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Riemann Integral on Fractal Structures

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Abstract: In this work we start developing a Riemann-type integration theory on spaces which are equipped with a fractal structure. These topological structures have a recursive nature, which allows us to guarantee a good approximation to the true value of a certain integral with respect to some measure defined on the Borel σ-algebra of the space. We give the notion of Darboux sums and lower and upper Riemann integrals of a bounded function when given a measure and a fractal structure. Furthermore, we give the notion of a Riemann-integrable function in this context and prove that each μ-measurable function is Riemann-integrable with respect to μ. Moreover, if μ is the Lebesgue measure, then the Lebesgue integral on a bounded set of $\mathbb{R}^n$ meets the Riemann integral with respect to the Lebesgue measure in the context of measures and fractal structures. Finally, we give some examples showing that we can calculate improper integrals and integrals on fractal sets.

Keywords: Riemann integral; Riemann-integrable; fractal structure; measure

MSC: 54E15; 28A25; 28A80

1. Introduction

Fractal structures were introduced in [1] to study non-Archimedean quasimetrization, although it is true that they have a wide range of applications. Some of them can be found in [2] and include metrization, topological and fractal dimension, filling curves, completeness, transitive quasi-uniformities, and inverse limits of partially ordered sets.

One of the most recent applications of fractal structures is to construct a probability measure by taking advantage of their recursive nature. For some reference on this topic, we refer the reader to [3,4]. The idea of this construction is defining a pre-measure on the elements of a fractal structure or some topological structures induced by it, so that, from several sufficient conditions and characterizations, we have extended that pre-measure to a probability measure on the Borel σ-algebra of the space. Indeed, the authors proved that each probability measure defined in a space with a fractal structure can be constructed by following the procedure mentioned.

On the other hand, the classical theory of Riemann-type integration starts from a bounded function on a compact rectangle of $\mathbb{R}^n$ and a collection of almost disjoint compact sets whose union is the said rectangle, which we refer to by the name partition. From the function and the partition, the lower and upper Darboux sums are defined, and by taking the supremum and the infimum of these sums over all the possible partitions, we get the lower and upper Riemann integrals, respectively. Section 2.3 recalls, in more detail, some different basic results and notions of this theory. Talking about a partition in the environment of the calculation of Riemann-type integrals suggests considering a fractal structure so that, based on each of its levels, we can obtain the lower and upper sums. Consequently, it will make sense to talk about a Riemann integral, although the volume can be replaced by a measure that is defined in the σ-algebra of the space in which we are working. Furthermore, it makes sense to think that considering a higher level of the fractal structure can guarantee a better approximation to the true value of the integral. Thus, interest arises in studying the application of fractal structures to the development of

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a Riemann-type integration theory, with respect to a certain measure defined in the space, and that is the main objective of this work. For this purpose, we first give the notion of Darboux sums with respect to a measure and a fractal structure in Section 3. After that, we introduce the notion of a Riemann-integrable function with respect to a measure and a fractal structure on a certain space in Section 4 and prove a Riemann theorem in this context (see Section 5). Moreover, in Section 6 we prove that if a bounded function is Riemann-integrable, its integral does not depend on the chosen fractal structure, so we just have to refer to the measure defined on the σ-algebra of the space. It is also shown that the integral introduced coincides with the Riemann integral in $\mathbb{R}^n$ and with the Lebesgue integral with respect to the measure. In the last section, we show some examples to illustrate this theory. Finally, it is worth highlighting that in the literature there are already other works related to the calculation of Riemann-type integrals on other types of spaces. For example, in [5–8].

2. Preliminaries

2.1. Fractal Structures

Despite being introduced in [1] for a topological space, fractal structures can be defined in a set, and this will be the definition we use in this work, as has been used previously in other works.

First, recall that a cover Γ_2 is a strong refinement of another cover Γ_1, written as $\Gamma_2 \prec\prec \Gamma_1$, if Γ_2 is a refinement of Γ_1 (that is, each element of Γ_2 is contained in some element of Γ_1), denoted by $\Gamma_2 \prec \Gamma_1$, and for each $B \in \Gamma_1$, it holds that $B = \bigcup\{A \in \Gamma_2 : A \subseteq B\}$ (equivalently, for each $B \in \Gamma_1$ and $x \in B$ there exists $A \in \Gamma_2$ such that $x \in A \subseteq B$). The definition of a fractal structure is as follows.

Definition 1. *A fractal structure on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\Gamma_{n+1} \prec\prec \Gamma_n$. The cover Γ_n is called the level n of the fractal structure.*

A fractal structure is said to be finite if each level is a finite covering.

In what follows, we introduce two simple examples of fractal structures. The first is defined in $[0,1]$ and its levels are given by $\Gamma_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}] : k = 0, \ldots, 2^n - 1\}$ for each $n \in \mathbb{N}$. Note that the previous fractal structure is finite (since it has a finite number of elements at each level). However, if we consider the Euclidean space $\mathbb{R}$, it is defined as the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_n = \{[\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}] : k \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$. In both cases, $\mathbf{\Gamma}$ is known as the natural fractal structure.

A fractal structure induces (see [1]) a transitive base of a quasi-uniformity given by $\{U_{\Gamma_n} : n \in \mathbb{N}\}$, where $U_\Gamma = \{(x,y) \in X \times X : y \notin \bigcup\{A \in \Gamma : x \notin A\}\}$ for each cover Γ.

If $\mathbf{\Gamma}$ is a fractal structure on a set X and $A \subseteq X$, the fractal structure induced on A (see [1]) is defined as $\mathbf{\Gamma}_A = \{\Gamma_n^A : n \in \mathbb{N}\}$, where $\Gamma_n^A = \{B \cap A : B \in \Gamma_n\}$ for each $n \in \mathbb{N}$.

2.2. Measure Theory

Now we recall some definitions related to measure theory from [9]. Let X be a set, then there are several classes of sets of X. If $\mathcal{R}$ is a non-empty collection of subsets of X, we say that $\mathcal{R}$ is a ring if it is closed under complement and finite union. Furthermore, given $\mathcal{Q}$ is a non-empty collection of subsets of X, it is said to be an algebra if it is a ring such that $X \in \mathcal{Q}$. Moreover, a non-empty collection of subsets of X, $\mathcal{A}$, is a σ-algebra if it is closed under complement and countable union and $X \in \mathcal{A}$. If $\mathcal{A}$ is a σ-algebra on X, then the pair $(X, \mathcal{A})$ is called a measurable space.

For a given topological space, (X, τ), $\mathcal{B} = \sigma(\tau)$ is the Borel σ-algebra of the space, that is, it is the σ-algebra generated by the open sets of X.

A set mapping is said to be σ-additive if $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for each countable collection $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$.

Definition 2 ([9], Section 7)**.** *Given a measurable space* $(\Omega, \mathcal{A})$*, a measure* μ *is a non-negative and* σ*-additive set mapping defined in* $\mathcal{A}$ *such that* $\mu(\emptyset) = 0$*. The triple* $(\Omega, \mathcal{A}, \mu)$ *is called a measure space.*

A measure is monotonic (which means that if $A, B \in \mathcal{A}$ are such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$). It is also continuous from below: if (A_n) is a monotonically non-decreasing sequence of sets (which means that $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$), then $\mu(A_n) \to \mu(\bigcup_{n\in\mathbb{N}} A_n)$. Moreover, it is continuous from above: if (A_n) is monotonically non-increasing (which means that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$) and $\mu(A_1) < \infty$, then $\mu(A_n) \to \mu(\bigcap_{n\in\mathbb{N}} A_n)$. Finally, it is sub-$\sigma$-additive (which means that $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$ for each countable collection $\{A_n\}_{n=1}^{\infty}$).

2.3. Riemann Integration Theory

In this subsection, we base on [10] in order to give a generalization of the n-dimensional Riemann integration theory.

A compact interval in the n-dimensional Euclidean space $\mathbb{R}^n$ is a product $J = [a_1, b_1] \times \ldots \times [a_n, b_n]$ where $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$ for each $i = 1, \ldots, n$. A partition $\mathcal{D}$ of this interval is an n-tuple $(\mathcal{D}_1, \ldots, \mathcal{D}_n)$ where $\mathcal{D}_i$ is a partition of $[a_i, b_i]$ for each $i = 1, \ldots, n$, that is, a sequence $t_{i1}, t_{i2}, \ldots, t_{in}$ such that $a_i = t_{i1} < t_{i2} < \ldots < t_{in} = b_i$, which can also be seen as the sequence of compact intervals $[a_i, t_{i2}], [t_{i2}, t_{i3}], \ldots, [t_{in_{i-1}}, b_i]$. The partition norm is defined as $||\mathcal{D}|| = \max\{t_{ik+1} - t_{ik} : k = 1, \ldots, n_i; i = 1, \ldots, n\}$.

The partition $\mathcal{D}$ is called a refinement of a partition $\mathcal{D}' = (\mathcal{D}'_1, \ldots, \mathcal{D}'_n)$ if the sequences on $\mathcal{D}$ are subsequences of the sequences $a_i = t'_{i1} < t'_{i2} < \ldots < t'_{in} = b_i$. Note that two partitions always have a common refinement.

Moreover, we can define the volume of an interval $J = [a_1, b_1] \times \ldots \times [a_n, b_n]$ as the number $\mathrm{vol}(J) = \prod_{i=1}^{n}(b_i - a_i)$. Let f be a bounded function on an interval J and let D be a partition of J. The lower and upper Darboux sums of f in D are defined, respectively, by

$$s(f; \mathcal{D}) = \sum_{K\in|\mathcal{D}|} m_K \cdot \mathrm{vol}(K) \quad \text{and} \quad S(f; \mathcal{D}) = \sum_{K\in|\mathcal{D}|} M_k \cdot \mathrm{vol}(K)$$

where

$$m_K = \inf\{f(x) : x \in K\} \quad \text{and} \quad M_k = \sup\{f(x) : x \in K\}$$

and $|\mathcal{D}|$ denotes the family of all sets of the partition $\mathcal{D}$. Note that if $\mathcal{D}$ is a refinement of $\mathcal{D}'$, then $s(f; \mathcal{D}) \geq s(f; \mathcal{D}')$ and $S(f; \mathcal{D}) \leq S(f; \mathcal{D}')$ and, if we consider a common refinement, it can be proved that $s(f, \mathcal{D}) \leq S(f; \mathcal{D}')$ for each pair of partitions $\mathcal{D}$ and $\mathcal{D}'$. Now, we recall the definition of the lower and upper Riemann integrals of f over J.

Definition 3. *Let* f *be a bounded function on an interval* $J = [a_1, b_1] \times \ldots \times [a_n, b_n]$ *and* $\mathcal{D}$ *be a partition of* J*. Then the lower and upper sums of* f *over* J *are defined, respectively, by*

$$\underline{\int_J} f = \sup_{\mathcal{D}} s(f; \mathcal{D}) \quad \textit{and} \quad \overline{\int_J} f = \inf_{\mathcal{D}} S(f; \mathcal{D})$$

and, in case that both values coincide, we refer to that number by the name of the Riemann integral of f *over* J *and denote it by*

$$\int_J f$$

Two of the most well-known theorems in the classical theory of Riemann integral are the following ones:

Theorem 1. *A function* f *is Riemann-integrable if and only if for each* $\varepsilon > 0$*, there exists a partition* $\mathcal{D}$ *such that*

$$S(f; \mathcal{D}) - s(f; \mathcal{D}) < \varepsilon$$

A selection for a partition $\mathcal{D}$ is a collection of points $\xi = (x_D)_{D\in\mathcal{D}}$ such that $x_D \in D$ for each $D \in \mathcal{D}$. The Riemann sum for a function f relative to a partition $\mathcal{D}$ and a selection $\xi = (x_D)_{D\in\mathcal{D}}$ is defined as $S(f;\mathcal{D};\xi) := \sum_{D\in\mathcal{D}} f(x_D)\cdot \mathrm{vol}(D)$.

The next theorem is sometimes referred to as Riemann's theorem (see, for example, [Th. 7.1.11] [11]).

Theorem 2. *A function f is Riemann-integrable if and only if there exists a number $L \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|S(f;\mathcal{D};\xi) - L| < \varepsilon$ for each partition $\mathcal{D}$ with $||\mathcal{D}|| < \delta$ and for each selection $\xi = (x_D)_{D\in\mathcal{D}}$ for $\mathcal{D}$. Moreover, if f is Riemann-integrable, then $L = \int f$.*

3. Darboux Sums with Respect to a Measure and a Fractal Structure

In this section, we see how to define the Darboux sums from a measure defined on a space with a fractal structure. This measure plays a similar role to that played by the Lebesgue measure in the classical theory of Riemann integrals when defining Darboux sums. For that purpose, we first need to give some conditions on the fractal structure we define on the space.

Definition 4. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathbf{\Gamma}$ be a fractal structure on X. $\mathbf{\Gamma}$ is said to be μ-disjoint if the following conditions hold:*

1. *$\Gamma_n \subseteq \mathcal{S}$ is countable for each $n \in \mathbb{N}$.*
2. *$\mu(B \cap J) = 0$ for each $B, J \in \Gamma_n$ such that $B \neq J$ and each $n \in \mathbb{N}$.*
3. *$\mu(A) < \infty$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.*

Darboux sums are defined for each level of a fractal structure in a space as follows:

Definition 5. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure, and $f : X \to \mathbb{R}$ be a bounded function. Then, for each $J \in \Gamma_n$, we set*

$$m(f;J) = \inf\{f(x) : x \in J\}$$

$$M(f;J) = \sup\{f(x) : x \in J\}$$

so that the lower and upper Darboux sums with respect to μ for each level of the fractal structure are given by

$$L(f;\Gamma_n,\mu) = \sum_{J\in\Gamma_n} m(f;J)\mu(J)$$

and

$$U(f;\Gamma_n,\mu) = \sum_{J\in\Gamma_n} M(f;J)\mu(J)$$

respectively, when the series are absolutely convergent.

Next, we see that the first condition in Definition 4 allows us to calculate both the Darboux sums and the measure of each element used in them, while the second condition means that overlapping is not a problem.

Proposition 1. *Let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ and $\mathbf{\Gamma}^* = \{\Gamma_n^* : n \in \mathbb{N}\}$ be two fractal structures on X and $\overline{\mathbf{\Gamma}} = \mathbf{\Gamma} \vee \mathbf{\Gamma}^* = \{\overline{\Gamma}_n : n \in \mathbb{N}\}$ the family given by $\overline{\Gamma}_n = \{B \cap J : B \in \Gamma_n, J \in \Gamma_n^*\}$ for each $n \in \mathbb{N}$. Then $\overline{\Gamma}_n \prec\prec \Gamma_n, \Gamma_n^*$ for each $n \in \mathbb{N}$ and $\overline{\mathbf{\Gamma}}$ is a fractal structure.*

Proof. First, we prove that $\overline{\Gamma}_n \prec\prec \Gamma_n$. Given $A \in \overline{\Gamma}_n$, then there exist $B \in \Gamma_n$ and $C \in \Gamma_n^*$ such that $A = B \cap C$. It is clear that $A \subseteq B$ and hence $\overline{\Gamma}_n \prec \Gamma_n$. On the other hand, given $A \in \Gamma_n$ and $x \in A$, since Γ_n^* is a covering, there exists $B \in \Gamma_n^*$ such that $x \in B$ and therefore $x \in A \cap B \subseteq A$ and $A \cap B \in \overline{\Gamma}_n$. It follows that $\overline{\Gamma}_n \prec\prec \Gamma_n$. Analogously, it can be shown that $\overline{\Gamma}_n \prec\prec \Gamma_n^*$.

Now we see that $\overline{\mathbf{\Gamma}}$ is a fractal structure on X, that is, it is a countable family of coverings of X such that $\overline{\Gamma}_{n+1} \prec\prec \overline{\Gamma}_n$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Given $x \in X$, there exist $B \in \Gamma_n$ and $J \in \Gamma_n^*$ such that $x \in B$ and $x \in J$, since Γ_n and Γ_n^* are both coverings of X. Hence, $x \in B \cap J \in \overline{\Gamma}_n$, which means that $\overline{\Gamma}_n$ is a covering of X.

On the other hand, let $H = B \cap J \in \overline{\Gamma}_{n+1}$ be such that $B \in \Gamma_{n+1}$ and $J \in \Gamma_{n+1}^*$. Then there exist $B' \in \Gamma_n$ such that $B \subset B'$ and $J' \in \Gamma_n^*$ such that $J \subset J'$. Therefore, $B \cap J \subset B' \cap J' \in \overline{\Gamma}_n$ and hence $\overline{\Gamma}_{n+1} \prec \overline{\Gamma}_n$.

Finally, let $H' = B' \cap J' \in \overline{\Gamma}_n$ be such that $B' \in \Gamma_n$ and $J' \in \Gamma_n^*$ and $x \in H' = B' \cap J'$. Since $\Gamma_{n+1} \prec\prec \Gamma_n$ and $\Gamma_{n+1}^* \prec\prec \Gamma_n^*$, there exist $B \in \Gamma_{n+1}$ such that $x \in B \subseteq B'$ and $J \in \Gamma_{n+1}^*$ such that $x \in J \subseteq J'$. It follows that $x \in B \cap J \subseteq B' \cap J' = H'$ and $B \cap J \in \overline{\Gamma}_n$. Therefore, $\overline{\Gamma}_{n+1} \prec\prec \overline{\Gamma}_n$, and $\overline{\mathbf{\Gamma}}$ is a fractal structure. $\square$

Remark 1. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathbf{\Gamma}$ and $\mathbf{\Gamma}^*$ be two μ-disjoint fractal structures on X. Then $\mathbf{\Gamma} \vee \mathbf{\Gamma}^*$ is a μ-disjoint fractal structure on X.*

Proof. If $A, B \in \Gamma_n$ and $C, D \in \Gamma_n^*$ then $(A \cap C) \cap (B \cap D) \subseteq A \cap B$ and $(A \cap C) \cap (B \cap D) \subseteq C \cap D$. If $A \cap C \neq B \cap D$ then $A \neq B$ or $C \neq D$ and hence $\mu(A \cap B) = 0$ or $\mu(C \cap D) = 0$, since $\mathbf{\Gamma}$ and $\mathbf{\Gamma}^*$ are μ-disjoint. It follows that $\mu((A \cap C) \cap (B \cap D)) = 0$ by the monotonicity of the measure. $\square$

Lemma 1. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathbf{\Gamma}$ be a μ-disjoint fractal structure on X. Then:*

1. *$\mu(A \cup B) = \mu(A) + \mu(B)$ for each $A, B \in \Gamma_n$ such that $A \neq B$.*
2. *Given k different elements of Γ_n, $A_1, ..., A_k$, then $\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$.*
3. *Let $\{A_i : i \in \mathbb{N}\}$ be a countable family of different elements of Γ_n. Then $\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.*

Proof.

1. Let $A, B \in \Gamma_n$ with $A \neq B$. Then we can write $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$. Since $\mu(A \cap B) = 0$ by hypothesis, it follows that $\mu(A \cup B) = \mu(A) + \mu(B)$.
2. Let $A_1, \ldots, A_k \in \Gamma_n$ be such that they are all different. Reasoning by induction on k, we prove that $\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$ for each $k \in \mathbb{N}$. For $k = 1$, it is clear. Suppose that the equality holds for a certain $k \in \mathbb{N}$. Let us see that it also holds for $k + 1$: First, we have $\bigcup_{i=1}^{k+1} A_i = \left(\bigcup_{i=1}^{k} A_i\right) \bigcup A_{k+1}$. Moreover, the induction hypothesis lets us write

$$\mu\left(\left(\bigcup_{i=1}^{k} A_i\right) \bigcup A_{k+1}\right) + \mu\left(\left(\bigcup_{i=1}^{k} A_i\right) \bigcap A_{k+1}\right) = \mu\left(\bigcup_{i=1}^{k} A_i\right) + \mu(A_{k+1}) = \sum_{i=1}^{k+1} \mu(A_i).$$

The fact that μ is sub-σ-additive implies that

$$\mu\left(\left(\bigcup_{i=1}^{k} A_i\right) \bigcap A_{k+1}\right) = \mu\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right) \leq \sum_{i=1}^{k} \mu(A_i \cap A_{k+1})$$

and the fact that $\mathbf{\Gamma}$ is μ-disjoint means that $\sum_{i=1}^{k} \mu(A_i \cap A_{k+1}) = 0$. Consequently, $\mu\left(\left(\bigcup_{i=1}^{k} A_i\right) \bigcap A_{k+1}\right) = 0$.

If we join all the previous equalities, we conclude that

$$\mu\left(\bigcup_{i=1}^{k+1} A_i\right) = \mu\left(\left(\bigcup_{i=1}^{k} A_i\right) \bigcup A_{k+1}\right) = \sum_{i=1}^{k+1} \mu(A_i).$$

3. Let $\{A_i : i \in \mathbb{N}\}$ be a countable family of different elements of Γ_n. Since $\bigcup_{i=1}^{k} A_i \overset{k\to\infty}{\to} \bigcup_{i\in\mathbb{N}} A_i$ and μ is continuous from below, it holds that

$$\sum_{i=1}^{k} \mu(A_i) = \mu\left(\bigcup_{i=1}^{k} A_i\right) \overset{k\to\infty}{\to} \mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

where we have taken into account the previous item in the first equality.

□

The next proposition gathers some relationships between both Darboux sums with respect to two μ-disjoint fractal structures defined on a space.

Proposition 2. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ and $\mathbf{\Gamma}^* = \{\Gamma_n^* : n \in \mathbb{N}\}$ be two μ-disjoint fractal structures on X. Let $f : X \to \mathbb{R}$ be a bounded function and $n, m \in \mathbb{N}$. Then:*

1. $L(f;\Gamma_n;\mu) \leq U(f;\Gamma_n;\mu)$.
2. *If $\Gamma_m^* \prec\prec \Gamma_n$, then $L(f;\Gamma_n;\mu) \leq L(f;\Gamma_m^*;\mu) \leq U(f;\Gamma_m^*;\mu) \leq U(f;\Gamma_n;\mu)$. In particular, it holds that $L(f;\Gamma_n;\mu) \leq L(f;\Gamma_m;\mu) \leq U(f;\Gamma_m;\mu) \leq U(f;\Gamma_n;\mu)$, if $n \leq m$.*
3. $L(f;\Gamma_n;\mu) \leq U(f;\Gamma_m^*;\mu)$.

Proof.

1. It is clear since $m(f;J) \leq M(f,J)$ and $\mu(J) \geq 0$ for each $J \in \Gamma_n$.
2. Since $\Gamma_m^* \prec\prec \Gamma_n$, if $J \in \Gamma_n$, then we have that:
   (a) $\mu(J) = \mu(\bigcup\{H \in \Gamma_m^* : H \subseteq J\}) = \sum_{H\in\Gamma_m^*, H\subseteq J} \mu(H)$ by Lemma 1.
   (b) If $H \in \Gamma_m^*$ and $H \subseteq J$, then $m(f;J) \leq m(f;H) \leq M(f;H) \leq M(f;J)$.
   (c) Each $H \in \Gamma_m^*$ such that $\mu(H) \neq 0$ is contained in exactly one $J \in \Gamma_n$.

   For the proof of (c), note that H is contained in some $J \in \Gamma_n$, since $\Gamma_m^* \prec \Gamma_n$. Suppose that $H \subset J_1, J_2$, where $J_1, J_2 \in \Gamma_n$ with $J_1 \neq J_2$. Then $H \subset J_1 \cap J_2$ and since μ is monotonic and $\mu(J_1 \cap J_2) = 0$ (because the fractal structure is μ-disjoint), we have $\mu(H) = 0$, a contradiction.

   Item (a) lets us write

$$L(f;\Gamma_n;\mu) = \sum_{J\in\Gamma_n} m(f;J)\mu(J) = \sum_{J\in\Gamma_n} m(f;J)\left[\sum_{H\in\Gamma_m^*, H\subset J} \mu(H)\right]$$

   and, by item (b), it follows that

$$\sum_{J\in\Gamma_n} m(f;J)\left[\sum_{H\in\Gamma_m^*, H\subset J} \mu(H)\right] \leq \sum_{J\in\Gamma_n} \sum_{H\in\Gamma_m^*, H\subset J} m(f;H)\mu(H)$$

   Now, by item (c),

$$\sum_{J\in\Gamma_n} \sum_{H\in\Gamma_m^*, H\subset J} m(f;H)\mu(H) = \sum_{H\in\Gamma_m^*} m(f;H)\mu(H) = L(f;\Gamma_m^*;\mu)$$

and, by the first item, it holds that

$$L(f;\Gamma_m^*;\mu) \le U(f;\Gamma_m^*;\mu) = \sum_{H\in\Gamma_m^*} M(f;H)\mu(H)$$

Now we use item (c), so that

$$\sum_{H\in\Gamma_m^*} M(f;H)\mu(H) = \sum_{J\in\Gamma_n}\sum_{H\in\Gamma_m^*, H\subset J} M(f;H)\mu(H)$$

and item (b) lets us write

$$\sum_{J\in\Gamma_n}\sum_{H\in\Gamma_m^*, H\subset J} M(f;H)\mu(H) \le \sum_{J\in\Gamma_n} M(f;J)\left[\sum_{H\in\Gamma_m^*, \mu(H)>0, H\subset J}\mu(H)\right]$$

Finally, item (a) means that

$$\sum_{J\in\Gamma_n} M(f;J)\left[\sum_{H\in\Gamma_m^*, \mu(H)>0, H\subset J}\mu(H)\right] = \sum_{J\in\Gamma_n} M(f;J)\mu(J) = U(f;\Gamma_n;\mu).$$

3. Let $\overline{\mathbf{\Gamma}} = \mathbf{\Gamma} \vee \mathbf{\Gamma}^*$ and $p = \max\{n, m\}$. Then $\overline{\Gamma}_p \prec\prec \Gamma_n, \Gamma_m^*$. Note that $\overline{\mathbf{\Gamma}}$ is μ-disjoint by Remark 1. By the previous items, we have that

$$L(f;\Gamma_n;\mu) \overset{(2)}{\le} L(f;\overline{\Gamma}_p;\mu) \overset{(1)}{\le} U(f;\overline{\Gamma}_p;\mu) \overset{(2)}{\le} U(f;\Gamma_m^*;\mu).$$

□

We can also observe the next result from the previous proposition.

Remark 2. *Under the hypothesis of the previous proposition, it follows that* $L(f;\Gamma_n;\mu) \le L(f;\Gamma_{n+1};\mu) \le U(f;\Gamma_{n+1};\mu) \le U(f;\Gamma_n;\mu)$ *for each* $n \in \mathbb{N}$*, which means that* $\lim_n U(f;\Gamma_n;\mu) = \inf\{U(f;\Gamma_n;\mu) : n \in \mathbb{N}\}$ *and* $\lim_n L(f;\Gamma_n) = \sup\{L(f;\Gamma_n) : n \in \mathbb{N}\}$.

4. Riemann Integral with Respect to a Measure and a Fractal Structure

Once we know how to define the lower and upper Darboux sums when given a bounded function, a measure μ, and a μ-disjoint fractal structure on a space X, the next step is defining the lower and upper Riemann integrals with respect to the measure and the fractal structure. Moreover, we can give some conditions so that both integrals coincide.

Definition 6. *Let* $(X, \mathcal{S}, \mu)$ *be a measure space,* $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ *be a* μ*-disjoint fractal structure on* X*, and* $f : X \to \mathbb{R}$ *be a bounded function. We define the lower and upper Riemann integrals of* f *with respect to* μ *and* $\mathbf{\Gamma}$ *on* X *as follows:*

1. *Upper Riemann integral of* f *with respect to* μ *and* $\mathbf{\Gamma}$*:*

$$\overline{\int_X}^{(\mu,\mathbf{\Gamma})} f := \inf\{U(f;\Gamma_n;\mu) : n \in \mathbb{N}\} = \lim_n U(f;\Gamma_n;\mu).$$

2. *Lower Riemann integral of* f *with respect to* μ *and* $\mathbf{\Gamma}$*:*

$$\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f := \sup\{L(f;\Gamma_n;\mu) : n \in \mathbb{N}\} = \lim_n L(f;\Gamma_n;\mu).$$

Remark 3. *Note that, by Proposition 2,* $\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f \le \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$.

Definition 7. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X and $f : X \to \mathbb{R}$ be a bounded function. f is said to be Riemann-integrable with respect to μ and $\mathbf{\Gamma}$ on X if $\overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$ is finite and $\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$.*

If f is Riemann-integrable with respect to μ and $\mathbf{\Gamma}$ on X, we define the Riemann integral of f with respect to μ and $\mathbf{\Gamma}$ on X, $\int_X^{(\mu,\mathbf{\Gamma})} f$, by

$$\int_X^{(\mu,\mathbf{\Gamma})} f = \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f.$$

We denote by $R(X; \mu; \mathbf{\Gamma})$ the set of Riemann-integrable functions with respect to μ and $\mathbf{\Gamma}$ on X.

Remark 4. *If $\mu(X) = 0$, then $R(X; \mu; \mathbf{\Gamma}) = \{f : X \to \mathbb{R} : f \text{ is bounded}\}$ and $\int_X^{(\mu,\mathbf{\Gamma})} f = 0$ for each $f \in R(X; \mu; \mathbf{\Gamma})$.*

Proposition 3. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X and $f : X \to \mathbb{R}$ be a bounded function. The following statements are equivalent:*

1. *$f \in R(X; \mu; \mathbf{\Gamma})$.*
2. *For each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $U(f; \Gamma_n; \mu) - L(f; \Gamma_n; \mu) \leq \varepsilon$.*
3. *For each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $U(f; \Gamma_n; \mu) - L(f; \Gamma_n; \mu) \leq \varepsilon$ for each $n \geq n_0$.*

Proof. $(1 \Leftrightarrow 3)$ By definition of Riemann integral, we have that

$$\begin{aligned} f \in R(X; \mu; \mathbf{\Gamma}) &\Leftrightarrow \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f \\ &\Leftrightarrow \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f - \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = 0 \\ &\Leftrightarrow \lim_n U(f; \Gamma_n; \mu) - \lim_n L(f; \Gamma_n; \mu) = 0 \end{aligned} \tag{1}$$

what is equivalent, in terms of convergence, to claim that for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $U(f; \Gamma_n; \mu) - L(f; \Gamma_n; \mu) \leq \varepsilon$ for each $n \geq n_0$.

$(2 \Rightarrow 3)$ Suppose that for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $U(f; \Gamma_{n_0}; \mu) - L(f; \Gamma_{n_0}; \mu) \leq \varepsilon$. Let $n \geq n_0$. Then, by Proposition 2, it follows that $U(f; \Gamma_n; \mu) - L(f; \Gamma_n; \mu) \leq U(f; \Gamma_{n_0}; \mu) - L(f; \Gamma_{n_0}; \mu) \leq \varepsilon$.

$(3 \Rightarrow 2)$ It is immediate. □

Note that the third condition in the previous proposition is equivalent to

$$\lim_{n \to \infty} (U(f; \Gamma_{n_0}; \mu) - L(f; \Gamma_{n_0}; \mu)) = 0$$

Corollary 1. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, $\mathbf{\Gamma}^* = \{\Gamma_n^* : n \in \mathbb{N}\}$ be two μ-disjoint fractal structures on X such that $\Gamma_n^* \prec\prec \Gamma_n$ for each $n \in \mathbb{N}$. Then $R(X; \mu, \mathbf{\Gamma}) \subseteq R(X; \mu, \mathbf{\Gamma}^*)$.*

Proof. Let $f \in R(X; \mu, \mathbf{\Gamma})$. By Proposition 3, $\lim_{n\to\infty}(U(f; \Gamma_n; \mu) - L(f; \Gamma_n; \mu)) = 0$. By Proposition 2, $L(f; \Gamma_n; \mu) \leq L(f; \Gamma_n^*; \mu) \leq U(f; \Gamma_n^*; \mu) \leq U(f; \Gamma_n; \mu)$ for each $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty}(U(f; \Gamma_n^*; \mu) - L(f; \Gamma_n^*; \mu)) = 0$ and hence, by Proposition 3 again, $f \in R(X; \mu, \mathbf{\Gamma}^*)$. □

5. Riemann Theorem for Fractal Structures

In what follows, we prove a theorem which is analogous to the Riemann theorem in $\mathbb{R}^n$, but for bounded functions defined on a space with a μ-disjoint fractal structure. This is one of the main results of this work.

Definition 8. *Let $\mathbf{\Gamma}$ be a fractal structure on a space X such that Γ_n is countable for each $n \in \mathbb{N}$. A selection for Γ_n is a collection of points $\xi := (x_A)_{A \in \Gamma_n}$ such that $x_A \in A$ for each $A \in \Gamma_n$.*

Definition 9. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X and $f : X \to \mathbb{R}$ be a bounded function. Let $n \in \mathbb{N}$ and $\xi = (x_A)_{A \in \Gamma_n}$ be a selection for Γ_n. The Riemann sum for f relative to Γ_n, ξ and μ is denoted by $S(f; \Gamma_n; \xi; \mu)$ and is defined as follows:*

$$S(f; \Gamma_n; \xi; \mu) := \sum_{A \in \Gamma_n} f(x_A)\mu(A).$$

Theorem 3 (Riemann's Theorem)**.** *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X, $f : X \to \mathbb{R}$ be a bounded function and $C \in \mathbb{R}$. The following statements are equivalent:*

1. *$f \in R(X; \mu; \mathbf{\Gamma})$ and $\int_X^{(\mu,\mathbf{\Gamma})} f = C$.*
2. *Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|C - S(f; \Gamma_n; \xi_n; \mu)| < \varepsilon$ for each $n \geq n_0$ and each selection for Γ_n, ξ_n.*
3. *Given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|C - S(f; \Gamma_n; \xi; \mu)| < \varepsilon$ for each selection for Γ_n, ξ.*
4. *$S(f; \Gamma_m; \xi_m; \mu) \overset{m \to \infty}{\longrightarrow} C$ for each sequence (ξ_m) such that ξ_m is a selection for Γ_m for each $m \in \mathbb{N}$.*

Proof. $(1 \Rightarrow 2)$ Suppose that $f \in R(X; \mu; \mathbf{\Gamma})$ and $\int_X^{(\mu,\mathbf{\Gamma})} f = C$. Then

$$\int_X^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \lim_n U(f; \Gamma_n; \mu) = \lim_n L(f; \Gamma_n; \mu) = C.$$

Let $\varepsilon > 0$. By Proposition 3, there exists $n_0 \in \mathbb{N}$ such that $U(f; \Gamma_m; \mu) - L(F; \Gamma_m; \mu) < \varepsilon$ for each $m \geq n_0$.

Now let $m \geq n_0$. Note that $U(f; \Gamma_m; \mu) \geq S(f; \Gamma_m; \xi_n; \mu) \geq L(f; \Gamma_m; \mu)$ for each selection for Γ_n, ξ_n. Moreover, since $\lim_n U(f; \Gamma_n; \mu) = \lim_n L(f; \Gamma_n; \mu) = C$, by Proposition 2 it holds that $U(f; \Gamma_m; \mu) \geq C \geq L(f; \Gamma_m; \mu)$.

Suppose that ξ_m is a selection for Γ_m and $m \geq n_0$. It follows that

$$|C - S(f; \Gamma_m; \xi_m; \mu)| \leq U(f; \Gamma_m; \mu) - L(F; \Gamma_m; \mu) < \varepsilon$$

$(3 \Rightarrow 1)$ It is enough to prove that $\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f = C$. Let $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $|C - S(f; \Gamma_n; \xi; \mu)| < \frac{\varepsilon}{2}$ for each ξ, selection for Γ_n. We distinguish two cases:

1. $\Gamma_n = \{A_1, ..., A_{s_n}\}$ is finite. Let $\xi_{n,1} = (y_{A_1}, ..., y_{A_{s_n}})$, $\xi_{n,2} = (z_{A_1}, ..., z_{A_{s_n}})$ be two selections for Γ_n such that

$$|M(f; A) - f(y_A)|\mu(A) \leq \frac{\varepsilon}{2s_n} \quad \text{for each} A \in \Gamma_n,$$
$$|m(f; A) - f(z_A)|\mu(A) \leq \frac{\varepsilon}{2s_n} \quad \text{for each} A \in \Gamma_n.$$

Then we have that

$$|U(f; \Gamma_n; \mu) - S(f; \Gamma_n; \xi_{n,1})| \leq \sum_{i=1}^{s_n} \frac{\varepsilon}{2s_n} = \frac{\varepsilon}{2},$$
$$|L(f; \Gamma_n; \mu) - S(f; \Gamma_n; \xi_{n,2})| \leq \sum_{i=1}^{s_n} \frac{\varepsilon}{2s_n} = \frac{\varepsilon}{2}.$$

2. $\Gamma_n = \{A_i : i \in \mathbb{N}\}$ is infinitely countable. Let $\xi_{n,1} = (y_{A_i})_{A_i \in \Gamma_n}$, $\xi_{n,2} = (z_{A_i})_{A_i \in \Gamma_n}$ be two selections for Γ_n such that

$$|M(f;A_i) - f(y_{A_i})|\mu(A_i) \leq \frac{\varepsilon}{2^{i+1}} \quad \text{for each } i \in \mathbb{N},$$
$$|m(f;A_i) - f(z_{A_i})|\mu(A_i) \leq \frac{\varepsilon}{2^{i+1}} \quad \text{for each } i \in \mathbb{N}.$$

Then we have that

$$|U(f;\Gamma_n;\mu) - S(f;\Gamma_n;\xi_{n,1})| \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2},$$
$$|L(f;\Gamma_n;\mu) - S(f;\Gamma_n;\xi_{n,2})| \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2}.$$

Hence, in both cases, we can write

$$|U(f;\Gamma_n;\mu) - C| = |U(f;\Gamma_n;\mu) - S(f;\Gamma_n;\xi_{n,1}) + S(f;\Gamma_n;\xi_{n,1}) - C| \leq$$
$$\leq |U(f;\Gamma_n;\mu) - S(f;\Gamma_n;\xi_{n,1})| + |S(f;\Gamma_n;\xi_{n,1}) - C| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies that

$$\overline{\int_X}^{(\mu,\mathbf{\Gamma})} f \leq U(f;\Gamma_n) < C + \varepsilon.$$

What is more,

$$|C - L(f;\Gamma_n;\mu)| = |C - S(f;\Gamma_n;\xi_{n,2}) + S(f;\Gamma_n;\xi_{n,2}) - L(f;\Gamma_n;\mu)| \leq$$
$$\leq |C - S(f;\Gamma_n;\xi_{n,2})| + |S(f;\Gamma_n;\xi_{n,2}) - L(f;\Gamma_n;\mu)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that

$$\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f \geq L(f;\Gamma_n;\mu) > C - \varepsilon$$

It follows that

$$C - \varepsilon < \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f \leq \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f < C + \varepsilon$$

for each $\varepsilon > 0$. The arbitrariness of $\varepsilon > 0$ leads us to conclude that $f \in R(X;\mu;\mathbf{\Gamma})$ and $\int_X^{(\mu,\mathbf{\Gamma})} f = C = \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$.

$(2 \Leftrightarrow 4)$ It is immediate.

$(2 \Rightarrow 3)$ It is immediate.
□

6. Riemann Integral with Respect to a Measure

The next result allows us to claim that the Riemann integral of a bounded function with respect to a measure and a fractal structure, in fact, does not depend on the fractal structure.

Proposition 4. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ and $\mathbf{\Gamma}^* = \{\Gamma_n^* : n \in \mathbb{N}\}$ be two μ-disjoint fractal structures on X and $f : X \to \mathbb{R}$ be a bounded function. If $f \in R(X;\mu;\mathbf{\Gamma})$ and $f \in R(X;\mu;\mathbf{\Gamma}^*)$, then $\int_X^{(\mu,\mathbf{\Gamma})} f = \int_X^{(\mu,\mathbf{\Gamma}^*)} f$.*

Proof. Let $C = \int_X^{(\mu,\mathbf{\Gamma})} f$, and $D = \int_X^{(\mu,\mathbf{\Gamma}^*)} f$, and suppose that $C < D$. Then

$$\lim_n U(f;\Gamma_n;\mu) = \lim_n L(f;\Gamma_n;\mu) = C < D = \lim_n U(f;\Gamma_n^*;\mu) = \lim_n L(f;\Gamma_n^*;\mu).$$

Then

$$\lim_n U(f;\Gamma_n;\mu) < \lim_n L(f;\Gamma_n^*;\mu),$$

which is a contradiction with Proposition 2, since $U(f;\Gamma_n;\mu) \geq L(f;\Gamma_n^*;\mu)$ for each $n \in \mathbb{N}$. $\square$

Therefore, if a bounded function is Riemann-integrable with respect to a measure, μ, and different μ-disjoint fractal structures, then all the integrals have the same value. Therefore, it makes sense to introduce the following concept:

Definition 10. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a bounded function. f is said to be μ-Riemann-integrable if there exists a μ-disjoint fractal structure $\mathbf{\Gamma}$ on X such that f is Riemann-integrable on X with respect to μ and $\mathbf{\Gamma}$. Moreover, if so, the integral is defined as*

$$\int_X^{\mu} f = \int_X^{(\mu,\mathbf{\Gamma})} f.$$

From now on, $R(X;\mu)$ will denote the set of all bounded functions that are μ-Riemann-integrable on X.

The proof of the following result is straightforward.

Lemma 2. *Let $\mathbf{\Gamma}$ be a fractal structure on a set Y, X be a set and $f : X \to Y$ be a map. Then $f^{-1}(\mathbf{\Gamma}) = \{f^{-1}(\Gamma_n) : n \in \mathbb{N}\}$ is a fractal structure on X, where $f^{-1}(\Gamma_n) = \{f^{-1}(A) : A \in \Gamma_n\}$ for each $n \in \mathbb{N}$.*

Once we know that the Riemann integral does not depend on the chosen fractal structure, we give some sufficient conditions to ensure that a function is Riemann-integrable with respect to a measure.

Proposition 5. *Let $(X, \mathcal{S}, \mu)$ be a finite measure space and $f : X \to \mathbb{R}$ be a bounded measurable function. Then $f \in R(X;\mu)$ and $\int_X^{\mu} f = \int f d\mu$.*

Proof. Let $\mathbf{\Delta} = \{\Delta_n : n \in \mathbb{N}\}$ be the fractal structure in $\mathbb{R}$ given by $\Delta_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}[: k \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$, and let $\mathbf{\Gamma} = f^{-1}(\mathbf{\Delta})$. Note that $\mathbf{\Gamma}$ is a fractal structure by the previous lemma and it is μ-disjoint since f is measurable, X has finite measure and $A \cap B = \emptyset$ for each $A, B \in \Gamma_n$ with $A \neq B$ and each $n \in \mathbb{N}$.

Now, we prove that f is Riemann-integrable with respect to μ and $\mathbf{\Gamma}$.

Given $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, let $E_i^n = f^{-1}([\frac{i}{2^n}, \frac{i+1}{2^n}[)$ and consider the simple functions (since f is bounded) $l_n(x) = \sum_{i\in\mathbb{Z}} \frac{i}{2^n}\chi_{E_i^n}(x)$ and $u_n(x) = \sum_{i\in\mathbb{Z}} \frac{i+1}{2^n}\chi_{E_i^n}(x)$, where χ_A is the characteristic function of A. Then it is clear that $l_n(x) \leq f(x) \leq u_n(x)$ for each $x \in X$.

Given $n \in \mathbb{N}$, it follows that $\int l_n d\mu = \sum_{i\in\mathbb{Z}} \frac{i}{2^n}\mu(E_i^n) \leq \sum_{i\in\mathbb{Z}} m(f;E_i^n)\mu(E_i^n) = L(f;\Gamma_n;\mu) \leq U(f;\Gamma_n;\mu) = \sum_{i\in\mathbb{Z}} M(f;E_i^n)\mu(E_i^n) \leq \sum_{i\in\mathbb{Z}} \frac{i+1}{2^n}\mu(E_i^n) = \int u_n d\mu$. Since $l_n \leq f \leq u_n$, $U(f;\Gamma_n;\mu) - L(f;\Gamma_n;\mu) \leq \int(u_n - l_n)d\mu \leq \sum_{i\in\mathbb{Z}} \frac{1}{2^n}\mu(E_i^n) = \frac{1}{2^n}\mu(X)$ and X has finite measure, then f is Lebesgue integrable and $\int f d\mu = \lim_{n\to\infty} \int l_n d\mu = \lim_{n\to\infty} \int u_n d\mu$. It follows from Proposition 3 that f is integrable with respect to μ and $\mathbf{\Gamma}$ and $\int_X^{\mu} f = \int_X^{(\mu,\mathbf{\Gamma})} f = \int f d\mu$. $\square$

The previous result states that, for bounded functions and finite measure spaces, the Riemann integral with respect to a measure is the same as the classic Lebesgue integral with respect to that measure. An open question is if this result is still true for non-finite measure spaces.

Another interesting interpretation of the previous result is that the Lebesgue integral with respect to a measure can be calculated by choosing some simple and easy fractal structure, since the calculation of the Riemann integral with respect to that fractal structure and the measure is easier since it only involves the calculation of the Darboux sums and some limits. This is particularly true when it is easy to calculate the measure of the elements of the fractal structure.

An obvious consequence of the previous proposition is that continuous maps are Riemann-integrable with respect to any measure on the Borel σ-algebra.

Corollary 2. *Let (X, τ) be a topological space, μ be a finite measure on the Borel σ-algebra and $f : X \to \mathbb{R}$ be a bounded continuous map. Then $f \in R(X; \mu)$.*

Riemann Integrability vs. Riemann Integrability with Respect to the Lebesgue Measure

Functions that are Riemann-integrable (in the classic sense) in a rectangle in $\mathbb{R}^N$ are also Riemann-integrable with respect to the Lebesgue measure, and both integrals coincide.

Proposition 6. *Let $X = [a_1, b_1] \times \ldots \times [a_N, b_N]$, $f : X \to \mathbb{R}$ be a bounded function, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be the natural fractal structure on $\mathbb{R}^N$ induced on X. Then f is Riemann-integrable (in the classic sense) if and only if it is Riemann-integrable on X with respect to the Lebesgue measure λ and $\mathbf{\Gamma}$. Moreover, if f is Riemann-integrable on X, both integrals coincide.*

Proof. On the one hand, suppose that f is Riemann-integrable (in the classic sense). Let $\varepsilon > 0$. Then, by Theorem 2, there exists $\delta > 0$ such that $|S(f; \mathcal{D}; \xi) - \int_X f| < \varepsilon$ for each partition $\mathcal{D}$ with $||\mathcal{D}|| < \delta$ and for each selection $\xi = (x_D)_{D \in \mathcal{D}}$ for $\mathcal{D}$.

Let $n \in \mathbb{N}$ be such that $\frac{1}{2^n} < \delta$ and ξ be a selection for Γ_n. Then it is clear that $|S(f; \Gamma_n; \xi; \lambda) - \int_X f| = |S(f; \Gamma_n; \xi) - \int_X f| < \varepsilon$, since Γ_n is a partition of X with norm $\frac{1}{2^n}$, which is less than δ. It follows from Theorem 3 that f is Riemann-integrable on X with respect to λ and $\mathbf{\Gamma}$ and $\int_X^{(\lambda, \mathbf{\Gamma})} f = \int_X f$.

On the other hand, suppose that f is Riemann-integrable on X with respect to λ and $\mathbf{\Gamma}$ and let $\varepsilon > 0$. By Proposition 3 there exists $n \in \mathbb{N}$ such that $U(f; \Gamma_n; \lambda) - L(f; \Gamma_n; \lambda) < \varepsilon$. Since Γ_n is a partition, it follows from Theorem 1 that f is Riemann-integrable (in the classic sense).

Finally, by definition, it is clear that $\underline{\int}_X^{(\lambda, \mathbf{\Gamma})} f \leq \underline{\int}_X f \leq \overline{\int}_X f \leq \overline{\int}_X^{(\lambda, \mathbf{\Gamma})} f$. Hence, if f is Riemann-integrable on X, then it is Riemann-integrable with respect to λ and $\mathbf{\Gamma}$ and it follows that $\int_X^{(\lambda, \mathbf{\Gamma})} f = \int_X f$. $\square$

Corollary 3. *Let $X = [a_1, b_1] \times \ldots \times [a_N, b_N]$ and $f : X \to \mathbb{R}$ be a Riemann-integrable function (in the classic sense), then f is λ-Riemann-integrable and both integral coincide, where λ is the Lebesgue measure.*

7. Examples

In the previous section, we have shown (Proposition 6 and Corollary 3) that the classic Riemann integral is a particular case of the theory, since it is the Riemann integral with respect to the natural fractal structure and the Lebesgue measure.

Also, we have shown (Proposition 5) that, for bounded functions on finite measure spaces, the classic Lebesgue integral with respect to the measure is a particular case of the theory, since it coincides with the Riemann integral with respect to a certain fractal structure and the measure. In this case, the fractal structure depends on the function, while in the classic Riemann integral, we can always use the natural fractal structure for any function.

In this section, we give three examples in which an integral is calculated according to the theory that has been developed before.

In Corollary 3 it was shown that each Riemann-integrable function (in the classic sense) is Riemann-integrable with respect to the Lebesgue measure. The first is an example of a function that is not Riemann-integrable (in the classic sense), but it is Riemann-integrable with respect to the Lebesgue measure.

7.1. Example 1

Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0 & x \in [0,1] - \mathbb{Q} \\ 1 & x \in \mathbb{Q} \cap [0,1] \end{cases}$$

First, we prove that f is not Riemann-integrable with respect to a certain fractal structure when considering the Lebesgue measure. Let $\boldsymbol{\Gamma_{nat}}$ be the natural fractal structure on $[0,1]$. Then,

$$\Gamma_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right] : k \in \{0, \ldots, 2^n - 1\} \right\}$$

for each $n \in \mathbb{N}$. Note that given $n \in \mathbb{N}$ and $J \in \Gamma_n$, we have that $M(f;J) = 1$ and $m(f;J) = 0$. Hence, the lower and upper Darboux sums are, respectively,

$$U(f;\Gamma_n) = \sum_{J \in \Gamma_n} M(f;J)\lambda(J) = \sum_{J \in \Gamma_n} \lambda(J) = \frac{1}{2^n} \sum_{k=0}^{2^n - 1} 1 = 1$$

and

$$L(f;\Gamma_n) = \sum_{J \in \Gamma_n} m(f;J)\lambda(J) = 0$$

for each $n \in \mathbb{N}$. It follows that

$$\overline{\int}_{[0,1]}^{\boldsymbol{\Gamma_{nat}}} f = 1 \neq 0 = \underline{\int}_{[0,1]}^{\boldsymbol{\Gamma_{nat}}} f$$

which means that f is not Riemann-integrable on $[0,1]$ with respect to the natural fractal structure and the Lebesgue measure.

However, let $\boldsymbol{\Delta}$ be the fractal structure defined by

$$\Delta_n = \{\{x\} : x \in \mathbb{Q} \cap [0,1]\} \cup \{[0,1] - \mathbb{Q}\}$$

for each $n \in \mathbb{N}$. Since for each $n \in \mathbb{N}$, it holds that $B \cap J = \emptyset$ for each $B, J \in \Delta_n$ such that $B \neq J$, it follows that $\lambda(B \cap J) = \lambda(\emptyset) = 0$ for each $J, B \in \Delta_n$ such that $B \neq J$ and each $n \in \mathbb{N}$. It follows that $\boldsymbol{\Delta}$ is λ-disjoint. Now, let $n \in \mathbb{N}$ and $J \in \Delta_n$. Then

$$M(f;J) = m(f;J) = \begin{cases} 0 & J = [0,1] - \mathbb{Q} \\ 1 & J = \{x\} \text{ where } x \in \mathbb{Q} \cap [0,1]. \end{cases}$$

Since $\lambda(\{x\}) = 0$ for each $x \in \mathbb{Q} \cap [0,1]$, we have that

$$U(f;\Delta_n;\lambda) = \sum_{J \in \Delta_n} M(f;J)\lambda(J) = M(f;[0,1] - \mathbb{Q})\lambda([0,1] - \mathbb{Q}) = 0$$

and

$$L(f;\Delta_n;\lambda) = \sum_{J \in \Delta_n} m(f;J)\lambda(J) = m(f;[0,1] - \mathbb{Q})\lambda([0,1] - \mathbb{Q}) = 0.$$

We conclude that

$$\int_{[0,1]}^{\lambda} f = \int_{[0,1]}^{(\lambda,\boldsymbol{\Delta})} f = \overline{\int}_{[0,1]}^{(\lambda,\boldsymbol{\Delta})} f = \underline{\int}_{[0,1]}^{(\lambda,\boldsymbol{\Delta})} f = 0$$

and, hence, $f \in R(X;\lambda)$.

7.2. Example 2

In [3,4] it is shown how to define a finite measure from the elements of a fractal structure. This case is particularly interesting, since you know the measure of the elements of the fractal structure. By the results in [3,4] you can prove that a pre-measure defined in the elements of the fractal structure can be extended to the Borel σ-algebra. But if you are only interested in the calculation of integrals, you do not need to bother about how the extension is done or how to calculate the measure of other sets, since you only need the measure of the elements of the fractal structure in order to calculate integrals. This is similar to the case of the classic Riemann integral where you only need to know the measure of an interval in order to calculate integrals. Next, we present a simple example.

The next example shows that there exist Riemann-integrable functions with respect to a certain measure on fractal sets. Indeed, in the following, we work on the Cantor set in order to calculate integrals.

Let $f_0, f_2 : [0,1] \to \mathbb{R}$ be the functions given by $f_0(x) = \frac{x}{3}$ and $f_2(x) = \frac{x}{3} + \frac{2}{3}$. Recall that the Cantor set, C, is defined as the unique compact subset on $[0,1]$ such that $C = f_0(C) \cup f_2(C)$. Now let $g : [0,1] \to [0,1]$ be a function defined by the following rule: given $x \in [0,1]$, we write it in base 3. Next, we truncate it by the first 1 (if it is not the case, we consider the whole expression of x in base 3). In the resulting expression, we exchange twos by ones. Then we have a number in base 2 whose decimal value is $g(x)$ for some $x \in [0,1]$. This function is known as devil's staircase (see, for example, [12]), and its graph can be seen in Figure 1. We are interested in the integration of the restriction of this function to the Cantor set.

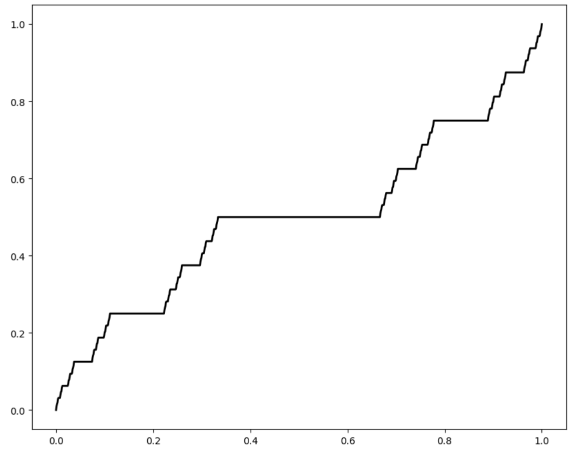

Figure 1. Devil's staircase.

Now, let $\mathbf{\Gamma}$ be the natural fractal structure as a self similar set (see [13]), defined by the following levels:

$$\Gamma_1 = \{f_0(C), f_2(C)\}$$

$$\Gamma_{n+1} = \{f_i(J) : i = 0,2; J \in \Gamma_n\}.$$

Let $J = f_{a_1} f_{a_2} \dots f_{a_n}(C) \in \Gamma_n$ be such that $a_i \in \{0,2\}$ for each $i = 1, \dots, n$. Then

$$J = [(0.a_1...a_n)_3, (0.a_1...a_n\overline{2})_3] \cap C = \left[\sum_{i=1}^{n} \frac{a_i}{3^i}, \sum_{i=1}^{n} \frac{a_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{2}{3^i}\right] \cap C = \left[\sum_{i=1}^{n} \frac{a_i}{3^i}, \frac{1}{3^n} + \sum_{i=1}^{n} \frac{a_i}{3^i}\right] \cap C.$$

Note that $g(J) = [(0.x_1 \dots x_n)_2, (0.x_1 \dots x_n\overline{1})_2]$, where $a_i = 2x_i$ for each $i = 1, \dots, n$, and hence $M(g|_C; J) = (0.x_1 \dots x_n)_2 + \frac{1}{2^n}$ and $m(g|_C; J) = (0.x_1 \dots x_n)_2$.

Let $J \in \Gamma_n$ for some $n \in \mathbb{N}$. We define the set function μ by

$$\mu(J) = \frac{1}{2^n}.$$

By [4], it is known that μ can be extended to a measure on the Borel σ-algebra. Consequently,

$$U(g|_C;\Gamma_n;\mu) = \sum_{J\in\Gamma_n} M(g|_C;J)\mu(J) = \frac{1}{2^n}\sum_{m=0}^{2^n-1}\left(\frac{m}{2^n}+\frac{1}{2^n}\right) = \frac{(2^n-1)2^n}{2^{2n+1}}+\frac{1}{2^n}$$
$$= \frac{1}{2}-\frac{1}{2^{n+1}}+\frac{1}{2^n} \to \frac{1}{2}$$

and

$$L(g|_C;\Gamma_n;\mu) = \sum_{J\in\Gamma_n} m(g|_C;J)\mu(J) = \frac{1}{2^n}\sum_{m=0}^{2^n-1}\frac{m}{2^n} = \frac{(2^n-1)2^n}{2^{2n+1}}$$
$$= \frac{1}{2}-\frac{1}{2^{n+1}} \to \frac{1}{2}.$$

Hence, $\lim_n U(g|_C;\Gamma_n;\mu) = \lim_n L(g|_C;\Gamma_n;\mu) = \frac{1}{2}$.

We conclude that $g|_C \in R(C;\mu)$ and $\displaystyle\int_C^\mu g|_C = \frac{1}{2}$.

7.3. Example 3

Let $X =]-\infty,0]$, λ be the Lebesgue measure, $f : X \to \mathbb{R}$ be the map defined by $f(x) = e^x$ and $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be the natural fractal structure induced on X. Then $L(f;\Gamma_n;\lambda) = \sum_{i=1}^{\infty}\frac{1}{2^n}e^{-\frac{i}{2^n}}$ and $U(f;\Gamma_n;\lambda) = \sum_{i=0}^{\infty}\frac{1}{2^n}e^{-\frac{i}{2^n}}$. Therefore, $U(f;\Gamma_n,\lambda)-L(f;\Gamma_n;\lambda) = \frac{1}{2^n}$ for each $n\in\mathbb{N}$. It follows from Proposition 3 that f is Riemann-integrable with respect to λ and $\mathbf{\Gamma}$ and $\int_X^{(\lambda,\mathbf{\Gamma})} f = \lim_{n\to\infty}\sum_{i=0}^{\infty}\frac{1}{2^n}e^{-\frac{i}{2^n}} = \lim_{n\to\infty}\frac{1}{2^n}\sum_{i=0}^{\infty}e^{-\frac{i}{2^n}} = \lim_{n\to\infty}\frac{1}{2^n}\frac{1}{1-e^{-\frac{1}{2^n}}} = 1$.

It also follows that f is Riemann-integrable with respect to λ and $\int_X^\lambda f = 1$. Note that the integral coincides with the improper classic Riemann integral.

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