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A New Distance-Type Fuzzy Inference Method Based on Characteristic Parameters

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Abstract: Reasoning is a cognitive activity that leverages knowledge to generate solutions to problems. Knowledge representations in the brain require both symbolic and graphical information since visual information is figurative and conveys a large amount of information. Consequently, graphical knowledge representation is often employed in reasoning. Distance-type fuzzy inference utilizes the distance information between the antecedent and the set of facts as the basis for inference. Compared to Mamdani inference, the distance-type fuzzy inference method not only satisfies the convexity and asymptotic properties of the inference results but also adheres to the separation rule (modus ponens), a fundamental principle in inference. This paper discusses extensions of distance-type fuzzy inference methods to handle spatial figures. In this paper, we first explain the distance-type fuzzy inference method. Then, we discuss the concept representation in the feature space and independent parameters that can completely express the characteristics of a figure in space, which are defined as "characteristic parameters". Furthermore, we describe the correspondence between figures and vectors in the feature space, propose a new distance-type fuzzy inference method based on characteristic parameters and describe its characteristics. Finally, an example is used to demonstrate the inference results of this new distance-type fuzzy inference method.

Keywords: characteristic parameters; feature space; distance-type fuzzy inference; convexity of inference results; separation rules; graphical information

MSC: 68T27; 68T37

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1. Introduction

Reasoning is a cognitive activity that leverages knowledge to generate solutions to problems [1,2]. There are various forms of knowledge expression, with graphical knowledge expression often employed in reasoning [3,4]. This is because visual information is figurative and offers a large amount of information for representing knowledge in the brain; hence, not only symbolic but also graphical information is important. In order to achieve more human-like reasoning, research on inference methods capable of handling graphical information has been recognized in the fields of artificial intelligence and linguistics; however, effective inference algorithms for this purpose have yet to be developed [5].

Fuzzy sets represent vague concepts and are characterized by membership functions [6–8]. By observing the shape of a fuzzy set drawn using a membership function, it is possible to visually recognize whether it is a triangle, trapezoid, or some other shape not easily expressible in words [9,10]. Ambiguous concepts can be quantified using graphical information, with the horizontal axis taking values in the closed interval [0, 1]. If we consider the figure drawn by the membership function as a general figure, the fuzzy inference method can be seen as an inference method for figures on the horizontal axis taking values in the closed interval [0, 1]. Recently, research has been carried out on the novel picture fuzzy set (PFS) [11].

The author previously proposed a distance-type fuzzy inference method based on the distance information between fuzzy sets [12]. Compared to Mamdani inference, distance-type fuzzy inference satisfies not only the convexity and asymptotic properties of the inference results but also the separation rule (modus ponens), a major principle in inference [13–15]. Consequently, AI systems constructed using distance-type fuzzy inference methods can be mathematically rigorous, as the validity of their antecedent-affirmative inference is guaranteed, and the inference results are fuzzy numbers, extensions of real numbers [16]. The latest applications of Mamdani fuzzy inference include intelligent adjustment of weather conditions [17], short-term traffic flow prediction [18], intelligent clinical decision support systems [19], improved information confidentiality [20], and improved aircraft controllers [21], etc. The effectiveness of distance-type fuzzy inference methods has been confirmed in applications, including, walking training using an omnidirectional robot [22], independent life support robot for the lower-limb handicapped [23], identification of the physiological needs of bedridden elderly [24], local path panning for construction robots [25], obstacle avoidance [26], etc. This paper discusses extensions of distance-type fuzzy inference methods that can handle not only fuzzy sets but also spatial figures.

On a plane or in a space, meanings and concepts can be represented through information about the shape and position of figures. Since the coordinate values of a figure can quantify a concept, they correspond to a membership function that acts as a quantification of an ambiguous concept. In other words, if mutually independent parameters can represent the properties of figures in planes and in spaces, then a set of figures whose elements are parameter values can be considered an extension of a fuzzy set. In this paper, independent parameters that can completely represent the properties of figures in spaces are called "characteristic parameters", and the Euclidean space consisting of the same number of independent variables as the characteristic parameters is called the "feature space". In this paper, we propose a distance-type fuzzy inference method based on the representation of concepts in the feature space.

First, we describe a distance-type fuzzy inference method [12] based on distance information between fuzzy sets. Next, we discuss the representation of concepts in the feature space. Furthermore, we describe the correspondence between figures and vectors in the feature space and propose a distance-type fuzzy inference method in the feature space for handling spatial figures. This inference method inherits the main features of distance-type fuzzy inference methods, allowing inference with less computational effort and application to vague concepts quantified using graphics. Finally, a real-world example is used to demonstrate the inference results of the distance-type fuzzy inference method in the feature space.

2. Distance-Type Fuzzy Inference Method

In this section, we elucidate the algorithm of the distance-type fuzzy inference method [12] along with its features.

We denote by R the whole set of real numbers and let $F(R)$ be the entire fuzzy set in R. It is important to note that when no subscript is added to R, $F(R)$ represents general objects without distinction. However, when different concepts are represented in a distinguishable manner, a corresponding subscript is added to the lower right of R. We denote by $\overline{F}(R)$ the entire bounded convex fuzzy set. We denote by $\overline{F}_n(R)$ the whole regular fuzzy set in $\overline{F}(R)$. That is, $\overline{F}_n(R)$ represents all bounded and normal convex fuzzy sets. Clearly, the relation, $\overline{F}_n(R) \subseteq \overline{F}(R) \subseteq F(R)$ exists. We consider n fuzzy rules $R^i (i = 1, 2, \ldots, n)$ for m antecedents and one consequent as

$$\begin{array}{c} R^i:\ x_1 = A^{i1}, x_2 = A^{i2}, \cdots, x_m = A^{im} \Rightarrow y = B^i \\ \text{Fact}:\ x_1 = A^1, x_2 = A^2, \cdots, x_m = A^m \\ \hline \text{Conclusion}: \qquad y = B \end{array} \tag{1}$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. A^{ij}, B^i, A^j and B represent the antecedent and consequent, given the fact and inference results, respectively, and $A^{ij} \in \overline{F}(R_{Aj}), B^i \in \overline{F_n}(R_B), A^j \in \overline{F}(R_{Aj})$. Additionally, an inference was made under the following condition:

In A^{ij}, A^j, and rules $R^1 \sim R^n$, there is no rule whose antecedents are exactly the same. That is, for $\forall q_1, q_2 \in \{1, 2, \ldots, n\}$, and $q_1 \neq q_2$,

$$\sum_{l=1}^{m} d(A^{q_1 l}, A^{q_2 l}) \neq 0$$

This condition is used to eliminate rules that contradict each other.

The distance-type fuzzy inference method comprises the following three steps.

STEP 1: Using the general Formula (A1) shown in the Appendix A, calculate $d_1 \sim d_n$ with Equation (2) using the distance $d_{ij}(A^{ij}, A^j)$ between the given A^{ij} and factual A^j. If the fuzzy sets are triangular or pi-type fuzzy sets, as is commonly used, the general formula in the Appendix A is simplified, and the distance between the fuzzy sets can be easily calculated [12].

$$d_i = \sum_{j=1}^{m} d_{ij}\left(A^{ij}, A^j\right) \; i = 1, 2, ., n. \tag{2}$$

The obtained distances $d_1 \sim d_n$ satisfy the axiom of distance. Furthermore, according to the inference condition, there is no case where two or more of $d_1 \sim d_n$ are 0.

STEP 2: Find the α-level set B_α for the inference result B as

$$B_\alpha = [inf(B_\alpha), sup(B_\alpha)] \tag{3}$$

$$inf(B_\alpha) = \frac{\sum_{i=1}^{n}\left[inf(B_\alpha^i) \prod_{j=1, j\neq i}^{n} d_j\right]}{\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} d_j} \tag{4}$$

$$sup(B_\alpha) = \frac{\sum_{i=1}^{n}\left[sup(B_\alpha^i) \prod_{j=1, j\neq i}^{n} d_j\right]}{\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} d_j} \tag{5}$$

where $sup(B_\alpha)$ is the upper limit of B_α and $inf(B_\alpha)$ is the lower limit. We write $sup(B_\alpha)$ as $\overline{B_\alpha}$ and $inf(B_\alpha)$ as $\underline{B_\alpha}$, except where this may cause confusion.

$$B_\alpha = \frac{\sum_{i=1}^{n}\left[B_\alpha^i \prod_{j=1, j\neq i}^{n} d_j\right]}{\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} d_j} = \left[\frac{\sum_{i=1}^{n}\left[\underline{B_\alpha^i} \prod_{j=1, j\neq i}^{n} d_j\right]}{\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} d_j}, \frac{\sum_{i=1}^{n}\left[\overline{B_\alpha^i} \prod_{j=1, j\neq i}^{n} d_j\right]}{\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} d_j}\right] \tag{6}$$

STEP 3: We obtain inference result B using the decomposition theorem below.

$$B = \bigcup_\alpha \alpha \cdot B_\alpha \tag{7}$$

Next, we describe the characteristics of distance-type fuzzy inference methods.

Feature 1: If $A^j = A^{qj}$ for $\exists q \in \{1, 2, \ldots, n\}$ and $\forall j \in \{1, 2, \ldots, m\}$, then $B = B^q$ holds for the inference result. That is, the distance-type fuzzy inference method satisfies the separation rule (modus ponens).

Feature 2: Inference result B from distance-type fuzzy inference is a regular fuzzy set.

Feature 3: The inference result B obtained from the distance-type fuzzy inference method is a convex fuzzy set.

Feature 4: Let B be the inference result when the distances are $d_1, d_2, \ldots, d_q, \ldots, d_n$, and B' be the inference result when the distance is $d_1, d_2, \ldots, d'_q, \ldots, d_n$. For $\forall k \in \{1, 2, \ldots, n\} - \{q\}$, assuming $d_k \neq 0$, if $d_q < d'_q$, then $d(B, B^q) < d(B', B^q)$ holds. However, B^q is the result of inference when $d_q = 0$.

This property means that the closer a given fact is to the antecedent of a rule, the closer the result of inference is to the consequent of that rule. Features 1 and 4 together are called the "asymptotic properties" of inference.

Feature 5: For $\forall i \in \{1, 2, \ldots, n\}$, if the consequent $B' \in \overline{F}_n(R_B)$ is a triangular fuzzy set, the inference result $B \in \overline{F}_n(R_B)$ is also a triangular fuzzy set.

3. Representation of Concepts in Feature Space

3.1. Quantification of Concepts Using Graphical Information

Fuzzy sets, delineating vague concepts, are defined by membership functions [6–8]. Ambiguous concepts are quantified through graphical information along a horizontal axis, with values spanning the closed interval [0, 1]. For instance, in Figure 1, the graphical representation of two triangles on a horizontal axis quantifies the ambiguous concepts of "medium" and "slightly large".

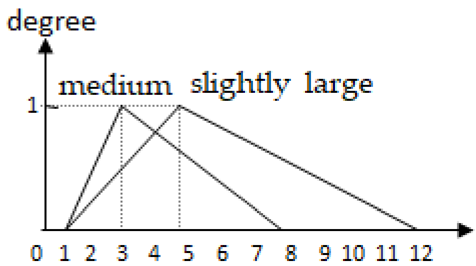

Figure 1. Fuzzy set of size.

On a plane or in a space, meanings and concepts can be conveyed through information about the shape and position of a figure, along with other graphical details [27,28]. For instance, concepts like "doll", "animal", "star", "building", "truck", and "airplane" can be visually represented by manipulating the coordinate values of line segments in a 20-sided polygon. The information within these figures, represented by the coordinates of the polygonal line segments, can then be utilized to quantify the aforementioned concepts. Therefore, the coordinate values of the line segments serve to quantify the concept and correspond to the function of the membership function used for quantifying ambiguous concepts. In essence, if the characteristics of a figure in a plane or space can be expressed through mutually independent parameters, a set of figures whose elements correspond to these parameter values can be considered an extension of a fuzzy set.

3.2. Characteristic Parameters and Feature Space

The triangle in Figure 1 represents a regular triangular fuzzy set that can be entirely characterized using three parameters $\left(p_A^1, p_A^2, p_A^3\right)$, as shown in Figure 2. No additional parameters are needed to articulate the features of this figure. Similarly, a polygon with 20 line segments can be fully expressed through the coordinates of its line segments, requiring 40 parameters $\left(p_D^1, \ldots, p_D^{40}\right)$. No extra parameters are necessary to convey the characteristics depicted in this figure. In other words, there exists a one-to-one relationship between the parameters $\left(p_A^1, p_A^2, p_A^3\right)$ and the regular triangular fuzzy set. Likewise, there is a one-to-one relationship between the parameters $\left(p_D^1, \ldots, p_D^{40}\right)$ and a polygon with 20 line segments.

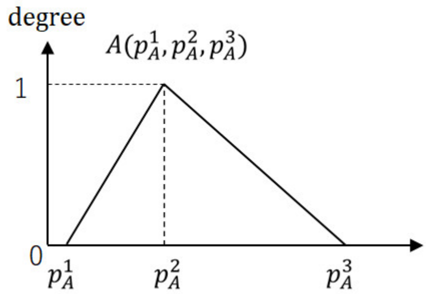

Figure 2. Regular triangular fuzzy set.

In this paper, parameters that are mutually independent and can fully characterize the features of a figure in a plane or in a space are termed "characteristic parameters". A Euclidean space with the same number of independent variables as characteristic parameters is referred to as a "feature space". When a figure representing a specific meaning is projected onto the feature space, it transforms into a singular vector. In other words, this vector can completely express the characteristics of the original figure. For instance, by substituting the values (1, 3, 8) and (1, 5, 12) for the characteristic parameters $\left(p_A^1, p_A^2, p_A^3\right)$, two triangles, A(1, 3, 8) and A(1, 5, 12), representing "medium" and "slightly large", respectively, are formed, as depicted in Figure 1. When these two triangles are mapped to the feature space, two vectors in the feature space, as illustrated in Figure 3, are generated. These vectors entirely capture the characteristics of A(1, 3, 8) and A(1, 5, 12). Although the feature space of a polygon with 20 line segments has more than four dimensions, making its vectors challenging to visually confirm, it possesses the essential properties of a vector and serves as a faithful representation of the original shape.

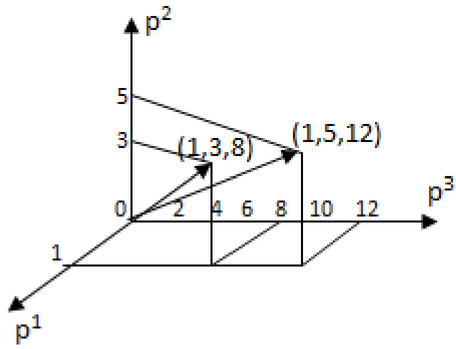

Figure 3. Regular triangular feature space.

Therefore, when a figure representing a specific concept is projected onto its feature space, it materializes as a single vector within the mapping space. This concept is symbolized by vectors in the feature space. A polygon with 20 line segments representing the concepts of "doll", "animal", "star", "building", "truck", and "airplane", as described above, transforms into vectors in the mapping space when projected onto the feature space. These vectors can effectively articulate the concepts of "doll", "animal", "star", "building", "truck", and "plane".

In general, a planar polygon with n line segments has $2n$ vertex coordinates. The planar polygon is completely represented using $2n$ independent parameters $\left(p_D^1, \ldots, p_D^{2n}\right)$. No additional parameters are needed to articulate the features of this plane polygon. That is, there is a one-to-one relationship between the parameters $\left(p_D^1, \ldots, p_D^{2n}\right)$ and a polygon with a plane polygon. Here, we define the parameters $\left(p_D^1, \ldots, p_D^{2n}\right)$ as characteristic parameters and the Euclidean space stretched by $2n$ variables as the feature space. The results of the analysis in the feature space exhibit an equivalent correspondence, as demonstrated in the figure. Therefore, depending on the problem, it is often more convenient to discuss the feature space rather than directly study the geometry itself.

4. Distance-Type Fuzzy Inference Method Based on Characteristic Parameters

Humans frequently resort to drawing diagrams during the process of reasoning to solve problems [29]. Drawing diagrams serves a crucial role not only in creating easily understandable representations but also in acquiring information that may not be explicitly presented in problems and in regulating inferences [30]. Consequently, graphical knowledge representation is often employed in reasoning [3,4]. In this section, we propose a distance-type fuzzy inference method based on characteristic parameters using distance information between figures. Since the fuzzy set represented by the membership function is a form of the figure, the distance-type fuzzy inference method in the feature space can be considered an extension of the method described in Section 2.

Here, we consider n inference rules for m antecedents and one consequent as follows:

$$\frac{\begin{array}{c} R^i: \ x_1 = C^{i1}, x_2 = C^{i2}, \cdots, x_m = C^{im} \Rightarrow y = D^i \\ Fact: \ x_1 = C^1, x_2 = C^2, \cdots, x_m = C^m \end{array}}{Conclusion: \qquad y = D} \tag{8}$$

where $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$. The antecedent C^{ij}, consequent D^i, fact C^j and conclusion D are sets of figures that are expressed by their characteristic parameters as follows:

$$\begin{array}{c} C^{ij} = \left(p^1_{C^{ij}}, p^2_{C^{ij}}, \ldots, p^{n_{C^j}}_{C^{ij}}\right) \\ D^i = \left(p^1_{D^i}, p^2_{D^i}, \ldots, p^{n_D}_{n^i}\right) \\ C^j = \left(p^1_{C^j}, p^2_{C^j}, \ldots, p^{n_{C^j}}_{C^j}\right) \\ D = \left(p^1_D, p^2_D, \ldots, p^{n_D}_D\right) \end{array}$$

Let $F_{C^j}\{p^1, p^2, \ldots, p^{n_{C^j}}, R^{n_{C^j}}\}$ be the entire set of figures in the n_{C^j}-dimensional feature space $R^{n_{C^j}}$ and $F_D\{p^1, p^2, \ldots, p^{n_D}, R^{n_D}\}$ be the entire set of figures in the n_D-dimensional feature space R^{n_D}. The antecedent figure set C^{ij}, fact figure set C^j, consequent figure set D^i and conclusion figure set D are the vectors in the feature space, as follows. In other words, the number of sides of the j-th fact figure set C^j and j-th antecedent figure set C^{ij} are the same, and the number of sides of the consequent figure set D^i and conclusion set D are the same.

$$\begin{array}{c} C^j, C^{ij} \in F_{C^j}\{p^1, p^2, \ldots, p^{n_{C^j}}, R^{n_{C^j}}\} \\ D^i, D \in F_D\{p^1, p^2, \ldots, p^{n_D}, R^{n_D}\} \\ i = 1,2,\ldots,n, \ j = 1,2,\ldots,m. \end{array}$$

Accordingly, the distance-type fuzzy inference method in the feature space comprises the following three steps:

STEP 1: In the feature space $R^{n_{C^j}}$, we calculate the distance $d\left(C^{ij}, C^j\right)$ between the j-th fact variable figure set C^j and the j-th antecedent variable figure set C^{ij} for the i-th rule, where $1 \le p < \infty$ and $|\cdot|$ denotes the absolute value.

$$d\left(C^{ij}, C^j\right) = \left[\sum_{q=1}^{n_{C^j}} \left|\left(p^q_{C^{ij}} - p^q_{C^j}\right)\right|^p\right]^{\frac{1}{p}} \tag{9}$$

STEP 2: Using $d\left(C^{ij}, C^j\right)$, we calculate the distance $d_1 \sim d_n$ between the fact and each rule using (10).

$$d_k = \frac{\prod\limits_{i=1, i\neq k}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)}{\sum\limits_{s=1}^{n} \prod\limits_{i=1, i\neq s}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)} \tag{10}$$

where $k = 1,2,\ldots,n$.

STEP 3: We calculate the set of shapes D of the inference result in feature space using Equation (11).

$$D = [PH]^{\mathrm{T}} \tag{11}$$

where

$$H = (d_1, d_2, \ldots, d_n)^{\mathrm{T}}, \quad P = \begin{pmatrix} p^1_{D^1}, p^1_{D^2}, \cdots, p^1_{D^n} \\ p^2_{D^1}, p^2_{D^2}, \cdots, p^2_{D^n} \\ \cdots, \cdots, \cdots, \cdots \\ p^{n_D}_{D^1}, p^{n_D}_{D^2}, \cdots, p^{n_D}_{D^n} \end{pmatrix}$$
$$i = 1, 2, \ldots, n, \ \mathrm{T}: \ \text{transposition}$$

Column i $\left(p^1_{D^i}, p^2_{D^i}, \ldots, p^{n_D}_{D^i}\right)$ in matrix P represents the vector of the consequent D_i of the i-th rule, and $(d_1, d_2, \ldots, d_n)$ represents the distance between the fact and each rule.

Because the process of the distance-type fuzzy inference method in the feature space relies on computations between vectors in the feature space, there is no need to employ a composition theorem, such as the one presented in Equation (7). When applying the distance-type fuzzy inference method in the feature space to ordinary fuzzy sets, the computational complexity is reduced compared to the method described in Section 2.

The distance-type fuzzy inference method of [12] is only applicable to fuzzy sets. The new distance-type fuzzy inference method proposed in this paper, which is also based on characteristic parameters, can be applied not only to fuzzy sets but also to shapes. However, the shape is not an arbitrary shape but only a shape that can be represented by a characteristic parameter. This inference method computes inferences in a characteristic space. The characteristic space is a multidimensional Euclidean space stretched by characteristic parameters.

5. Characteristics of This Inference Method

This section discusses the characteristics of distance-type fuzzy inference methods in the feature space.

Theorem 1. *The characteristic parameter representing the set of figures D resulting from the inference method is bounded. Explicitly, for* $\forall q \in \{1, 2, \ldots, n_D\}$, *if*

$$p_{qmin} := \min\left\{p^q_{D^1}, p^q_{D^2}, \ldots, p^q_{D^n}\right\}$$
$$p_{qmax} := \max\left\{p^q_{D^1}, p^q_{D^2}, \ldots, p^q_{D^n}\right\}$$

then Equation (12) holds.

$$p_{q\mathrm{min}} \le p^q_D \le p_{q\mathrm{max}} \tag{12}$$

Proof. For $\forall q \in \{1, 2, \ldots, n_D\}$, Inequality (13) is obtained from Equations (10) and (11).

$$\begin{aligned} p^q_D &= \sum_{k=1}^{n} p^q_{D^k} \frac{\prod\limits_{i=1, i\neq k}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)}{\sum\limits_{s=1}^{n} \prod\limits_{i=1, i\neq s}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)} \\ &\le p_{q\mathrm{max}} \sum_{k=1}^{n} \frac{\prod\limits_{i=1, i\neq k}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)}{\sum\limits_{s=1}^{n} \prod\limits_{i=1, i\neq s}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)} = p_{q\mathrm{max}} \end{aligned} \tag{13}$$

Similarly, Inequality (14) is obtained.

$$p_D^q \geq p_{q\min} \sum_{k=1}^{n} \frac{\prod\limits_{i=1, i \neq k}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)}{\sum\limits_{s=1}^{n} \prod\limits_{i=1, i \neq s}^{n} \sum\limits_{j=1}^{m} d\left(C^{ij}, C^j\right)} = p_{q\min} \tag{14}$$

Therefore, Inequality (12) holds true. □

Theorem 2. *If for* $\exists k \in \{1, 2, \ldots, n\}$ *and* $\forall j \in \{1, 2, \ldots, m\}$, $C^j = C^{kj}$, *then for* $\forall q \in \{1, 2, \ldots, n_D\}$, *the equation* $P_D^q = P_{D^k}^q$ *holds true. That is, the inference result is* $D = D^k$, *which satisfies the separation rule (modus ponens).*

Proof. From $C^j = C^{kj}$, we obtain $d_s = 0$ and $d_k = 0$ for $\forall s \in \{1, 2, \ldots, n\} - \{k\}$. Therefore, from Equation (11) it can be seen that $\forall q \in \{1, 2, \ldots, n_D\}$ and $\vec{P_q^0} = \vec{P_q^k}$, that is, $D = D^k$. □

Theorem 3. *For the inference result, if* $\exists k \in \{1, 2, \ldots, n\}$ *and* $D = D^k$, *then if vectors* $(P_{D^1} - P_{D^k})$, $(P_{D^2} - P_{D^k})$, …, $(P_{D^{k-1}} - P_{D^k})$, $(P_{D^{k+1}} - P_{D^k})$, …, $(P_{D^n} - P_{D^k})$ *are linearly independent, then for* $\forall j \in \{1, 2, \ldots, m\}$, *a given fact* $C^j = C^{kj}$ *must be true.*

Proof. $D = D^k$ is equivalent to the following equation.

$$P_D = \left(p_{D^k}^1, p_{D^k}^2, \ldots, p_{D^k}^{n_D}\right) \tag{15}$$

By substituting Equation (15) into Equation (11) and rearranging it, we get

$$\sum_{s=1, s \neq k}^{n} (P_{D^s} - P_{D^k}) \prod_{i=1, i \neq s}^{n} \sum_{j=1}^{m} d\left(C^{ij}, C^j\right) = 0 \tag{16}$$

Furthermore, because $(P_{D^1} - P_{D^k})$, $(P_{D^2} - P_{D^k})$, …, $(P_{D^{k-1}} - P_{D^k})$, $(P_{D^{k+1}} - P_{D^k})$, …, $(P_{D^n} - P_{D^k})$ are linearly independent vectors, and Equation (17) holds true from Equation (16).

$$\prod_{i=1, i \neq s}^{n} \sum_{j=1}^{m} d\left(C^{ij}, C^j\right) = 0 \tag{17}$$

Additionally, because there are no mutually contradictory rules, we obtain

$$\sum_{j=1}^{m} d\left(C^{kj}, C^j\right) = 0 \tag{18}$$

By the distance axiom, we can see that the equation $C^j = C^{kj}$ holds for $\forall j \in \{1, 2, \ldots, m\}$. □

Theorems 1–3 denote characteristics of distance-type fuzzy inference methods in the feature space applicable to all general spatial figures. When applied to fuzzy sets, this inference method exhibits the following two characteristics.

Theorem 4. *If the consequents* $D^1 \sim D^n$ *are all regular fuzzy sets, then the result* D *deduced using this inference method is also a regular fuzzy set.*

Proof. In Equation (11), let the q-th element p_D^q in the vector representing the inference result D represent the maximum value of the membership function $\mu_D(y)$ of D. If the consequents $D^1 \sim D^n$ are regular fuzzy sets, then $\forall k \in \{1, 2, \ldots, n\}$, $p_{D^k}^q = 1$ holds. Therefore, using Equation (11), $p_D^q = 1$. □

Theorem 5. *If the consequents $D^1 \sim D^n$ are all convex fuzzy sets, then the result D deduced using this inference method is also a convex fuzzy set.*

Proof. If all consequents are convex fuzzy sets, then the vectors must satisfy particular topological relations in the feature space. Without loss of generality, by reordering, we assume the following topological relationship:

$$P^1_{D^k} \leq P^2_{D^k} \leq \ldots \leq P^{n_B-1}_{D^k} \leq P^{n_B}_{D^k} k = \{1, 2, \ldots, n\} \tag{19}$$

Using Equation (11), the q_k-th element p^q_D and the $(q+1)$-th element p^{q+1}_D in the vector expressing the inference result D can be expressed as

$$p^q_D = \sum_{k=1}^{n} d_k p^q_{D^k} \tag{20}$$

$$\begin{gathered} p^{q+1}_D = \sum_{k=1}^{n} d_k p^{q+1}_{D^k} \\ q = 1, 2, \ldots, (n_D - 1) \end{gathered} \tag{21}$$

From Inequality (19), Equations (20) and (21) result in

$$p^{q+1}_D - p^q_D = \sum_{k=1}^{n} d_k \left(p^{q+1}_{D^k} - p^q_{D^k} \right) \geq 0 \tag{22}$$

That is, because the elements in the vector expressing the inference result D satisfy the following inequality, D is also a convex fuzzy set.

$$p^1_D \leq p^2_D \leq \ldots \leq p^{n_D-1}_D \leq p^{n_D}_D$$

□

6. Specific Example

In this section, we use a specific example to show the inference results of the distance-type fuzzy inference method in a feature space.

The Mamdani-type inference method uses the maximum membership of the product set as the basis for inference; the product set of two sets is not applicable to the case of an empty set. The distance-type fuzzy inference method uses distance information between two sets as the basis for inference, so it can be applied even if the intersection set is empty or not. This paper is an extension of the distance-type fuzzy inference method to shapes, and it can be applied to fuzzy sets and shapes that are far from each other. The following example is for figures that are far from each other.

Consider the following case: one triangular figure input, one pentagonal figure output and three rules (Figure 4):

$$\begin{array}{c} R^1 : x = \text{triangle } C^1 \;\Rightarrow\; y = \text{pentagon } D^1 \\ R^2 : x = \text{triangle } C^2 \;\Rightarrow\; y = \text{pentagon } D^2 \\ R^3 : x = \text{triangle } C^3 \;\Rightarrow\; y = \text{pentagon } D^3 \\ \text{Fact} : x = \text{triangle } C \\ \hline \text{Conclusion} : y = \text{pentagon } D \end{array}$$

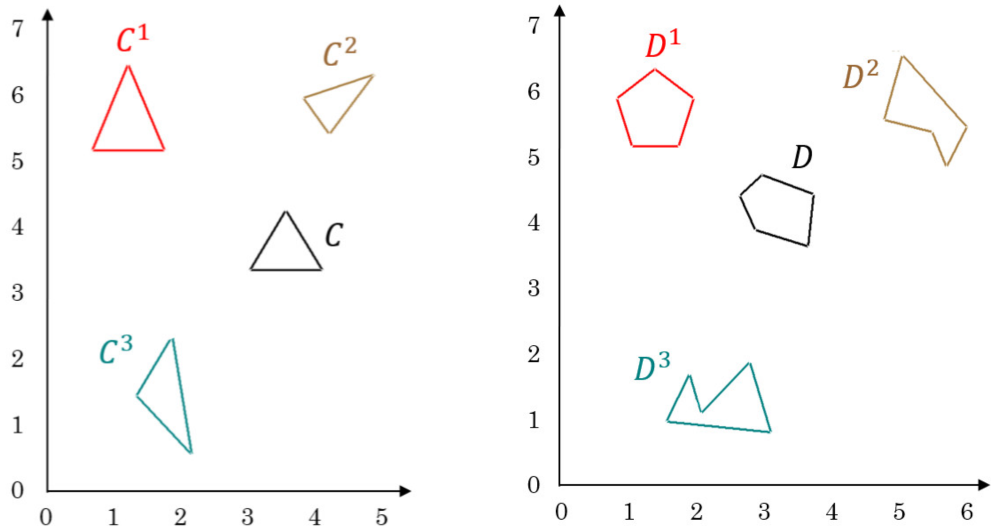

Figure 4. One triangular input, one pentagonal output and three rules of inference.

In this scenario, the fact figure C is moved towards the antecedent figure C^3 and superimposed, so that the two have exactly the same shape. As the result of the inference, the set of figures D approaches and overlaps the pentagon D^3 through the movement of the fact figure C, as shown in Figures 5–7, and becomes exactly the same shape.

$$\begin{array}{c}R^1: x = \text{triangle } C^1 \ \Rightarrow \ y = \text{pentagon } D^1 \\ R^2: x = \text{triangle } C^2 \ \Rightarrow \ y = \text{pentagon } D^2 \\ R^3: x = \text{triangle } C^3 \ \Rightarrow \ y = \text{pentagon } D^3 \\ \text{Fact}: x = \mathit{Triangle\ C\ approaching}\ C^3 \\ \hline \text{Conclusion}: y = \mathit{Pentagon\ D\ approaching}\ D^3\end{array}$$

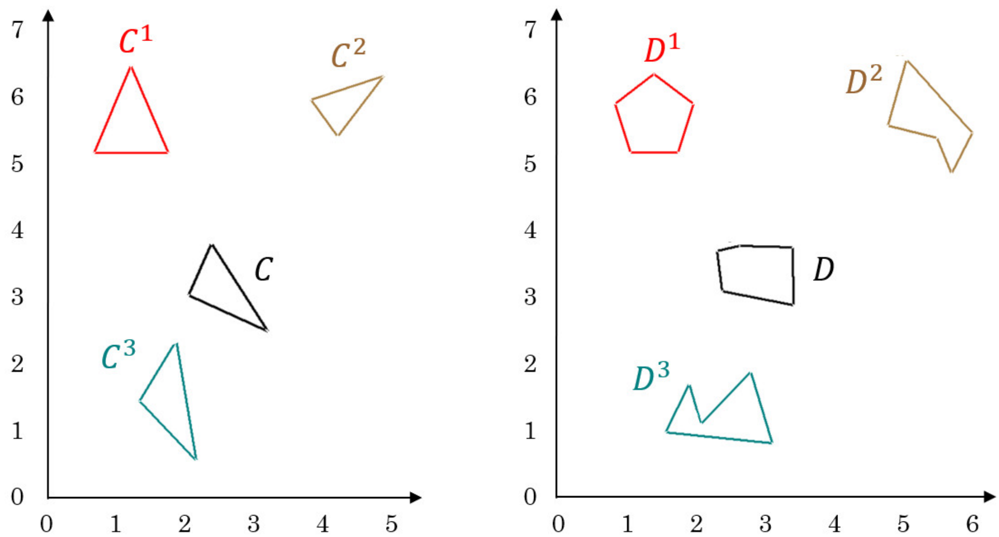

Figure 5. Inference results when fact C approaches antecedent C^3.

$$\begin{array}{c}R^1: x = \text{triangle } C^1 \ \Rightarrow \ y = \text{pentagon } D^1 \\ R^2: x = \text{triangle } C^2 \ \Rightarrow \ y = \text{pentagon } D^2 \\ R^3: x = \text{triangle } C^3 \ \Rightarrow \ y = \text{pentagon } D^3 \\ \text{Fact}: x = \mathit{Triangle\ C\ superimposed\ on}\ C^3 \\ \hline \text{Conclusion}: y = \mathit{Pentagon\ D\ superimposed\ on}\ D^3\end{array}$$

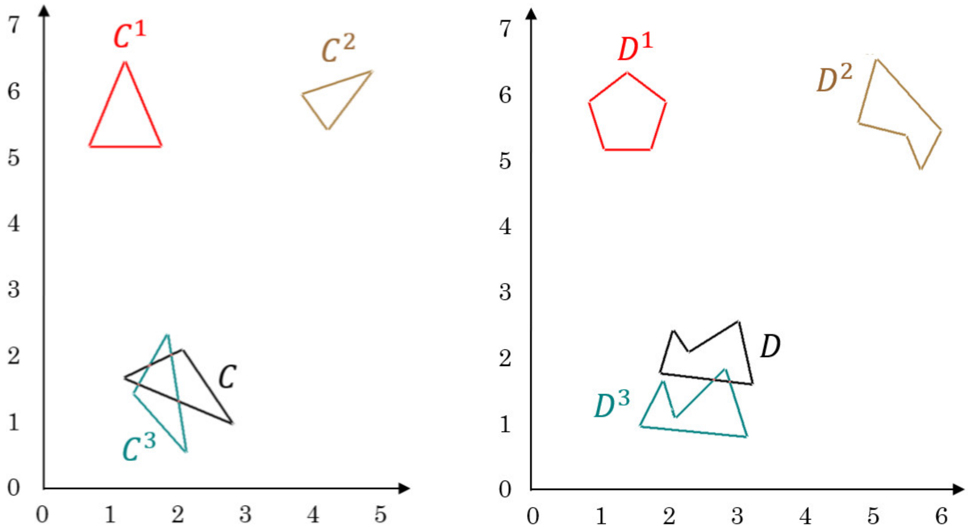

Figure 6. Inference results when fact C is superimposed on the antecedent C^3.

$$\begin{array}{c} R^1 : x = \text{triangle } C^1 \;\Rightarrow\; y = \text{pentagon } D^1 \\ R^2 : x = \text{triangle } C^2 \;\Rightarrow\; y = \text{pentagon } D^2 \\ R^3 : x = \text{triangle } C^3 \;\Rightarrow\; y = \text{pentagon } D^3 \\ \text{Fact} : x = C^3 \textit{ and the same shape as triangle } C \\ \hline \text{Conclusion} : y = D^3 \textit{ and the same pentagon } D \end{array}$$

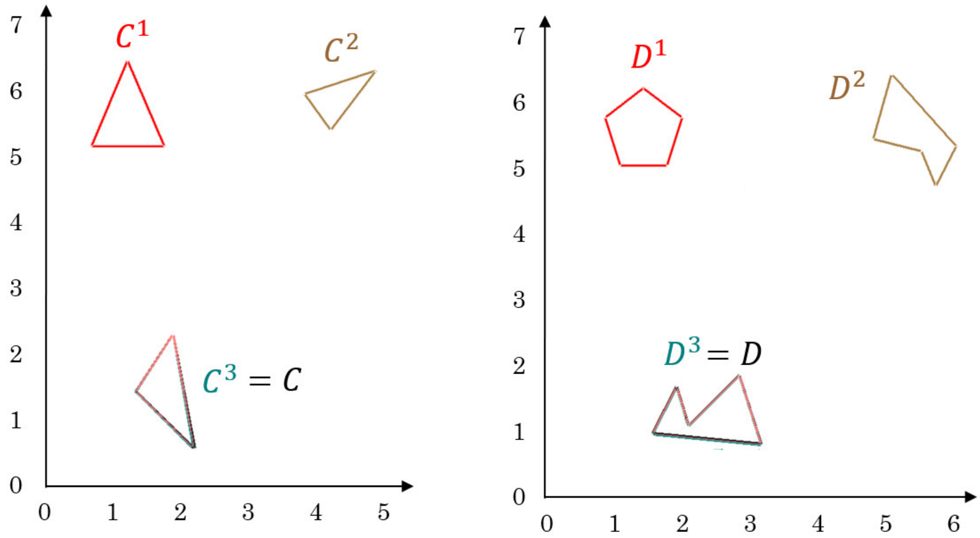

Figure 7. Inference result when fact C is the same as the antecedent C^3.

Compared to Mamdani inference, distance-type fuzzy inference satisfies not only the convexity and asymptotic properties of the inference results but also the separation rule (modus ponens), a major principle in inference. The new distance-type fuzzy inference method proposed in this paper inherits the above characteristics. From the above example, we can confirm that the results of this inference method satisfy the separation rule and asymptotic properties.

7. Conclusions

This paper discusses the extensions of distance-type fuzzy inference methods to handle spatial figures. To represent knowledge in the brain, both symbolic and graphical information is crucial, as graphical information is figurative and conveys a substantial amount of information. On a plane or in a space, meanings and concepts can be expressed using information about the shape, position of a figure, and other graphical details. The coordinate values of a figure play a crucial role in quantifying a concept, serving as a membership function for ambiguous concepts. In this paper, we initially explored the quantification of concepts using graphical information and the representation of concepts in feature spaces. Subsequently, we introduced a distance-based fuzzy inference method in the feature space, utilizing distance information between figures and the correspondence between figures and vectors in the feature space. We then further delineated the characteristics of this inference method. Finally, an example was employed to demonstrate the inference results of the distance-type fuzzy inference method based on characteristic parameters.

Compared to Mamdani inference, distance-type fuzzy inference satisfies not only the convexity and asymptotic properties of the inference results but also the separation rule (modus ponens), a major principle in inference. The new distance-type fuzzy inference method proposed in this paper inherits the above characteristics. Furthermore, since the Mamdani-type inference method uses the maximum membership of the product set as the basis for inference, the product set of two sets is not applicable to the case of an empty set. The distance type fuzzy inference method uses distance information between two sets as the basis for inference, so it can be applied whether the intersection set is empty or not. This paper is an extension of the distance-type fuzzy inference method to shapes; it can be applied to fuzzy sets and shapes that are far from each other.

This inference method retains the key characteristic of distance-type fuzzy inference methods while necessitating lower computational complexity. Possible applications of this inference method include automatic generation of figures, elucidation of brain mechanisms of figure processing processes, production of animations and generation of facial expressions.

This inference method can only be applied to figures that can be represented using a finite number of feature parameters. In other words, the limitation of this method is that it cannot be applied to figures that cannot be represented by a finite number of feature parameters. In addition, in actual expert and other AI system configurations, there may be a huge number of rules and factual uncertainties. In this case, tactics such as rule prioritization are necessary.

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Appendix A. Method for Calculating the Distance Between Fuzzy Sets [12]

For any fuzzy set $A, B \subset \overline{F}(R)$, the real-valued function $d(A, B)$ defined by the following equation, is a distance function of $\overline{F}(R)$.

$$\begin{aligned} d(A,B) := &\left[\int_0^1 \left| infA_{M\alpha} - infB_{M\alpha} \right|^p d\alpha\right]^{\frac{1}{p}} \\ &+ \left[\int_0^1 \left| supA_{M\alpha} - supB_{M\alpha} \right|^p d\alpha\right]^{\frac{1}{p}} \\ &+ \left[\int_0^1 \left| \left(\frac{1}{M_A} - 1\right)\mu_A(x) - \left(\frac{1}{M_B} - 1\right)\mu_B(x) \right|^p dx\right]^{\frac{1}{p}} \end{aligned} \quad (A1)$$

where $1 \leq p < \infty$ and $| \cdot |$ denotes the absolute value. A_M denotes the fuzzy set normalized by the maximum value M_A of the membership function of A. $supA_{M\alpha}$ and $infA_{M\alpha}$ represent the upper and lower bounds of the α-level set $A_{M\alpha}$ of the fuzzy set A_M, respectively. The same is true for the B_M.

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