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Quasi-Compactness of Operators for General Markov Chains and Finitely Additive Measures

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Abstract: We study Markov operators T, A, and T^* of general Markov chains on an arbitrary measurable space. The operator, T, is defined on the Banach space of all bounded measurable functions. The operator A is defined on the Banach space of all bounded countably additive measures. We construct an operator T^*, topologically conjugate to the operator T, acting in the space of all bounded finitely additive measures. We prove the main result of the paper that, in general, a Markov operator T^* is quasi-compact if and only if T is quasi-compact. It is proved that the conjugate operator T^* is quasi-compact if and only if the Doeblin condition (D) is satisfied. It is shown that the quasi-compactness conditions for all three Markov operators T, A, and T^* are equivalent to each other. In addition, we obtain that, for an operator T^* to be quasi-compact, it is necessary and sufficient that it does not have invariant purely finitely additive measures. A strong uniform reversible ergodic theorem is proved for the quasi-compact Markov operator T^* in the space of finitely additive measures. We give all the proofs for the most general case. A detailed analysis of Lin's counterexample is provided.

Keywords: general Markov chains; Markov operators; quasi-compactness conditions; finitely additive measures; invariant measures; ergodic theorem; probability theory

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1. Introduction

In this paper, we investigate general Markov chains (MC) on arbitrary measurable spaces. The main methodological feature of our work is that we actively use finitely additive measures. We present a continuation of our research in [1–3]. We apply the operator-theoretical treatment that was introduced by Kryloff and Bogoliouboff in [4]. In this article, the authors introduced a new notion of quasi-compactness and gave a condition for the quasi-compactness of Markov operators on spaces of bounded measurable functions. Under this condition, the Markov operators have good ergodic properties, as was shown in [4]. Yosida and Kakutani in [5] applied this quasi-compactness condition to Markov operators defined on spaces of bounded countably additive measures. This allowed the authors in [5] to prove several new ergodic theorems for general Markov operators with certain additional conditions. Later, the operator approach to the study of general and special MCs developed rapidly. In particular, various authors studied the question of the relationship between the conditions of quasi-compactness of MC and the Doeblin condition of ergodicity of MC [6]. The method developed in [6] was called the direct point-set-theoretical (or probabilistic) method in [5]. The operator approach was also used to study MCs, with operators defined on spaces of integrable functions (see, for example, Horowitz [7]). This methodology was also successfully used to study the ergodic properties of topological MCs (see, for example, Foguel [8–10]). We actively use the operator results for MCs presented in several papers by Lin [11–13]. The operator approach to studying Markov chains is well and thoroughly presented in the monographs Neveu [14] and Revuz [15].

Hernandez-Lerma and Lasserre [16] also studied Markov chains using both probabilistic and operator approaches.

In the recently published paper, Mebarki, Messirdi, and Benharrat [17] construct a general theory of quasi-compact operators in arbitrary and special Banach spaces. Corollaries for Markov chains are given. Probabilistic and operator approaches are also used to describe Markov chains in [17].

In those cases where our constructions and theorems in this article have points of contact with the results of other works listed above (and many others), we provide the corresponding references, remarks, and comments in the main text of the article. In many works using the classical operator approach to studying general Markov operators, two Markov operators are considered. These are the operator T in the space of bounded measurable functions and the operator A in the space of bounded countably additive measures. These operators are dual in a sense, but they are not topologically conjugate. In this paper (as well as in [1–3]), we construct an extension of the operator A to the space of bounded finitely additive measures, and we obtain the corresponding operator T^* conjugate to the operator T. In this way, we obtain a functionally closed construction of spaces and operators. We emphasize that in this construction, the transition function of the Markov chain remains countably additive in the second argument.

The main results of the paper are presented in Sections 4 and 5. In Section 4, in Theorem 4, we prove, in the general case, that a Markov operator T in the space of measurable functions is quasi-compact if and only if its conjugate operator T^* in the space of finitely additive measures is quasi-compact. Furthermore, in Theorem 5, it is proved that if the operator T in an arbitrary Banach space is quasi-compact, then its restriction to an invariant Banach subspace is also quasi-compact. In Corollary 1, it is obtained that if the conjugate Markov operator T^* on the space of finitely additive measures is quasi-compact, then the Markov operator A on the space of countably additive measures is also quasi-compact. In Theorem 6, we prove, in the general case, that the conjugate Markov operator T^* in the space of finitely additive measures is quasi-compact if and only if the Doeblin condition (D) is satisfied. Proposition 1 presents a summary logical diagram on the pairwise equivalence of the quasi-compactness conditions for all three Markov operators T, A, and T^* and on their equivalence to the Doeblin condition (D).

In Theorems 8 and 9, it is proved that for an arbitrary MC, its Markov operator T^* is quasi-compact if and only if the condition $(*)$ is satisfied: all its invariant finitely additive measures are countably additive, i.e., the operator T^* does not have invariant purely finitely additive measures. In Theorem 10, it is proved that if the operator T is quasi-compact, then the set of all its invariant finitely additive measures is finite-dimensional (and all such invariant measures are countably additive). Theorem 11 gives a conversion of Theorem 10 to the case when the operator T has a unique invariant finitely additive measure.

In Section 5, in Theorem 12, we prove, in the general case, a strong uniform reversible ergodic theorem for the quasi-compact Markov operator T^* in the space of finitely additive measures. In Section 6, we study in detail Lin's counterexample [11] of a Markov chain on a countable phase space with Markov operators P and P^* defined on the spaces L_1 and L_∞, respectively. In [11], the author showed that in this construction, the operator P is quasi-compact, but its conjugate operator P^* is not quasi-compact. In our construction, the Markov operators T and T^* for the same Markov chain are defined on other spaces: on the spaces $B(X,\Sigma)$ and $B^*(X,\Sigma) = ba(X,\Sigma)$, respectively. In Section 6, we prove by direct calculation that in this scheme, the conjugate operator T^* for a given MC is also quasi-compact. This does not refute but confirms our Theorem 4.

2. Definitions, Notations and Some Information

Here are some of the basic definitions and concepts we use and their symbolism, focusing on [18,19]. We also use the notation system from our article [1]. Everywhere below, $R = R^1$ is the set of real numbers (number line), and N is the set of natural numbers. Let X be an arbitrary infinite set, and Σ be some sigma-algebra of its subsets. We denote by

$B(X, \Sigma)$ the Banach space of bounded Σ-measurable functions $f : X \to R$ with the sup-norm $\|f\| = sup|f(x)|$.

Definition 1 (see [19]). *A finitely additive non-negative measure $\mu : \Sigma \to R$ is called purely finitely additive (pure charge, pure mean) if any countably additive measure λ satisfying the condition $0 \leq \lambda \leq \mu$, is identically zero. The alternating measure μ is called purely finitely additive if in its Jordan decomposition $\mu = \mu^+ - \mu^-$ both non-negative measures μ^+ and μ^- are purely finitely additive.*

If the measure μ is purely finitely additive, then it is equal to zero on every one-point set: $\mu(\{x\}) = 0, \forall x \in X$ (see [20] [Lemma 1]). The converse, generally speaking, is not true (for example, for the Lebesgue measure on the segment $[0, 1]$).

Theorem 1 (Yosida-Hewitt decomposition, see [19]). *Any finitely additive measure μ can be uniquely decomposed into the sum $\mu = \mu_1 + \mu_2$, where μ_1 is countably additive, and μ_2 is a purely finitely additive measure.*

In this article, we consider Banach spaces of bounded measures $\mu : \Sigma \to R$, with a norm equal to the total variation of the measure μ (but an equivalent sup-norm can also be used): $ba(X, \Sigma)$ is the space of finitely additive measures, $ca(X, \Sigma)$ is the space of countably additive measures, $pfa(X, \Sigma)$ is the space of purely finitely additive measures. If $\mu \geq 0$, then the norm is $\|\mu\| = \mu(X)$. Purely finitely additive measures also form a Banach space $pfa(X, \Sigma)$ with the same norm and $ba(X, \Sigma) = ca(X, \Sigma) \oplus pfa(X, \Sigma)$. We denote the sets of measures: $S_{ba} = \{\mu \in ba(X, \Sigma) : \mu \geq 0, ||\mu|| = 1\}$, $S_{ca} = \{\mu \in ca(X, \Sigma) : \mu \geq 0, ||\mu|| = 1\}$, $S_{pfa} = \{\mu \in pfa(X, \Sigma) : \mu \geq 0, ||\mu|| = 1\}$. All measures from these sets will be called probabilistic. We also denote $S_B = \{f \in B(X, \Sigma), \ 0 \leq f(x) \leq 1, ||f|| = 1\}$. A detailed exposition of the foundations of the general theory of finitely additive measures is contained in the monograph K.P.S.B. Rao, and M.B. Rao [21], in which such measures are called charges.

3. Markov Operators

Markov chains (MCs) on a (phase) measurable space (X, Σ) are given by their transition function (probability) $p(x, E), x \in X, E \in \Sigma$, under ordinary conditions:

(1) $0 \leq p(x, E) \leq 1, p(x, X) = 1, \forall x \in X, \forall E \in \Sigma$;
(2) $p(\cdot, E) \in B(X, \Sigma), \forall E \in \Sigma$;
(3) $p(x, \cdot) \in ca(X, \Sigma), \forall x \in X$.

We emphasize that the transition function is a countably additive measure with respect to the second argument, i.e., we consider classical Markov chains. The transition function generates two Markov linear bounded positive integral operators:

$$T : B(X, \Sigma) \to B(X, \Sigma), (Tf)(x) = Tf(x) = \int_X f(y)p(x, dy), \forall f \in B(X, \Sigma), \forall x \in X;$$

$$A : ca(X, \Sigma) \to ca(X, \Sigma), (A\mu)(E) = A\mu(E) = \int_X p(x, E)\mu(dx), \forall \mu \in ca(X, \Sigma), \forall E \in \Sigma.$$

Let $\mu_0 \in S_{ca}$ be the initial measure. Then the iterative sequence of countably additive probability measures $\mu_n = A\mu_{n-1} \in S_{ca}, n \in N$, is usually identified with the Markov chain. We will call $\{\mu_n\}$ a Markov sequence of measures. Topologically conjugate to the space $B(X, \Sigma)$ is the (isomorphic) space of finitely additive measures: $B^*(X, \Sigma) = ba(X, \Sigma)$ (see, for example, [18]). In this case, the operator $T^* : ba(X, \Sigma) \to ba(X, \Sigma)$ serves as topologically conjugate to the operator T, which is uniquely determined by the well-known rule:

$$T^*\mu(E) = \int_X p(x, E)\mu(dx), \forall \mu \in ba(X, \Sigma), \forall E \in \Sigma.$$

The operator T^* is the only bounded continuation of the operator A to the entire space $ba(X,\Sigma)$ while preserving its analytic form. The operator T^* has its own invariant subspace $ca(X,\Sigma)$, i.e., $T^*[ca(X,\Sigma)] \subset ca(X,\Sigma)$, on which it matches the original A operator. The construction of the Markov operators T and T^* is now functionally closed. We will use the notation $(T^*)_{ca}$, $(T^*)_{ca} = A$, to restrict the operator T^* to the subspace $ca(X,\Sigma)$. The norms of Markov operators on the corresponding spaces are $\|T\| = 1$, $\|A\| = 1$, $\|T^*\| = 1$. In such a setting, it is natural to admit to consideration the Markov sequences of probabilistic finitely additive measures: $\mu_0 \in S_{ba}, \mu_n = T^*\mu_{n-1} \in S_{ba}, n \in N$, keeping the countable additivity of the transition function $p(x,\cdot)$ with respect to the second argument. Such a Markov chain can have cycles consisting of finitely additive measures. The properties of such cycles Markov chain are considered in detail in our paper [22]. You can change the problem statement and allow the transition function $p(x,\cdot)$ itself to be just a finitely additive measure with respect to the second argument. Such Markov chains are also studied (see [20,23]) and are called "finitely additive Markov chains". In this article, we do not consider such MC. Thus, in our case the following terminology is appropriate: we study countably additive Markov chains with operators defined on the space of finitely additive measures. Let us denote the sets of invariant probability measures of the Markov chain:

$$\Delta_{ba} = \{\mu \in S_{ba} : \mu = T^*\mu\}, \Delta_{ca} = \{\mu \in S_{ca} : \mu = A\mu\}, \Delta_{pfa} = \{\mu \in S_{pfa} : \mu = T^*\mu\}.$$

Let M_{ba} be the linear subspace of invariant measures of the Markov chain in the space $ba(X,\Sigma)$. Obviously, M_{ba} is generated by the set Δ_{ba}: $M_{ba} = Sp\Delta_{ba}$. We will also use the notation M_{ca} and M_{pfa} with similar meaning. The linear dimension of the set Δ_{ba} will mean the algebraic dimension of the linear space M_{ba} generated by it and denote it by $dim\Delta_{ba} = dimM_{ba}$. Similarly, we talk about the dimension of the sets Δ_{ca} and Δ_{pfa}. The classical countably additive Markov chain may or may not have invariant probability countably additive measures, i.e., possibly $\Delta_{ca} = \emptyset$ (for example, for a symmetric walk on Z). Šidak was one of the first to extend the Markov operator A to the space of finitely additive measures in the framework of the operator approach and proved the following two important theorems in [24] (1962).

Theorem 2 (**Šidak [24] [Theorem 2.2]**). *Any countably additive Markov chain on an arbitrary measurable space (X,Σ) has at least one invariant finitely additive probability measure, i.e., always $\Delta_{ba} \neq \emptyset$.*

This result was then briefly proved in the author's paper [3] as a simple consequence of the Krein–Rutman theorem ([25] [Theorem 3.1]).

Theorem 3 (**Šidak [24] [Theorem 2.5]**). *If a finitely additive measure μ for an arbitrary Markov chain is invariant $A\mu = \mu$, and $\mu = \mu_1 + \mu_2$ is its decomposition into countably additive and purely finitely additive components, then each of them is also invariant: $A\mu_1 = \mu_1$, $A\mu_2 = \mu_2$.*

4. Quasi-Compactness Conditions and Finitely Additive Measures

Definition 2 (**see, for example, [18] [Definition VI.5.1]**). *A linear operator F, which transfers the Banach space Y to itself, is called compact (completely continuous) if it transfers each bounded set of space Y into a pre-compact (relatively compact) set, i.e., such that its closure is compact in Y.*

It is well known (see, for example, [18] [Theorem VI.5.2, Schauder's Theorem]) that the operator $F\colon Y \to X$ is compact if and only if its conjugate operator $F^*\colon Y^* \to X^*$ is compact.

Definition 3. *A linear operator $F\colon Y \to Y$, where Y is an arbitrary Banach space, is called quasi-compact (quasi-completely continuous) if the following condition is satisfied*

$$(K_F)\begin{cases} \textit{there is a compact (completely continuous) operator } F_1 \\ \textit{and an integer } k \geq 1 \textit{ such that } ||F^k - F_1|| < 1 \end{cases}.$$

The transition function of a general Markov chain generates three different Markov operators:

$T\colon B(X,\Sigma) \to B(X,\Sigma)$, $A\colon ca(X,\Sigma) \to ca(X,\Sigma)$ and $T^*\colon ba(X,\Sigma) \to ba(X,\Sigma)$. The quasi-compactness condition (K_F) can be applied to each of them with the appropriate replacement of the symbol F by the symbols T, A, and T^*. Denote such conditions (K_T), (K_A) and (K_{T^*}), respectively. Next, we use one construction from [18]), which we present here in our symbolism and edition. Let $\mathfrak{X}$ be an arbitrary Banach space, $B(\mathfrak{X})$ be the space of all linear continuous bounded operators $F\colon \mathfrak{X} \to \mathfrak{X}$.

We denote:

$\mathfrak{X}^*$ is a Banach space topologically conjugate to $\mathfrak{X}$, F^* is the operator $F^*\colon \mathfrak{X}^* \to \mathfrak{X}^*$, topologically conjugate to the operator F and $B(\mathfrak{X}^*)$ is the space of all conjugate operators on $\mathfrak{X}^*$ (see [18] [Definition VI.2.1]). We introduce the mapping $\psi\colon B(\mathfrak{X}) \to B(\mathfrak{X}^*)$, $\psi(F) = F^*$, $F \in B(\mathfrak{X})$ (note that in [18] no symbol is introduced for such a mapping). In [18] [Lemma VI.2.2], it is proved that the mapping ψ is an isometric isomorphism of the space $B(\mathfrak{X})$ into the space $B(\mathfrak{X}^*)$. According to [18] [Definition II.3.17], the isometric isomorphism ψ is assumed to be a linear mapping.

The mapping ψ, generally speaking, does not map the space $B(\mathfrak{X})$ onto the entire space $B(\mathfrak{X}^*)$, which is implied in the above Lemma VI.2.2 ([18]). Let $\tilde{B}(\mathfrak{X}^*)$ be the subspace of $B(\mathfrak{X}^*)$ of all operators $G\colon \mathfrak{X}^* \to \mathfrak{X}^*$ for which there exists a pre-conjugate operator $F\colon \mathfrak{X} \to \mathfrak{X}$ such that $F^* = G$. Then, obviously, $\psi(B(\mathfrak{X})) = \tilde{B}(\mathfrak{X}^*)$ and on $\tilde{B}(\mathfrak{X}^*)$ there exists an inverse isometric and linear isomorphism $\psi^{-1}\colon \tilde{B}(\mathfrak{X}^*) \to B(\mathfrak{X})$. Moreover, $\psi^{-1}(\tilde{B}(\mathfrak{X}^*)) = B(\mathfrak{X})$. We will also call the set $\tilde{B}(\mathfrak{X}^*)$ the scope of definition of the isomorphism ψ^{-1}. Therefore, we prove the following general statement for arbitrary Markov operators.

Theorem 4. *Let (X,Σ) be an arbitrary measurable space. An arbitrary Markov operator $T\colon B(X,\Sigma) \to B(X,\Sigma)$ is quasi-compact if and only if its conjugate Markov operator $T^*\colon ba(X,\Sigma) \to ba(X,\Sigma)$ is quasi-compact, i.e., the conditions (K_T) and (K_{T^*}) are equivalent: $(K_T) \Leftrightarrow (K_{T^*})$.*

Proof. First, we show that $(K_T) \Rightarrow (K_{T^*})$. Let the operator T be quasi-compact. Then there are $k \in N$ and a compact operator T_1 such that $\|T^k - T_1\| < 1$. Let us rewrite this inequality in the following form: $T^k = T_1 + V$, where $\|V\| < 1$. We pass to conjugate operators and obtain $(T^k)^* = T_1^* + V^*$. By Schauder's Theorem ([18]), the operator T_1^* conjugate to a compact operator T_1 is also compact.

According to [18] (Lemma VI.2.2) and our analysis presented above the mapping ψ that takes the original operators $T\colon Y \to Y$ (here Y is an arbitrary Banach space) to the conjugate operators $T^*\colon Y^* \to Y^*$ is a linear isometric isomorphism, i.e., preserves the norms of the operators. Therefore, in our case, from the inequality $\|V\| < 1$ it follows that $\|V^*\| = \|V\| < 1$.

Next, we will show that the operator $(T^k)^*$ can have its superscripts swapped. To do this, we will use Lemma from [18] [Lemma VI.2.4]. It proves that for two linear operators Q_1 and Q_2 acting on the corresponding Banach spaces, the equality $(Q_1 Q_2)^* = Q_2^* Q_1^*$ holds. In our case, we take the operators $Q_1 = T$ and $Q_2 = T$ and get the equality $(T^2)^* = (T^*)^2$. From here, by induction, we obtain the formula for arbitrary $k \in N$: $(T^k)^* = (T^*)^k$. Let us substitute the right side of this equality into the general formula and obtain: $(T^*)^k = T_1^* + V^*$. Thus, we have obtained that the operator T^* conjugate to T is also quasi-compact. This means that $(K_T) \Rightarrow (K_{T^*})$.

Let us prove the converse statement. Let for some Markov operator T its conjugate operator T^* be quasi-compact. This means that there are $k \in N$ and a compact operator S_1 such that $\|(T^*)^k - S_1\| < 1$. We write this inequality in the following form: $(T^*)^k = S_1 + U$, where $\|U\| < 1$. The pre-conjugate operator to the operator T^* is, by assumption, the operator T, i.e., $\psi(T) = T^*$. Since $(T^k)^* = (T^*)^k$, then the pre-conjugate to the operator $(T^*)^k$ is the operator T^k, i.e., $\psi(T^k) = (T^*)^k$.

Now we use the inverse isometric isomorphism ψ^{-1}. Since the isomorphism ψ^{-1} is a linear mapping and the operator $(T^*)^k = (T^k)^*$ is included in the scope of definition $\tilde{B}(\mathfrak{X}^*)$ of mapping ψ^{-1}, then we can do the following transformations:

$$T^k = \psi^{-1}((T^k)^*) = \psi^{-1}(S_1 + U) = \psi^{-1}(S_1) + \psi^{-1}(U).$$

Thus, the operators S_1 and U also fall into the scope of definition $\tilde{B}(\mathfrak{X}^*)$ of the inverse isomorphism ψ^{-1}. We denote the pre-conjugate operators $T_1 = \psi^{-1}(S_1)$ and $V = \psi^{-1}(U)$, where ${T_1}^* = S_1$ and $V^* = U$.

We have obtained the decomposition $T^k = T_1 + V$.

Since, by assumption, the operator S_1 is compact, then, by the already mentioned Schauder's Theorem ([18]), its pre-conjugate operator $T_1 = \psi^{-1}(S_1)$ is also compact. The mapping ψ takes the original operator T to the conjugate operator T^* and is an isometric isomorphism. Therefore, if $\|U\| < 1$, then for the pre-conjugate operator $V = \psi^{-1}(U)$, $V^* = U$, $\|V\| = \|U\| < 1$ holds. Thus, we have obtained that the operator T pre-conjugate to the quasi-compact operator T^* is also quasi-compact, i.e., $(K_{T^*}) \Rightarrow (K_T)$. This completes the proof. □

Remark 1. *In our work [2], at the beginning of §12, it is noted that the quasi-compactness condition (K_{T^*}) for the operator T^* is equivalent to the quasi-compactness condition (K_T) for the operator T. However, proof of this fact is not given there.*

Remark 2. *Herkenrach ([26] [Lemma 3.2]) showed (under certain conditions) that if some linear operator special type $F\colon Y \to Y$ on a Banach space Y is quasi-compact, then the conjugate to Therefore, the operator $F^*\colon Y^* \to Y^*$ is also quasi-compact. However, the proof of this particular statement is not given in the work [26]. There is no converse statement there, either.*

Remark 3. *Lin [11] considered Markov chains with operators P and P^* defined on the left and right sides of L_1 and L_∞, respectively. In Remark [11] [Section 2], it is argued that if P is quasi-compact, then its conjugate P^* may not be quasi-compact. A corresponding counterexample is given there to support this. In this paper, in Section 6, we study this counterexample in detail. We show that if a given concrete Markov chain is described using the construction of our Markov operators T, A, and T^*, then the Markov operator T^* is quasi-compact. Thus, this example of Lin does not refute our proved Theorem 4 but only confirms it.*

Let us prove one more general statement for any operators, not just Markov ones.

Theorem 5. *Let a linear bounded operator $F\colon Y \to Y$ in a Banach space Y have a non-degenerate invariant Banach subspace $M \subset Y$, $F(M) \subset M$. Then, if the operator F in the space Y is quasi-compact, its restriction $(F)_M$ to the subspace M is also quasi-compact.*

Proof. Let the operator $F\colon Y \to Y$ be quasi-compact. Then, by Definition 3, for some $n \in N$ and a compact operator F_1, $F^n = F_1 + V$ holds, where $\|V\| < 1$. We restrict these operators to the invariant subspace M and obtain $(F^n)_M = (F_1)_M + (V)_M$. From this equality, it follows that for any set $E \subset M$, we have

$$(F^n)_M(E) = (F_1)_M(E) + (V)_M(E).$$

In addition, for any $E \subset M$ the following holds: $(F_1)_M(E) = F_1(E)$. It is obvious that an arbitrary bounded set $E \subset M$ will remain bounded even if it is embedded in Y. A norm in the subspace M is, by general definition, induced from the space Y (i.e., coinciding with the norm from Y on the subspace M). Consequently, the closures of the sets $(F_1)_M(E)$ and $F_1(E)$, taken in their coinciding norms, coincide, i.e.,

$$\overline{(F_1)_M(E)} = \overline{F_1(E)}.$$

Note that, by the assumptions of the theorem, the subspace M is Banach and therefore closed in Y. By construction, the operator F_1 is compact. Therefore, by Definition 2, for a bounded set $E \subset M$, the set $\overline{F_1(E)}$ is compact in the norm of the space Y. This means that the set $\overline{(F_1)_M(E)}$ is compact in the norm of the subspace M induced from Y. Thus, the operator $(F_1)_M$ is compact in the subspace M. Let us now turn to the operator V in the expansion of the operator $F^n = F_1 + V$ and to its restriction $(V)_M$ to the space M. The norms of operators in normed spaces are calculated using the formulas:

$$\|V\| = \sup \|V(x)\|, \text{where } \|x\| \leq 1, x \in Y;$$

$$\|(V)_M\| = \sup \|(V)_M(x)\|, \text{ where } \|x\| \leq 1, x \in M.$$

Since the subspace M is part of the space Y, then $\|(V)_M\| \leq \|V\|$. By construction (and by condition), $\|V\| < 1$. Consequently, $\|(V)_M\| < 1$. Thus, the restriction of the operator $(F)_M$ to the subspace M is quasi-compact. The proof is complete. □

Corollary 1. *Consider a general Markov chain on an arbitrary measurable space (X, Σ). If the Markov operator $T^* : ba(X, \Sigma) \to ba(X, \Sigma)$, conjugate to the operator $T : B(X) \to B(X)$ is quasi-compact, then the Markov operator $A : ca(X, \Sigma) \to ca(X, \Sigma)$ (here $A = (T^*)_{ca}$) is also quasi-compact.*

Now we write Doeblin condition (D) of ergodicity of the MC (see, for example, [27]):

$$(D)\begin{cases} \text{there is a bounded measure } \varphi \in ca(X,\Sigma),\ \varphi \geq 0, \varepsilon > 0 \text{ and } k \in N, k \geq 1, \\ \text{such that from } \varphi(E) \leq \varepsilon, E \in \Sigma, \text{ should } p^k(x,E) \leq 1 - \varepsilon \text{ for all } x \in X. \end{cases}.$$

Remark 4. *The superscript k in p^k denotes the order of the integral convolution (iteration) of the transition function, not its degree.*

Theorem 6. *Let (X, Σ) be an arbitrary measurable space on which a general Markov chain is defined. Its Markov operator $T^* \colon ba(X, \Sigma) \to ba(X, \Sigma)$ is quasi-compact, if and only if the Doeblin condition (D) is satisfied, i.e., the conditions (K_{T^*}) and (D) are equivalent: $(K_{T^*}) \Leftrightarrow (D)$.*

Proof. Neveu in the book [14] [V.3.2] showed a scheme of proof that the condition K_T of quasi-compactness of the operator $T \colon B(X, \Sigma) \to B(X, \Sigma)$ is equivalent to the Doeblin condition (D). In ([26] [Theorem 2.1]), Herkenrath gives his full proof (with reference to Yosida and Kakutani [5] and Neveu [14]) that the Doeblin condition (D) is equivalent to the quasi-compactness condition (K_T) for the operator $T \colon B(X, \Sigma) \to B(X, \Sigma)$, i.e., $(D) \Leftrightarrow (K_T)$. In our Theorem 4 above, it is proved that the quasi-compactness condition (K_T) is equivalent to the quasi-compactness condition (K_{T^*}). Therefore, $(K_{T^*}) \Leftrightarrow (D)$. The theorem is proved. □

The following statement is obtained from the Herkenrath Theorem 2.2 [26] and Theorem 2.1 [26].

Theorem 7 (**[26]**)**.** *The operator $A \colon ca(X, \Sigma) \to ca(X, \Sigma)$ is quasi-compact if and only if the operator $T \colon B(X, \Sigma) \to B(X, \Sigma)$ is quasi-compact, i.e., $(K_A) \Leftrightarrow (K_T)$.*

Now we can formulate a summary statement obtained through the efforts of many authors (including ours).

Proposition 1. *For a general Markov chain on an arbitrary measurable space (X, Σ), the conditions for the quasi-compactness of its three operators are equivalent to the Doeblin condition (D):*

$$(K_T) \Leftrightarrow (K_A) \Leftrightarrow (K_{T^*}) \Leftrightarrow (D).$$

We emphasize that there are no restrictions or additional conditions for Markov chains for the resulting diagram. If at least one of the three Markov operators T, A, or T^* of MC is quasi-compact, then we will call the Markov chain itself quasi-compact. We will now use some of the results obtained in our paper [1].

In our articles [1,2] the following condition $(*)$ was introduced:

$$(*) \qquad \Delta_{ba} \subset ca(X, \Sigma),$$

it means that all invariant finitely additive measures of the operator T^* (and such measures exist for any Markov chain) are countably additive.

Theorem 8. *For an arbitrary MC, its Markov operator T^* is quasi-compact if and only if the condition $(*)$ is satisfied.*

Proof. According to [1] [Theorem 7] the condition $(*)$ is equivalent to the Doeblin condition (D). In Theorem 6 of this paper, it is proved that the Doeblin condition (D) is equivalent to the quasi-compactness condition of the operator T^*. Therefore, $(K_{T^*}) \Leftrightarrow (*)$. The theorem is proved. □

We introduce one more condition:

$$(\tilde{*}) \quad \Delta_{pfa} = \emptyset.$$

The fulfillment of the condition $(\tilde{*})$ means that the Markov chain does not have invariant purely finitely additive measures. Obviously, these conditions $(*)$ and $(\tilde{*})$ coincide.

Let us repeat the previous Theorem 8 in a new formulation.

Theorem 9. *For an arbitrary MC, its Markov operator T^* is quasi-compact if and only if it does not have invariant purely finitely additive measures.*

Corollary 2. *Conditions $(*)$ and $(\tilde{*})$ are equivalent to the condition $\Delta_{ba} = \Delta_{ca}$ and the condition $M_{ba} = M_{ca}$.*

Now, we can present a new version of theorem [1] [Theorem 9]—Theorem 10.

Theorem 10. *Let a general MC be given on an arbitrary (X, Σ). Then, if the Markov operator T^* is quasi-compact, then the set of its finitely additive invariant measures Δ_{ba} is finite-dimensional, i.e., $dim\Delta_{ba} = n$, where $1 \leq n < \infty$.*

Proof. In our article [3] in Theorems 8.1 and 8.2, it is proved that if the condition $(*)$ is satisfied, then $dim\Delta_{ba} = n$, where $1 \leq n < \infty$. According to the conditions of the theorem, the operator T^* is quasi-compact. Therefore, according to Theorem 8, the condition $(*)$ is satisfied. The theorem is proved. □

Now we will prove the reversal of Theorem 10 in the one-dimensional case for $n = 1$ (see, also, [1] [Theorem 10]).

Theorem 11. *Let on an arbitrary (X, Σ) a MC be given for which the dimension $dim\Delta_{ba} = 1$, i.e., its Markov operator T^* has in S_{ba} a unique invariant finitely additive measure $\mu = T^*\mu \in S_{ba}$, $\Delta_{ba} = \{\mu\}$. Then this invariant measure μ is countably additive, the Markov operator T^* is quasi-compact and the conditions (K_{T^*}), (K_T) and (K_A), as well as (D), $(*)$ and $(\tilde{*})$ are satisfied for the considered MC.*

Proof. In our theorem [3] [Theorem 8.3] we proved that if for some general MC, we have $dim\Delta_{ba} = 1$ and $\Delta_{ba} = \{\mu\}$, $\mu \in S_{ba}$, then this invariant measure μ is countably additive, i.e., $\mu \in S_{ca}$. This means that for this MC the condition $(*) : \Delta_{ba} \subset ca(X, \Sigma)$ is satisfied.

Then, by Theorem 8, the Markov operator $(T*)$ of the considered MC is quasi-compact, i.e., the condition (K_{T^*}) is satisfied for it. From our Theorems 4 and 6–9 (see above), it follows that the remaining conditions listed in this theorem are also satisfied for this MC. The theorem is proved. □

Please note that the proof of the theorem used above [3] [Theorem 8.3] is not simple; it is carried out using the technique of weak topologies in the space $ba(X, \Sigma)$.

Lemma 1. *Let an arbitrary Markov chain be given on a measurable space (X, Σ). If for any finitely additive measure $\mu \in S_{ba}$ that is invariant for a Markov operator T^* there exists a point $x_\mu \in X$ such that $\mu(\{x_\mu\}) > 0$, then the operator T^* is quasi-compact and the condition (K_{T^*}) is satisfied.*

Proof. Let the conditions of the lemma be satisfied. Suppose that there exists a purely finitely additive measure $\mu \in \Delta_{pfa}$ that is invariant for the Markov operator T^*. Then by Lemma 1 from [20] on any one-point set $\mu(\{x\}) = 0, x \in X$. This contradicts the condition of the Lemma about the existence of a point $x_\mu \in X$ such that $\mu(\{x_\mu\}) > 0$. Therefore, this Markov chain does not have invariant purely finitely additive measures, i.e., $\Delta_{pfa} = \emptyset$. It follows from our Theorems 8 and 9 that such a Markov chain satisfies the condition (K_{T^*}) and it is quasi-compact. The Lemma is proven. □

Corollary 3. *If the condition of the Lemma 1 is satisfied, then all invariant finitely additive measures of the operator T^* are countably additive, i.e., $\Delta_{ba} = \Delta_{ca}$ and $dim\Delta_{ba} = n < \infty$.*

Remark 5. *Lemma 1, generally speaking, cannot be reversed. If the operator T^* is quasi-compact and if $\mu = T^*\mu, \mu \in S_{ca}$ is its invariant countably additive measure, then it is possible (for an infinite X) that $\mu(\{x\}) = 0$ for all $x \in X$.*

5. Ergodic Theorem

In the book by Dunford and Schwartz [18], in Chapter VIII, paragraphs 4–6 and 8, the history of ergodic theory and ergodic theorems is described in detail. Specific problems in mechanics, physics and chemistry are given that can be solved using ergodic theorems of various types. There, in Chapter VIII, Markov processes and Markov chains are singled out in several places, and their classical ergodic properties and theorems are briefly analyzed. We add that today, ergodic theorems for Markov chains are widely used in economics, statistics and in the planning of experiments. In [18] and in many other sources, the following non-rigorous interpretation of the ergodicity of a particular system is given: the average over space is equal to the average over time. Please note that from a mathematical point of view, ergodic theorems are more convenient (and useful) to formulate and prove in the form of limit theorems, using current averages (Cesaro averages) over time. This is what we do below in our Theorem 12.

The literature that studies various aspects of ergodic theory and its applications is very extensive. However, to solve the problems in this section, we only need to use the monograph [18] and several articles cited below.

Let us consider some ergodic properties of Markov operators following from the statements obtained above for T, A, and T^* (see, also, [17]). Lin [13] [Theorem 1] proved a general strong uniform ergodic theorem for arbitrary linear positive quasi-compact operators on a Banach lattice. Theorem [13] [Theorem 1] can also be applied to Markov operators with different domains of their definition, which is noted by Lin [13]. An important feature of this theorem is that it also contains a conversion (under certain conditions) of classical uniform ergodic theorems: from the ergodic asymptotics of the operator under consideration, it follows that it is quasi-compact.

In 1937, for the Markov operator $T: B(X, \Sigma) \to B(X, \Sigma)$, the quasi-compactness conditions were introduced in the first paper on this topic by Kryloff and Bogoliouboff [4]. The first uniform ergodic theorem was also proved for this operator there. For this reason, further study of quasi-compact Markov operators was most often carried out in function spaces. After the paper by Yosida and Kakutani ([5], 1941), quite a lot of papers appeared in which quasi-compactness conditions were considered, and the corresponding ergodic theorems were proved for Markov operators $A\colon ca(X,\Sigma) \to ca(X,\Sigma)$ defined on spaces of countably additive measures (see, for example, [18] [VIII.8]). The Markov operators $T^*\colon ba(X,\Sigma) \to ba(X,\Sigma)$, defined on the space of finitely additive measures, are still mentioned only in a few works. Taking this into account, in the next theorem, we will consider the results of applying the theorem Lin [13] [Theorem 1] only for the Markov operators T^* and A and compare their ergodic properties.

Theorem 12. *Let the general MC be given on an arbitrary measurable space (X,Σ). Then the following three statements are true:*

1. *The Markov operator $T^*\colon ba(X,\Sigma) \to ba(X,\Sigma)$ is quasi-compact (i.e., the condition (K_{T^*}) is satisfied) if and only if the following ergodic condition is satisfied:*
$$(Erg, T^*)\begin{cases} \text{there exists a finite-dimensional projection } G_{T^*} \\ \text{onto the space of all invariant measures of the operator } T^* \\ \text{such that for } n \to \infty \text{ the following holds:} \\ |||\frac{1}{n}\sum\limits_{i=1}^{n}(T^*)^i - G_{T^*}||| = \sup\limits_{\mu \in S_{ba}} \|\frac{1}{n}\sum\limits_{i=1}^{n}(T^*)^i(\mu) - G_{T^*}(\mu)\| \to 0. \end{cases}$$
2. *The Markov operator $A\colon ca(X,\Sigma) \to ca(X,\Sigma)$ is quasi-compact (i.e., the condition (K_A) is satisfied) if and only if the following ergodic condition is satisfied:*
$$(Erg, A)\begin{cases} \text{there exists a finite-dimensional projection } G_A \\ \text{onto the space of all invariant measures of the operator } A \\ \text{such that for } n \to \infty \text{ the following holds:} \\ |||\frac{1}{n}\sum\limits_{i=1}^{n}A^i - G_A||| = \sup\limits_{\mu \in S_{ca}} \|\frac{1}{n}\sum\limits_{i=1}^{n}A^i(\mu) - G_A(\mu)\| \to 0. \end{cases}$$
3. *Ergodicity conditions (Erg, T^*) and (Erg, A) are equivalent, i.e.,*
$$(Erg, T^*) \Leftrightarrow (Erg, A).$$

Proof. Theorem [13] [Theorem 1] requires that the linear operator under consideration be positive on some Banach lattice. Our measure spaces $ba(X,\Sigma)$ and $ca(X,\Sigma)$ are linear and are vector Banach lattices (see, for example, [19]). The Markov operators T^* and A are positive by construction. These conditions are satisfied for T^* and A. In addition, the conditions of [13] [Theorem 1] require that the operators T^* and A satisfy the following properties:
$$\frac{\|(T^*)^n\|}{n} \to 0 \text{ and } \frac{\|A^n\|}{n} \to 0 \text{ for } n \to \infty.$$
Since for Markov operators it is always true that $\|T^*\| = 1, \|(T^*)^n\| = 1$ and $\|A\| = 1$, $\|A^n\| = 1$, for all $n \in N$, then the above conditions are also satisfied for them. Therefore, the asymptotic formulas and statements [13] [Theorem 1] is also true for both operators T^* and A (in our notation). According to our Theorem 4 the conditions (K_T) and (K_{T^*}) are equivalent, i.e., $(K_T) \Leftrightarrow (K_{T^*})$. By Theorem 7 the equivalence of $(K_T) \Leftrightarrow (K_A)$ also holds. Therefore, $(K_{T^*}) \Leftrightarrow (K_A)$. It follows that the conditions (Erg, T^*) and (Erg, A) are also equivalent, and statement 3 of this theorem is true. The theorem is proved. □

Remark 6. *Theorem 12, the operator $G_{T^*}\colon ba(X,\Sigma) \to ba(X,\Sigma)$ projects the space $ba(X,\Sigma)$ onto the subspace of all its invariant probability finitely additive measures M_{ba}, i.e., $G_{T^*}[ba(X,\Sigma)] = M_{ba}$ and $G^2_{T^*} = G_{T^*}$. According to Theorem 8 and Corollary 2 we have $M_{ba} = M_{ca}$. Therefore, the restriction G_{ca} of the projector G_{T^*} to the space $ca(X,\Sigma)$ is also a projector of the space $ca(X,\Sigma)$ to the subspace M_{ca}, i.e., $G_{ca}[ca(X,\Sigma)] = M_{ca}$ and $G^2_{ca} = G_{ca}$.*

Remark 7. *The substantive difference between conditions (Erg, T^*) and (Erg, A) is that in the first condition, the supremum is taken over all initial finitely additive probability measures $\mu \in S_{ba}$, including purely finitely additive probability measures $\mu \in S_{pfa}$. And in the second condition (Erg, A), the supremum is taken only over initial countably additive probability measures $\mu \in S_{ca}$. However, the limit projections G_{T^*} and G_A coincide in this case.*

Below, in Section 6, when analyzing Lin's Example, we will show how this looks in a simple case.

6. Example Michael Lin

6.1. Description of Special Markov Chains

The work by Michael Lin [11] considers the linear positive contraction operator P in the space $L_1(X, \Sigma, \mu)$ and its adjoint operator P^* in the space $L_\infty(X, \Sigma, \mu)$, where $L_\infty = L_1{}^*$. The operator P is called Markovian. A general classical definition of a quasi-compact operator is given. The equivalent Doeblin condition is also mentioned there with references to Neveu [14] and to Doob [27]. The said work contains a Remark at the end of paragraph 2 ([11], p. 467), which states that "If P is quasi-compact, then P^* may fail to be quasi-compact". A corresponding counterexample is also given there. Theorem 4, which we proved above, states that if the Markov operator T defined on the space $B(X, \Sigma)$ is quasi-compact, then so is the operator T^* conjugate to it on the space $ba(X, \Sigma)$, where $ba(X, \Sigma) = B^*(X, \Sigma)$, is also quasi-compact. The opposite is also true. Now, we will show, using the same example, that there is no contradiction here. Studying this example will also allow us to illustrate the possible applications of our theorems and methods. Let us give this example from Lin [11]. Given a Markov chain on a countable measurable space $X = \{0, 1, 2, \ldots\} = \{0\} \cup N$ (in fact, the example considers $\Sigma = 2^X$) with an explicit the form of its transition function:

$$P(i, \{0\}) = \frac{1}{2}, \text{ for all } i \in X, \text{ including } P(0, \{0\}) = \frac{1}{2}, \tag{1}$$

$$P(i, \{i+1\}) = \frac{1}{2}, \text{ for all } i \in X, \text{ including } P(0, \{1\}) = \frac{1}{2}. \tag{2}$$

The transition function of this Markov chain can be represented in other forms. For any $i \in X$ and $E \subset X$, we write:

$$P(i, E) = \frac{1}{2}\delta_{i+1}(E) + \frac{1}{2}\delta_0(E), \tag{3}$$

$$P(i, E) = \frac{1}{2}\chi_E(i+1) + \frac{1}{2}\chi_E(0), \tag{4}$$

where δ_i is the Dirac measure at point i, and χ_E is the characteristic function of the set E. This definition of the transition function makes it possible to use our Markov operators T, A, and T^* on the spaces $B(X, \Sigma)$, $ca(X, \Sigma)$ and $ba(X, \Sigma)$, respectively. As we have already noted, in the work of Lin [11], other spaces L_1 and L_∞ are used, and, accordingly, other Markov operators P and P^*. Next, we will consider step by step all three Markov operators of this Markov chain T, A, and T^*, show their quasi-compactness, and reveal some of their other interesting properties.

6.2. Operator T

Recall that the operator $T\colon B(X, \Sigma) \to B(X, \Sigma)$ in the general case is given by the following integral formula:

$$g(x) = Tf(x) = \int_X f(y) P(x, dy), \text{ where } f, g \in B(X, \Sigma), x \in X. \tag{5}$$

Let us substitute the formula (3) into equality (5), and after simple transformations, we obtain for our MC:

$$g(i) = Tf(i) = \frac{1}{2}f(i+1) + \frac{1}{2}f(0), i \in X. \tag{6}$$

It is known that the unit function $h(x) \equiv 1$ is invariant for any MC and for the corresponding operator $Th(x) = h(x), \forall x \in X$ (this can be verified by simple substitution). Invariant functions of the operator, T, are also called harmonic. Therefore, $\Delta_B \neq \oslash$. We use (6) to find all invariant functions of the operator T and solve the equation $f(i) = Tf(i)$, $i \in X$. Simple transformations give us the answer: $f(i) = f(0)$, for all $i \in X$. Let us rewrite this equality in the equivalent form:

$$f(i) = f(0)h(i), f \in S_B, i \in X. \tag{7}$$

So, we have obtained that all invariant functions of a given operator T differ from the function h only by a numerical coefficient. Consequently, the operator T has only a one-dimensional space M_h of all its invariant functions

$$M_h = \{\alpha \cdot h \colon \alpha \in R^1\}, \dim M_h = 1. \tag{8}$$

Conclusion 1. We can consider function h to be the unique invariant function of a given MC up to a numerical coefficient α. It is also obvious that this operator T does not have cycles. Let $f_0 \in S_B$ and $f_n = Tf_{n-1} = T^n f_0, n \in N$. Using Equality (6), we obtain for all $i \in X$:

$$f_1(i) = Tf_0(i) = \frac{1}{2} f_0(i+1) + \frac{1}{2} f_0(0) \cdot h(i), \tag{9}$$

$$f_2(i) = Tf_1(i) = \frac{1}{2^2} f_0(i+2) + \frac{3}{2^2} f_0(0) \cdot h(i), \tag{10}$$

Let us introduce the projector $Q_T \colon B(X, \Sigma) \to B(X, \Sigma)$ onto the one-dimensional space M_h of invariant functions of operator T.

According to [18] [VI.9.18], there are infinitely many such projectors. For all such operators the equality $Q_T f = f(0) \cdot h$ holds, where $f \in S_B$ and $Q_T^2 = Q_T$, i.e., $Q_T(f(0) \cdot h) = f(0) \cdot h$. Every finite-dimensional projector is a compact operator ([18] [VI.9.20]).

Now we make the following estimates for all $i \in X$, $f_0 \in S_B$:

$$\begin{aligned} &|(T^2 f_0)(i) - (Q_T f_0)(i)| = |f_2(i) - f_0(0) \cdot h(i)| \\ &= |\frac{1}{2^2} f_0(i+2) - \frac{2^2-1}{2^2} f_0(0) \cdot h(i) - f_0(0) \cdot h(i)| \\ &\leq \frac{1}{2^2} |f_0(i+2)| + \frac{1}{2^2} |f_0(0) \cdot h(i)| \\ &\leq \frac{1}{2^2} \|f_0\| + \frac{1}{2^2} \|f_0\| \|h\| \leq \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}. \end{aligned} \tag{11}$$

The right side of this inequality does not depend on the initial function f_0 and the current ordinate i. Hence, in the operator norm we obtain

$$|||T^2 - Q_T||| = \sup_{f_0 \in S_b} \sup_{i \in X} |(T^2 f_0)(i) - (Q_T f_0)(i)| \leq \frac{1}{2}. \tag{12}$$

Conclusion 2. The Markov operator T for a given MC is quasi-compact.

6.3. Operator A

In [11], an invariant countably additive measure for this Markov chain is given. In [11], it is not specified which measure is invariant. However, it is easy to check that for our operator A it is invariant. However, within the framework of our approach, we need all invariant (probabilistic) finitely additive measures $\mu \in S_{ba}$, including purely finitely additive measures, if they exist. To do this, we need to solve the equation $T^*\mu = \mu$ in the class of finitely additive measures $\mu \in S_{ba}$. Let us write this equation in more detail:

$$\mu(E) = T^*\mu(E) = \int_X P(i, E) d\mu(i), \forall E \in \Sigma.$$

In this notation, we cannot go from the integral to an infinite sum (to a series) since the measure μ can be purely finitely additive. We take a concrete set $E = \{0\} \in \Sigma$. Then, from the general equation, we obtain:

$$\mu(\{0\}) = \int_X P(i, \{0\}) d\mu(i) = \int_X \frac{1}{2} \cdot d\mu(i) = \frac{1}{2} \cdot \mu(X) = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Suppose that the invariant measure $\mu \in S_{ba}$ is purely finitely additive. Then, on the one-point set, we have $\mu(\{0\}) = 0$. However, above we obtained $\mu(\{0\}) = \frac{1}{2}$. From this contradiction, we obtained a conclusion.

Conclusion 3. This MC does not have invariant purely finitely additive measures and all its invariant measures are countably additive. Now, we need to make sure that the MC has a unique invariant countably additive measure. We will do this by constructing it directly. Given the above Conclusion 3, we can construct this measure in the class S_{ca} using the operator A instead of the operator T^*. We continue solving the equation:

$$\mu(E) = A\mu(E) = \int_X P(i, E) d\mu(i), \forall E \in \Sigma.$$

For $E = \{0\}$ we have already received $\mu(\{0\}) = \dfrac{1}{2} = \dfrac{1}{2^1}$.

Now we will find $\mu(\{1\})$:

$$\mu(\{1\}) = A\mu(\{1\}) = \int_X P(i, \{1\}) d\mu(i) = \int_{\{0\}} + \int_N$$

$$= P(0, \{1\})\mu(\{0\}) + \int_N P(i, \{1\}) d\mu(\{i\}) = \frac{1}{2} \cdot \mu(\{0\}) + 0 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}.$$

For an arbitrary $n \in N$, by induction, we obtain

$$\mu(\{n\}) = \frac{1}{2} \cdot \mu(\{n-1\}) = \frac{1}{2^{n+1}}.$$

We need to make sure that the sum of these quantities is equal to one. Since the invariant measures μ are countably additive, then

$$\mu(X) = \mu(\cup_{i=0}^{\infty}\{i\}) = \sum_{i=0}^{\infty} \mu(\{i\}) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = 1.$$

Conclusion 4. From the construction it follows that the found invariant measure μ for the operators A and T^* is unique, i.e., $\Delta_{ba} = \Delta_{ca} = \{\mu\}$.

Lin [11] also gives the constructed invariant measure for a given Markov chain (in our case, for operators A and T^*). We repeat this measure here to indicate the operators to which it applies and to show its uniqueness in the class of finitely additive measures and in the class of countably additive measures, which is what we have done. Now, we prove the quasi-compactness of the operator A by direct methods without resorting to our theorems and corresponding theorems from other sources. Above, we have already found out that the operator A has a unique invariant countably additive measure $\mu^* = A\mu^*$. This measure is the only invariant measure for the operator $T^* \colon ba(X, \Sigma) \to ba(X, \Sigma)$ in the class of finitely additive measures.

We introduce the following notation: $M_m = \{0, 1, 2, \ldots, m-1\}$, $N_m = \{m, m+1, m+2, \ldots\}$.

Obviously, $M_m \cup N_m = X = \{0, 1, 2, \ldots\}$, $X \setminus M_m = N_m$, $X \setminus N_m = M_m$ and $M_1 = \{0\}$, $N_1 = N$. Let an arbitrary initial countably additive measure $\mu_0 \in S_{ca}$ be given. It generates a Markov sequence of measures $\mu_n = A\mu_{n-1} = A^n\mu_0, \mu_n \in S_{ca}, n \in N$. We will only need the first two measures μ_1 and μ_2 from this sequence. Let us study these measures. Let $\mu_0 \in S_{ca}$, and $\mu_1 = A\mu_0$. Then for any $E \subset X$ the following formula holds true:

$$\mu_1(E) = A\mu_0(E) = \int_X p(i, E)d\mu_0(i).$$

We consider step by step the values of $\mu_1(E)$ on some of the sets $E \subset X$ we need.

(1). Let $E = \{0\}$. Then

$$\mu_1(\{0\}) = \int_X p(i, \{0\})d\mu_0(i) = \int_X \frac{1}{2}d\mu_0(i) = \frac{1}{2}\mu_0(X) = \frac{1}{2^1}, \text{ i.e., } \mu_1(\{0\}) = \frac{1}{2}.$$

(2). Let $E = \{1\}$. Then

$$\mu_1(\{1\}) = \int_X p(i, \{1\})d\mu_0(i) = \int_{\{0\}} + \int_N = p(0, \{1\}) \cdot \mu_0(\{0\}) + \int_N 0 \cdot d\mu_0(i)$$

$$= \frac{1}{2}\mu_0(\{0\}) + 0, \text{ i.e., } \mu_1(\{1\}) = \frac{1}{2}\mu_0(\{0\}).$$

(3). Let $E = \{2\}$. Then $\mu_1(\{2\}) = \frac{1}{2}\mu_0(\{1\})$.

(4). Let $E = \{m\}, m \in N$. Then, by induction, $\mu_1(\{m\}) = \frac{1}{2}\mu_0(\{m-1\})$.

(5). $\mu_1(N) = \frac{1}{2} \cdot \mu_0(X) = \frac{1}{2}$.

Remark 8. *For any initial measure $\mu_0 \in S_{ca}$ we have $\mu_0(X) = 1$. Consequently, the equality $\mu_1(\{0\}) = \frac{1}{2}$ holds for any initial measure, i.e., they do not depend on μ_0. This is also true for the equality $\mu_1(N) = \frac{1}{2}$.*

Now we will consider the second iteration in the Markov sequence of measures $\mu_2 = A\mu_1 = A^2\mu_0$, where $\mu_0 \in S_{ca}$ and

$$\mu_2(E) = A\mu_1(E) = \int_X p(i, E)d\mu_1(i), \forall E \subset X.$$

Let us take some concrete sets $E \in \Sigma$:

(1). Let $E = \{0\}$. Then

$$\mu_2(\{0\}) = \int_X p(i, \{0\})d\mu_1(i) = \int_X \frac{1}{2}d\mu_1(i) = \frac{1}{2}\mu_1(X) = \frac{1}{2} \cdot 1, \text{ i.e., } \mu_2(\{0\}) = \frac{1}{2}\mu_1(X) = \frac{1}{2}.$$

(2). Let $E = \{1\}$. Then

$$\mu_2(\{1\}) = \int_X p(i, \{1\})d\mu_1(i) = \int_{\{1\}} + \int_N = p(0, \{1\}) \cdot \mu_1(\{0\}) + \int_N 1 \cdot d\mu_1(i)$$

$$= \frac{1}{2}\mu_1(\{0\}) = \frac{1}{2} \cdot \frac{1}{2} + 0 = \frac{1}{2^2}, \text{ i.e., } \mu_2(\{1\}) = \frac{1}{2^2} = \frac{1}{4}.$$

(3). Furthermore, by induction, for $m \geq 2$ we obtain: $\mu_2(\{m\}) = \frac{1}{2^2}\mu_0(\{m-2\})$.

(4). $\mu_2(N) = \frac{1}{2}\mu_0(X) = \frac{1}{2}$.

(5). $\mu_2(M_2) = \frac{1}{2^1} + \frac{1}{2^2} = \frac{3}{2^2}$ and $\mu_2(N_2) = \frac{1}{2^2}$.

Remark 9. *Here the values of the measures $\mu_2(\{0\}) = \frac{1}{2^1}$ and $\mu_2(\{1\}) = \frac{1}{2^2}$ are also not dependent on the initial measure $\mu_0 \in S_{ca}$. Similarly, the values of $\mu_2(M_2) = \frac{3}{2^2}$ and $\mu_2(N_2) = \frac{1}{2^2}$ do not depend on μ_0.*

Please note that $\mu_1(\{0\}) = \mu^*(\{0\}) = \frac{1}{2^1}$, $\mu_2(\{0\}) = \mu^*(\{0\}) = \frac{1}{2^1}$, and $\mu_2(\{1\}) = \mu^*(\{1\}) = \frac{1}{2^2}$, where μ^* is the invariant measure of the operator A. This process of substitution of values $\mu_n(m)$ by values $\mu^*(m)$ (at $m \le n$) continues to occur at $n \to \infty$. However, now it is enough for us to track such substitutions only for $n = 1$ and $n = 2$, which we have already done. Thus, $|\mu_2(\{0\}) - \mu^*(\{0\})| = 0$ and $|\mu_2(\{1\}) - \mu^*(\{1\})| = 0$, i.e., $|\mu_2(E) - \mu^*(E)| = 0$ when $E = M_2$. We estimate the norms of measures:

$$\|\mu_2 - \mu^*\| = \sup_{E\subset X} |\mu_2(E) - \mu^*(E)| = \sup_{E\subset N_2} |\mu_2(E) - \mu^*(E)|$$

$$\le \sup_{E\subset N_2} [|\mu_2(E)| + |\mu^*(E)|] \le \sup_{E\subset N_2} |\mu_2(E)| + \sup_{E\subset N_2} |\mu^*(E)|$$

$$= \mu_2(N_2) + \mu^*(N_2) = \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}.$$

From the previous remarks, it follows that

$$\|\mu_2 - \mu^*\| = \sup_{\mu_0\in S_{ca}} \|\mu_2 - \mu^*\| \le \frac{1}{2},$$

i.e.,

$$\|A^2\mu_0 - \mu^*\| = \sup_{\mu_0\in S_{ca}} \|A^2\mu_0 - \mu^*\| \le \frac{1}{2}.$$

Let us introduce the projector $Q_A\colon ca(X,\Sigma) \to ca(X,\Sigma)$ onto the one-dimensional space $L = \{\alpha \cdot \mu^*, \alpha \in R^1\}$ of invariant measures for operator A.

According to [18] [VI.9.18], there are infinitely many such projectors. For all such operators the equality $Q_A\mu = \mu^*$ holds, where $\mu \in S_{ca}$ and $Q_A^2 = Q_A$. Every finite-dimensional projector is a compact operator ([18] [VI.9.20]).

Now, from the obtained inequalities, it follows that in the operator norm, we obtained

$$|||A^2 - Q_A||| \le \frac{1}{2}.$$

Conclusion 5. The quasi-compactness condition from Definition 3 is satisfied. This means that the Markov operator $A\colon ca(X,\Sigma) \to ca(X,\Sigma)$ of a given Markov chain is quasi-compact.

We proved this without using our theorems from this article. For the operator A, all classical ergodic theorems are true.

6.4. Purely Finitely Additive Initial Measure

In the present subsection, we investigate the behavior of the Markov sequence of measures $\mu_n = T^*\mu_{n-1}, n \in N$ for the case when the initial measure μ_0 is purely finitely additive, i.e., $\mu_0 \in S_{pfa}$. First, we note that such a measure μ_0, like any other purely finitely additive measure $\mu \in S_{pfa}$ on a given measurable space (X,Σ), has the following properties:

$$\mu_0(\{m\}) = 0, \quad \mu_0(M_m) = 0, \mu_0(N_m) = 1$$

for any $m \in X$, $\mu_0(X) = \mu_0(N) = 1$ and $\mu_0(E) = 0$, for any finite set $E \subset X$. Let us take the measure $\mu_1 = T^*\mu_0$ and find its value $\mu_1(E)$ for some important sets $E \subset X$. For an arbitrary set $E \subset X$ the following holds:

$$\mu_1(E) = T^*\mu_0(E) = \int_X P(i,E)d\mu_0(i) = \int_{\{0\}} + \int_N$$
$$= P(0,E)\mu_0(\{0\}) + \int_N = 0 + \int_N P(i,E)d\mu_0(i).$$

Let $E = \{0\}$. Then

$$\mu_1(\{0\}) = \int_X P(i, \{0\}) d\mu_0(i) = \int_X \frac{1}{2} d\mu_0(i) = \frac{1}{2}\mu_0(X) = \frac{1}{2} = \frac{1}{2^1}.$$

Let $E = \{1\}$. Then

$$\mu_1(\{1\}) = \int_X P(x, \{2\})\mu_0(dx) = P(0, \{1\})\mu_0(\{0\}) = 0.$$

Let $E = \{m\}, m \in N$. Then

$$\mu_1(\{m\}) = \int_X P(i, \{m\}) d\mu_0(i) = P(m-1, \{m\})\mu_0(\{m-1\}) = 0.$$

Let $E = N$. Then

$$\mu_1(N) = \int_X P(i, N) d\mu_0(i) = \frac{1}{2} \cdot \mu_0(N) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}.$$

Let us now consider the second iteration of the Markov sequence of measures $\mu_2 = T^*\mu_1 = (T^*)^2\mu_0$, where $\mu_0 \in S_{pfa}$. We find the value of the measure μ_2 for some sets $E \subset X$ using the same scheme as for μ_1 (omitting simple transformations):

$$\mu_2(\{0\}) = \frac{1}{2^1}, \quad \mu_2(\{1\}) = \frac{1}{2^2} \quad \text{and} \quad \mu_2(\{m\}) = 0, \text{ for all } m \geq 2.$$

Let $E = N_m$, where $m \geq 2$. Then $\mu_2(N_m) = \dfrac{1}{2^2}$, for all $m \geq 2$. Now, let us consider the general case of the n-th iteration of a Markov sequence of measures $\{\mu_n\}$ for $n \geq 3$ and $\mu_0 \in S_{pfa}$: $\mu_n = T^*\mu_{n-1} = (T^*)^n\mu_0$. From the formulas for μ_1 and μ_2 we obtain by induction the following values for $\mu_n(E)$:

$$\mu_n(\{0\}) = \frac{1}{2^1}, \quad \mu_n(\{1\}) = \frac{1}{2^2}, \quad \cdots, \quad \mu_n(\{n-1\}) = \frac{1}{2^n},$$

$$\mu_n(\{m\}) = 0 \text{ for all } m \geq n.$$

Since the measures $\{\mu_n\}$ are finitely additive, then

$$\mu_n(M_n) = \sum_{k=0}^{k=n-1} \mu_n(\{k\}) = \sum_{k=1}^{k=n} \frac{1}{2^k} = 1 - \frac{1}{2^{n+1}}.$$

From this, we obtain that $\mu_n(N_n) = \frac{1}{2^{n+1}}$. Obviously, if $m \geq n$, then $\mu_n(M_m) = \mu_n(M_n)$ and $\mu_n(N_m) = \mu_n(N_n)$. We decompose finitely additive measures $\mu_n(E)$ into countably additive components $(\mu_n)_{ca}$ and purely finitely additive components $(\mu_n)_{pfa}$, for which $\mu_n = (\mu_n)_{ca} + (\mu_n)_{pfa}$. In this case, $\mu_0 = (\mu_0)_{pfa}$ and $(\mu_0)_{ca} = 0$, by condition. From the values of the measures $\mu_n(E)$ obtained above on some sets $E \subset X$, it is easy to see that

$$(\mu_1)_{ca} = \frac{1}{2}\delta_0 \quad \text{and} \quad (\mu_1)_{pfa} = \frac{1}{2}\eta_1,$$

where δ_0 is a countably additive Dirac measure at the point 0, and η_1 is some purely finitely additive measure (generated by the measure μ_0), $\eta_1 \in S_{pfa}$. Next,

$$(\mu_2)_{ca} = \frac{1}{2}\delta_0 + \frac{1}{2^2}\delta_1 \quad \text{and} \quad (\mu_2)_{pfa} = \frac{1}{2^2}\eta_2,$$

where η_2 is also some purely finitely additive measure, $\eta_2 \in S_{pfa}$. Then,

$$(\mu_n)_{ca} = \frac{1}{2}\delta_0 + \frac{1}{2^2}\delta_1 + \cdots + \frac{1}{2^n}\delta_{n-1} \text{ and } (\mu_n)_{pfa} = \frac{1}{2^n}\eta_n,$$

where $\eta_n \in S_{pfa}$. Let us find the norms of the resulting components of the measures $\mu_n(E)$. Since all the measures used in these components are positive, the Dirac measures are pairwise singular and $\|\delta_n\| = 1, n \in N$, then

$$\|(\mu_n)_{ca}\| = \frac{1}{2} \cdot 1 + \frac{1}{2^2} \cdot 1 + \cdots + \frac{1}{2^n} \cdot 1 = 1 - \frac{1}{2^n}.$$

$$\|(\mu_n)_{pfa}\| = \frac{1}{2^n}.$$

Conclusion 6. In the considered MC, for any initial purely finitely additive measure $\mu_0 \in S_{pfa}$, the Markov sequence of measures μ_n generated by the measure μ_0 has its countably additive components tending to one in the norm, while its purely finitely additive components tend to zero in the norm as $n \to \infty$. Obviously, in both cases, this convergence is uniform in the initial measure $\mu_0 \in S_{pfa}$:

$$\sup_{\mu_0 \in S_{pfa}} \|(\mu_n)_{ca}\| \to 1 \quad \text{and} \quad \sup_{\mu_0 \in S_{pfa}} \|(\mu_n)_{pfa}\| \to 0.$$

We also note that this convergence is exponentially fast.

6.5. Conjugate Operator T^*

Now, we describe the operator T^* of our Markov chain, in general, defined on the entire space of finitely additive measures $ba(X, \Sigma)$. We want to prove by direct methods that the Markov operator T^* of this Markov chain is quasi-compact. Let the initial finitely additive measure $\mu_0 \in S_{ba}$ be arbitrary. This measure generates a Markov sequence of measures $\mu_n = T^*\mu_{n-1} = (T^*)^n\mu_0, \mu_n \in S_{ba}, n \in N$. Each such measure can be decomposed into a countably additive component $(\mu_n)_{ca}$ and a purely finitely additive component $(\mu_n)_{pfa}$: $\mu_n = (\mu_n)_{ca} + (\mu_n)_{pfa}$. Above, we have already considered Markov sequences of measures μ_n for the cases where $\mu_0 \in ca(X, \Sigma)$ and $\mu_0 \in pfa(X, \Sigma)$. The same decomposition can be used to consider the general case. However, it is more convenient to go the other way. To prove the quasi-compactness of the operator T^*, we will use almost verbatim the proof of the quasi-compactness of the operator $A = (T^*)_{ca}$ in Section 6.3. We replace the condition $\mu_0 \in S_{ca}$ with the condition $\mu_0 \in S_{ba}$ in the corresponding formulas in Section 6.3 for μ_1 and μ_2. We also replace the notation of the operator A with the notation of the operator T^*. It is easy to check that in all the corresponding transformations in Section 6.3, we never used the countable additivity of the measures μ_1 and μ_2, but only the finite additivity of these measures. As a result, we obtain the following statements: For $\mu_0 \in S_{ba}$, $\mu_1 = T^*\mu_0$, $\mu_2 = T^*\mu_1$, we have:

$$\mu_1(\{0\}) = \frac{1}{2}, \mu_1(\{1\}) = \frac{1}{2}\mu_0(\{0\}), \mu_1(N) = \frac{1}{2},$$

$$\mu_2(\{0\}) = \frac{1}{2}, \mu_2(\{1\}) = \frac{1}{2^2},$$

$$\mu_2(M_2) = \mu_2(\{0\} \cup \{1\}) = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{2^2}, \mu_2(N_2) = \frac{1}{2^2}.$$

Note that the values of the measure μ_2 do not depend on the initial measure on the indicated sets $\mu_0 \in S_{ba}$. Recall that the measure $\mu^*(m) = \frac{1}{2^{m+1}}$, for any $m \in X$, is invariant for the operator A and for the operator T^*. The measure μ^* is countably additive, but we did not use this in our transformations above. Obviously, $\mu_2(\{0\}) = \mu^*(\{0\}) = \frac{1}{2}$, $\mu_2(\{1\}) = \mu^*(\{1\}) = \frac{1}{2^2}$ and $\mu_2(M_2) = \frac{3}{2^2}$, $\mu_2(N_2) = \frac{1}{2^2}$. Now we can obtain estimates for the norm $\|\mu_2 - \mu^*\|$ for any initial measure $\mu_0 \in S_{ba}$, repeating the corresponding estimates for the same norm for any $\mu_0 \in S_{ca}$ (we omit the details):

$$\|\mu_2 - \mu^*\| \le \mu_2(N_2) + \mu^*(N_2) = \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}.$$

Since this estimate does not depend on the initial measure $\mu_0 \in S_{ba}$, then

$$\|\mu_2 - \mu^*\| = \sup_{\mu_0 \in S_{ba}} \|\mu_2 - \mu^*\| \le \frac{1}{2},$$

i.e.,

$$\|(T^*)^2\mu_0 - \mu^*\| = \sup_{\mu_0 \in S_{ba}} \|(T^*)^2\mu_0 - \mu^*\| \le \frac{1}{2}.$$

Exactly as before for the operator $A\colon ca(X,\Sigma) \to ca(X,\Sigma)$, we introduce the projector $Q_{T^*}\colon ba(X,\Sigma) \to ba(X,\Sigma)$ onto the one-dimensional space $L = \{\alpha\mu^*\colon \alpha \in R^1\}$ of invariant measures for the operator T^*. Obviously, a given projector Q_{T^*} is a projector onto the same one-dimensional space when it is restricted from the space $ba(X,\Sigma)$ to the space $ca(X,\Sigma)$. From the estimates obtained above, we arrive at the operator norm

$$|||(T^*)^2 - Q_{T^*}||| \le \frac{1}{2}.$$

Conclusion 7. According to Definition 3, the operator T^* is quasi-compact on the space $ba(X,\Sigma) = B^*(X,\Sigma)$ for the considered MC. From classical general ergodic theorems, we obtain that for the considered MC, the general strong uniform ergodic theorem is true:

$$(T^*)^n \Rightarrow Q_{T^*} \text{ at } n \to \infty,$$

in the class of all bounded finitely additive measures (and in the class of all bounded countably additive measures). General Conclusion. An example of a Markov chain from Lin [11], if its operators T and T^* are considered on the spaces $B(X,\Sigma)$ and $B^*(X,\Sigma) = ba(X,\Sigma)$, respectively, does not contradict our Theorem 4 proved above, and confirms it.

7. Discussion

1. The work of Foguel [8] (1966) is one of the first in which the following construction of Markov operators is constructed for general Markov chains: $P\colon B(X,\Sigma) \to B(X,\Sigma)$ and $P^*\colon ba(X,\Sigma) \to ba(X,\Sigma)$, where $ba(X,\Sigma) = B^*(X,\Sigma)$ is the space of finitely additive measures (Foguel refers on [18]). We also use this construction in this paper and in our other works. In our present paper, we have not found any direct intersections with the results of [8]. In many of Fogel's subsequent works (see, for example, [9,10]), the author focuses on topological spaces X and Markov operators in the spaces L_1 and L_∞. We do not consider such constructions in the present paper.

2. Horowitz [7] considers the construction of Markov operators $P\colon L_1 \to L_1$ (left) and $P^*\colon L_\infty \to L_\infty$ (right). Theorem 4.1 in [7] proves the equivalence of 10 different conditions for ergodic and conservative Markov chains. In particular, it is proved that the condition "(a) there exists no invariant pure charge" is equivalent to the condition "(h) P is a quasi-compact operator on L_∞". In the present paper, we use a different construction of Markov operators. It is difficult to compare theorems proved within these two different approaches. Therefore, we can say that our Theorems 8 and 9 are distant analogs of Horowitz Theorem 4.1 [7].

3. We want to compare the results of the present paper with those of Lin [12]. We have already performed such an analysis in our paper [1] since the results of [1] are also close to the results of [12] [Theorem 5]. We consider it appropriate to give here a large corrected quotation from [1] [Section 7].

In the second part of [12] (1975), Lin considered an arbitrary measurable space (X, Σ) and investigated the properties of the Markov operator P defined on the space of functions $B(X, \Sigma)$ and the properties of the conjugate operator P^* defined on the conjugate space. It should be noted that the symbols $B^*(X, \Sigma)$, $ba(X, \Sigma)$, and also the isomorphism $B^*(X, \Sigma) = ba(X, \Sigma)$ were not used explicitly in the text of article [12]. We are also using this formulation of the problem with the corresponding addition.

In Theorem 5 of [12], a number of assertions under some hard a priori condition of "ergodicity" on the Markov chain are considered. In particular, it is proved that the Doeblin condition (viii) is equivalent to the condition (vi): "The space of P^* invariant functionals (i.e., finitely additive measures) is one-dimensional". But this is true only due to the a priori "ergodicity" condition in [12] [Theorem 5]. In the general case, only the finite dimensionality of the space of invariant finitely additive measures, and hence also of invariant countably additive measures, follows from the Doeblin condition (see, for example, [27], and also our theorems in Sections 4 and 5 from [1]).

In the same Theorem 5 from [12], it was proved (under the same a priori ergodicity condition) that the Doeblin condition (viii) and the condition (vi) are equivalent to the condition (v): "Every P^*-invariant functionals is a measure", here it is countably additive measures.

If we translate these statements into the language we use, then we obtain a special case of our Theorem 7 [1], but for a one-dimensional space of invariant finitely additive measures.

Theorems 10 and 11 from [1] also generalize the corresponding assertions Theorem 5 from [12] since they do not assume that the Markov chain is ergodic (end of quote).

We can now add one more new remark to those given above. In our Theorem 4, it was proved that the condition "the operator T^* is quasi-compact" is equivalent to the condition "the operator T is quasi-compact" (in [12] these operators are denoted by the symbols P^* and P). There is no such statement in [12] [Theorem 5]. In [12], in point (i), it is written only that the operator P is quasi-compact.

Thus, our Theorem 4 significantly strengthens Theorem 5 from [12].

4. Hernández-Lerma and Lasserre proved in [16] [Theorem 6.3.1] (2003) that for a Markov chain defined on a separable metrizable phase space (X, Σ), under certain assumptions, there exists an invariant finite additive measure. It is also shown that if a finitely additive measure is invariant, then both its countably additive and purely finitely additive components are invariant. In this article, we consider general Markov chains and do not separately single out the particular case of the topological phase space.

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