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Lifts of a Semi-Symmetric Metric Connection from Sasakian Statistical Manifolds to Tangent Bundle

Rajesh Kumar [1], Sameh Shenawy [2], Nasser Bin Turki [3,*], Lalnunenga Colney [1,4] and Uday Chand De [5]

[1] Department of Mathematics, Pachhunga University College, Mizoram University, Aizawl 796004, India; rajesh@pucollege.edu.in (R.K.); lalnunenga_official@yahoo.com (L.C.)
[2] Basic Science Department, Modern Academy for Engineering and Technology, Maadi 4410242, Egypt; drssshenawy@eng.modern-academy.edu.eg
[3] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[4] Department of Mathematics and Computer Science, Mizoram University, Aizawl 796004, India
[5] Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata 700019, India; ucdpm@caluniv.ac.in
* Correspondence: nassert@ksu.edu.sa

Abstract: The lifts of Sasakian statistical manifolds associated with a semi-symmetric metric connection in the tangent bundle are characterized in the current research. The relationship between the complete lifts of a statistical manifold with semi-symmetric metric connections and Sasakian statistical manifolds with a semi-symmetric metric connection in the tangent bundle is investigated. We also discuss the classification of Sasakian statistical manifolds with respect to semi-symmetric metric connections in the tangent bundle. Finally, we derive an example of the lifts of Sasakian statistical manifolds to the tangent bundle.

Keywords: Sasakian statistical manifolds; statistical manifolds; vertical and complete lifts; tangent bundle; semi-symmetric metric connection; partial differential equations; mathematical operators

MSC: 53C25; 53C15; 53B12; 58A30; 53C05

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1. Introduction

Friedmann and Schouten [1] proposed the concept of a semi-symmetric connection (SSC) on a differentiable manifold. If a linear connection satisfies the expression:

$$\ddot{T}(X_0, Y_0) = \omega_0(Y_0)X_0 - \omega_0(X_0)Y_0,$$

and is not torsion-free, it is referred to as a semi-symmetric connection (SSC), where $\ddot{T}$ is the torsion tensor, ω_0 is a one-form, and X_0, Y_0 are vector fields. Hayden [2] introduced the notion of the metric connection, which was called the Hayden connection. The semi-symmetric metric connection (SSMC) satisfies the semi-symmetric condition and is known to exist if $\ddot{\nabla} g_0 = 0$; otherwise, it is a semi-symmetric non-metric connection (SSNMC), and it was further studied in [2,3]. Amari [4] was the first to study statistical inference problems in information geometry, which was the concept of statistical structure. Every point on a statistical manifold, a differentiable manifold, depicts a probability distribution. A statistical manifold with infinite dimensions can be found in the collection of all probability measurements. Furuhata et al. investigated the concepts of the Sasakian and Kenmotsu statistical structures [5–7]. Kurose [8] studied the concept of the holomorphic statistical structure as a generalization of Kahler's structure. Kazan and Kazan [9] investigated the SSMC on Sasakian statistical manifolds. In [10,11], the authors investigated connections on statistical manifolds. Numerous geometers have explored the tangent bundle of differential geometry, including Yano and Kobayashi [12], Yano and Ishihara [13], Tani [14], and Pandey and Chaturvedi [15]. Yano and Ishihara [13] established the lifts of the manifold, as well as the connection in the tangent bundle. Different manifolds associated with different connections in the tangent bundle were studied in [16–24]. Kumar et al. [25] recently studied the lifts of the semi-symmetric non-metric connection (SSNMC) from statistical manifolds to the tangent bundle.

This Introduction is followed by a section on the preliminary materials. Section 3 investigates the statistical manifold and Sasakian statistical manifolds' lifts to its tangent bundle, and Section 4 computes the SSMC in the tangent bundle, whereas Section 5 is concerned with the investigation of the lifts of the curvature tensor of a statistical manifold with the SSMC in the tangent bundle. Section 6 investigates the lifts of some curvature tensors of Sasakian statistical manifolds with the SSMC in the tangent bundle and proves some theorems. Finally, in Section 7, an example is provided to demonstrate the lifts of Sasakian statistical manifolds in the tangent bundle.

2. Preliminaries

In a differentiable manifold M, let $T_0M = \bigcup_{p\in M} T_{0p}M$ be the tangent bundle, where $T_{0p}M$ is the tangent space at point $p \in M$ and $\pi : T_0M \to M$ is the natural bundle structure of T_0M over M. For any coordinate system (Q, x^h) in M, where (x^h) is a local coordinate system in the neighborhood Q, $(\pi^{-1}(Q), x^h, y^h)$ is the coordinate system in T_0M, where (x^h, y^h) is an induced coordinate system in $\pi^{-1}(Q)$ from (x^h) [13].

2.1. Vertical and Complete Lifts

Let us define a vector field X_0, a tensor field F_0 of type $(1,1)$, a function f_0, a one-form ω_0, and affine connection $\ddot{\nabla}$ in M; its vertical and complete lifts are given by f_0^v, X_0^v, ω_0^v, F_0^v, $\ddot{\nabla}^v$ and f_0^c, X_0^c, ω_0^c, F_0^c, $\ddot{\nabla}^c$, respectively. The following formulas for complete and vertical lifts were defined by [13]:

$$(f_0X_0)^v = f_0^vX_0^v,\ \ (f_0X_0)^c = f_0^cX_0^v + f_0^vX_0^c, \tag{1}$$

$$X_0^vf_0^v = 0,\ \ X_0^vf_0^c = X_0^cf_0^v = (X_0f_0)^v,\ \ X_0^cf_0^c = (X_0f_0)^c, \tag{2}$$

$$\omega_0(f_0^v) = 0,\ \ \omega_0^v(X_0^c) = \omega_0^c(X_0^v) = \omega_0(X_0)^v,\ \ \omega_0^c(X_0^c) = \omega_0(X_0)^c, \tag{3}$$

$$F_0^vX_0^c = (F_0X_0)^v,\ \ F_0^cX_0^c = (F_0X_0)^c, \tag{4}$$

$$[X_0, Y_0]^v = [X_0^c, Y_0^v] = [X_0^v, Y_0^c],\ \ [X_0, Y_0]^c = [X_0^c, Y_0^c], \tag{5}$$

$$\ddot{\nabla}^c_{X_0^c}Y_0^c = (\ddot{\nabla}_{X_0}Y_0)^c,\ \ \ddot{\nabla}^c_{X_0^c}Y_0^v = (\ddot{\nabla}_{X_0}Y_0)^v. \tag{6}$$

2.2. Statistical Manifold

In an n-dimensional Riemannian manifold (M, g_0) with Riemannian metric g_0, we consider $\ddot{\nabla}$ as an affine connection and $\tilde{\ddot{\nabla}}$ as its Levi-Civita connection. The structure $(M, \ddot{\nabla}, g_0)$ is known as a statistical manifold if $\ddot{\nabla}$ satisfy an affine and torsion-free connection and $\ddot{\nabla}g_0$ satisfies the Codazzi equation:

$$(\ddot{\nabla}_{X_0}g_0)(Y_0, Z_0) = (\ddot{\nabla}_{Y_0}g_0)(X_0, Z_0), \tag{7}$$

for all $X_0, Y_0, Z_0 \in \chi_0(M)$, where $\chi_0(M)$ is the set of all tangent vector fields on M. We know that there exists an affine connection $\ddot{\nabla}^*$, which is the dual of $\ddot{\nabla}$ with respect to g_0 such that

$$X_0g_0(Y_0, Z_0) = g_0(\ddot{\nabla}_{X_0}, Z_0) + g_0(Y_0, \ddot{\nabla}^*_{X_0}Z_0). \tag{8}$$

Also, the pair of connections $\ddot{\nabla}$ and $\ddot{\nabla}^*$ satisfies $(\ddot{\nabla}^*)^* = \ddot{\nabla}$; one can obtain

$$\tilde{\ddot{\nabla}} = \frac{1}{2}(\ddot{\nabla} + \ddot{\nabla}^*). \tag{9}$$

The tensor field K_0 of type $(1,2)$ on $(M, \ddot{\nabla}, g_0)$ is defined by

$$K_0(X_0, Y_0) = \ddot{\nabla}_{X_0} Y_0 - \widetilde{\nabla}_{X_0} Y_0, K_0(X_0, Y_0) = \frac{1}{2}\left(\ddot{\nabla}_{X_0} Y_0 - \ddot{\nabla}^*_{X_0} Y_0\right), \tag{10}$$

and K_0 is symmetric, which gives

$$g_0(K_0(X_0, Y_0), Z_0) = g_0(K_0(X_0, Z_0), Y_0), \;\; K_0(X_0, Y_0) = K_0(Y_0, X_0). \tag{11}$$

The statistical curvature tensor field associated with $\ddot{\nabla}$ is defined as

$$\ddot{R}(X_0, Y_0)Z_0 = \ddot{\nabla}_{X_0} \ddot{\nabla}_{Y_0} Z_0 - \ddot{\nabla}_{Y_0} \ddot{\nabla}_{X_0} Z_0 - \ddot{\nabla}_{[X_0, Y_0]} Z_0. \tag{12}$$

By replacing $\ddot{\nabla}$ with $\ddot{\nabla}^*$, we can obtain the statistical curvature tensor field $\ddot{R}^*$. The curvature tensor fields $\ddot{R}$ and $\ddot{R}^*$ satisfy

$$\ddot{R}(X_0, Y_0)Z_0 = -\ddot{R}(Y_0, X_0)Z_0, \;\; \ddot{R}^*(X_0, Y_0)Z_0 = -\ddot{R}^*(Y_0, X_0)Z_0, \tag{13}$$

$$g_0(\ddot{R}(X_0, Y_0)Z_0, W_0) = -g_0(\ddot{R}^*(X_0, Y_0)W_0, Z_0), \tag{14}$$

$$\ddot{R}(X_0, Y_0)Z_0 + \ddot{R}(Y_0, Z_0)X_0 + \ddot{R}(Z_0, X_0)Y_0 = 0. \tag{15}$$

2.3. Sasakian Statistical Manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold, and it is known to admit an almost contact Riemannian structure $(\Phi_0, \omega_0, A, g_0)$, where Φ_0 is a $(1,1)$ tensor field, A is a vector field, ω_0 is a one-form, and g_0 is a Riemannian metric on M such that

$$\Phi_0 A = 0, \;\; \omega_0(A) = 1, \;\; g(A, X_0) = \omega_0(X_0), \tag{16}$$

$$\Phi_0^2 X_0 = -X_0 + \omega_0(X_0)A, \tag{17}$$

$$g_0(\Phi_0 X_0, \Phi_0 Y_0) = g_0(X_0, Y_0) - \omega_0(X_0)\omega_0(Y_0), \tag{18}$$

for all vector fields X_0, Y_0 on M. Also, if $(\Phi_0, \omega_0, A, g_0)$ satisfy

$$d\omega_0 = 0, \;\; \widetilde{\nabla}_{X_0} A = \Phi_0 X_0, \tag{19}$$

$$(\widetilde{\nabla}_{X_0} \Phi_0)Y_0 = \omega_0(Y_0)X_0 - g_0(X_0, Y_0)A, \tag{20}$$

then M is called a Sasakian manifold [6,26].

A quadruple $(\ddot{\nabla}, g_0, \Phi_0, A)$ is known as a Sasakian statistical structure on M, if $(\ddot{\nabla}, g_0)$ is a statistical structure and (g_0, Φ_0, A) is a Sasakian structure on M and the formula:

$$K_0(X_0, \Phi_0 Y_0) + \Phi_0 K_0(X_0, Y_0) = 0, \tag{21}$$

holds for any vector fields X_0 and Y_0 on M [6]. In a statistical structure $(\ddot{\nabla}, g_0)$ and an almost-contact metric structure (g_0, Φ_0, A) on M, the structure $(\ddot{\nabla}, g_0, \Phi_0, A)$ is known as a Sasakian statistical structure if and only if it satisfies the following formulas [9,27]:

$$\ddot{\nabla}_{X_0} \Phi_0 Y_0 - \Phi_0 \ddot{\nabla}^*_{X_0} Y_0 = g_0(Y_0, A)X_0 - g_0(Y_0, X_0)A, \tag{22}$$

$$\ddot{\nabla}_{X_0} A = \Phi_0 X_0 + g_0(\ddot{\nabla}_{X_0} A, A)A. \tag{23}$$

3. Statistical Manifold and Sasakian Statistical Manifolds in the Tangent Bundle

In this section, we obtain the complete lifts of the statistical manifolds and Sasakian statistical manifolds to the tangent bundle.

Suppose T_0M is the tangent bundle and $X_0 = X_0^i \frac{\partial}{\partial x^i}$ is a local vector field on M; then, its vertical and complete lifts in terms of partial differential equations are:

$$X_0^v = X_0^i \frac{\partial}{\partial y^i}, \tag{24}$$

$$X_0^c = X_0^i \frac{\partial}{\partial x^i} + \frac{\partial X_0^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \tag{25}$$

Let the tangent bundle be denoted by T_0M. Then, obtaining the complete lifts on Equations (7)–(15) by mathematical operators, we obtain

$$(\ddot{\nabla}^c_{X_0^c} g_0^c)(Y_0^c, Z_0^c) = (\ddot{\nabla}^c_{Y_0^c} g_0^c)(X_0^c, Z_0^c), \tag{26}$$

$$X_0^c g_0^c(Y_0^c, Z_0^c) = g_0^c(\ddot{\nabla}^c_{X_0^c} Y_0^c, Z_0^c) + g_0^c(Y_0^c, \ddot{\nabla}^{*c}_{X_0^c} Z_0^c), \tag{27}$$

$$\widetilde{\nabla}^c = \frac{1}{2}(\ddot{\nabla}^c + \nabla^{*c}), \tag{28}$$

$$K_0^c(X_0^c, Y_0^c) = \ddot{\nabla}^c_{X_0^c} Y_0^c - \widetilde{\nabla}^c_{X_0^c} Y_0^c, \tag{29}$$

$$K_0^c(X_0^c, Y_0^c) = \frac{1}{2}(\ddot{\nabla}^c_{X_0^c} Y_0^c - \ddot{\nabla}^{*c}_{X_0^c} Y_0^c), \tag{30}$$

$$g_0^c(K_0^c(X_0^c, Y_0^c), Z_0^c) = g_0^c(K_0^c(X_0^c, Z_0^c), Y_0^c), \;\; K_0^c(X_0^c, Y_0^c) = K_0^c(Y_0^c, X_0^c), \tag{31}$$

$$\ddot{R}^c(X_0^c, Y_0^c)Z_0^c = \ddot{\nabla}^c_{X_0^c}\ddot{\nabla}^c_{Y_0^c} Z_0^c - \ddot{\nabla}^c_{Y_0^c}\ddot{\nabla}^c_{X_0^c} Z_0^c - \ddot{\nabla}^c_{[X_0^c, Y_0^c]}, \tag{32}$$

$$\ddot{R}^c(X_0^c, Y_0^c)Z_0^c = -\ddot{R}^c(Y_0^c, X_0^c)Z_0^c, \;\; \ddot{R}^{*c}(X_0^c, Y_0^c)Z_0^c = -\ddot{R}^{*c}(Y_0^c, X_0^c)Z_0^c, \tag{33}$$

$$g_0^c(\ddot{R}^c(X_0^c, Y_0^c)Z_0^c, W_0^c) = -g_0^c(\ddot{R}^{*c}(X_0^c, Y_0^c)W_0^c, Z_0^c), \tag{34}$$

$$\ddot{R}^c(X_0^c, Y_0^c)Z_0^c + \ddot{R}^c(Y_0^c, Z_0^c)X_0^c + \ddot{R}^c(Z_0^c, X_0^c)Y_0^c = 0. \tag{35}$$

Similarly, obtaining complete lifts on Equations (16)–(23) by mathematical operators, we obtain

$$(\Phi_0 A)^c = 0, \;\; \omega_0^c(A^c) = 1, \;\; g_0^c(A^c, X_0^c) = \omega_0^c(X_0^c), \tag{36}$$

$$(\Phi_0^2 X_0)^c = -X_0^c + \omega_0^c(X_0^c)A^v + \omega_0(X_0^c)A^c, \tag{37}$$

$$\begin{aligned} g_0^c\big((\Phi_0 X_0)^c, (\Phi_0 Y_0)^c\big) \;\; &= \;\; g_0^c(X_0^c, Y_0^c) - \omega_0^c(X_0^c)\omega_0^v(Y_0^c) \\ &\quad -\omega_0^v(X_0^c)\omega_0^c(Y_0^c), \end{aligned} \tag{38}$$

$$d\omega_0^c = 0, \;\; \widetilde{\nabla}^c_{X_0^c} A^c = (\Phi_0 X_0)^c, \tag{39}$$

$$\begin{aligned} (\widetilde{\nabla}^c_{X_0^c} \Phi_0^c) Y_0^c \;\; &= \;\; \omega_0^c(Y_0^c)X_0^v + \omega_0^v(Y_0^c)X_0^c - g_0^c(X_0^c, Y_0^c)A^v \\ &\quad -g_0^v(X_0^c, Y_0^c)A^c, \end{aligned} \tag{40}$$

$$K_0^c\big(X_0^c, (\Phi_0 Y_0)^c\big) + \Phi_0^c K_0^c(X_0^c, Y_0^c) = 0, \tag{41}$$

$$\begin{aligned} \ddot{\nabla}^c_{X_0^c}\Phi_0^c Y_0^c - \Phi_0^c \ddot{\nabla}^{*c}_{X_0^c} Y_0^c \;\; &= \;\; g_0^c(Y_0^c, A^c)X_0^v + g_0^c(Y_0^v, A^c)X_0^c \\ &\quad -g_0^c(Y_0^c, X_0^c)A^v - g_0^c(Y_0^v, X_0^c)A^c, \end{aligned} \tag{42}$$

$$\ddot{\nabla}^c_{X_0^c} A^c = \Phi_0^c X_0^c + g_0^c(A^c, \ddot{\nabla}^c_{X_0^c} A^c)A^v + g_0^c(A^v, \ddot{\nabla}^c_{X_0^c} A^c)A^c. \tag{43}$$

4. Semi-Symmetric Metric Connection in the Tangent Bundle

Let M be an n-dimensional Riemannian manifold; the linear connection $\check{\nabla}$ on M is given by [3]

$$\check{\nabla}_{X_0} Y_0 = \widetilde{\nabla}_{X_0} Y_0 + \omega_0(Y_0)X_0 - g_0(X_0, Y_0)A - K_0(X_0, Y_0), \tag{44}$$

for all vector fields X_0 and $Y_0 \in \chi(M^n)$, and ω_0 is a one-form associated with vector field A and defined by

$$\omega_0(X_0) = g_0(X_0, A). \tag{45}$$

By using (10) in (44), we obtain

$$\check{\nabla}_{X_0} Y_0 = \widetilde{\nabla}^*_{X_0} Y_0 + \omega_0(Y_0)X_0 - g_0(X_0, Y_0)A + K_0(X_0, Y_0), \tag{46}$$

The torsion tensor is given by

$$\check{T}(X_0, Y_0) = \omega_0(Y_0)X_0 - \omega_0(X_0)Y_0. \tag{47}$$

A linear connection satisfying (47) is called a semi-symmetric connection (SSC). For any vector fields X_0, Y_0, Z_0 on M, we have

$$(\check{\nabla}_{X_0} g_0)(Y_0, Z_0) = X_0 g_0(Y_0, Z_0) - g_0(\check{\nabla}_{X_0} Y_0, Z_0) - g_0(Y_0, \check{\nabla}_{X_0} Z_0). \tag{48}$$

Using (8), (11), and (44), we obtain

$$(\check{\nabla}_{X_0} g_0)(Y_0, Z_0) = 0. \tag{49}$$

Hence, a linear connection $\check{\nabla}$ defined by (44) satisfies (47) and (49) and is called an SSMC.

Let the tangent bundle be denoted by T_0M, and we obtain the complete lifts of Equations (44)–(49) by mathematical operators; we obtain

$$\begin{aligned} \check{\nabla}^c_{X^c_0} Y^c_0 \quad &= \quad \widetilde{\nabla}^c_{X^c_0} Y^c_0 + \omega^c_0(Y^c_0) X^v_0 + \omega^v_0(Y^c_0) X^c_0 - g^c_0(X^c_0, Y^c_0) A^v \\ &\quad -g^c_0(X^v_0, Y^c_0) A^c - K^c_0(X^c_0, Y^c_0), \end{aligned} \tag{50}$$

$$\omega^c_0(X^c_0) = g^c_0(X^c_0, A^c), \tag{51}$$

$$\begin{aligned} \check{\nabla}^c_{X^c_0} Y^c_0 \quad &= \quad \widetilde{\nabla}^{*c}_{X^c_0} Y^c_0 + \omega^c_0(Y^c_0) X^v_0 + \omega^v_0(Y^c_0) X^c_0 - g^c_0(X^c_0, Y^c_0) A^v \\ &\quad -g^c_0(X^v_0, Y^c_0) A^c + K^c_0(X^c_0, Y^c_0), \end{aligned} \tag{52}$$

$$\begin{aligned} \check{T}^c(X^c_0, Y^c_0) \quad &= \quad \omega^c_0(Y^c_0) X^v_0 + \omega^v_0(Y^c_0) X^c_0 - \omega^c_0(X^c_0) Y^v_0 \\ &\quad -\omega^v_0(X^c_0) Y^c_0, \end{aligned} \tag{53}$$

$$\begin{aligned} (\check{\nabla}^c_{X^c_0} g^c)(Y^c_0, Z^c_0) &= X^c_0 g^c(Y^v_0, Z^c_0) + X^v_0 g^c(Y^c_0, Z^c_0) \\ &\quad - g^c(\check{\nabla}^c_{X^c_0} Y^c_0, Z^c_0) - g^c(Y^c_0, \check{\nabla}^c_{X^c_0} Z^c_0), \end{aligned} \tag{54}$$

$$(\check{\nabla}^c_{X^c_0} g^c_0)(Y^c_0, Z^c_0) = 0. \tag{55}$$

Theorem 1. *Let M be a Sasakian statistical manifold admitting an SSMC in the tangent bundle T_0M, which satisfies (53) and (55), then the SSMC of the Sasakian statistical manifold in the tangent bundle is given by (50) and satisfies (52).*

Proof. Let ($\check{\nabla}^c$ be a metric connection in the tangent bundle satisfying (53) on a Sasakian statistical manifold M defined by

$$\check{\nabla}^c_{X^c_0} Y^c_0 = \ddot{\nabla}^c_{X^c_0} Y^c_0 + H^c(X^c_0, Y^c_0), \tag{56}$$

where $\ddot{\nabla}^c$ is a torsion-free connection and H^c is the tensor field of type $(1, 2)$ in the tangent bundle. Using (27) and (56), we obtain

$$\begin{aligned} 0 &= (\check{\nabla}^c_{X^c_0} g^c)(Y^c_0, Z^c_0) \\ &= X^c_0 g^c(Y^c_0, Z^c_0) - g^c(\check{\nabla}^c_{X^c_0} Y^c_0, Z^c_0) - g^c(Y^c_0, \check{\nabla}^c_{X^c_0} Z^c_0) \\ &= X^c_0 g^c(Y^c_0, Z^c_0) - g^c\left(\check{\nabla}^c_{X^c_0} Y^c_0 + H^c(X^c_0, Y^c_0), Z^c_0\right) \\ &- g^c\left(Y^c_0, \check{\nabla}^c_{X^c_0} Z^c_0 + H^c(X^c_0, Z^c_0)\right) \\ &= -2g^c\left(K^c_0(X^c_0, Z^c_0), Y^c_0\right) - g^c\left(H^c(X^c_0, Y^c_0), Z^c_0\right) \\ &- g^c\left(H^c(X^c, Z^c), Y^c_0\right). \end{aligned} \tag{57}$$

So,

$$g^c\left(H^c(X^c,Y^c),Z^c\right)+g^c\left(H^c(X_0^c,Z_0^c),Y_0^c\right)=-2g^c\left(K_0^c(X_0^c,Z_0^c),Y_0^c\right). \tag{58}$$

Now, using (56), we obtain

$$\breve{T}^c(X_0^c,Y_0^c)=H^c(X_0^c,Y_0^c)-H^c(Y_0^c,X_0^c). \tag{59}$$

By using (53), we have

$$\begin{aligned}
&g^c\left(\breve{T}^c(X_0^c,Y_0^c),Z_0^c\right)+g^c\left(\breve{T}^c(Z_0^c,X_0^c),Y_0^c\right)+g^c\left(\breve{T}^c(Z_0^c,Y_0^c),X_0^c\right)\\
&=g^c\left(H^c(X_0^c,Y_0^c)-H^c(Y_0^c,X_0^c),Z_0^c\right)+g^c\left(H^c(Z_0^c,X_0^c)-H^c(X_0^c,Z_0^c),Y_0^c\right)\\
&+g^c\left(H^c(Z_0^c,Y_0^c)-H^c(Y_0^c,Z_0^c),X_0^c\right)\\
&=2\Big(g^c(H^c(X_0^c,Y_0^c),Z_0^c)+g^c(K_0^c(X_0^c,Z_0^c),Y_0^c)\Big).
\end{aligned} \tag{60}$$

Using (53) in (60), we obtain

$$\begin{aligned}
g^c\left(H^c(X_0^c,Y_0^c),Z_0^c\right)=\frac{1}{2}\Big[&g^c\left(\omega^c(Y_0^c)X_0^v+\omega^v(Y_0^c)X_0^c\right.\\
&\left.-\omega^c(X_0^c)Y_0^v-\omega^v(X_0^c)Y_0^c,Z_0^c\right)\\
&+g^c\left(\omega^c(X_0^c)Z_0^v+\omega^v(X_0^c)Z_0^c\right.\\
&\left.-\omega^c(Z_0^c)X_0^v-\omega^v(Z_0^c)X_0^c,Y_0^c\right)\\
&+g^c\left(\omega^c(Y_0^c)Z_0^v+\omega^v(Y_0^c)Z_0^c\right.\\
&\left.-\omega^c(Z_0^c)Y_0^v-\omega^v(Z_0^c)Y_0^c,X_0^c\right)\Big]\\
&-g^c\left(K_0^c(X_0^c,Z_0^c),Y_0^c\right).
\end{aligned} \tag{61}$$

Hence, we obtain

$$H^c(X_0^c,Y_0^c)=\omega^c(Y_0^c)X_0^v+\omega^v(Y_0^c)X_0^c-g^c(X_0^c,Y_0^c)A^v-g^c(X_0^v,Y_0^c)A^c-K_0^c(X_0^c,Y_0^c). \tag{62}$$

Further, by using Equations (29), (30), and (50), we obtain (52). □

5. Curvature Tensor of Semi-Symmetric Metric Connection on Statistical Manifolds in the Tangent Bundle

In an n-dimensional Riemannian manifold M and the statistical structure $(\ddot{\nabla}^c, g_0^c)$ on M in the tangent bundle T_0M, we have the relation between the complete lifts of the SSMC $\breve{\nabla}^c$ and torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle from (29) and (50) by

$$\begin{aligned}
\breve{\nabla}^c_{X_0^c}Y_0^c=\ddot{\nabla}^c_{X_0^c}Y_0^c+\omega_0^c(Y_0^c)X_0^v+\omega_0^v(Y_0^c)X_0^c\\
-g_0^c(X_0^c,Y_0^c)A^v-g_0^c(X_0^v,Y_0^c)A^c-K_0^c(X_0^c,Y_0^c).
\end{aligned} \tag{63}$$

The complete lifts of the Riemannian curvature tensor $\breve{R}^c$ of M associated with the SSMC $\breve{\nabla}^c$ in the tangent bundle are given by

$$\breve{R}^c(X_0^c,Y_0^c)Z_0^c=\breve{\nabla}^c_{X_0^c}\breve{\nabla}^c_{Y_0^c}Z_0^c-\breve{\nabla}^c_{Y_0^c}\breve{\nabla}^c_{X_0^c}Z_0^c-\breve{\nabla}^c_{[X_0^c,Y_0^c]}Z_0^c, \tag{64}$$

Using Equation (63) in (64), we obtain

$$
\begin{aligned}
\breve{R}^c(X_0^c, Y_0^c)Z_0^c \quad & = \ddot{R}^c(X_0^c, Y_0^c)Z_0^c + \Big(\omega_0^c(X_0^c)A^v + \omega_0^v(X_0^c)A^c \\
& -\omega_0^c(A_0^c)X_0^v - \omega_0^v(A_0^c)X_0^c - \ddot{\nabla}^c_{X_0^c}A^c + K_0^c(X_0^c, A^c)\Big) \\
& g_0^c(Y_0^c, Z_0^c) - \Big(\omega_0^c(Y_0^c)A^v + \omega_0^v(Y_0^c)A^c - \omega_0^c(A_0^c)Y_0^v \\
& -\omega_0^v(A_0^c)Y_0^c - \ddot{\nabla}^c_{Y_0^c}A^c + K_0^c(Y_0^c, A^c)\Big)g_0^c(X_0^c, Z_0^c) \\
& -g_0^c\Big(\omega_0^c(X_0^c)A^v + \omega_0^v(X_0^c)A^c - \ddot{\nabla}^c_{X_0^c}A^c + K_0^c(X_0^c, A^c) \\
& , Z_0^c\Big)Y_0^c + g_0^c\Big(\omega_0^c(Y_0^c)A^v + \omega_0^v(Y_0^c)A^c - \ddot{\nabla}^c_{Y_0^c}A^c \\
& +K_0^c(Y_0^c, A^c), Z_0^c\Big)X_0^c - (\ddot{\nabla}^c_{X_0^c}K_0^c)(Y_0^c, Z_0^c) \\
& +(\ddot{\nabla}^c_{Y_0^c}K_0^c)(X_0^c, Z_0^c) + K_0^c\Big(X_0^c, K_0^c(Y_0^c, Z_0^c)\Big) \\
& -K_0^c\Big(Y_0^c, K_0^c(X_0^c, Z_0^c)\Big),
\end{aligned} \tag{65}
$$

where $\ddot{R}^c$ is the complete lifts of the curvature tensor of M associated with torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle T_0M and is defined as

$$
\ddot{R}^c(X_0^c, Y_0^c)Z_0^c = \ddot{\nabla}^c_{X_0^c}\ddot{\nabla}^c_{Y_0^c}Z_0^c - \ddot{\nabla}^c_{Y_0^c}\ddot{\nabla}^c_{X_0^c}Z_0^c - \ddot{\nabla}^c_{[X_0^c, Y_0^c]}Z_0^c, \tag{66}
$$

Similarly, we obtain the complete lifts of the relation between the SSMC $\breve{\nabla}^c$ and dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle from Equations (30) and (63) as

$$
\begin{aligned}
\breve{\nabla}^c_{X_0^c}Y_0^c = {} & \ddot{\nabla}^{*c}_{X_0^c}Y_0^c + \omega_0^c(Y_0^c)X_0^v + \omega_0^v(Y_0^c)X_0^c - g_0^c(X_0^c, Y_0^c)A^v \\
& - g_0^c(X_0^v, Y_0^c)A^c + K_0^c(X_0^c, Y_0^c).
\end{aligned} \tag{67}
$$

The relation between the complete lifts of the Riemannian curvature tensor $\breve{R}^c$ associated with the SSMC $\breve{\nabla}^c$ and Riemannian curvature tensor $\breve{R}^{*c}$ associated with the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M is obtain by using Equation (67) in (64) by

$$
\begin{aligned}
\breve{R}^c(X_0^c, Y_0^c)Z_0^c = {} & \ddot{R}^{*c}(X_0^c, Y_0^c)Z_0^c + \Big(\omega_0^c(X_0^c)A^v + \omega_0^v(X_0^c)A^c \\
& - \omega_0^c(A_0^c)X_0^v - \omega_0^v(A_0^c)X_0^c - \ddot{\nabla}^{*c}_{X_0^c}A^c - K_0^c(X_0^c, A^c)\Big) \\
& g_0^c(Y_0^c, Z_0^c) - \Big(\omega_0^c(Y_0^c)A^v + \omega_0^v(Y_0^c)A^c - \omega_0^c(A_0^c)Y_0^v \\
& - \omega_0^v(A_0^c)Y_0^c - \ddot{\nabla}^{*c}_{Y_0^c}A^c - K_0^c(Y_0^c, A^c)\Big)g_0^c(X_0^c, Z_0^c) \\
& - g_0^c\Big(\omega_0^c(X_0^c)A^v + \omega_0^v(X_0^c)A^c - \ddot{\nabla}^{*c}_{X_0^c}A^c - K_0^c(X_0^c, A^c) \\
& , Z_0^c\Big)Y_0^c + g_0^c\Big(\omega_0^c(Y_0^c)A^v + \omega_0^v(Y_0^c)A^c - \ddot{\nabla}^{*c}_{Y_0^c}A^c \\
& - K_0^c(Y_0^c, A^c), Z_0^c\Big)X_0^c + (\ddot{\nabla}^{*c}_{X_0^c}K_0^c)(Y_0^c, Z_0^c) \\
& - (\ddot{\nabla}^{*c}_{Y_0^c}K_0^c)(X_0^c, Z_0^c) + K_0^c\Big(X_0^c, K_0^c(Y_0^c, Z_0^c)\Big) \\
& - K_0^c\Big(Y_0^c, K_0^c(X_0^c, Z_0^c)\Big),
\end{aligned} \tag{68}
$$

where $\ddot{R}^{*c}$ is the complete lifts of the Riemannian curvature tensor of M associated with the lift of the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M and is defined as

$$
\ddot{R}^{*c}(X_0^c, Y_0^c)Z_0^c = \ddot{\nabla}^{*c}_{X_0^c}\ddot{\nabla}^{*c}_{Y_0^c}Z_0^c - \ddot{\nabla}^{*c}_{Y_0^c}\ddot{\nabla}^{*c}_{X_0^c}Z_0^c - \ddot{\nabla}^{*c}_{[X_0^c, Y_0^c]}Z_0^c. \tag{69}
$$

Theorem 2. *In a Riemannian manifold M, let the complete lifts of the statistical structure be $(\ddot{\nabla}^c, g_0^c)$ in the tangent bundle T_0M. Then:*

1. *The relation between the lifts of the Riemannian curvature tensor $\check{R}^c$ of SSMC $\check{\nabla}^c$ and the Riemannian curvature tensor $\ddot{R}^c$ of torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle T_0M is given by Equation (65).*
2. *The relation between the lifts of the Riemannian curvature tensor $\check{R}^c$ of SSMC $\check{\nabla}^c$ and the Riemannian curvature tensor $\ddot{R}^{*c}$ of dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M is given by Equation (68).*

Proposition 1. *For a statistical manifold in the tangent bundle, the following relations hold:*

(i) $-g_0^c(\ddot{\nabla}^c_{X_0^c} A_0^c, W_0^c) + g_0^c(K_0^c(X_0^c, A_0^c), W_0^c) = -g_0^c(\ddot{\nabla}^{*c}_{X_0^c} A_0^c, W_0^c) + g_0^c(K_0^c(X_0^c, A_0^c), W_0^c)$.

(ii) $g_0^c((\ddot{\nabla}^c_{Y_0^c} K_0^c)(X_0^c, Z_0^c), W_0^c) - g_0^c((\ddot{\nabla}^c_{X_0^c} K_0^c)(Y_0^c, Z_0^c), W_0^c) = g_0^c((\ddot{\nabla}^{*c}_{Y_0^c} K_0^c)(X_0^c, W_0^c), Z_0^c) - g_0^c((\ddot{\nabla}^{*c}_{X_0^c} K_0^c)(Y_0^c, W_0^c), Z_0^c)$.

(iii) $g_0^c(K_0^c(X_0^c, K_0^c(Y_0^c, Z_0^c)), W_0^c) = g_0^c(K_0^c(Y_0^c, K_0^c(X_0^c, W_0^c)), Z_0^c)$.

Proof. Using (29) and (30), we obtain (i). For (ii), we use (31) and prove that

$$\begin{aligned} g_0^c(K_0^c(\ddot{\nabla}^c_{Y_0^c} X_0^c, Z_0^c), W_0^c) - g_0^c(K_0^c(\ddot{\nabla}^c_{X_0^c} Y_0^c, Z_0^c), W_0^c) &= g_0^c(K_0^c([Y_0^c, X_0^c], Z_0^c), W_0^c) \\ &= g_0^c(K_0^c([Y_0^c, X_0^c], W_0^c), Z_0^c) \\ &= g_0^c(K_0^c(\ddot{\nabla}^{*c}_{Y_0^c} X_0^c, W_0^c), Z_0^c) - g_0^c(K_0^c(\ddot{\nabla}^{*c}_{X_0^c} Y_0^c, W_0^c), Z_0^c). \end{aligned} \tag{70}$$

Now, using (27) and (31), we obtain

$$\begin{aligned} & g_0^c((\ddot{\nabla}^c_{Y_0^c} K_0^c)(X_0^c, Z_0^c), W_0^c) \\ &= g_0^c(\ddot{\nabla}^c_{Y_0^c} K_0^c(X_0^c, Z_0^c), W_0^c) - g_0^c(K_0^c(X_0^c, \ddot{\nabla}^c_{Y_0^c} Z_0^c), W_0^c) - g_0^c(K_0^c(\ddot{\nabla}^c_{Y_0^c} X_0^c, Z_0^c), W_0^c) \\ &= Y_0^c g_0^c(W_0^v, K_0^c(X_0^c, Z_0^c)) + Y_0^v g_0^c(W_0^c, K_0^c(X_0^c, Z_0^c)) - g_0^c(K_0^c(X_0^c, Z_0^c), \ddot{\nabla}^{*c}_{Y_0^c} W_0^c) \\ &- g_0^c(K_0^c(X_0^c, W_0^c), \ddot{\nabla}^c_{Y_0^c} Z_0^c) - g_0^c(K_0^c(\ddot{\nabla}^c_{Y_0^c} X_0^c, Z_0^c), W_0^c) \\ &= Y_0^c g_0^c(W_0^v, K_0^c(X_0^c, Z_0^c)) + Y_0^v g_0^c(W_0^c, K_0^c(X_0^c, Z_0^c)) - g_0^c(K_0^c(X_0^c, \ddot{\nabla}^{*c}_{Y_0^c} W_0^c), Z_0^c) \\ &- Y_0^c g_0^c(Z_0^v, K_0^c(X_0^c, W_0^c)) - Y_0^v g_0^c(Z_0^c, K_0^c(X_0^c, W_0^c)) + g_0^c(Z_0^c, \ddot{\nabla}^{*c} K_0^c(X_0^c, W_0^c)) \\ &- g^c(K_0^c(\ddot{\nabla}^c_{Y_0^c} X_0^c, Z_0^c), W_0^c) \\ &= g^c(Z_0^c, \ddot{\nabla}^{*c}_{Y_0^c} K_0^c(X_0^c, W_0^c)) - g^c(K_0^c(X_0^c, \ddot{\nabla}^{*c}_{Y_0^c} W_0^c), Z_0^c) - g_0^c(K_0^c(\ddot{\nabla}^c_{Y_0^c} X_0^c, Z_0^c), W_0^c). \end{aligned} \tag{71}$$

From (70) and (71), we obtain

$$\begin{aligned} & g_0^c((\ddot{\nabla}^c_{Y_0^c} K_0^c)(X_0^c, Z_0^c), W_0^c) - g_0^c((\ddot{\nabla}^c_{X_0^c} K_0^c)(Y_0^c, Z_0^c), W_0^c) \\ &= g_0^c(Z_0^c, \ddot{\nabla}^{*c}_{Y_0^c} K_0^c(X_0^c, W_0^c)) - g_0^c(K_0^c(X_0^c, \ddot{\nabla}^{*c}_{Y_0^c} W_0^c), Z_0^c) - g_0^c(K_0^c(\ddot{\nabla}^{*c}_{Y_0^c} X_0^c, W_0^c), Z_0^c) \\ &- g_0^c(Z_0^c, \ddot{\nabla}^{*c}_{X_0^c} K_0^c(Y_0^c, W_0^c)) + g_0^c(K_0^c(Y_0^c, \ddot{\nabla}^{*c}_{X_0^c} W_0^c), Z_0^c) + g_0^c(K_0^c(\ddot{\nabla}^{*c}_{X_0^c} Y_0^c, W_0^c), Z_0^c) \\ &= g_0^c((\ddot{\nabla}^{*c}_{Y_0^c} K_0^c)(X_0^c, W_0^c), Z_0^c) - g_0^c((\ddot{\nabla}^{*c}_{X_0^c} K_0^c)(Y_0^c, W_0^c), Z_0^c). \end{aligned} \tag{72}$$

For (iii), from the symmetry property of $\ddot{K}^c$, we deduce

$$\begin{aligned} g_0^c(K_0^c(X_0^c, K_0^c(Y_0^c, Z_0^c)), W_0^c) &= g_0^c(K_0^c(X_0^c, W_0^c), K_0^c(Y_0^c, Z_0^c)) \\ &= g_0^c(K_0^c(Y_0^c, K_0^c(X_0^c, W_0^c)), Z_0^c). \end{aligned} \tag{73}$$

□

6. Curvature Tensor of Semi-Symmetric Metric Connection on Sasakian Statistical Manifolds in The Tangent Bundle

Let $(M, \ddot{\nabla}^c, g_0^c, \Phi_0^c, \omega_0^c, A^c)$ be the complete lifts of $(2n+1)$-dimensional Sasakian statistical manifolds in the tangent bundle T_0M. Then, the complete lifts of the curvature tensors $\check{R}^c$ associated with SSMC $\check{\nabla}^c$ in the tangent bundle are given by

$$\check{R}^c(X_0^c, Y_0^c)Z_0^c = \check{\nabla}^c_{X_0^c}\check{\nabla}^c_{Y_0^c}Z_0^c - \check{\nabla}^c_{Y_0^c}\check{\nabla}^c_{X_0^c}Z_0^c - \check{\nabla}^c_{[X_0^c, Y_0^c]}Z_0^c. \tag{74}$$

Using Equations (36)–(43) and (63) in Equation (74), we can obtain the relation of the complete lifts of the curvature tensor $\check{R}^c$ of the SSMC $\check{\nabla}^c$ and the curvature tensor $\ddot{R}^c$ of the torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle T_0M as

$$\begin{aligned}\check{R}^c(X_0^c, Y_0^c)Z_0^c = {} & \ddot{R}^c(X_0^c, Y_0^c)Z_0^c + \Big((\Phi_0^2X_0)^c - (\Phi_0X_0)^c\Big)g_0^c(Y_0^c, Z_0^c) \\ & - \Big((\Phi_0^2Y_0)^c - (\Phi_0Y_0)^c\Big)g_0^c(X_0^c, Z_0^c) + g_0^c\Big((\Phi_0X_0)^c, Z_0^c\Big)Y_0^v \\ & + g_0^c\Big((\Phi_0X_0)^v, Z_0^c\Big)Y_0^c - g_0^c\Big((\Phi_0Y_0)^c, Z_0^c\Big)X_0^v \\ & - g_0^c\Big((\Phi_0Y_0)^v, Z_0^c\Big)X_0^c - \omega_0^c(X_0^c)\omega_0^c(Z_0^c)Y_0^v \\ & - \omega_0^c(X_0^c)\omega_0^v(Z_0^c)Y_0^c - \omega_0^v(X_0^c)\omega_0^c(Z_0^c)Y_0^c \\ & + \omega_0^c(Y_0^c)\omega_0^c(Z_0^c)X_0^v + \omega_0^c(Y_0^c)\omega_0^v(Z_0^c)X_0^c \\ & + \omega_0^v(Y_0^c)\omega_0^c(Z_0^c)X_0^c - (\ddot{\nabla}^c_{X_0^c}K_0^c)(Y_0^c, Z_0^c) \\ & + (\ddot{\nabla}^c_{Y_0^c}K_0^c)(X_0^c, Z_0^c) + K_0^c\Big(X_0^c, K_0^c(Y_0^c, Z_0^c)\Big) \\ & - K_0^c\Big(Y_0^c, K_0^c(X_0^c, Z_0^c)\Big).\end{aligned} \tag{75}$$

Similarly, we obtain the relation of the complete lifts of the curvature tensor $\check{R}^c$ of the SSMC $\check{\nabla}^c$ and the curvature tensor $\ddot{R}^{*c}$ of the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M by using Equations (36)–(43) and (67) in (74) as

$$\begin{aligned}\check{R}^c(X_0^c, Y_0^c)Z_0^c = {} & \ddot{R}^{*c}(X_0^c, Y_0^c)Z_0^c + \Big((\Phi_0^2X_0)^c - (\Phi_0X_0)^c\Big)g_0^c(Y_0^c, Z_0^c) \\ & - \Big((\Phi_0^2Y_0)^c - (\Phi_0Y_0)^c\Big)g_0^c(X_0^c, Z_0^c) + g_0^c\Big((\Phi_0X_0)^c, Z_0^c\Big)Y_0^v \\ & + g_0^c\Big((\Phi_0X_0)^v, Z_0^c\Big)Y_0^c - g_0^c\Big((\Phi_0Y_0)^c, Z_0^c\Big)X_0^v \\ & - g_0^c\Big((\Phi_0Y_0)^v, Z_0^c\Big)X_0^c - \omega_0^c(X_0^c)\omega_0^c(Z_0^c)Y_0^v \\ & - \omega_0^c(X_0^c)\omega_0^v(Z_0^c)Y_0^c - \omega_0^v(X_0^c)\omega_0^c(Z_0^c)Y_0^c \\ & + \omega_0^c(Y_0^c)\omega_0^c(Z_0^c)X_0^v + \omega_0^c(Y_0^c)\omega_0^v(Z_0^c)X_0^c \\ & + \omega_0^v(Y_0^c)\omega_0^c(Z_0^c)X_0^c + (\ddot{\nabla}^{*c}_{X_0^c}K_0^c)(Y_0^c, Z_0^c) \\ & - (\ddot{\nabla}^{*c}_{Y_0^c}K_0^c)(X_0^c, Z_0^c) + K_0^c\Big(X_0^c, K_0^c(Y_0^c, Z_0^c)\Big) \\ & - K_0^c\Big(Y_0^c, K_0^c(X_0^c, Z_0^c)\Big).\end{aligned} \tag{76}$$

Proposition 2. *Let $(M, \ddot{\nabla}^c, g_0^c, \Phi_0^c, \omega_0^c, A^c)$ be the complete lifts of a $(2n+1)$-dimensional Sasakian statistical manifold in the tangent bundle T_0M. Then, we have:*

1. $\check{R}^c(X_0^c, Y_0^c)A^c = \ddot{R}^c(X_0^c, Y_0^c)A^c + \omega_0^v(X_0^c)(\Phi_0Y_0)^c + \omega_0^c(X_0^c)(\Phi_0Y_0)^v - \omega_0^v(Y_0^c)(\Phi_0X_0)^c - \omega_0^c(Y_0^c)(\Phi_0X_0)^v - (\ddot{\nabla}^c_{X_0^c}K_0^c)(Y_0^c, A^c) + (\ddot{\nabla}^c_{Y_0^c}K_0^c)(X_0^c, A^c).$
2. $\check{R}^c(X_0^c, Y_0^c)A^c = \ddot{R}^{*c}(X_0^c, Y_0^c)A^c + \omega_0^v(X_0^c)(\Phi_0Y_0)^c + \omega_0^c(X_0^c)(\Phi_0Y_0)^v - \omega_0^v(Y_0^c)(\Phi_0X_0)^c - \omega_0^c(Y_0^c)(\Phi_0X_0)^v + (\ddot{\nabla}^{*c}_{X_0^c}K_0^c)(Y_0^c, A^c) - (\ddot{\nabla}^{*c}_{Y_0^c}K_0^c)(X_0^c, A^c).$

3. $\breve{R}^c(A^c, X_0^c)Y_0^c = \ddot{R}^c(A^c, X_0^c)Y_0^c + \omega_0^c(Y_0^c)(\Phi_0 X_0)^v + \omega_0^v(Y_0^c)(\Phi_0 X_0)^c - g_0^c\Big((\Phi_0 X_0)^c, Y_0^c\Big)A^v - g_0^c\Big((\Phi_0 X_0)^v, Y_0^c\Big)A^c - (\ddot{\nabla}^c_{A^c} K_0^c)(X_0^c, Y_0^c) + (\ddot{\nabla}^c_{X_0^c} K_0^c)(A_0^c, Y_0^c) + K_0^c\Big(A^c, K_0^c(X_0^c, Y_0^c)\Big) - K_0^c\Big(X_0^c, K_0^c(A^c, Y_0^c)\Big)$.
4. $\breve{R}^c(A^c, X_0^c)Y_0^c = \ddot{R}^{*c}(A^c, X_0^c)Y_0^c + \omega_0^c(Y_0^c)(\Phi_0 X_0)^v + \omega_0^v(Y_0^c)(\Phi_0 X_0)^c - g_0^c\Big((\Phi_0 X_0)^c, Y_0^c\Big)A^v - g_0^c\Big((\Phi_0 X_0)^v, Y_0^c\Big)A^c + (\ddot{\nabla}^{*c}_{A^c} K_0^c)(X_0^c, Y_0^c) - (\ddot{\nabla}^{*c}_{X_0^c} K_0^c)(A_0^c, Y_0^c) + K_0^c\Big(A^c, K_0^c(X_0^c, Y_0^c)\Big) - K_0^c\Big(X_0^c, K_0^c(A^c, Y_0^c)\Big)$.

Proof. In a Sasakian manifold, we have from [6]

$$\ddot{\nabla}_{X_0} A = \Phi_0 X_0 + \omega_0(\ddot{\nabla}_{X_0} A)A. \tag{77}$$

Obtaining the complete lifts of Equation (77), we have

$$\ddot{\nabla}^c_{X_0^c} A^c = (\Phi_0 X_0)^c + \omega_0^c(\ddot{\nabla}^c_{X_0^c} A^c)A^v + \omega_0^v(\ddot{\nabla}^c_{X_0^c} A^c)A^c. \tag{78}$$

From Equation (29), we have

$$K_0^c(X_0^c, A^c) = \omega_0^c(\ddot{\nabla}^c_{X_0^c} A^c)A^v + \omega_0^v(\ddot{\nabla}^c_{X_0^c} A^c)A^c. \tag{79}$$

So,

$$\begin{aligned} K_0^c\Big(X_0^c, K_0^c(Y_0^c, A^c)\Big) &= \omega_0^c(\ddot{\nabla}^c_{X_0^c} A^c)\omega_0^c(\ddot{\nabla}^c_{Y_0^c} A^c)A^v \\ &+ \omega_0^c(\ddot{\nabla}^c_{X_0^c} A^c)\omega_0^v(\ddot{\nabla}^c_{Y_0^c} A^c)A^c \\ &+ \omega_0^v(\ddot{\nabla}^c_{X_0^c} A^c)\omega_0^c(\ddot{\nabla}^c_{Y_0^c} A^c)A^c. \end{aligned} \tag{80}$$

Hence, we obtain

$$K_0^c\Big(X_0^c, K_0^c(Y_0^c, A^c)\Big) = K_0^c\Big(Y_0^c, K_0^c(X_0^c, A^c)\Big). \tag{81}$$

Using Equations (36)–(38) and (81) in Equations (75) and (76), we obtain (1)–(4). □

6.1. *Ricci Tensor Associated with Semi-Symmetric Metric Connection of Sasakian Statistical Manifolds in the Tangent Bundle*

The complete lifts of the Ricci tensor $\breve{S}^c$ associated with the connection $\breve{\nabla}^c$ in the tangent bundle $T_0 M$ are given as

$$\breve{S}^c(X_0^c, Y_0^c) = \text{Trace of the map: } Z_0^c \to \breve{R}^c(X_0^c, Y_0^c)Z_0^c. \tag{82}$$

Then, from (75) and (76), we have

$$\begin{aligned} \breve{S}^c(X_0^c, Y_0^c) = \ddot{S}^c(X_0^c, Y_0^c) &- (2n+1)g_0^c\Big((\Phi_0 X_0)^c, (\Phi_0 Y_0)^c + Y_0^c\Big) \\ &- \sum_{i=1}^{2n+1} g_0^c\Big((\ddot{\nabla}^c_{X_0^c} K_0^c)(e_i^c, e_i^c) - (\ddot{\nabla}^c_{e_i^c} K_0^c)(X_0^c, e_i^c) \\ &- K_0^c(X_0^c, K_0^c(e_i^c, e_i^c)) + K_0^c(e_i^c, K_0^c(X_0^c, e_i^c)), Y_0^c\Big), \end{aligned} \tag{83}$$

and

$$\breve{S}^c(X_0^c, Y_0^c) = \ddot{S}^{*c}(X_0^c, Y_0^c) - (2n+1)g_0^c\Big((\Phi_0 X_0)^c, (\Phi_0 Y_0)^c + Y_0^c\Big) - \sum_{i=1}^{2n+1} g_0^c\Big((\ddot{\nabla}^{*c}_{X_0^c} K_0^c)(e_i^c, e_i^c) - (\ddot{\nabla}^{*c}_{e_i^c} K_0^c)(X_0^c, e_i^c) + K_0^c\big(X_0^c, K_0^c(e_i^c, e_i^c)\big) - K_0^c\big(e_i^c, K_0^c(X_0^c, e_i^c)\big), Y_0^c\Big). \tag{84}$$

Theorem 3. *Let $(M, \ddot{\nabla}^c, g_0^c, \Phi_0^c, \omega_0^c, A^c)$ be the complete lifts of a $(2n+1)$-dimensional Sasakian statistical manifold in the tangent bundle T_0M. Then:*

1. *The relation between the Ricci tensor $\breve{S}^c$ of the SSMC $\breve{\nabla}^c$ and the Ricci tensor $\ddot{S}^c$ of the torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle T_0M is given by Equation (83).*
2. *The relation between the Ricci tensor $\breve{S}^c$ of the SSMC $\breve{\nabla}^c$ and the Ricci tensor $\ddot{S}^{*c}$ of the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M is given by Equation (84).*

6.2. Scalar Curvature Associated with Semi-Symmetric Metric Connection of Sasakian Statistical Manifolds in the Tangent Bundle

From Equations (83) and (84), it follows that

$$\breve{r}^c = \ddot{r}^c + 2n - 4n^2 - \sum_{i,j=1}^{2n+1} g_0^c\Big((\ddot{\nabla}^c_{e_j^c} K_0^c)(e_i^c, e_i^c) - (\ddot{\nabla}^c_{e_i^c} K_0^c)(e_j^c, e_i^c) - K_0^c\big(e_j^c, K_0^c(e_i^c, e_i^c)\big) + K_0^c\big(e_i^c, K_0^c(e_j^c, e_i^c)\big), e_j^c\Big), \tag{85}$$

and

$$\breve{r}^c = \ddot{r}^{*c} + 2n - 4n^2 + \sum_{i,j=1}^{2n+1} g_0^c\Big((\ddot{\nabla}^{*c}_{e_j^c} K_0^c)(e_i^c, e_i^c) - (\ddot{\nabla}^{*c}_{e_i^c} K_0^c)(e_j^c, e_i^c) + K_0^c\big(e_j^c, K_0^c(e_i^c, e_i^c)\big) - K_0^c\big(e_i^c, K_0^c(e_j^c, e_i^c)\big), e_j^c\Big). \tag{86}$$

where $\breve{r}^c$, $\ddot{r}^c$, and $\ddot{r}^{*c}$ are the complete lifts of the scalar curvatures associated with the SSMC $\breve{\nabla}^c$, torsion-free connection $\ddot{\nabla}^c$, and dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M, respectively.

Theorem 4. *Let $(M, \ddot{\nabla}^c, g_0^c, \Phi_0^c, \omega_0^c, A^c)$ be the complete lifts of a $(2n+1)$-dimensional Sasakian statistical manifold in the tangent bundle T_0M. Then:*

1. *The relation between the scalar curvature $\breve{r}^c$ of the SSMC $\breve{\nabla}^c$ and the scalar curvature $\ddot{r}^c$ of the torsion-free connection $\ddot{\nabla}^c$ in the tangent bundle T_0M is given by Equation (85).*
2. *The relation between the scalar curvature $\breve{r}^c$ of the SSMC $\breve{\nabla}^c$ and the scalar curvature $\ddot{r}^{*c}$ of the dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M is given by Equation (86).*

Theorem 5. *In a $(2n+1)$-dimensional Sasakian statistical manifold in the tangent bundle T_0M, the complete lift of the Ricci tensor of the Sasakian statistical manifold associated with the SSMC in the tangent bundle T_0M is said to be Ricci flat if the complete lift of the scalar curvature with respect to torsion-free connection $\ddot{\nabla}^c$ and dual connection $\ddot{\nabla}^{*c}$ in the tangent bundle T_0M satisfies*

$$\ddot{r}^c = -2n + 4n^2 + \sum_{i,j=1}^{2n+1} g^c\Big((\ddot{\nabla}^c_{e_j^c} K_0^c)(e_i^c, e_i^c) - (\ddot{\nabla}^c_{e_i^c} K_0^c)(e_j^c, e_i^c) - K_0^c\big(e_j^c, K_0^c(e_i^c, e_i^c)\big) + K_0^c\big(e_i^c, K_0^c(e_j^c, e_i^c)\big), e_j^c\Big), \tag{87}$$

and

$$\ddot{r}^{*c} = -2n + 4n^2 - \sum_{i,j=1}^{2n+1} g^c \Big((\ddot{\nabla}^{*c}_{e^c_j} K^c_0)(e^c_i, e^c_i) - (\ddot{\nabla}^{*c}_{e^c_i} K^c_0)(e^c_j, e^c_i) + K^c_0\big(e^c_j, K^c_0(e^c_i, e^c_i)\big) - K^c_0\big(e^c_i, K^c_0(e^c_j, e^c_i)\big), e^c_j \Big). \tag{88}$$

Proof. Treating $\check{S}^c(X^c_0, Y^c_0) = 0$ in Equations (83) and (84), then contracting it with respect to X^c_0 and Y^c_0 in the tangent bundle T_0M, we obtain the results. □

7. Example

In this section, we shall show an example of the lifts of a Sasakian statistical manifold in the tangent bundle T_0M. Let us consider M to be a three-dimensional manifold, which is defined as

$$M = \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^3; x_3 \neq 0 \Big\}, \tag{89}$$

where $\mathbb{R}$ is the set of real numbers. Let x_1, x_2, x_3 be given by

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = -x_1\Big(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\Big) + \frac{\partial}{\partial x_2}, \quad e_3 = \frac{1}{2}\frac{\partial}{\partial x_3},$$

where $\{e_1, e_2, e_3\}$ are the linearly independent global frame on M. Let T_0M be the tangent bundle, and let the one-form ω_0 be given by

$$\omega_0(X_0) = g_0(X_0, e_3).$$

The Riemannian metric g_0 is defined by

$$g_0(e_i, e_j) = \begin{cases} 1, & i = j = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Let Φ_0 be the tensor field defined by

$$\Phi_0 e_i = \begin{cases} -e_2, & i = 1 \\ e_1, & i = 2 \\ 0, & i = 3. \end{cases}$$

Using the linearity of Φ_0 and g_0, we acquire $\omega_0(e_3) = 1$, $\Phi_0^2 X_0 = -X_0 + \omega_0(X_0)e_3$ and $g_0(\Phi_0 X_0, \Phi_0 Y_0) = g_0(X_0, Y_0) - \omega_0(X_0)\omega_0(Y_0)$. Thus, for $e_3 = A$, the structure $(\Phi_0, A, \omega_0, g_0))$ is an almost-contact metric structure on M. In addition, M satisfies

$$(\widetilde{\nabla}_{X_0}\Phi_0)Y_0 = \omega_0(Y_0)X_0 - g_0(X_0, Y_0)e_3.$$

Thus, for $e_3 = A$, M is a Sasakian manifold.

Also, M satisfies

$$K_0(X_0, \widetilde{\nabla}_{Y_0} e_3) + \Phi_0 K_0(X_0, Y_0) = 0, \tag{90}$$

$$\ddot{\nabla}_{X_0}(\widetilde{\nabla}_{Y_0} e_3) - \Phi_0 \ddot{\nabla}^*_{X_0} Y_0 = g_0(Y_0, e_3)X_0 - g_0(Y_0, X_0)e_3, \tag{91}$$

$$\ddot{\nabla}_{X_0} e_3 = \widetilde{\nabla}_{X_0} e_3 + g_0(\ddot{\nabla}_{X_0} e_3, e_3)e_3. \tag{92}$$

Then, M is called a Sasakian statistical manifold.

In tangent bundle T_0M, let the complete and vertical lifts of e_1, e_2, e_3 be e^c_1, e^c_2, e^c_3 and e^v_1, e^v_2, e^v_3 on M, and let g^c_0 be the complete lift of the Riemannian metric g_0 on T_0M such that

$$g^c(X_0^v, e_3^c) = \Big(g^c(X_0, e_3)\Big)^v = \Big(\omega_0(X_0)\Big)^v \tag{93}$$

$$g^c(X_0^c, e_3^c) = \Big(g^c(X_0, e_3)\Big)^c = \Big(\omega_0(X_0)\Big)^c \tag{94}$$

$$g^c(e_3^c, e_3^c) = 1, \qquad g^v(X_0^v, e_3^c) = 0, \qquad g^v(e_3^v, e_3^c) = 0, \tag{95}$$

and so on. Let Φ_0^c and Φ_0^v be the complete and vertical lifts of the $(1,1)$ tensor field Φ_0 defined by

$$\Phi_0^v(e_3^v) = \Phi_0^c(e_3^c) = 0, \tag{96}$$

$$\Phi_0^v(e_1^v) = e_1^v, \qquad \Phi_0^c(e_1^c) = e_1^c, \tag{97}$$

$$\Phi_0^v(e_2^v) = e_2^v, \qquad \Phi_0^c(e_2^c) = e_2^c, \tag{98}$$

$$\Phi_0^v(e_3^v) = e_3^v, \qquad \Phi_0^c(e_3^c) = e_3^c. \tag{99}$$

Using the linearity of Φ_0 and g_0, we infer that

$$(\Phi_0^2 Y_0)^c = -Y_0^c + \omega_0^c(Y_0)e_3^v + \omega_0^v(Y_0)e_3^c, \tag{100}$$

$$g^c\Big((\Phi_0 e_3)^c, (\Phi_0 e_2)^c\Big) = g_0^c(e_3^c, e_2^c) - \omega_0^c(e_3^c)\omega_0^v(e_2^c) - \omega_0^v(e_3^c)\omega_0^c(e_2^c). \tag{101}$$

$$(\widetilde{\nabla}^c_{e_3^c}\Phi_0^c)e_2^c = \omega_0^c(e_2^c)e_3^v + \omega_0^v(e_2^c)e_3^c - g_0^c(e_3^c, e_2^c)A^v - g_0^c(e_3^v, e_2^c)A^c. \tag{102}$$

Also,

$$K_0^c(e_3^c, \widetilde{\nabla}^c_{e_2^c}A^c) + \Phi_0^c K_0^c(e_3^c, e_2^c) = 0, \tag{103}$$

$$\begin{aligned}\ddot{\nabla}^c_{e_3^c}(\widetilde{\nabla}^c_{e_2^c}A^c) - \Phi_0^c\ddot{\nabla}^{*c}_{e_3^c}e_2^c = g_0^c(e_2^c, A^c)e_3^v + g_0^c(e_2^v, A^c)e_3^c - g_0^c(e_2^c, e_3^c)A^c \\ - g_0^c(e_2^v, e_3^c)A^c,\end{aligned} \tag{104}$$

$$\ddot{\nabla}^c_{e_3^c}A^c = \widetilde{\nabla}^c_{e_3^c}A^c + g_0^c(A^c, \ddot{\nabla}^c_{e_3^c}A^c)A^v + g_0^c(A^v, \ddot{\nabla}^c_{e_3^c}A^c)A^v. \tag{105}$$

Then, for $e_3 = A$ in Equations (93)–(100), the structure $(\Phi_0^c, A^c, \omega_0^c, g_0^c)$ is an almost-contact metric structure on T_0M and satisfies the relation (102). Then, $(\Phi_0^c, A^c, \omega_0^c, g_0^c, T_0^M)$ is a Sasakian manifold. Also, this structure satisfies the relation (103)–(105) and is called a Sasakian statistical manifold.

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