho+\mu)}, \\ D_1 = \dfrac{C_1 C_2 \varphi^2}{(1+\theta)\rho c}, \\ C_2 = \dfrac{[\lambda+\theta(2\lambda-1)]\eta a^2}{4b(\rho+\mu)}, \\ D_2 = \dfrac{C_2^2 \varphi^2}{2(1+\theta)\rho c}. \end{cases} \tag{35}$$

Proof. According to the definition of the gap fairness concern model, the objective function for both the internet celebrity and the brand manufacturer are

$$\max_p J_M^{GF} = \int_0^\infty e^{-\rho t}[(1-\lambda)p(a-bp)\eta N(t)]dt,$$

$$\max_{u(t)} J_O^{GF} = \int_0^\infty e^{-\rho t}\left[[\lambda+\theta(2\lambda-1)]p(a-bp)\eta N(t) - \frac{1}{2}(1+\theta)cu^2(t)\right]dt.$$

Let V_M^{GF} and V_O^{GF} represent the profit functions of the brand manufacturer and the internet celebrity, respectively. We can derive the following HJB equations:

$$\rho V_M^{GF} = \max_p \left[(1-\lambda)p(a-bp)\eta N(t) + V_M^{GF'}[\varphi u(t) - \mu N(t)]\right], \tag{36}$$

$$\begin{aligned} \rho V_O^{GF} = \max_{u(t)} \Big[&[\lambda+\theta(2\lambda-1)]p(a-bp)\eta N(t) - \frac{1}{2}(1+\theta)cu^2(t) \\ &+ V_O^{GF'}[\varphi u(t) - \mu N(t)]\Big]. \end{aligned} \tag{37}$$

Taking the first-order derivatives of p and $u(t)$ in the HJB Equations (36) and (37) and setting them to zero, we can obtain the optimal pricing and effort for the internet celebrity:

$$p^{GF*}(t) = \frac{a}{2b}, \tag{38}$$

$$u^{GF*}(t) = \frac{\varphi V_O^{GF'}}{(1+\theta)c}. \tag{39}$$

Substituting the optimal pricing (38) and effort level for the internet celebrity (39) into HJB Equations (36) and (37), we can obtain

$$\rho V_M^{GF} = \left[\frac{(1-\lambda)\eta a^2}{4b} - \mu V_M^{GF'}\right]N(t) + \frac{\varphi^2 V_M^{GF'} V_O^{GF'}}{(1+\theta)c}, \tag{40}$$

$$\rho V_O^{GF} = \left[\frac{[\lambda+\theta(2\lambda-1)]\eta a^2}{4b} - \mu V_O^{GF'}\right]N(t) + \frac{(\varphi V_O^{GF'})^2}{2(1+\theta)c}. \tag{41}$$

By observing the structure of Equations (40) and (41), we can infer that the optimal values V_M^{GF} and V_O^{GF} in the HJB equations are linear functions of the internet celebrity live-streaming traffic N. Therefore, we set

$$V_M^{GF} = C_1 N^{GF*}(t) + D_1, V_M^{GF'} = E_1, \tag{42}$$

$$V_O^{GF} = C_2 N^{GF*}(t) + D_2, V_O^{GF'} = E_2. \tag{43}$$

Substituting Equations (42) and (43) into (40) and (41), and using the method of undetermined coefficients, we can solve for the coefficients:

$$\begin{cases} C_1 = \dfrac{(1-\lambda)\eta a^2}{4b(\rho+\mu)}, \\ D_1 = \dfrac{C_1 C_2 \varphi^2}{(1+\theta)\rho c}, \\ C_2 = \dfrac{[\lambda+\theta(2\lambda-1)]\eta a^2}{4b(\rho+\mu)}, \\ D_2 = \dfrac{C_2^2 \varphi^2}{2(1+\theta)\rho c}. \end{cases} \tag{44}$$

Substituting coefficients (44) into the expressions for $u^{GF*}(t)$ in (39), we can obtain

$$u^{GF*}(t) = \frac{[\lambda+\theta(2\lambda-1)]\eta\varphi a^2}{4(1+\theta)bc(\rho+\mu)}.$$

According to (24), we can the derive that the live-streaming traffic under the self-due fairness concern model is

$$N^{SF*}(t) = \frac{\varphi u^{SF*}(t)}{\mu} + (N_0 - \frac{\varphi u^{SF*}(t)}{\mu})e^{-\rho t}. \tag{45}$$

The proposition is thus proved. We can derive the optimal pricing and effort level for the internet celebrity, as well as the live-streaming traffic at time t, and the optimal profits for both the brand manufacturer and the internet celebrity at time t under the gap fairness concern model. □

4.3. Self-Due Fairness Concern Model (SF)

Contrasting with gap fairness concerns, self-due fairness concerns use the internet celebrity's Nash bargaining profit as the reference point. If the internet celebrity's profit exceeds this reference point, it is deemed fair. Conversely, if the profit is below the reference point, it is perceived as unfair.

The profits of the internet celebrity and the brand manufacturer are discounted by the discount factor ρ over an infinite time interval. The objective functions of the internet celebrity and the brand manufacturer under the self-due fairness concern model are as follows:

$$\max_p J_M^{SF} = \int_0^\infty e^{-\rho t} R_M dt, \tag{46}$$

$$\max_{u(t)} J_O^{SF} = \int_0^\infty e^{-\rho t} \Big[R_O + \theta(R_O - \overline{R}_O) \Big] dt. \tag{47}$$

where $\theta(0 < \theta < 1)$ represents the fairness concern coefficient of the internet celebrity towards the brand manufacturer, R_O denotes the profit obtained by the internet celebrity from the brand manufacturer, and R_M denotes the profit obtained by the brand manufacturer.

Suppose the brand manufacturer is fairness neutral, and the internet celebrity has a fairness concern with fairness concern parameter θ, the Nash bargaining reference point [20] for the retailer is given by

$$\overline{R}_O = \frac{\gamma(1+\theta)}{1+\gamma\theta}(R_M + R_O). \tag{48}$$

Substituting Equations (3), (4), and (48) into Equations (46) and (47), we can obtain

$$\max_{p} J_M^{SF} = \int_0^{\infty} e^{-\rho t}[(1-\lambda)p(a-bp)\eta N(t)]dt, \tag{49}$$

$$\max_{u(t)} J_O^{SF} = \int_0^{\infty} e^{-\rho t}\left[[\lambda+\theta(\lambda-v)]p(a-bp)\eta N(t) - \frac{1}{2}(1+\theta-v\theta)cu^2(t)\right]dt. \tag{50}$$

where

$$v = \frac{\gamma(1+\theta)}{1+\gamma\theta}. \tag{51}$$

From Equations (49) and (50), we can obtain the optimal efforts, profits, and pricing strategies for both the internet celebrity and the brand manufacturer under the self-due fairness concern model:

Proposition 3. *Under the self-due fairness concern model, the differential game equilibrium strategies for the internet celebrity and the brand manufacturer are as follows:*

(1) The optimal pricing and effort level for the internet celebrity are

$$p^{SF*} = \frac{a}{2b}, \tag{52}$$

$$u^{SF*}(t) = \frac{[\lambda+\theta(\lambda-v)]\eta\varphi a^2}{4(1+\theta-v\theta)bc(\rho+\mu)}. \tag{53}$$

(2) The live-streaming traffic at time t is

$$N^{SF*}(t) = \frac{\varphi u^{SF*}(t)}{\mu} + (N_0 - \frac{\varphi u^{SF*}(t)}{\mu})e^{-\rho t}. \tag{54}$$

(3) The optimal profit for the brand manufacturer and the internet celebrity at time t are

$$V_M^{SF} = E_1 N^{SF*}(t) + F_1, \tag{55}$$

$$V_O^{SF} = E_2 N^{SF*}(t) + F_2. \tag{56}$$

where

$$\begin{cases} E_1 = \dfrac{(1-\lambda)\eta a^2}{4b(\rho+\mu)}, \\ F_1 = \dfrac{E_1 E_2 \varphi^2}{(1+\theta-v\theta)\rho c}, \\ E_2 = \dfrac{[\lambda+\theta(\lambda-v)]\eta a^2}{4b(\rho+\mu)}, \\ F_2 = \dfrac{E_2^2\varphi^2}{2(1+\theta-v\theta)\rho c}. \end{cases} \tag{57}$$

Proof. According to the definition of the fairness concern model, the objective function for both the internet celebrity and the brand manufacturer are

$$\max_{p} J_M^{SF} = \int_0^{\infty} e^{-\rho t}[(1-\lambda)p(a-bp)\eta N(t)]dt,$$

$$\max_{u(t)} J_O^{SF} = \int_0^{\infty} e^{-\rho t}\left[[\lambda+\theta(\lambda-v)]p(a-bp)\eta N(t) - \frac{1}{2}(1+\theta-v\theta)cu^2(t)\right]dt.$$

Let V_{SM}^{SF} and V_O^{SF} represent the profit functions of the brand manufacturer and the internet celebrity, respectively. We can derive the following HJB equations:

$$\rho V_M^{SF} = \max_p \left[(1-\lambda)p(a-bp)\eta N(t) + V_M^{SF'}[\varphi u(t) - \mu N(t)] \right], \tag{58}$$

$$\begin{aligned} \rho V_O^{SF} = \max_{u(t)} \Big[& [\lambda + \theta(\lambda - v)]p(a-bp)\eta N(t) - \frac{1}{2}(1+\theta-v\theta)cu^2(t) \\ & + V_O^{SF'}[\varphi u(t) - \mu N(t)] \Big]. \end{aligned} \tag{59}$$

Taking the first-order derivatives of p and $u(t)$ in the HJB Equations (58) and (59) and setting them to zero, we can obtain the optimal pricing and effort for the internet celebrity:

$$p^{SF*}(t) = \frac{a}{2b}, \tag{60}$$

$$u^{SF*}(t) = \frac{\varphi V_O^{F'}}{(1+\theta-v\theta)c}. \tag{61}$$

Substituting the optimal pricing (60) and the effort level for the internet celebrity (61) into HJB Equations (58) and (59), we can obtain

$$\rho V_M^{SF} = \left[\frac{(1-\lambda)\eta a^2}{4b} - \mu V_M^{SF'} \right] N(t) + \frac{\varphi^2 V_M^{SF'} V_O^{SF'}}{(1+\theta-v\theta)c}, \tag{62}$$

$$\rho V_O^{SF} = \left[\frac{[\lambda + \theta(2\lambda - 1)]\eta a^2}{4b} - \mu V_O^{SF'} \right] N(t) + \frac{\left(\varphi V_O^{SF'}\right)^2}{2(1+\theta-v\theta)c}. \tag{63}$$

By observing the structure of Equations (62) and (63), we can infer that the optimal values V_M^{SF} and V_O^{SF} in the HJB equations are linear functions of the internet celebrity live-streaming traffic N. Therefore, we set

$$V_M^{SF} = E_1 N + F_1, V_M^{SF'} = E_1, \tag{64}$$

$$V_O^{SF} = E_2 N + F_2, V_O^{SF'} = E_2. \tag{65}$$

Substituting Equations (64) and (65) into (62) and (63), and using the method of undetermined coefficients, we can solve for the coefficients:

$$\begin{cases} E_1 = \dfrac{(1-\lambda)\eta a^2}{4b(\rho+\mu)}, \\ F_1 = \dfrac{E_1 E_2 \varphi^2}{(1+\theta-v\theta)\rho c}, \\ E_2 = \dfrac{[\lambda + \theta(\lambda - v)]\eta a^2}{4b(\rho+\mu)}, \\ F_2 = \dfrac{E_2^2 \varphi^2}{2(1+\theta-v\theta)\rho c}. \end{cases} \tag{66}$$

Substituting coefficients (66) into the expressions for $u^{SF*}(t)$ in (61), we obtain

$$u^{SF*}(t) = \frac{[\lambda + \theta(\lambda - v)]\eta\varphi a^2}{4(1+\theta-v\theta)bc(\rho+\mu)}.$$

According to (24), we can the derive that the live-streaming traffic under the self-due fairness concern model is

$$N^{SF*}(t) = \frac{\varphi u^{SF*}(t)}{\mu} + (N_0 - \frac{\varphi u^{SF*}(t)}{\mu})e^{-\rho t}. \tag{67}$$

The proposition is thus proved. We can derive the optimal pricing and effort level for the internet celebrity, as well as the live-streaming traffic at time t, and the optimal profits for both the brand manufacturer and the internet celebrity at time t under the self-due fairness concern model. □

5. Numerical Analysis

In this section, we conducted numerical experiments using MATLAB R2016b to validate the theoretical results obtained in Section 5. The selection of basic parameter values was based on the research in [14,20], with appropriate adjustments to meet the model's assumptions. The specific settings were as follows: $a = 40, b = 0.8, c = 20, \varphi = 1$, $\mu = 1, \eta = 0.5, \lambda = 0.3, \theta = 0.3, \gamma = 0.1, \rho = 0.15, N_0 = 0$.

5.1. Comparison of Live-Streaming Traffic

Figure 2 demonstrates that the magnitude relationship of the internet celebrity traffic under the three collaboration modes was $u^{GF^*}(t) < u^{NF^*}(t) < u^{SF^*}(t)$. As time progressed, the internet celebrity traffic was maximized under the self-due fairness concern model. The internet celebrity traffic under the no fairness concern model was slightly higher than that under the gap fairness concern model.

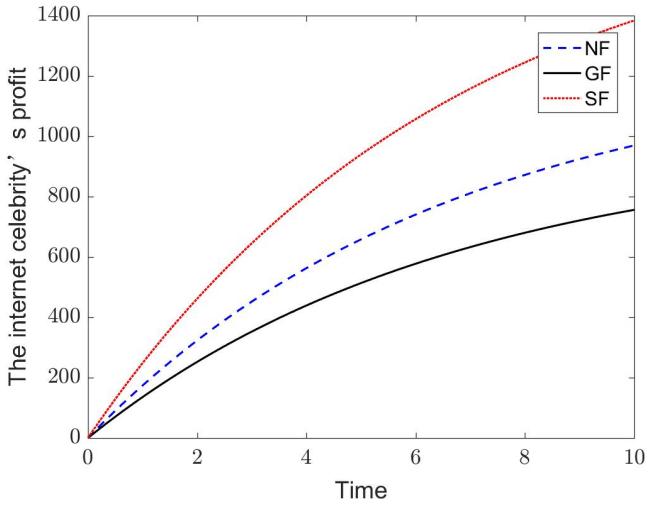

Figure 2. Comparison of live-streaming traffic over time under three models.

5.2. Comparison of Internet Celebrity's Profit

Figure 3 illustrates that the internet celebrity's profit was the highest under the self-due fairness concern model, followed by the no fairness concern model, and lowest under the gap fairness concern model. Given that the commission rate λ for the internet celebrity from the brand owner was 0.3, this resulted in the internet celebrity's profit being lower than that of the brand owner under the gap fairness concern model. This disparity induced a sense of unfairness in the internet celebrity, diminishing their enthusiasm for live-streaming, which, in turn, reduced the efficiency and profit. Conversely, the internet celebrity's sense of unfairness under the self-due fairness concern model was derived from their own Nash bargaining profit and was independent of the brand owner's profit, more closely aligning with real-world scenarios. The internet celebrity's profit motivated them when it exceeded their own Nash bargaining profit. This enhanced their live-streaming efficiency and encouraged consumers to purchase products, increasing their profit as a result.

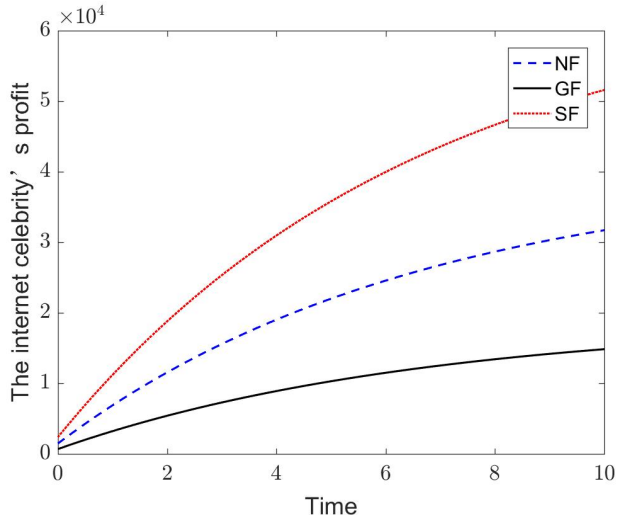

Figure 3. Comparison of internet celebrity's profit over time under three models.

5.3. Comparison of Brand Manufacturer's Profit

Figure 4 shows that the brand manufacturer's profit under the no fairness concern model was higher than under the self-due fairness concern and gap fairness concern models. This indicates that the brand manufacturer could achieve higher profits when there was no fairness concern between the brand manufacturer and the internet celebrity. However, the models incorporating fairness concerns also had significant advantages. Although the brand manufacturer's profit slightly decreased under the self-due fairness concern and gap fairness concern models, these models could enhance the internet celebrity's enthusiasm and efficiency because, when feeling fairly treated, they were more willing to invest effort into live-streaming, thereby attracting more consumers and increasing overall sales. In this situation, while the brand manufacturer's short-term profit decreases, the improvement in live-streaming efficiency and sales might lead to a larger market share and customer loyalty in the long run. Additionally, fairness concern models can promote long-term cooperative relationships between the brand manufacturer and the internet celebrity, reducing the risk of cooperation breakdown due to perceived unfairness.

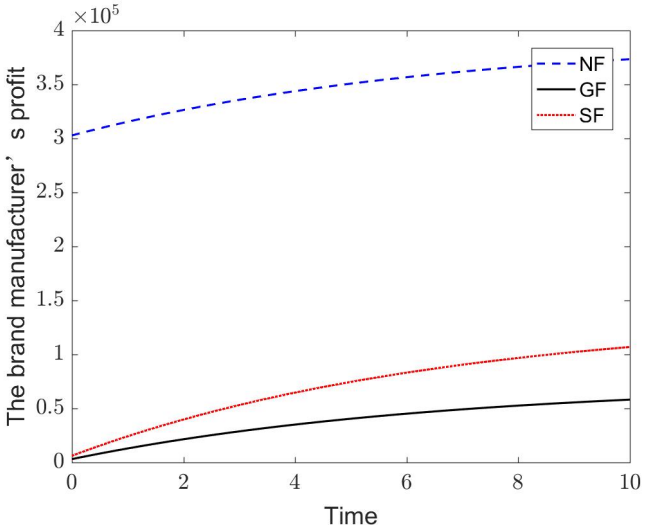

Figure 4. Comparison of brand manufacturer's profit over time under three models.

5.4. Impact of the Commission Rate

Figure 5 demonstrates that the internet celebrity's profit under the self-due fairness concern model exceeded that under the no fairness concern and gap fairness concern

models with a low commission rate. This was because the self-due fairness concern model emphasized the internet celebrity's focus on their own fairness and interests when cooperating with brand owners. The internet celebrity felt fairly treated under this model; feelings of exploitation or unfairness due to low commission rates were avoided, so a higher work motivation and engagement were maintained. In contrast, an internet celebrity may feel discontent or perceive unfairness due to low commission rates under the no fairness concern and gap fairness concern models. The internet celebrity's profits under the no fairness concern model may fall short of expectations because the cooperation does not consider their personal fairness perceptions. The internet celebrity's profits under the gap fairness concern model are constrained below the level of brand owners, potentially leading to feelings of exploitation or neglect, which could impact their work motivation and efficiency.

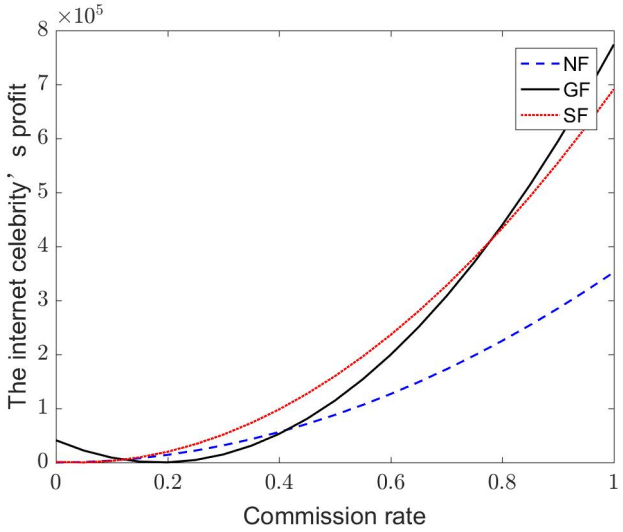

Figure 5. Comparison of internet celebrity's profit under three models with varying commission rate.

Figure 6 shows that when the commission rate was zero, the brand manufacturer's profit under the no fairness concern model was zero. This is because live-streaming traffic is positively correlated with the commission rate. A commission rate of zero means the live-streaming traffic is zero, resulting in zero profit. Under the self-due fairness concern and gap fairness concern models, the internet celebrity's sense of unfairness negatively impacted the brand manufacturer's profit, making it negative when the commission rate was zero. When the commission rate was 0.5, the profits were highest under all three models. This was because a commission rate of 0.5 provided a balanced distribution of interests between the internet celebrity and the brand manufacturer. The internet celebrity received half of the sales revenue as a commission, which incentivized them to improve their live-streaming quality and increase sales. This did not excessively sacrifice the brand manufacturer's profit, balancing both parties' gains within a reasonable range. When the commission rate was one, the brand manufacturer's entire sales revenue was paid to the internet celebrity, resulting in zero profit. Extremely low or high commission rates lead to undesirable outcomes. A low commission rate demotivates the internet celebrity, while a high commission rate reduces the brand manufacturer's net profit.

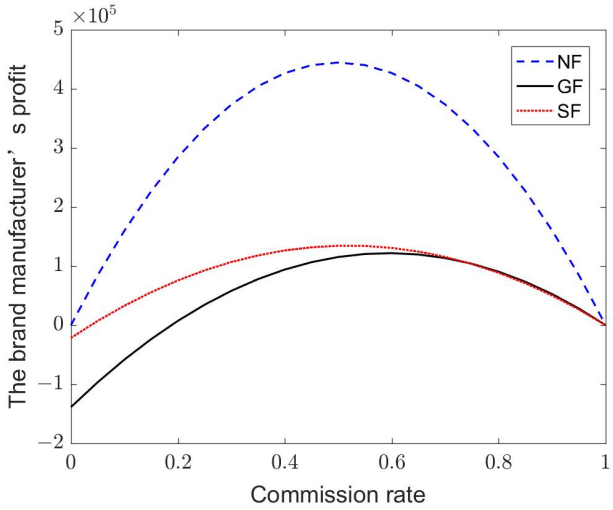

Figure 6. Comparison of brand manufacturer's profit under three models with varying commission rate.

5.5. Impact of the Fairness Concern Coefficient

Figures 7 and 8 demonstrate that the profits for both the internet celebrity and the brand manufacturer under the self-due fairness concern model were higher than under the gap fairness concern model, regardless of the changes in the fairness concern coefficient. Under the gap fairness concern model, with the commission rate set at 0.3, the internet celebrity's profit was lower than that of the brand manufacturer, leading to a sense of unfairness. This feeling of unfairness negatively impacted both parties' profits.

Under the self-due fairness concern model, the internet celebrity's sense of unfairness was based on their Nash bargaining profit, which is independent of the commission rate. It is influenced by the fairness concern coefficient and the Nash bargaining power parameter. Under the pure commission model, brand manufacturers often collaborate with ordinary internet celebrities, who typically have a lower Nash bargaining power. The internet celebrity's profit can be increased beyond the expected Nash bargaining profit by adjusting the Nash bargaining power parameter. This implies that the internet celebrity's sense of fairness is independent of the brand manufacturer's profit, thereby ensuring the brand manufacturer's profit remains unaffected. However, increasing the commission rate under the gap fairness concern model reduced the brand manufacturer's profit. Thus, the self-due fairness concern model was more advantageous and reasonable, ensuring higher profits for both parties and promoting a more equitable collaboration.

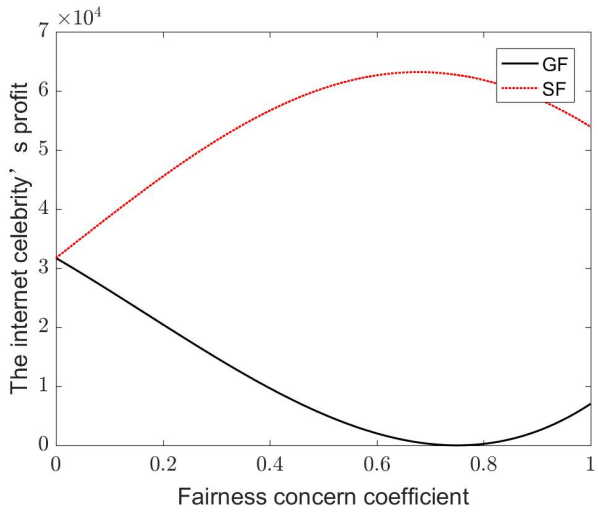

Figure 7. Comparison of internet celebrity's profit under GF and SF models with varying fairness concern coefficient.

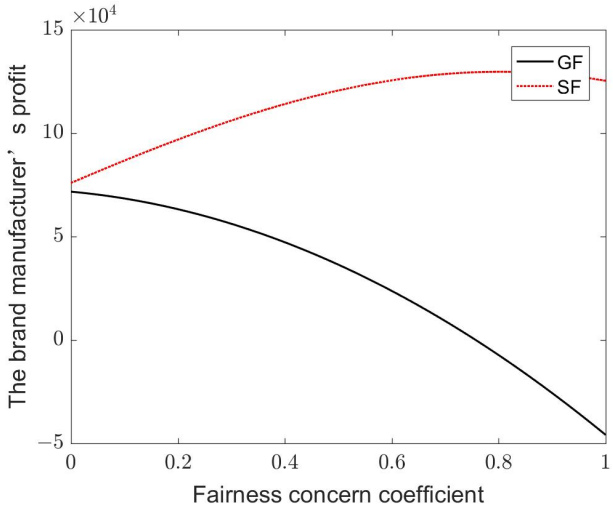

Figure 8. Comparison of brand manufacturer's profit under GF and SF models with varying fairness concern coefficient.

5.6. Impact of the Nash Bargaining Power Parameter

The Nash bargaining power parameter is only relevant to the self-due fairness concern model. Therefore, we focused on the Nash bargaining power parameter's impact on the profits of both the internet celebrity and the brand manufacturer within this model. Figures 9 and 10 show that the profits of both the internet celebrity and the brand manufacturer decreased as the Nash bargaining power parameter increased.

This phenomenon can be explained by the following mechanism: The Nash bargaining power parameter reflects the relative influence of each party on the profit distribution. An increase in this parameter indicates a significant increase in one party's influence, leading to a greater sense of imbalance in the profit distribution. Under the self-due fairness concern model, the internet celebrity's reference point is their self-due profit. Both parties will feel unfairly treated if changes in the bargaining power parameter cause discrepancies in the expectation of fair profit between the internet celebrity and the brand manufacturer. This sense of unfairness reduces the cooperation efficiency and motivation, ultimately leading to lower profits for both parties. Hence, while adjusting the Nash bargaining power parameter might temporarily change the profit distribution, excessive changes can cause long-term imbalances, reducing the overall cooperation efficiency and profits for both the internet celebrity and the brand manufacturer.

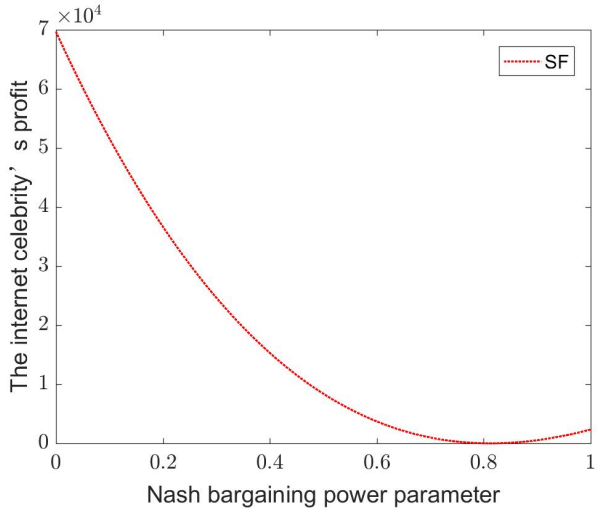

Figure 9. The internet celebrity's profit under the SF model with varying Nash bargaining power parameter.

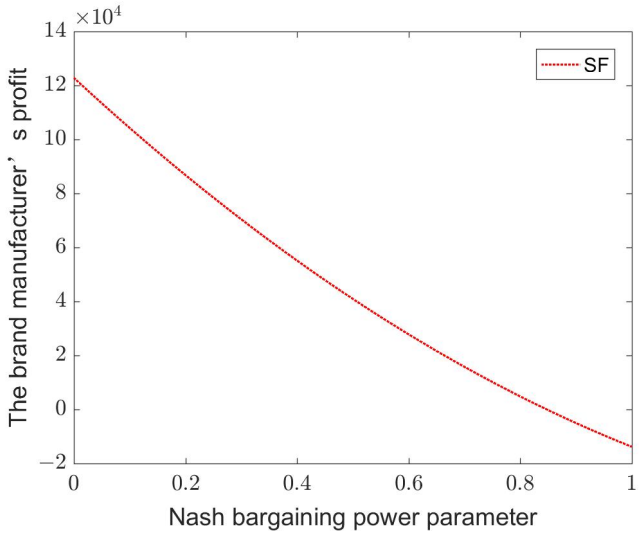

Figure 10. The brand manufacturer's profit under the SF model with varying Nash bargaining power parameter.

6. Conclusions

This study investigated the intricate dynamics between an internet celebrity and a brand manufacturer in social media live-stream selling. The application of differential game models and fairness concern theory demonstrated that the self-due fairness concern model significantly enhances cooperation efficiency and fosters long-term benefits for both parties. This model promotes sustainable and mutually beneficial collaborations by maintaining fairness perceptions and incentivizing motivation and efficiency. Conversely, the gap fairness concern model poses challenges, potentially leading to disparities in the profit distribution and impacting the collaboration stability and profitability over time. This underscores the importance of carefully balancing the interests of both internet celebrities and brand manufacturers, to sustain effective partnerships.

Moreover, this study advocates for future exploration into optimizing cooperation strategies that can adapt to evolving market dynamics and the dynamic landscape of social media platforms. Businesses can thus enhance the effectiveness and sustainability of their live-stream selling strategies, ensuring continued growth and competitiveness in the digital marketplace.

Furthermore, this conclusion emphasizes the practical implications of these findings for stakeholders in the digital economy. By understanding and addressing these dynamics, businesses can more effectively navigate the complexities of social media live-stream selling, capitalize on opportunities, and foster stronger customer engagement and market performance. This research contributes to a deeper understanding of strategic interactions in digital marketing and underscores the importance of fairness considerations in shaping successful collaborations.

Our study has several limitations that provide directions for future research. First, our study's basic parameter values were derived from the research in [14,20], with necessary adjustments to align with the model's assumptions. Future research could consider utilizing real data directly obtained from open statistical resources on live-streaming platforms, such as Douyin and Taobao, to validate the model's conclusions, ensuring broader applicability. Second, this paper addressed the optimal decision-making problem between an internet celebrity and a brand manufacturer under a pure commission model in continuous time. Building on this, we could incorporate random terminal time [37] and differential equations that account for dynamics and stochastic disturbances [20] to further deepen the research, which could be explored in future studies.

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