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# Compact Resolutions and Analyticity

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**Abstract:** We consider the large class $\mathfrak{G}$ of locally convex spaces that includes, among others, the classes of $(DF)$-spaces and $(LF)$-spaces. For a space $E$ in class $\mathfrak{G}$ we have characterized that a subspace $Y$ of $(E, \sigma(E, E'))$, endowed with the induced topology, is analytic if and only if $Y$ has a $\sigma(E, E')$-compact resolution and is contained in a $\sigma(E, E')$-separable subset of $E$. This result is applied to reprove a known important result (due to Cascales and Orihuela) about weak metrizability of weakly compact sets in spaces of class $\mathfrak{G}$. The mentioned characterization follows from the following analogous result: The space $C(X)$ of continuous real-valued functions on a completely regular Hausdorff space $X$ endowed with a topology $\xi$ stronger or equal than the pointwise topology $\tau_p$ of $C(X)$ is analytic iff $(C(X), \xi)$ is separable and is covered by a compact resolution.

**Keywords:** compact resolution; analytic space; locally convex space; weak metrizability; $C_p(X)$-spaces

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## 1. Introduction

A family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of sets covering a set $X$ is called a *resolution* of $X$ if $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}^\mathbb{N}$. A locally convex topological vector space $E$ belongs to class $\mathfrak{G}$ if there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ in $(E', \sigma(E', E))$ such that each sequence in any $A_\alpha$ is equicontinuous [1], and the resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is called a $\mathfrak{G}$-representation of $E'$.

The class $\mathfrak{G}$ is stable by taking subspaces, Hausdorff quotients, countable direct sums, and products. It contains "almost all" important classes of locally convex spaces, including $(LF)$-spaces and $(DF)$-spaces, hence it is indeed a very large class. We recall that this class $\mathfrak{G}$ of locally convex space was introduced in [1] motivated by particular results for $(LF)$-spaces and $(DF)$-spaces and common properties of the topological dual of each space of these two classes.

An interesting result from [1] states that a compact set $K$ is Talagrand compact if and only if it is homeomorphic to a subset of a locally convex space in class $\mathfrak{G}$. Therefore, dealing with Talagrand compact sets, one may ask when (weakly) compact sets in a locally convex space in class $\mathfrak{G}$ are (weakly) metrizable. Both questions were answered in [1,2], respectively, see also [3] (and references there). Additionally, in the theory of locally convex spaces working with compact sets of a locally convex space $E$ raise the questions about metrizability and weakly angelicity of compact subsets of $E$. In [1] and references therein, a list of positive results concerning both questions is provided, with $(LF)$-spaces and $(DF)$-spaces included in the list. For the spaces in class $\mathfrak{G}$, both above-mentioned problems have positive answers.

Nevertheless, as was proved in [4], the space $C_p(X)$ of continuous real-valued maps on a completely regular Hausdorff space $X$, endowed with the pointwise topology belongs to class $\mathfrak{G}$ if and only if $C_p(X)$ is metrizable.

All topological spaces are assumed to be completely regular. A topological space $X$ is *web-bounding* [5] (Note 3) if there is a family $\{A_\alpha : \alpha \in \Omega\}$ of subsets of $X$ for some non-empty $\Omega \subset \mathbb{N}^{\mathbb{N}}$ whose union $X_0$ is dense in $X$ and such that if $\alpha = (n_k) \in \Omega$ and $x_k \in C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \Omega, m_j = n_j, j = 1,\ldots,k\}$, then $(x_k)_k$ is functionally bounded. If the same holds for $X = X_0$, we call $X$ *strongly web-bounding*. The family $\{A_\alpha : \alpha \in \Omega\}$ is called, respectively, a web-bounding representation or a strongly web-bounding representation of $X$.

A topological space $X$ is called a *Lindelöf* $\Sigma$-space [6] (or a K-countably determined space [7]) if there is an upper semi-continuous compact-valued map from a non-empty subset $\Omega \subset N^{\mathbb{N}}$ covering $X$. If the same holds for $\Omega = \mathbb{N}^{\mathbb{N}}$, then $X$ is called *K-analytic*. $X$ is *quasi-Suslin* if there exists a set-valued map $T$ from $\mathbb{N}^{\mathbb{N}}$ into $X$ covering $X$ which is quasi-Suslin, i.e., if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$, then $(x_n)_n$ has a cluster point in $T(\alpha)$, see [8].

A topological space $X$ is *analytic* if it is a continuous image of the space $\mathbb{N}^{\mathbb{N}}$. Note that analytic $\Rightarrow$ K-analytic $\Leftrightarrow$ Lindelöf $\wedge$ quasi-Suslin, and K-analytic $\Rightarrow$ Lindelöf $\Sigma$. Every K-analytic space has a compact resolution, see [9], or [10], and every angelic space with a compact resolution is K-analytic, see [10] (Corollary 1.1).

Recall that topological spaces containing dense quasi-Suslin spaces are web-bounding [5]. Hence, every space containing a dense $\sigma$-compact space is web-bounding, in particular, separable spaces are web-bounding. Applying [1] (Theorem 1, Note 4) we have that a metrizable space is web-bounding if and only if it is separable. Additional information concerning K-analytic properties on spaces $C^b(X)$ and properties of weakly compact sets in $C(X)$ are developed in [11,12].

## 2. Main Results

The following theorems are the main results of this paper that provide two natural characterizations of analyticity. Theorem 1 characterizes when a non-empty subset $Y$ of a locally convex space $E$ in class $\mathfrak{G}$ is $\sigma(E, E')$-analytic and Theorem 3 characterizes when non-empty set $Y \subset C_p(X)$ is analytic, being $X$ a web-bounding space. Although spaces $C_p(X)$ of continuous real-valued maps on $X$ endowed with the pointwise topology $\tau_p$ do not belong to class $\mathfrak{G}$ for uncountable spaces $X$ (as we have mentioned above), the argument used in the proof of Theorem 3 applies to show the general Theorem 1.

**Theorem 1.** *A subset $Y$ of a locally convex space $E$ in class $\mathfrak{G}$ is $\sigma(E, E')$-analytic if and only if $Y$ has a $\sigma(E, E')$-compact resolution and is contained in a $\sigma(E, E')$-separable subset.*

Consequently, a locally convex space $E$ in class $\mathfrak{G}$ is weakly analytic if and only if $E$ is separable and admits a $\sigma(E, E')$-compact resolution. Note that the latter condition is equivalent to say that $E$ is weakly K-analytic (since $E$ is angelic by [1] (Theorem 11) and we apply [10] (Corollary 1.1)).

We prove that $C_p(X)$ is analytic if and only if $C_p(X)$ has a compact resolution and is separable, see Corollary 2.

Since every analytic compact set is metrizable [1] (Theorem 15), Theorem 1 yields the following result from [2].

**Corollary 1 (Cascales-Orihuela).** *A $\sigma(E, E')$-compact set $Y$ in a locally convex space $E$ in class $\mathfrak{G}$ is $\sigma(E, E')$-metrizable if and only if $Y$ is contained in a $\sigma(E, E')$-separable subset of $E$.*

Moreover, we provide a short proof of the following another interesting result of this type due to Cascales and Orihuela [1].

**Theorem 2 (Cascales-Orihuela).** *A precompact set $K$ in a locally convex space $E$ in class $\mathfrak{G}$ is metrizable.*

The following result uses some ideas from [1].

**Theorem 3.** *Let $X$ be a web-bounding space. A non-empty set $Y \subset C_p(X)$ is analytic if and only if $Y$ has a compact resolution and is contained in a separable subset of $C_p(X)$.*

## 3. Examples

**Example 1.** *In $\mathbb{R}^{\mathbb{N}}$ endowed with the product topology, let $E$ be the subspace of $\mathbb{R}^{\mathbb{N}}$ formed by the vectors with a finite number of non-null components. Every non-void closed subset $Y$ of $E$ is $\sigma(E, E')$-analytic.*

**Proof.** It is clear that the countable product $\mathbb{R}^{\mathbb{N}}$ belongs to class $\mathfrak{G}$, hence $E$ is also in class $\mathfrak{G}$. Let $y$ be an element of $Y$. For each $\alpha = (\alpha_i : i \in \mathbb{N}) \in \mathbb{N}^{\mathbb{N}}$ let

$$A_\alpha := \{y\} \cup \{(n_i : i \in \mathbb{N}) \in Y,\ n_i = 0 \text{ if } i > \alpha_1\}.$$

The family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution of $Y$. Moreover $Y$ is separable, because the topology of $\mathbb{R}^{\mathbb{N}}$ has a countable base. By Theorem 1, $Y$ is $\sigma(E, E')$-analytic. □

Theorem 2 is Theorem 2 in [1], where the authors provide a picture of possible applications of this Theorem, with detailed proofs concerning that:

- the inductive limits of increasing sequences of metrizable locally convex spaces;
- the generalized inductive limits

  $$E[\mathcal{T}] = \varinjlim(E_n[\mathcal{T}_n], A_n)$$

  of sequences of pairs $\{(E_n[\mathcal{T}_n], A_n) : n = 1, 2, \dots\}$, where every $A_n$ is $\mathcal{T}_n$-metrizable and every $E_n[\mathcal{T}_n]$ is locally convex;
- the locally convex $(DF)$-spaces;
- and the locally convex dual metric spaces;

  are in class $\mathfrak{G}$, hence, its precompact spaces are metrizable:

**Example 2.** *Let $X$ be the set $\mathbb{N}$ of natural number endowed with the discrete topology. A non-empty set $Y \subset C_p(X)$ is analytic if and only if $Y$ has a compact resolution.*

**Proof.** The space $X$ admits a compact resolution that it is a strongly web-bounding representation of $X$, hence the space $X$ is strongly web-bounding. If $Y$ has a compact resolution then the Theorem 3 implies that $Y$ is analytic, because the isomorphism between $\mathbb{R}^{\mathbb{N}}$ and $C_p(X)$ implies that $C_p(X)$ has a countable base. The converse is obvious because analytic $\Rightarrow$ $K$-analytic and every $K$-analytic space admits a compact resolution. □

## 4. Proofs

We need the following result [9].

**Proposition 1** (Talagrand)**.** *Let $(X, \xi)$ be a regular space which admits a stronger topology $\vartheta$ such that $(X, \vartheta)$ is a Lindelöf $\Sigma$-space. Then $d(X, \vartheta) \leq \omega(X, \xi)$, where $d(X)$ and $\omega(X)$ denote the density and the weight of $X$, respectively.*

### *4.1. Proof of Theorem 3*

Let us prove Theorem 3.

It is obvious that if $Y$ is analytic then $Y$ is separable and $K$-analytic, so Theorem 3 holds.

To prove the converse of the statement of this theorem it is enough to show that $Y$ admits a weaker metrizable topology because then, by [1] (Theorem 15), the space $Y$ is analytic.

Firstly we are going to check that to prove this converse we may suppose the additional condition that $X$ is a strongly web-bounding space.

In fact, let $X$ be a web-bounding space and suppose that there is a web-bounding representation $\{A_\alpha : \alpha \in \Omega\}$ of $X$ whose union $X_0$ is dense in $X$. Then the restriction map $\phi : C_p(X) \to C_p(X_0)$ defined by $\phi(f) := f|_{X_0}$ is an injective continuous linear map. Let $Y \subset C_p(X)$ be a subset with a compact resolution contained in a separable subset $L \subset C_p(X)$. Then for $\phi(Y)$ the assumptions are satisfied, so $\phi(Y)$ is analytic in the induced topology from $C_p(X_0)$ and consequently $\phi(Y)$ admits a weaker metrizable topology $\mathcal{T}$. Then $\{\phi^{-1}(A) : A \in \mathcal{T}\}$ is a weaker metrizable topology on $Y$. Therefore we may assume that $X$ is strongly web-bounding.

Hence, to finish the proof of Theorem 3 it is enough to prove the following Proposition.

**Proposition 2.** *Let $X$ be a strongly web-bounding space and let $Y$ be a non-empty subset of $C_p(X)$ such that $Y$ has a compact resolution and is contained in a separable subset of $C_p(X)$. Then $Y$ admits a weaker metrizable topology (hence, as was said before, $Y$ is analytic).*

**Proof.** Let $\upsilon X$ be the real-compactification of $X$. Since $X$ is strongly web-bounding, we apply [3] (Theorem 9.15) to deduce that $\upsilon X$ is Lindelöf $\Sigma$-space.

As a help to the reader we split the proof in two parts.

Step 1. Assume that $Y$ is a subset of $C_p(\upsilon X)$, $Y$ has a compact resolution and it is contained in a separable subset $L \subset C_p(\upsilon X)$. Now we prove that $L$ (and also $Y$) admits a weaker metrizable topology. Let $D$ be a countable dense subset of $L$. Let $\mathcal{T}_D$ and $\mathcal{T}_L$ be the weakest topologies on $\upsilon X$ that make continuous the functions of $D$ and $L$, respectively. By density $f(x) = f(y)$ for each $f \in D$ implies $f(x) = f(y)$ for each $f \in L$, hence the topological quotients $(\widehat{\upsilon X}, \widehat{\mathcal{T}_D})$ and $(\widehat{\upsilon X}, \widehat{\mathcal{T}_L})$ of $(\upsilon X, \mathcal{T}_D)$ and $(\upsilon X, \mathcal{T}_L)$ respect to the relations $x \sim y$ if $f(x) = f(y)$ for all $f$ of $D$ and $x \sim y$ if $f(x) = f(y)$ for all $f$ of $L$, respectively, are algebraically identical and we denote by $\varphi : \upsilon X \to \widehat{\upsilon X}$ is the quotient map.

If we define the map $F : (\upsilon X, \mathcal{T}_D) \to \mathbb{R}^D$ by $F(z) = \{f(z) : f \in D\}$, $z \in \upsilon X$, then clearly $F$ is continuous and $x \sim y$ if and only $F(x) = F(y)$. $(\widehat{\upsilon X}, \widehat{\mathcal{T}_D})$ is homeomorphic to a subspace of $\mathbb{R}^D$ and consequently $(\widehat{\upsilon X}, \widehat{\mathcal{T}_D})$ is metrizable and separable. On the other hand $(\widehat{\upsilon X}, \widehat{\mathcal{T}_L})$ is a Lindelöf $\Sigma$-space, since it is a continuous image of the Lindelöf $\Sigma$-space $\upsilon X$. It follows from Proposition 1 that the space $(\widehat{\upsilon X}, \widehat{\mathcal{T}_L})$ is separable.

Let $S = \{x_n : n \in \mathbb{N}\}$ be a countable subset of $\upsilon X$ such that the set $\varphi(S)$ is $\widehat{\mathcal{T}_L}$ dense in $\widehat{\upsilon X}$. For each $f \in L$ let $\widehat{f}$ be the element of $C_p(\widehat{\upsilon X})$ such that $f = \widehat{f}\varphi$. Let $f, g \in L$ be such that $f|_S = g|_S$. Then, from $\widehat{f}\varphi|_S = \widehat{g}\varphi|_S$ if follows that $\widehat{f}|_{\varphi(S)} = \widehat{g}|_{\varphi(S)}$ and the density condition implies that $\widehat{f} = \widehat{g}$. Therefore $f = \widehat{f}\varphi = \widehat{g}\varphi = g$. Consequently, if $f$ and $g$ are two different elements of $L$ there exists $m \in \mathbb{N}$ such that $f(x_m) \neq g(x_m)$. This means that the weaker topology on $L$ defined by the topology of the pointwise convergence on $S$ is metrizable.

Step 2. Let $Y \subset C_p(X)$ be equipped with a compact resolution and let $L$ be a separable set in $C_p(X)$ containing $Y$. Let $\psi : C_p(X) \to C_p(\upsilon X)$ be defined by $\psi(f) = f^{\,\upsilon}$ where $f^{\,\upsilon}$ is the unique continuous extension of $f$ to the whole $\upsilon X$. Since $\psi$ is continuous on each countable set, see [13] (Theorem 4.6(3)), $\psi(Y)$ has a resolution of countably compact sets. On the other hand, the space $C_p(\upsilon X)$ is angelic, see [5] (Theorem 3), so every countably compact set in $C_p(\upsilon X)$ is compact. Hence, $\psi(Y)$ has a compact resolution.

Let $\{f_n : n \in \mathbb{N}\}$ be a dense subset of $L$. Take any $\epsilon > 0$, any $f^{\,\upsilon} \in \psi(L)$ and let $U = \{u_1, u_2, \cdots, u_p\}$ be an arbitrary finite subset of $\upsilon X$. Then there is $f \in L$ with $\psi(f) = f^{\,\upsilon}$ and by [13] (Theorem 4.6(1)) for each $u_i \in U$ there exists $x_i \in X$ such that $f(x_i) = f^{\,\upsilon}(u_i)$ and $f_n(x_i) = {f_n}^{\upsilon}(u_i)$ for each $n \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $|f_m(x_i) - f(x_i)| < \epsilon$ for each $1 \leq i \leq p$. Hence,

$$|{f_m}^{\upsilon}(u_i) - f^{\,\upsilon}(u_i)| = |f_m(x_i) - f(x_i)| < \epsilon$$

for each $1 \leq i \leq p$. This shows that $\{{f_n}^{\upsilon} : n \in \mathbb{N}\}$ is a dense subset of $\psi(L)$, so that $\psi(L)$ is separable. By Step 1 we derive that $\psi(Y)$ is analytic in $C_p(\upsilon X)$. The continuity of the surjection $\psi^{-1} : C_p(\upsilon X) \to C_p(X)$ implies that $\psi^{-1}(\psi(Y)) = Y$ is also analytic. □

For a completely regular topological space $X$, Tkachuk proved in [14] that $C_p(X)$ is K-analytic if and only if it has a compact resolution. If $X$ is a separable metric space, then

$C_p(X)$ is analytic if and only if it admits a resolution consisting of bounded sets, see [15] (Corollary 2.5) and [16] (Proposition 1).

From the proof of Proposition 2 follows immediately the following claim that enables to get in Corollary 2 the following variant for analyticity of $C_p(X)$ for arbitrary $X$.

**Claim 1.** *Let $X$ be a topological space such that its real compactification $\upsilon X$ is Lindelöf $\Sigma$-space and let $Y$ be a non-empty subset of $C_p(X)$ such that $Y$ has a compact resolution and is contained in a separable subset of $C_p(X)$. Then, $Y$ admits a weaker metrizable topology (hence, as was said before, $Y$ is analytic).*

**Corollary 2.** *Let $\xi$ be a topology on $C(X)$ which is stronger or equal than the pointwise topology $\tau_p$ of $C(X)$. Then $(C(X), \xi)$ is analytic if and only if $(C(X), \xi)$ is separable and has a $\xi$-compact resolution.*

**Proof.** It is enough to prove this Corollary when $\xi = \tau_p$, because a submetrizable topological space is analytic if and only if it admits a compact resolution (see [1] (Theorem 15)). Assume that $C_p(X)$ is separable and has a compact resolution. Then by [17] (Corollary 23) the space $\upsilon X$ is a Lindelöf $\Sigma$-space. Now, Claim 1 for $Y = C_p(X)$ implies that $C_p(X)$ is analytic. The converse is clear. □

Hence, a separable space $C_p(X)$ admits a compact resolution if and only if it is analytic, or, equivalently, there is an upper semi-continuous compact-valued map from $\mathbb{N}^{\mathbb{N}}$ covering $C_p(X)$ if and only if $C_p(X)$ is a continuous image of $\mathbb{N}^{\mathbb{N}}$.

The following example shows that Corollary 2 does not work in general for the weak*-dual $L_p(X)$ of $C_p(X)$.

**Example 3.** *Corollary 2 fails for the weak*-dual $L_p([0,1]^{\mathbb{R}})$ of $C_p([0,1]^{\mathbb{R}})$.*

**Proof.** It is well known that the space $[0,1]^{\mathbb{R}}$ endowed with the product topology is K-analytic separable but not analytic. Consequently $L_p([0,1]^{\mathbb{R}})$ is K-analytic and separable by [6] (Proposition 0.5.14). $L_p([0,1]^{\mathbb{R}})$ is not analytic, since $[0,1]^{\mathbb{R}}$ is a closed subspace of $L_p([0,1]^{\mathbb{R}})$ and each closed subspace of an analytic space is analytic. □

### *4.2. Proofs of Theorems 1 and 2*

We are ready to prove Theorem 1.

**Proof.** Note that $(E', \sigma(E', E))$ is strongly web-bounding. Indeed, let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of $E'$. Then if $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and $x_k \in C_{n_1,n_2,\dots,n_k}$, $k \in \mathbb{N}$, there exists for each $k \in \mathbb{N}$ a $\beta_k \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \in A_{\beta_k}$ and $(\beta_{k1}, \beta_{k2}, \cdots, \beta_{kk}) = (n_1, n_2, \cdots, n_k)$. From these equalities for $k \in \mathbb{N}$ it follows that there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ with $\beta_k \leq \gamma$. Hence, $x_k \in A_\gamma$ for all $k \in \mathbb{N}$, yielding equicontinuity of $(x_k)_k$, so $(x_k)_k$ is functionally bounded. Finally, as $(E, \sigma(E, E'))$ is contained in $C_p(E', \sigma(E', E))$ the proof follows applying Theorem 3. □

We complete the paper with a short and elementary proof of Theorem 2. It is enough to make the proof for a compact subset $K$ of $E$, because the completion of a locally convex space $E$ in class $\mathfrak{G}$ belongs to class $\mathfrak{G}$ and the closure in the completion of a precompact subset of $E$ is a compact subset.

**Proof.** Let $\left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ be a $\mathfrak{G}$-representation of $E'$. By $\tau$ we denote the topology of $E$ and let $K$ be a compact of $E$. We say that a subset $M$ of $E'$ is $K^0$-separated if $\left(a + K^0\right) \cap M = \{a\}$, for each $a \in M$. By Zorn's lemma there exists a maximal $K^0$-separated subset $M_1$ of $E'$ and the maximal condition implies that $M_1 + K^0 = E'$.

Note that $M_1$ is countable. Indeed, otherwise, since $E' = \left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ and $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$, for $\alpha$, $\beta$ in $\mathbb{N}^{\mathbb{N}}$, we determine a sequence $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ such that each $C_{n_1,n_2,\dots,n_k}$, $k \in \mathbb{N}$, contains and uncountable subset of $M_1$ and then by a very easy standard argument we obtain countable infinite subset $P$ of $M_1$ and $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $P \subset A_\gamma$, see [3,10,18].

Since $E$ belongs to $\mathfrak{G}$, $P$ is equicontinuous, so, by Grothendieck theorem of polar topologies ([19] (Chapter IV, $21.7)) $P$ is precompact in the topology of uniform convergence on the $\tau$-precompact subsets of $E$. Therefore there exists a finite set $\{a_i : 1 \leq i \leq k\} \subset P$ such that $P \subset \bigcup\{a_i + K^0 : 1 \leq i \leq k\}$. Clearly there exists $1 \leq j \leq k$ such that the set $(a_j + K^0) \cap P$ is infinite, contradicting the hypothesis that $M_1$ ($\supset P$) is $K^0$-separated.

Let $M_n$ be a maximal subset of $E'$ that it is $n^{-1}K^0$-separated, for each $n \in \mathbb{N}$ . The set $M_0 := \bigcup\{M_n : n \in \mathbb{N}\}$ is countable. Let $\tau_{M_0}$ be the weakest topology on $K$ that makes continuous the functions of $M_0$. If $x \neq y$ are two points of $K$ then there exist $g \in E'$ and $n \in \mathbb{N}$ such that $|g(x) - g(y)| > 3n^{-1}$. Since $E' = M_n + n^{-1}K^0$, there exists $f \in M_n (\subset M_0)$ such that $g \in f + n^{-1}K^0$. Hence,

$$|f(x) - f(y)| = |g(x) - g(y) - g(x) + f(x) + g(y) - f(y)| > 3n^{-1} - 2n^{-1} = n^{-1}.$$

Therefore $(K, \tau_{M_0})$ is metrizable, so $K$ is metrizable. □

## 5. Conclusions

For a locally convex space $E$ in class $\mathfrak{G}$, we have characterized that a subset $Y$ of $(E, \sigma(E, E'))$, endowed with the induced topology, is $\sigma(E, E')$-analytic if and only if $Y$ has a $\sigma(E, E')$-compact resolution and is contained in a $\sigma(E, E')$-separable subset of $E$. If $X$ is a web-bounding space, then we have obtained that a non-empty subset $Y$ of $C_p(X)$ provided with the induced topology is analytic if and only if $Y$ has a compact resolution and is contained in a separable subset of $C_p(X)$. Moreover, for a topology $\xi$ on $C(X)$ which is stronger or equal to the pointwise topology $\tau_p$ of $C(X)$ we obtain that $(C(X), \xi)$ is analytic if and only if $(C(X), \xi)$ is separable and has a $\xi$-compact resolution. This last result suggests for future work to characterize the locally convex spaces $E$ in class $\mathfrak{G}$ that are analytic, being $\xi$ a topology stronger than the weak topology $\sigma(E, E')$.

Another direction of future research is to obtain similar characterizations for spaces in class $\mathfrak{G}$ and for spaces $C_p(X)$ replacing analytic by weaker properties like to be $K$-analytic or quasi-Suslin.

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